24.4. EXTENDING AFFINE MAPS TO LINEAR MAPS 757  
24.4。将仿射映射扩展到线性映射757

From a practical point of view, Proposition 24.6 shows us how to homogenize an affine map to turn it into a linear map between the two homogenized spaces. Assume that E and F are of finite dimension, that (a0,(u1,...,un)) is an affine frame of E with origin a0, and (b0,(v1,...,vm)) is an affine frame of F with origin b0. Then, with respect to the two bases (u1,...,un,a0) in Eb and (v1,...,vm,b0) in Fb, a linear map h: Eb → Fb is given by an  
从实践的角度来看，命题24.6向我们展示了如何使仿射映射同质化，使其成为两个同质化空间之间的线性映射。假设e和f是有限维的，（a0，（u1，…，un））是e的仿射框架，原点为a0，（b0，（v1，…，vm））是f的仿射框架，原点为b0。然后，对于eb中的两个碱基（u1，…，un，a0）和fb中的（v1，…，vm，b0），线性映射h:eb→fb由

+ 1) matrix A. Assume that this linear map h is equal to the homogenized  
+1）矩阵A。假设该线性映射H等于均匀化

version f of an affine map f. Since  
仿射映射f的版本f。自

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and since over the basis (u1,...,un,a0) in Eb, points are represented by vectors whose last coordinate is 1 and vectors are represented by vectors whose last coordinate is 0, the following properties hold.  
由于在eb的基（u1，…，un，a0）上，点由最后一个坐标为1的向量表示，而向量由最后一个坐标为0的向量表示，因此以下属性保持不变。

1. The last row of the matrix A = M(fb) with respect to the given bases is  
   矩阵A=m（fb）相对于给定基的最后一行是

(0,0,...,0,1)  
（0,0，…，0,1）

with n occurrences of 0.  
n次出现0。

1. The last column of A contains the coordinates  
   A的最后一列包含坐标

(µ1,...,µm,1)  
（μ1，…，μm，1）

of f(a0) with respect to the basis (v1,...,vm,b0).  
f（a0）的基础（v1，…，vm，b0）。

1. The submatrix of A obtained by deleting the last row and the last column is the matrix of the linear map →−f with respect to the bases (u1,...,un) and (v1,...,vm),  
   通过删除最后一行和最后一列得到的a的子矩阵是线性映射的矩阵→−f关于基（u1，…，un）和（v1，…，vm）。

Finally, since  
最后，因为

,  
，

given any x ∈ E and y ∈ F with coordinates (x1,...,xn,1) and (y1,...,ym, 1), for X =  
对于x，给定坐标（x1，…，xn，1）和（y1，…，ym，1）的x∈e和y∈f。=

(x1,...,xn,1)> and Y = (y1,...,ym,1)>, we have y = f(x) iff  
（x1，…，xn，1）>和y=（y1，…，ym，1）>，我们有y=f（x）iff

Y = AX.  
Y=最大值。

For example, consider the following affine map f : A2 → A2 defined as follows:  
例如，考虑如下定义的仿射映射f:a2→a2：

|  |  |  |
| --- | --- | --- |
| y1 Y1 | = = | ax1 + bx2 + µ1, ax1+bx2+祄1， |
| y2 Y2 | = = | cx1 + dx2 + µ2. cx1+dx2+μ2。 |

The matrix of fbis  
联邦调查局的矩阵

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CHAPTER 24. EMBEDDING AN AFFINE SPACE IN A VECTOR SPACE  
第24章。在向量空间中嵌入仿射空间

and we have  
我们有

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In Eb, we have  
在电子商务中，我们有

,  
，

which means that the homogeneous map fbis is obtained from f by “adding the variable of  
也就是说，通过“加上

|  |  |  |
| --- | --- | --- |
| homogeneity x3:” 同质性x3：“ |  |  |
| y1 Y1 | = = | ax1 + bx2 + µ1x3, ax1+bx2+礹1x3， |
| y2 Y2 | = = | cx1 + dx2 + µ2x3, cx1+dx2+礹2x3， |
| y3 Y3 | = = | x3. X3。 |

Chapter 25  
第二十五章

# Basics of Projective Geometry 射影几何基础

Think geometrically, prove algebraically.  
几何地思考，代数地证明。

—John Tate  
-约翰·泰特

25.1 Why Projective Spaces?  
25.1为什么是投影空间？

For a novice, projective geometry usually appears to be a bit odd, and it is not obvious to motivate why its introduction is inevitable and in fact fruitful. One of the main motivations arises from algebraic geometry.  
对于一个初学者来说，射影几何通常显得有点奇怪，并且不明显地激发了为什么它的引入是不可避免的，实际上是富有成效的。其中一个主要的动机来自代数几何。

The main goal of algebraic geometry is to study the properties of geometric objects, such as curves and surfaces, defined implicitly in terms of algebraic equations. For instance, the equation  
代数几何的主要目标是研究几何对象的性质，如曲线和曲面，这些对象是通过代数方程隐式定义的。例如，方程式

x2 + y2 − 1 = 0  
x2+y2−1=0

defines a circle in R2. More generally, we can consider the curves defined by general equations  
在r2中定义一个圆。一般来说，我们可以考虑由一般方程定义的曲线

ax2 + by2 + cxy + dx + ey + f = 0  
ax2+by2+cxy+dx+ey+f=0

of degree 2, known as conics. It is then natural to ask whether it is possible to classify these curves according to their generic geometric shape. This is indeed possible. Except for so-called singular cases, we get ellipses, parabolas, and hyperbolas. The same question can be asked for surfaces defined by quadratic equations, known as quadrics, and again, a classification is possible. However, these classifications are a bit artificial. For example, an ellipse and a hyperbola differ by the fact that a hyperbola has points at infinity, and yet, their geometric properties are identical, provided that points at infinity are handled properly.  
二度的，称为二次曲线的。然后自然会问，是否可以根据这些曲线的一般几何形状对它们进行分类。这确实是可能的。除了所谓的奇异情况，我们得到椭圆、抛物线和双曲线。同样的问题也可以问到由二次方程定义的曲面，即所谓的四次曲面，同样，分类也是可能的。然而，这些分类有点人为。例如，椭圆和双曲线的区别在于双曲线在无穷远处有点，但如果正确处理无穷远处的点，则它们的几何性质是相同的。

Another important problem is the study of intersection of geometric objects (defined algebraically). For example, given two curves C1 and C2 of degree m and n, respectively, what is the number of intersection points of C1 and C2? (by degree of the curve we mean the total degree of the defining polynomial).  
另一个重要问题是研究几何对象的交集（代数定义）。例如，给定m阶和n阶的两条曲线c1和c2，c1和c2的交点数是多少？（曲线的度数是指定义多项式的总度数）。

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七百五十九

Well, it depends! Even in the case of lines (when m = n = 1), there are three possibilities: either the lines coincide, or they are parallel, or there is a single intersection point. In general, we expect mn intersection points, but some of these points may be missing because they are at infinity, because they coincide, or because they are imaginary.  
好吧，看情况而定！即使在直线的情况下（当m=n=1时），也有三种可能：要么直线重合，要么它们平行，要么只有一个交叉点。一般来说，我们期望有mn交点，但其中一些可能会丢失，因为它们是无穷大的，因为它们是重合的，或者因为它们是虚构的。

What begins to transpire is that “points at infinity” cause trouble. They cause exceptions that invalidate geometric theorems (for example, consider the more general versions of the theorems of Pappus and Desargues), and make it difficult to classify geometric objects. Projective geometry is designed to deal with “points at infinity” and regular points in a uniform way, without making a distinction. Points at infinity are now just ordinary points, and many things become simpler. For example, the classification of conics and quadrics becomes simpler, and intersection theory becomes cleaner (although, to be honest, we need to consider complex projective spaces).  
开始出现的是“指向无限”会引起麻烦。它们会导致使几何定理失效的异常（例如，考虑pappus和desargues定理的更通用版本），并使几何对象的分类变得困难。射影几何是设计用来处理“无限点”和规则点的统一方式，而不作区分。无穷远处的点现在只是普通的点，许多事情变得简单了。例如，二次曲线和四次曲线的分类变得更简单，交集理论变得更清晰（老实说，我们需要考虑复杂的射影空间）。

Technically, projective geometry can be defined axiomatically, or by building upon linear algebra. Historically, the axiomatic approach came first (see Veblen and Young [177, 178], Emil Artin [6], and Coxeter [45, 46, 43, 44]). Although very beautiful and elegant, we believe that it is a harder approach than the linear algebraic approach. In the linear algebraic approach, all notions are considered up to a scalar. For example, a projective point is really a line through the origin. In terms of coordinates, this corresponds to “homogenizing.” For example, the homogeneous equation of a conic is  
从技术上讲，射影几何可以用公理定义，也可以建立在线性代数的基础上。历史上，公理化的方法首先出现（见Veblen and Young[177，178]、Emil Artin[6]和Coxeter[45，46，43，44]）。虽然非常漂亮和优雅，我们相信这是一个比线性代数方法更困难的方法。在线性代数方法中，所有的概念都被认为是一个标量。例如，投影点实际上是一条穿过原点的线。在坐标方面，这相当于“均匀化”，例如，二次曲线的齐次方程是

ax2 + by2 + cxy + dxz + eyz + fz2 = 0.  
ax2+by2+cxy+dxz+eyz+fz2=0。

Now, regular points are points of coordinates (x,y,z) with z = 06 , and points at infinity are points of coordinates (x,y,0) (with x, y, z not all null, and up to a scalar). There is a useful model (interpretation) of plane projective geometry in terms of the central projection in R3 from the origin onto the plane z = 1. Another useful model is the spherical (or the half-spherical) model. In the spherical model, a projective point corresponds to a pair of antipodal points on the sphere.  
现在，规则点是z=06的坐标点（x，y，z），无穷远处的点是坐标点（x，y，0）（x，y，z不都是空的，并且达到一个标量）。根据r3中从原点到平面z=1的中心投影，有一个有用的平面投影几何模型（解释）。另一个有用的模型是球形（或半球形）模型。在球面模型中，一个投影点对应于球面上的一对反极点。

As affine geometry is the study of properties invariant under affine bijections, projective geometry is the study of properties invariant under bijective projective maps. Roughly speaking, projective maps are linear maps up to a scalar. In analogy with our presentation of affine geometry, we will define projective spaces, projective subspaces, projective frames, and projective maps. The analogy will fade away when we define the projective completion of an affine space, and when we define duality.  
由于仿射几何是研究仿射双射下不变性质的，射影几何是研究双射射影映射下不变性质的。大致来说，射影映射是一个标量的线性映射。与我们对仿射几何的描述类似，我们将定义射影空间、射影子空间、射影框架和射影映射。当我们定义仿射空间的射影完备和定义对偶性时，这个类比就会消失。

One of the virtues of projective geometry is that it yields a very clean presentation of rational curves and rational surfaces. The general idea is that a plane rational curve is the projection of a simpler curve in a larger space, a polynomial curve in R3, onto the plane z = 1, as we now explain.  
射影几何的一个优点是它能产生非常清晰的有理曲线和有理曲面的表示。一般的观点是平面有理曲线是一条更简单曲线在更大的空间中的投影，一条r3中的多项式曲线，在平面z=1上，正如我们现在所解释的。

Polynomial curves are curves defined parametrically in terms of polynomials. More specifically, if E is an affine space of finite dimension n ≥ 2 and (a0,(e1,...,en)) is an affine frame  
多项式曲线是根据多项式参数定义的曲线。更具体地说，如果e是有限维n≥2的仿射空间，（a0，（e1，…，en））是仿射框架

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for E, a polynomial curve of degree m is a map F : A → E such that  
对于e，m次多项式曲线是f:a→e的映射，因此

F(t) = a0 + F1(t)e1 + ··· + Fn(t)en,  
f（t）=a0+f1（t）e1+····+fn（t）en，

for all t ∈ A, where F1(t),...,Fn(t) are polynomials of degree at most m.  
对于所有t∈a，其中f1（t），…，fn（t）最多为m的次数多项式。

Although many curves can be defined, it is somewhat embarassing that a circle cannot be defined in such a way. In fact, many interesting curves cannot be defined this way, for example, ellipses and hyperbolas. A rather simple way to extend the class of curves defined parametrically is to allow rational functions instead of polynomials. A parametric rational curve of degree m is a function F : A → E such that  
虽然可以定义许多曲线，但不能这样定义圆，这有点令人尴尬。事实上，许多有趣的曲线不能这样定义，例如椭圆和双曲线。扩展参数化定义的曲线类的一个相当简单的方法是允许有理函数而不是多项式。m阶的参数有理曲线是f:a→e的函数，因此

,  
，

for all t ∈ A, where F1(t),...,Fn(t),Fn+1(t) are polynomials of degree at most m. For example, a circle in A2 can be defined by the rational map  
对于所有t∈a，其中f1（t）、…、fn（t）、fn+1（t）是最多m的度多项式，例如a2中的圆可以用有理映射来定义。

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.

In terms of coordinates, the above curve is given by  
在坐标方面，上述曲线由

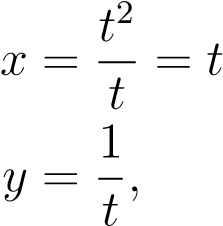
,  
，

and it is easily checked that x2 +y2 = 1. Note that the point (−1,0) is not achieved for any finite value of t, but it is for t = ∞.  
可以很容易地看出，x2+y2=1。注意，对于t的任何有限值（−1,0）都不能达到该点，但对于t=∞。

In the above example, the denominator F3(t) = 1 + t2 never takes the value 0 when t ranges over A, but consider the following curve in A2:  
在上述示例中，当t的范围超过a时，分母f3（t）=1+t2从不取0，但考虑a2中的以下曲线：

.  
.

Observe that G(0) is undefined. In terms of coordinates, the above curve is given by  
注意G（0）未定义。在坐标方面，上述曲线由



so we have y = 1/x. The curve defined above is a hyperbola, and for t close to 0, the point on the curve goes toward infinity in one of the two asymptotic directions.  
所以我们有y=1/x。上面定义的曲线是一条双曲线，对于t接近0，曲线上的点在两个渐近方向中的一个朝无穷大。

A clean way to handle the situation in which the denominator vanishes is to work in a projective space. Intuitively, this means viewing a rational curve in An as some appropriate projection of a polynomial curve in An+1, back onto An.  
处理分母消失的一种干净方法是在投影空间中工作。直观地说，这意味着在+1中的多项式曲线的适当投影中查看有理曲线，然后返回到。

Given an affine space E, for any hyperplane H in E and any point a0 not in H, the central projection (or conic projection, or perspective projection) of center a0 onto H, is the partial map p defined as follows: For every point x not in the hyperplane passing through a0 and parallel to H, we define p(x) as the intersection of the line defined by a0 and x with the hyperplane H; see Figure 25.1.  
给定一个仿射空间e，对于e中的任何超平面h和不在h中的任何点a0，中心a0到h的中心投影（或圆锥投影或透视投影）是部分映射p，定义如下：对于不在通过a0和平行于h的超平面中的每个点x，我们定义p（x）是由a0和x定义的线与超平面h的交点；见图25.1。

a

0

x

p(x)

H

Figure 25.1: A central projection in A3 through a0 onto the yellow hyperplane H. This central projection is not defined for any points in the peach hyperplane.  
图25.1:a3到a0中的中心投影到黄色超平面h上。此中心投影没有为桃超平面中的任何点定义。

For example, we can view G as a rational curve in A3 given by  
例如，我们可以将g看作a3中的有理曲线，由

G1(t) = a0 + t2e1 + e2 + te3.  
g1（t）=a0+t2e1+e2+te3。

If we project this curve G1 (in fact, a parabola in A3) using the central projection (perspective projection) of center a0 onto the plane of equation x3 = 1, we get the previous hyperbola; see Figure 25.2. For t = 0, the point G1(0) = a0 +e2 in A3 is in the plane of equation x3 = 0, and its projection is undefined. We can consider that G1(0) = a0 + e2 in A3 is projected to infinity in the direction of e2 in the plane x3 = 0. In the setting of projective spaces, this direction corresponds rigorously to a point at infinity; see Figure 25.2.  
如果我们使用中心a0的中心投影（透视投影）将曲线g1（实际上是a3中的抛物线）投影到方程x3=1的平面上，我们得到先前的双曲线；见图25.2。对于t=0，a3中的点g1（0）=a0+e2在式x3=0的平面上，其投影未定义。我们可以认为，在平面x3=0中，a3中的g1（0）=a0+e2沿e2的方向投影到无穷大。在射影空间的设置中，这个方向严格对应于无穷远处的一个点；见图25.2。

Let us verify that the central projection used in the previous example has the desired effect. Let us assume that E has dimension n + 1 and that (a0,(e1,...,en+1)) is an affine  
让我们验证前面示例中使用的中心投影是否具有所需的效果。假设e的维数为n+1，（a0，（e1，…，en+1））是仿射

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(1

,1,

1)

(0

,1,

0)

G (t)

1

a

0

Figure 25.2: A central projection in A3 through a0 of the parabola G1(t) onto the hyperplane x3 = 1.  
图25.2：抛物线g1（t）在a3到a0中的中心投影到超平面x3=1上。

frame for E. We want to determine the coordinates of the central projection p(x) of a point x ∈ E onto the hyperplane H of equation xn+1 = 1 (the center of projection being a0). If  
设为E的框架，我们要确定X点的中心投影P（X）∈E在方程Xn+1=1（投影中心为a0）的超平面H上的坐标。如果

x = a0 + x1e1 + ··· + xnen + xn+1en+1,  
x=a0+x1e1+·····+xnen+xn+1en+1，

assuming that xn+1 = 06 ; a point on the line passing through a0 and x has coordinates of the form (λx1,...,λxn+1); and p(x), the central projection of x onto the hyperplane H of equation xn+1 = 1, is the intersection of the line from a0 to x and this hyperplane H. Thus we must have λxn+1 = 1, and the coordinates of p(x) are  
假设xn+1=06；穿过a0和x的直线上的一点具有形式的坐标（λx 1，…，λxn+1）；和p（x），x在方程xn+1=1的超平面h上的中心投影是a0到x的直线与该超平面h的交点。因此，我们必须得到λxn+1=1，and p（x）的坐标为

.  
.

Note that p(x) is undefined when xn+1 = 0. In projective spaces, we can make sense of such points.  
注意，当xn+1=0时，p（x）是未定义的。在射影空间中，我们可以理解这些点。

The above calculation confirms that G(t) is a central projection of G1(t). Similarly, if we define the curve F1 in A3 by  
上述计算证实G（t）是g1（t）的中心投影。同样地，如果我们将a3中的曲线f1定义为

F1(t) = a0 + (1 − t2)e1 + 2te2 + (1 + t2)e3,  
f1（t）=a0+（1−t2）e1+2te2+（1+t2）e3，

the central projection of the polynomial curve F1 (again, a parabola in A3) onto the plane of equation x3 = 1 is the circle F.  
多项式曲线f1（同样是a3中的抛物线）在方程x3=1平面上的中心投影是圆f。

What we just sketched is a general method to deal with rational curves. We can use our “hat construction” to embed an affine space E into a vector space Eb having one more dimension, then construct the projective space P Eb. This turns out to be the “projective completion” of the affine space E. Then we can define a rational curve in P , basically as the central projection of a polynomial curve in Ebback onto P . The same approach can be used to deal with rational surfaces. Due to the lack of space, such a presentation is omitted. However, it can be found on the web; see http://www.cis.upenn.edu/ jean/gbooks/geom2.html.  
我们刚才画的是处理有理曲线的一般方法。我们可以利用我们的“帽构造”将仿射空间e嵌入到一个多维度的向量空间eb中，然后构造投影空间peb。这就是仿射空间e的“投影完成”，然后我们可以定义p中的有理曲线，基本上是ebback中多项式曲线对p的中心投影。同样的方法也可以用来处理有理曲面。由于缺乏空间，这种陈述被省略了。但是，可以在网上找到；请参阅http://www.cis.upenn.edu/jean/gbooks/geom2.html。

e  
e

More generally, the projective completion of an affine space is a very convenient tool to handle “points at infinity” in a clean fashion.  
更一般地说，仿射空间的投影完成是一个非常方便的工具，可以以干净的方式处理“无限点”。

This chapter contains a brief presentation of concepts of projective geometry. The following concepts are presented: projective spaces, projective frames, homogeneous coordinates, projective maps, projective hyperplanes, multiprojective maps, affine patches. The projective completion of an affine space is presented using the “hat construction.” The theorems of Pappus and Desargues are proved, using the method in which points are “sent to infinity.” We also discuss the cross-ratio and duality. The chapter ends with a very brief explanation of the use of the complexification of a projective space in order to define the notion of angle and orthogonality in a projective setting. We also include a short section on applications of projective geometry, notably to computer vision (camera calibration), efficient communication, and error-correcting codes.  
本章简要介绍射影几何的概念。给出了以下概念：射影空间、射影框架、齐次坐标、射影映射、射影超平面、多射影映射、仿射面片。利用“hat构造”给出了仿射空间的射影完备，利用点“送至无穷远”的方法，证明了pappus和desargues的定理，并讨论了交叉比和对偶性。本章最后对射影空间的复杂性的使用作了非常简短的解释，以便在射影环境中定义角度和正交性的概念。我们还简要介绍了射影几何的应用，特别是计算机视觉（相机校准）、有效通信和纠错代码。

## 25.2 Projective Spaces 25.2投影空间

As in the case of affine geometry, our presentation of projective geometry is rather sketchy. For a systematic treatment of projective geometry, we recommend Berger [11, 12], Samuel [138], Pedoe [132], Coxeter [45, 46, 43, 44], Beutelspacher and Rosenbaum [22], Fresnel [66], Sidler [156], Tisseron [170], Lehmann and Bkouche [112], Vienne [179], and the classical treatise by Veblen and Young [177, 178], which, although slightly old-fashioned, is definitely worth reading. Emil Artin’s famous book [6] contains, among other things, an axiomatic presentation of projective geometry, and a wealth of geometric material presented from an algebraic point of view. Other “oldies but goodies” include the beautiful books by Darboux [48] and Klein [101]. For a development of projective geometry addressing the delicate problem of orientation, see Stolfi [162], and for an approach geared towards computer graphics, see Penna and Patterson [133].  
在仿射几何中，我们对射影几何的描述相当粗略。对于射影几何的系统治疗，我们推荐Berger[11，12]、Samuel[138]、Pedoe[132]、Coxeter[45，46，43，44]、Beutelspacher和Rosenbaum[22]、Fresnel[66]、Sidler[156]、Tisseron[170]、Lehmann和Bkouche[112]、Vienne[179]和Veblen和Young的经典论文。[177178]虽然有些过时，但绝对值得一读。埃米尔·阿廷的著名著作[6]包括射影几何的公理化表述，以及从代数角度呈现的大量几何材料。其他的“古老而美好”包括达布克斯[48]和克莱恩[101]的美丽书籍。关于射影几何学的发展，解决了方向的微妙问题，见Stolfi[162]，关于面向计算机图形的方法，见Penna和Patterson[133]。

First, we define projective spaces, allowing the field K to be arbitrary (which does no harm, and is needed to allow finite and complex projective spaces). Roughly speaking, every projective concept is a linear–algebraic concept “up to a scalar.” For spaces, this is made precise as follows.  
首先，我们定义了射影空间，允许场k是任意的（这不会造成伤害，并且需要允许有限和复杂的射影空间）。粗略地说，每一个射影概念都是一个线性代数概念，“达到一个标量”。对于空间来说，精确到如下。

Definition 25.1. Given a vector space E over a field K, the projective space P(E) induced by E is the set (E − {0})/ ∼ of equivalence classes of nonzero vectors in E under the  
定义25.1.给定场k上的向量空间e，e引起的射影空间p（e）是e中非零向量在

25.2. PROJECTIVE SPACES equivalence relation ∼ defined such that for all u,v ∈ E − {0},  
25.2。射影空间等价关系～定义如下：对于所有u，v∈e−0，

u ∼ v iff v = λu, for some λ ∈ K − {0}.  
u～v iff v=λu，对于某些λ∈k−0。

The canonical projection p: (E − {0}) → P(E) is the function associating the equivalence class [u]∼ modulo ∼ to u = 06 . The dimension dim(P(E)) of P(E) is defined as follows: If E is of infinite dimension, then dim(P(E)) = dim(E), and if E has finite dimension, dim(E) = n ≥ 1 then dim(P(E)) = n − 1.  
规范投影p：（e−0）→p（e）是将等价类[u]模～u=06关联起来的函数。p（e）的维数dim（p（e））定义如下：如果e是无限维，那么dim（p（e））=dim（e），如果e是有限维，dim（e）=n≥1，那么dim（p（e））=n−1。

Mathematically, a projective space P(E) is a set of equivalence classes of vectors in E. The spirit of projective geometry is to view an equivalence class p(u) = [u]∼ as an “atomic” object, forgetting the internal structure of the equivalence class. For this reason, it is customary to call an equivalence class a = [u]∼ a point (the entire equivalence class [u]∼ is collapsed into a single object viewed as a point).  
在数学上，射影空间p（e）是e中向量的一组等价类。射影几何的精神是将等价类p（u）=[u]视为“原子”对象，忽略等价类的内部结构。出于这个原因，通常调用等价类A=[U]一个点（整个等价类[U]折叠成一个被视为点的单个对象）。

Remarks:  
评论：

(1) If we view E as an affine space, then for any nonnull vector u ∈ E, since  
（1）如果我们把e看作仿射空间，那么对于任何非零向量u∈e，因为

[u]∼ = {λu | λ ∈ K, λ = 06 },  
[u]λuλ∈k，λ=06，

letting  
出租

Ku = {λu | λ ∈ K}  
ku=λuλ∈k

denote the subspace of dimension 1 spanned by u, the map  
表示用u表示的维度1的子空间，即地图

[u]∼ 7→ Ku  
[u]7→ku

from P(E) to the set of one-dimensional subspaces of E is clearly a bijection, and since subspaces of dimension 1 correspond to lines through the origin in E, we can view P(E) as the set of lines in E passing through the origin. So, the projective space P(E) can be viewed as the set obtained from E when lines through the origin are treated as points.  
从p（e）到e的一维子空间集显然是一个双射，由于维1的子空间对应于e中穿过原点的线，我们可以将p（e）视为e中穿过原点的线集。因此，射影空间p（e）可以看作是通过原点的直线被视为点时从e得到的集合。

However, this is a somewhat deceptive view. Indeed, depending on the structure of the vector space E, a line (through the origin) in E may be a fairly complex object, and treating a line just as a point is really a mental game. For example, E may be the vector space of real homogeneous polynomials P(x,y,z) of degree 2 in three variables x,y,z (plus the null polynomial), and a “line” (through the origin) in E corresponds to an algebraic curve of degree 2. Lots of details need to be filled in, but roughly speaking, the curve defined by P is the “zero locus of P,” i.e., the set of points (x,y,z) ∈ P(R3) (or perhaps in P(C3)) for which P(x,y,z) = 0. We will come back to this point in Section 25.4 after having introduced homogeneous coordinates.  
然而，这是一种有点欺骗性的观点。实际上，根据向量空间e的结构，e中的一条线（通过原点）可能是一个相当复杂的对象，将一条线当作一个点来处理实际上是一种心理游戏。例如，e可以是二次实齐次多项式p（x，y，z）在三个变量x，y，z（加上零多项式）中的向量空间，e中的“线”（通过原点）对应二次代数曲线。需要填写很多细节，但粗略地说，由p定义的曲线是“p的零轨迹”，即点集（x，y，z）∈p（r3）（或可能在p（c3）），其中p（x，y，z）=0。在引入齐次坐标后，我们将在第25.4节中回到这一点。

More generally, E may be a vector space of homogeneous polynomials of degree m in 3 or more variables (plus the null polynomial), and the lines in E correspond to such objects as algebraic curves, algebraic surfaces, and algebraic varieties. The point of view where a complex object such as a curve or a surface is treated as a point in a (projective) space is actually very fruitful and is one of the themes of algebraic geometry (see Fulton [67] or Harris [86]).  
更一般地说，e可以是3个或更多变量（加上零多项式）中m次齐次多项式的向量空间，e中的线对应于代数曲线、代数曲面和代数变种等对象。把复杂物体（如曲线或曲面）视为（投影）空间中的一个点的观点实际上非常有效，是代数几何的主题之一（见Fulton[67]或Harris[86]）。

(2) When dim(E) = 1, we have dim(P(E)) = 0. When E = {0}, we have P(E) = ∅. By convention, we give it the dimension −1.  
（2）当dim（e）=1时，我们得到dim（p（e））=0。当e=0时，我们得到p（e）=∅。按照惯例，我们给它一个维度-1。

We denote the projective space P(Kn+1) by PnK. When K = R, we also denote PnR by RPn, and when K = C, we denote PnC by CPn. The projective space P0K is a (projective) point. The projective space P1K is called a projective line. The projective space P2K is called a projective plane.  
我们用pnk表示射影空间p（kn+1）。当k=r时，我们也用rpn表示pnr，当k=c时，我们用cpn表示pnc。投影空间p0k是一个（投影）点。投影空间p1k称为投影线。射影空间p2k称为射影平面。

The projective space P(E) can be visualized in the following way. For simplicity, assume that E = Rn+1, and thus P(E) = RPn (the same reasoning applies to E = Kn+1, where K is any field).  
射影空间p（e）可以用以下方式可视化。为了简单起见，假设e=rn+1，因此p（e）=rpn（同样的推理也适用于e=kn+1，其中k是任何字段）。

Let H be the affine hyperplane consisting of all points (x1,...,xn+1) such that xn+1 = 1. Every nonzero vector u in E determines a line D passing through the origin, and this line intersects the hyperplane H in a unique point a, unless D is parallel to H. When D is parallel to H, the line corresponding to the equivalence class of u can be thought of as a point at infinity, often denoted by u∞. Thus, the projective space P(E) can be viewed as the set of points in the hyperplane H, together with points at infinity associated with lines in the hyperplane H∞ of equation xn+1 = 0. We will come back to this point of view when we consider the projective completion of an affine space. Figure 25.3 illustrates the above representation of the projective space for E = R2 and E = R3.  
设h为由所有点（x1，…，xn+1）组成的仿射超平面，这样xn+1=1。e中的每一个非零向量u决定了一条穿过原点的线d，并且这条线与唯一点a中的超平面h相交，除非d与h平行。当d与h平行时，与u的等价类相对应的线可以被认为是无穷远的点，通常表示为通过u∞。因此，投影空间p（e）可以看作是超平面h中的一组点，以及方程xn+1=0的超平面h∞中与直线相关的无穷远点。当我们考虑仿射空间的射影完备时，我们将回到这个观点。图25.3说明了e=r2和e=r3的射影空间的上述表示。

y = 1

∞

u

[

u

]

~

v

[

v

]

~

(

i.

)

z = 1

[

u

]

~

v

[

]

~

u

∞

u

(ii.)  
（二）

Figure 25.3: The hyperplane model representations of RP1 and RP2.  
图25.3:RP1和RP2的超平面模型表示。

25.2. PROJECTIVE SPACES  
25.2。射影空间

We refer to the above model of P(E) as the hyperplane model. In this model some hyperplane H∞ (through the origin) in Rn+1 is singled out, and the points of P(E) arising from the hyperplane H∞ are declared to be “points at infinity.” The purpose of the affine hyperplane H parallel to H∞ and distinct from H∞ is to get images for the other points in P(E) (i.e., those that arise from lines not contained in H∞). It should be noted that the choice of which points should be considered as infinite is relative to the choice of H∞. Viewing certain points of P(E) as points at infinity is convenient for getting a mental picture of P(E), but there is nothing intrinsic about that. Points of P(E) are all equal, and unless some additional structure in introduced in P(E) (such as a hyperplane), a point in P(E) doesn’t know whether it is infinite! The notion of point at infinity is really an affine notion. This point will be made precise in Section 25.8.  
我们将上述P（E）模型称为超平面模型。在这个模型中，我们选取了Rn+1中的一些超平面H∞（通过原点），并将由超平面H∞产生的p（e）点声明为“无穷远点”，仿射超平面H与H∞平行，与H∞不同，其目的是为了得到其它点的图像。n p（e）（即由H∞中未包含的线产生的线）。值得注意的是，选择哪些点应被视为无穷大，与选择H∞有关。把p（e）的某些点看作无穷远处的点，可以方便地从精神上了解p（e），但这并不是什么内在的东西。p（e）的点都是相等的，除非p（e）中引入了一些附加的结构（例如超平面），p（e）中的点不知道它是否无穷大！无穷远点的概念实际上是仿射概念。这一点将在第25.8节中精确说明。

Again, for RPn = P(Rn+1), instead of considering the hyperplane H, we can consider the n-sphere Sn of center 0 and radius 1, i.e., the set of points (x1,...,xn+1) such that  
同样，对于rpn=p（rn+1），不考虑超平面h，我们可以考虑圆心0和半径1的n球sn，即点集（x1，…，xn+1），这样

.  
.

In this case, every line D through the center of the sphere intersects the sphere Sn in two antipodal pointsn by identifying antipodal pointsa+ and a−. The projective spacea+ RPandn ais the quotient space obtained from−. It is hard to visualize such an the sphere S object! We call this model of P(E) the spherical model. See Figure 25.4.  
在这种情况下，通过球体中心的每一条线d都会通过识别反极点sa+和a-，与两个反极点sn中的球体sn相交。投影空间a+rpandn是从−得到的商空间。很难想象这样一个球体的物体！我们称这个P（E）模型为球形模型。见图25.4。

x

x

y

y

(

i.

)

x

x

(ii.)  
（二）

Figure 25.4: The spherical model representations of RP1 and RP2.  
图25.4:rp1和rp2的球形模型表示。

A more subtle construction consists in considering the (upper) half-sphere instead of the sphere, where the upper half-sphere is set of points on the sphere Sn such that xn+1 ≥ 0. This time, every line through the center intersects the (upper) half-sphere in a single point, except on the boundary of the half-sphere, where it intersects in two antipodal points a+ and a−. Thus, the projective space RPn is the quotient space obtained from the (upper) half-sphere by identifying antipodal points a+ and a− on the boundary of the half-sphere. We call this model of P(E) the half-spherical model; see Figure 25.5.  
更微妙的结构是考虑（上）半球体而不是球体，其中上半球体是球体sn上的一组点，因此xn+1≥0。这一次，穿过中心的每一条线与（上）半球体在一个点上相交，除了在半球体的边界上，在那里它与两个反极点A+和A-相交。因此，射影空间Rpn是通过识别半球体边界上的对极点A+和A-从（上）半球体获得的商空间。我们称这个P（E）模型为半球面模型；见图25.5。

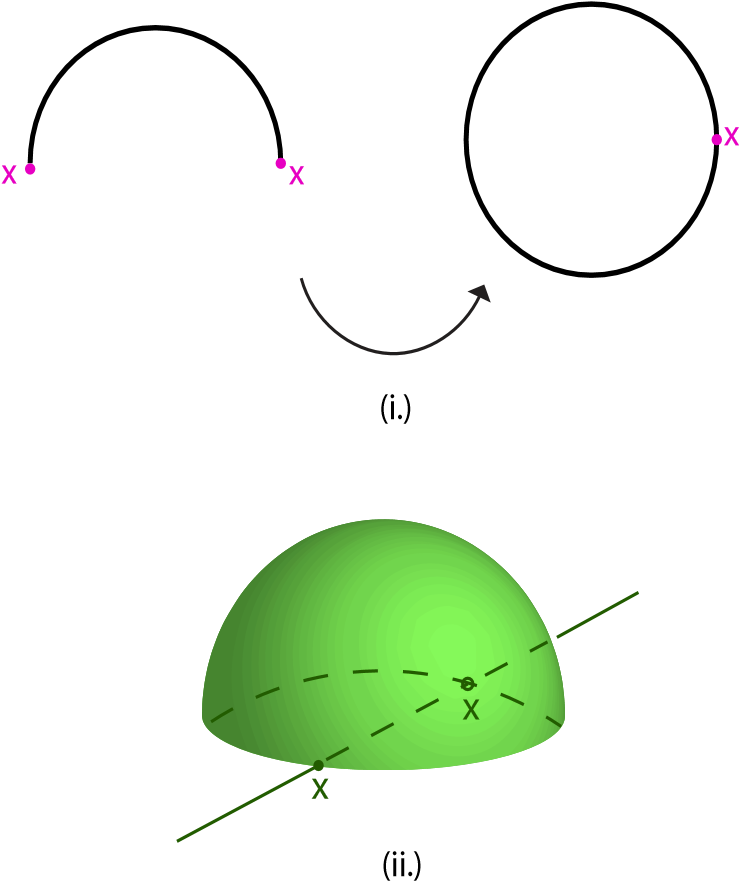


Figure 25.5: The half-spherical model representations of RP1 and RP2.  
图25.5:rp1和rp2的半球面模型表示。

When n = 2, we get a circle. When n = 3, the upper half-sphere is homeomorphic to a closed disk (say, by orthogonal projection onto the xy-plane), and RP2 is in bijection with a closed disk in which antipodal points on its boundary (a unit circle) have been identified. This is hard to visualize! In this model of the real projective space, projective lines are great semicircles on the upper half-sphere, with antipodal points on the boundary identified. Boundary points correspond to points at infinity. By orthogonal projection, these great semicircles correspond to semiellipses, with antipodal points on the boundary identified. Traveling along such a projective “line,” when we reach a boundary point, we “wrap around”! In general, the upper half-sphere is homeomorphic to the closed unit ball in Rn, whose boundary is the (nSn−1. For example, the projective space RP3 is in bijection with the closed unit ball in , with antipodal points on its boundary (the sphere S2) identified!  
当n=2时，我们得到一个圆。当n=3时，上半球体同构于一个封闭圆盘（例如，通过在xy平面上的正交投影），并且rp2与一个封闭圆盘处于双射状态，在该封闭圆盘中，其边界上的反极点（单位圆）已被识别。这很难想象！在实射影空间的模型中，射影线是上半球面上的大半圆，边界上的反极点是确定的。边界点对应于无穷远处的点。通过正交投影，这些大半圆对应于半椭圆，并在边界上识别出对极点。沿着这样一条投射的“线”行进，当我们到达一个边界点时，我们就“环绕”！一般来说，上半球体与RN中的闭合单元球同胚，其边界为（nsn-1）。例如，投影空间rp3是双射的，封闭的单位球在里面，在它的边界（球体s2）上识别出了反极点！

Remarks:  
评论：

1. A projective space P(E) has been defined as a set without any topological structure. When the field K is either the field R of reals or the field C of complex numbers, the vector space E is a topological space. Thus, the projection map p: (E −{0}) → P(E) induces a topology on the projective space P(E), namely the quotient topology. This means that a subset V of P(E) is open iff p−1(V ) is an open set in E. Then, for example, it turns out that the real projective space RPn is homeomorphic to the space  
   射影空间p（e）被定义为一个没有任何拓扑结构的集合。当K域是实数的R域或复数的C域时，向量空间E是拓扑空间。因此，投影图p:（e−0）→p（e）在投影空间p（e）上诱导拓扑，即商拓扑。这意味着p（e）的一个子集v是开的，如果p−1（v）是e中的一个开集，那么，举例来说，实际的射影空间rpn与空间同构。

25.3. PROJECTIVE SUBSPACES  
25.3。射影子空间

obtained by taking the quotient of the (upper) half-sphere , by the equivalence  
通过取（上）半球体的商，通过等价

Another interesting fact is that the complex projective linerelation identifying antipodal points a+ and a− on the boundary of the half-sphere.CP1 = P(C2) is homeomorphic to the (real) 2-sphere S2, and that the real projective space RP3 is homeomorphic to the group of rotations SO(3) of R3.  
另一个有趣的事实是，在半球体的边界上识别反极点A+和A−的复杂射影线性关系。cp1=p（c2）与（实数）2球体s2同胚，而实数射影空间rp3与旋转群so（3）同胚。

1. If H is a hyperplane in E, recall from Proposition 10.4 that there is some nonnull linear form f ∈ E∗ such that H = Kerf. Also, given any nonnull linear form f ∈ E∗, its kernel H = Kerf = f−1(0) is a hyperplane, and if Kerf = Kerg = H, then g = λf for some λ = 06 . These facts can be concisely stated by saying that the map  
   如果h是e中的超平面，从命题10.4中回忆，有一些非零线性形式f∈e，这样h=kerf。另外，对于任何非空线性形式f∈e，其核h=kerf=f−1（0）是一个超平面，如果kerf=kerg=h，那么对于一些λ=06，g=λf。这些事实可以通过说

[f]∼ 7→ Kerf  
[f]7→切口

mapping the equivalence class [f]∼ = {λf | λ = 06 } of a nonnull linear form f ∈ E∗ to the hyperplane H = Kerf in E is a bijection between the projective space P(E∗) and the set of hyperplanes in E. When E is of finite dimension, this bijection yields a useful duality, which will be investigated in Section 25.12.  
将非零线性形式f∈e的等价类[f]λfλ=06映射到e中的超平面h=kerf是投影空间p（e）与e中的超平面集之间的双射。当e是有限维时，该双射产生一个有用的对偶性，将被转化为第25.12节中的TIG。

We now define projective subspaces.  
我们现在定义射影子空间。

## 25.3 Projective Subspaces 25.3投影子空间

Projective subspaces of a projective space P(E) are induced by subspaces of the vector space E.  
投影空间p（e）的投影子空间由向量空间e的子空间诱导而成。

Definition 25.2. Given a nontrivial vector space E, a projective subspace (or linear projective variety) of P(E) is any subset W of P(E) such that there is some subspace V =6 {0} of E with W = p(V − {0}). The dimension dim(W) of W is defined as follows: If V is of infinite dimension, then dim(W) = dim(V ), and if dim(V ) = p ≥ 1, then dim(W) = p − 1. We say that a family (ai)i∈I of points of P(E) is projectively independent if there is a linearly independent family (ui)i∈I in E such that ai = p(ui) for every i ∈ I.  
定义25.2.对于非平凡向量空间e，p（e）的射影子空间（或线性射影变体）是p（e）的任何子空间w，因此存在一些子空间v=6 0 e，w=p（v−0）。W的尺寸dim（w）定义如下：如果v是无限尺寸，那么dim（w）=dim（v），如果dim（v）=p≥1，那么dim（w）=p−1。我们假设p（e）点的族（a i）i∈i是投影独立的，如果在e中有一个线性独立的族（ui）i∈i，那么ai=p（ui）对于每个i∈i。

Remark: If we allow the empty subset to be a projective subspace, then if assign the empty subset to the trivial subspace {0}, we obtain a bijection between the subspaces of E and the projective subspaces of P(E). If P(V ) is the projective space induced by the vector space V , we also denote p(V − {0}) by P(V ), or even by p(V ), even though p(0) is undefined.  
注：如果我们允许空子集是投影子空间，那么如果将空子集赋给平凡子空间0，我们得到e的子空间与p（e）的投影子空间之间的双射。如果p（v）是向量空间v诱导的投影空间，我们也用p（v）表示p（v−0），甚至用p（v），即使p（0）未定义。

A projective subspace of dimension 0 is a called a (projective) point. A projective subspace of dimension 1 is called a (projective) line, and a projective subspace of dimension 2 is called a (projective) plane. If H is a hyperplane in E, then P(H) is called a projective hyperplane. It is easily verified that any arbitrary intersection of projective subspaces is a projective subspace.  
维度0的投影子空间称为（投影）点。维数1的射影子空间称为（射影）线，维数2的射影子空间称为（射影）平面。如果h是e中的超平面，则p（h）称为投影超平面。很容易证明射影子空间的任意交集都是射影子空间。

A single point is projectively independent. Two points a,b are projectively independent if a =6 b. Two distinct points define a (unique) projective line. Three points a,b,c are projectively independent if they are distinct, and neither belongs to the projective line defined by the other two. Three projectively independent points define a (unique) projective plane.  
单点是投影独立的。如果a=6b，两点a，b是投影独立的。两个不同的点定义了一条（唯一的）投影线。三个点A、B、C如果是不同的，则它们是投影独立的，并且都不属于其他两点定义的投影线。三个投影独立的点定义了一个（唯一的）投影平面。

A closer look at projective subspaces will show some of the advantages of projective geometry: In considering intersection properties, there are no exceptions due to parallelism, as in affine spaces.  
仔细观察射影子空间将显示射影几何的一些优点：在考虑交集性质时，不存在由于平行性而产生的例外情况，如仿射空间。

Let E be a nontrivial vector space. Given any nontrivial subset S of E, the subset S defines a subset U = p(S − {0}) of the projective space P(E), and if hSi denotes the subspace of E spanned by S, it is immediately verified that P(hSi) is the intersection of all projective subspaces containing U, and this projective subspace is denoted by hUi. Then n ≥ 2 point a1,...,an ∈ P(E) are projectively independent iff for all i = 1,...,n the point ai does not belong to the projective subspace ha1,...,ai−1,ai+1,...,ani spanned by {a1,...,ai−1,ai+1,...,an}.  
设e为非平凡向量空间。对于e的任何非平凡子集，子集s定义了射影空间p（e）的子集u=p（s−0），如果hsi表示e的子空间，则立即验证p（hsi）是包含u的所有射影子空间的交集，并且该射影子空间是由辉指出。那么n≥2点a1，…，an∈p（e）对所有i=1，…，都是投影独立的iff，n点aI不属于投影子空间ha1，…，ai−1，ai+1，…，ani，其范围为a1，…，ai−1，ai+1，…，an。

Given any subspaces M and N of E, recall from Proposition 23.15 that we have the  
考虑到e的m和n的任何子空间，从23.15号提案中回忆起，我们有

Grassmann relation  
格拉斯曼关系

dim(M) + dim(N) = dim(M + N) + dim(M ∩ N).  
尺寸（m）+尺寸（n）=dim（m+n）+尺寸（m n）。

Then the following proposition is easily shown.  
那么下面的命题就很容易地显示出来了。

Proposition 25.1. Given a projective space P(E), for any two projective subspaces U,V of  
提案25.1.给定射影空间p（e），对于任意两个射影子空间u，v

P(E), we have dim(U) + dim(V ) = dim(hU ∪ V i) + dim(U ∩ V ).  
p（e），我们有dim（u）+dim（v）=dim（hu v i）+dim（u v）。

Furthermore, if dim(U)+dim(V ) ≥ dim(P(E)), then U∩V is nonempty and if dim(P(E)) = n, then:  
此外，如果dim（u）+dim（v）≥dim（p（e）），则u v为非空，如果dim（p（e））=n，则：

1. The intersection of any n hyperplanes is nonempty.  
   任何n个超平面的交集都是非空的。
2. For every hyperplane H and every point a /∈ H, every line D containing a intersects H in a unique point.  
   对于每一个超平面h和每一个点a/∈h，每一条包含一个相交点h的线d。
3. In a projective plane, every two distinct lines intersect in a unique point.  
   在射影平面中，每两条不同的线在一个唯一的点上相交。

As a corollary, in 3D projective space (dim(P(E)) = 3), for every plane H, every line not contained in H intersects H in a unique point.  
作为推论，在三维投影空间（dim（p（e））=3）中，对于每个平面h，h中不包含的每一条线与h在一个唯一点相交。

It is often useful to deal with projective hyperplanes in terms of nonnull linear forms and equations. Recall that the map  
用非零线性形式和方程来处理射影超平面通常是有用的。回想一下地图

[f]∼ 7→ Kerf  
[f]7→切口

25.3. PROJECTIVE SUBSPACES  
25.3。射影子空间

is a bijection between P(E∗) and the set of hyperplanes in E, mapping the equivalence class [f]∼ = {λf | λ = 06 } of a nonnull linear form f ∈ E∗ to the hyperplane H = Kerf. Furthermore, if u ∼ v, which means that u = λv for some λ = 06 , we have  
是p（e）和e中超平面集之间的双射，将非零线性形式f e的等价类[f]λfλ=06映射到超平面h=kerf。此外，如果u～v，也就是说，对于某些λ=06，u=λv，我们有

f(u) = 0 iff f(v) = 0,  
f（u）=0 iff（v）=0，

since f(v) = λf(u) and λ = 06 . Thus, there is a bijection  
因为f（v）=λf（u）和λ=06。因此，有一个双射

{λf | λ = 06 } 7→ P(Kerf)  
λfλ=06\_7→p（切口）

mapping points in P(E∗) to hyperplanes in P(E). Any nonnull linear form f associated with some hyperplane P(H) in the above bijection (i.e., H = Kerf) is called an equation of the projective hyperplane P(H). We also say that f = 0 is the equation of the hyperplane P(H).  
将p（e）中的点映射到p（e）中的超平面。任何与上述双射（即H=kerf）中某些超平面P（H）相关的非零线性形式F称为投影超平面P（H）的方程。我们还说f=0是超平面p（h）的方程。

Before ending this section, we give an example of a projective space where lines have a nontrivial geometric interpretation, namely as “pencils of lines.” If E = R3, recall that the dual space E∗ is the set of all linear maps f : R3 → R. As we have just explained, there is a bijection p(f) 7→ P(Kerf)  
在结束这一节之前，我们给出一个投影空间的例子，其中线条有一个非平凡的几何解释，即“线条的铅笔”。如果e=r3，回想一下双空间e是所有线性映射的集合f:r3→r。正如我们刚才解释的，有一个双射p（f）。7→P（切口）

between P(E∗) and the set of lines in P(E), mapping every point a∗ = p(f) to the line Da∗ = P(Kerf).  
在p（e）和p（e）中的一组线之间，将每个点a=p（f）映射到线da=p（切口）。

Is there a way to give a geometric interpretation in P(E) of a line ∆ in P(E∗)? Well, a line ∆ in P(E∗) is defined by two distinct points a∗ = p(f) and b∗ = p(g), where f,g ∈ E∗ are two linearly independent linear forms. But f and g define two distinct planes H1 = Kerf and H2 = Kerg through the origin (in E = R3), and H1 and H2 define two distinct lines  
有没有办法用p（e）来解释∆in p（e）线的几何意义？那么，p（e）中的线∆由两个不同的点a=p（f）和b=p（g）定义，其中f，g e是两个线性无关的线性形式。但是f和g定义了两个不同的平面h1=kerf和h2=kerg通过原点（e=r3），h1和h2定义了两条不同的线

D1 = p(H1) and D2 = p(H2) in P(E). The line ∆ in P(E∗) is of the form ∆ = p(V ), where  
d1=p（h1），d2=p（h2）in p（e）。p（e）中的线∆的形式为∆=p（v），其中

V = {λf + µg | λ,µ ∈ R}  
V=λf+\_gλ，\_

is the plane in E∗ spanned by f,g. Every nonnull linear form λf + µg ∈ V defines a plane H = Ker(λf + µg) in E, and since H1 and H2 (in E) are distinct, they intersect in a line L that is also contained in every plane H as above. Thus, the set of planes in E associated with nonnull linear forms in V is just the set of all planes containing the line L. Passing to P(E) using the projection p, the line L in E corresponds to the point c = p(L) in P(E), which is just the intersection of the lines D1 and D2. Thus, every point of the line ∆ in P(E∗) corresponds to a line in P(E) passing through c (the intersection of the lines D1 and D2), and this correspondence is bijective.  
是e中被f，g所跨越的平面。每一个非零线性形式λf+μg∈v定义了e中的平面h=ker（λf+μg），并且由于h1和h2（e中）是不同的，它们相交于L线，也包含在上述每个平面h中。因此，与v中的非零线性形式相关联的e中的一组平面只是包含线l的所有平面的集合。通过投影p传递到p（e），e中的线l对应于p（e）中的点c=p（l），即线d1和d2的交点。因此，直线∆in p（e）的每一点对应于通过c（直线d1和d2的交点）的P（e）中的一条直线，并且这种对应是双射的。

In summary, a line ∆ in P(E∗) corresponds to the set of all lines in P(E) through some given point. Such sets of lines are called pencils of lines and are illustrated in Figure 25.6.  
总之，p（e）中的一行∆对应于p（e）中通过某个给定点的所有行的集合。这组线条称为铅笔线条，如图25.6所示。

The above discussion can be generalized to higher dimensions and is discussed quite extensively in Section 25.12. In brief, letting E = Rn+1, there is a bijection mapping points in P(E∗) to hyperplanes in P(E). A line in P(E∗) corresponds to a pencil of hyperplanes in  
上述讨论可概括为更高的维度，并在第25.12节中进行了广泛讨论。简而言之，假设e=rn+1，在p（e）中有一个双射映射点到p（e）中的超平面。p（e）中的一条线对应于

O

D

1

H

1

H

D

2

2

H

-

c

z

=

1

Figure 25.6: A pencil of lines through c in the hyperplane model of RP2  
图25.6:rp2超平面模型中穿过c的一束线

P(E), i.e., the set of all hyperplanes containing some given projective subspace W = p(V ) of dimension n − 2. For n = 3, a pencil of planes in RP3 = P(R4) is the set of all planes (in RP3) containing some given line W. Other examples of unusual projective spaces and pencils will be given in Section 25.4.  
p（e），即包含一些给定投影子空间的所有超平面的集合，w=p（v），尺寸n-2。对于n=3，rp3=p（r4）中的平面铅笔是包含一些给定的线w的所有平面的集合（在rp3中）。第25.4节将给出其他不寻常投影空间和铅笔的示例。

Next, we define the projective analogues of bases (or frames) and linear maps.  
接下来，我们定义基（或帧）和线性映射的投影类似物。

## 25.4 Projective Frames 25.4投影框架

As all good notions in projective geometry, the concept of a projective frame turns out to be uniquely defined up to a scalar.  
正如射影几何中所有好的概念一样，射影框架的概念被唯一地定义为一个标量。

Definition 25.3. Given a nontrivial vector space E of dimension n+1, a family (ai)1≤i≤n+2 of n + 2 points of the projective space P(E) is a projective frame (or basis) of P(E) if there exists some basis (e1,...,en+1) of E such that ai = p(ei) for 1 ≤ i ≤ n + 1, and an+2 = p(e1 + ··· + en+1). Any basis with the above property is said to be associated with the projective frame (ai)1≤i≤n+2.  
定义25.3.给定一个维数n+1的非平凡向量空间e，射影空间p（e）的n+2点的族（a i）1≤i≤n+2是p（e）的射影框架（或基），如果e存在一些基（e1，…，en+1），使得ai=p（ei）对于1≤i≤n+1，an+2=p（e1+·····+en+1）。具有上述性质的任何基被称为与射影框架（ai）1≤i≤n+2相关。

The justification of Definition 25.3 is given by the following proposition.  
定义25.3的理由由以下命题给出。

Proposition 25.2. If (ai)1≤i≤n+2 is a projective frame of P(E), for any two bases (u1,..., un+1), (v1,...,vn+1) of E such that ai = p(ui) = p(vi) for 1 ≤ i ≤ n + 1, and an+2 = p(u1 + ··· + un+1) = p(v1 + ··· + vn+1), there is a nonzero scalar λ ∈ K such that vi = λui, for all i, 1 ≤ i ≤ n + 1.  
提案25.2.如果（a i）1≤i≤n+2是p（e）的投影框架，对于e的任意两个基（u1，…，un+1），（v1，…，vn+1），使得ai=p（ui）=p（vi）对于1≤i≤n+1，and an+2=p（u1+·············+un+1）=p（v1+············+vn+1），存在一个非零的标量λ∈k，使得vi=λui，对于所有i，1≤i≤n

Proof. Since p(ui) = p(vi) for 1 ≤ i ≤ n + 1, there exist some nonzero scalars λi ∈ K such that vi = λiui for all i, 1 ≤ i ≤ n + 1. Since we must have  
证据。由于p（ui）=p（vi）对于1≤i≤n+1，存在一些非零标度λi∈k，因此对于所有i，1≤i≤n+1，vi=λiui。因为我们必须

p(u1 + ··· + un+1) = p(v1 + ··· + vn+1),  
p（u1+·····+un+1）=p（v1+····+vn+1）

there is some λ = 06 such that  
有一些λ=06这样

λ(u1 + ··· + un+1) = v1 + ··· + vn+1 = λ1u1 + ··· + λn+1un+1,  
λ（u1+····+un+1）=v1+····+vn+1=λ1u1+···+λn+1un+1，

and thus we have  
因此我们有

(λ − λ1)u1 + ··· + (λ − λn+1)un+1 = 0,  
（λ−λ1）u1+····+（λ−λn+1）un+1=0，

and since (u1,...,un+1) is a basis, we have λi = λ for all i, 1 ≤ i ≤ n + 1, which implies λ1 = ··· = λn+1 = λ.   
因为（u1，…，un+1）是一个基础，我们得到所有i的λi=λ，1≤i≤n+1，这意味着λ1=····=λn+1=λ。

Proposition 25.2 shows that a projective frame determines a unique basis of E, up to a (nonzero) scalar. This would not necessarily be the case if we did not have a point an+2 such that an+2 = p(u1 + ··· + un+1).  
命题25.2表明，射影框架决定了e的唯一基础，达到（非零）标量。如果我们没有一个点An+2，这样an+2=p（u1+·····+un+1），情况就不一定是这样了。

When n = 0, the projective space consists of a single point a, and there is only one projective frame, the pair (a,a). When n = 1, the projective space is a line, and a projective frame consists of any three pairwise distinct points a,b,c on this line. When n = 2, the projective space is a plane, and a projective frame consists of any four distinct points a,b,c,d such that a,b,c are the vertices of a nondegenerate triangle and d is not on any of the lines determined by the sides of this triangle. These examples of projective frames are illustrated in Figure 25.7. The reader can easily generalize to higher dimensions.  
当n=0时，射影空间由单点A组成，只有一个射影帧，即对（A，A）。当n=1时，射影空间是一条直线，射影框架由这条直线上任意三对不同的点A、B、C组成。当n=2时，射影空间是一个平面，射影框架由四个不同的点a、b、c、d组成，这样a、b、c是非退化三角形的顶点，d不在由三角形边确定的任何直线上。这些投影帧的例子如图25.7所示。读者可以很容易地归纳出更高的维度。

Given a projective frame (ai)1≤i≤n+2 of P(E), let (u1,...,un+1) be a basis of E associated with (ai)1≤i≤n+2. For every a ∈ P(E), there is some u ∈ E − {0} such that  
给定P（e）的投影帧（a i）1≤i≤n+2，设（u1，…，un+1）为与（ai）1≤i≤n+2相关的e的基础。对于每一个a∈p（e），有一些u∈e−0这样

a = [u]∼ = {λu | λ ∈ K − {0}},  
a=[u]λuλ∈k−0，

the equivalence class of u, and the set  
u的等价类和集合

{(x1,...,xn+1) ∈ Kn+1 | v = x1u1 + ··· + xn+1un+1, v ∈ [u]∼ = a}  
（x1，…

of coordinates of all the vectors in the equivalence class [u]∼ is called the set of homogeneous coordinates of a over the basis (u1,...,un+1).  
等价类[u]中所有向量的坐标称为A超基齐次坐标集（u1，…，un+1）。

Note that for each homogeneous coordinate (x1,...,xn+1) we must have xi = 06 for some i, 1 ≤ i ≤ n + 1, and any two homogeneous coordinates (x1,...,xn+1) and (y1,...,yn+1) for a differ by a nonzero scalar, i.e., there is some λ = 06 such that yi = λxi, 1 ≤ i ≤ n + 1. Homogeneous coordinates (x1,...,xn+1) are sometimes denoted by (x1 : ··· : xn+1), for instance in algebraic geometry.  
请注意，对于每个齐次坐标（x1，…，xn+1），对于某些i，1±i＝n＋1，以及任何两个齐次坐标（x1，…，xn+1）和（y1，…，yn+1），都必须有一个非零标量，也就是说，有一些α＝06，所以i＝1，1，i＝n+1。齐次坐标（x1，…，xn+1）有时用（x1：···：xn+1）表示，例如在代数几何中。

By Proposition 25.2, any other basis (v1,...,vn+1) associated with the projective frame  
根据25.2号提案，与投影框架相关的任何其他基础（v1，…，vn+1）

(ai)1≤i≤n+2 differs from (u1,...,un+1) by a nonzero scalar, which implies that the set of homogeneous coordinates of a ∈ P(E) over the basis (v1,...,vn+1) is identical to the set of homogeneous coordinates of a ∈ P(E) over the basis (u1,...,un+1). Consequently, we can associate a unique set of homogeneous coordinates to every point a ∈ P(E) with respect to the projective frame (ai)1≤i≤n+2. With respect to this projective frame, note that an+2 has homogeneous coordinates (1,...,1), and that ai has homogeneous coordinates (0,...,1,...,0), where the 1 is in the ith position, where 1 ≤ i ≤ n + 1. We summarize the above discussion in the following definition.  
（a i）1≤i≤n+2不同于（u1，…，un+1）的非零标量，这意味着在基（v1，…，vn+1）上a∈p（e）的齐次坐标集与在基（u1，…，un+1）上a∈p（e）的齐次坐标集相同。因此，我们可以将一组唯一的齐次坐标与射影框架（a i）1≤i≤n+2的每一点a∈p（e）关联起来。关于这个射影框架，注意+2有齐次坐标（1，…，1），而ai有齐次坐标（0，…，1，…，0），其中1在第i个位置，其中1≤i≤n+1。我们将上述讨论概括为以下定义。

P0  
P0

K a  
K A

P

y = 1

∞

u

u

1

a

1

u

2

a

2

a

3

P

K

1

∞

u

z = 1

a

1

u

1

u

2

a

2

z = 1

u

3

a

3

a

4

u

1

u

2

u

3

K

2

Figure 25.7: The projective frames for projective spaces of dimension 1, 2, and 3.  
图25.7：尺寸1、2和3的投影空间的投影框架。

Definition 25.4. Given a nontrivial vector space E of dimension n + 1, for any projective frame (ai)1≤i≤n+2 of P(E) and for any point a ∈ P(E), the set of homogeneous coordinates of a with respect to (ai)1≤i≤n+2 is the set of (n + 1)-tuples  
定义25.4.给定一个维数为n+1的非平凡向量空间e，对于p（e）的任何投影框（a i）1≤i≤n+2，对于任意点a∈p（e），a相对于（ai）1≤i≤n+2的齐次坐标集是（n+1）-元组集。

{(λx1,...,λxn+1) ∈ Kn+1 | xi = 06 for some i, λ = 06 , a = p(x1u1 + ··· + xn+1un+1)}, where (u1,...,un+1) is any basis of E associated with (ai)1≤i≤n+2.  
{（S1x1，…，Lax xn+ 1）kn+1＝1＝06，对于一些i，s＝06，a= p（x1u1+···+xn+1un+1）}，其中（u1，…，un+1）是与（ai）1 i i＝n+2相关联的E的任何基础。

Given a projective frame (ai)1≤i≤n+2 for P(E), if (x1,...,xn+1) are homogeneous coordinates of a point a ∈ P(E), we write a = (x1,...,xn+1), and with a slight abuse of language, we may even talk about a point (x1,...,xn+1) in P(E) and write (x1,...,xn+1) ∈ P(E).  
对于p（e），如果（x1，…，xn+1）是点A∈p（e）的齐次坐标，我们可以在p（e）中写a=（x1，…，xn+1），并且稍微滥用语言，我们甚至可以在p（e）中谈论点（x1，…，xn+1）并写（x1，…，xn+1）∈p（e）。

The special case of the projective line P1K is worth examining. The projective line P1K consists of all equivalence classes [x,y] of pairs (x,y) ∈ K2 such that (x,y) = (06 ,0), under the equivalence relation ∼ defined such that  
投影线p1k的特殊情况值得研究。投影线p1k由对（x，y）的所有等价类[x，y]组成，因此（x，y）=（06，0），在等价关系下定义如下：

(x1,y1) ∼ (x2,y2) iff x2 = λx1 and y2 = λy1,  
（x1，y1）（x2，y2）iff x2=λx1和y2=λy1，

for some λ ∈ K−{0}. When y = 06 , the equivalence class of (x,y) contains the representative (xy−1,1), and when y = 0, the equivalence class of (x,0) contains the representative (1,0).  
对于某些λ∈k−0。当y=06时，（x，y）的等价类包含代表（xy−1,1），当y=0时，（x，0）的等价类包含代表（1,0）。

Thus, there is a bijection between K and the set of equivalence classes containing some representative of the form (x,1), and we denote the class [x,1] by x. The equivalence class [1,0] is denoted by ∞ and it is called the point at infinity. Thus, the projective line P1K is in bijection with K ∪ {∞}. The three points ∞ = [1,0], 0 = [0,1], and 1 = [1,1], form a projective frame for P1K. The projective frame (∞,0,1) is often called the canonical frame of P1K.  
因此，在k和包含形式（x，1）的一些代表性的等价类集合之间有一个双射，我们用x表示类[x，1]。等价类[1,0]用∞表示，它被称为无穷远处的点。因此，投影线p1k是K∞的双射。三个点∞=[1,0]、0=[0,1]和1=[1,1]构成p1k的投影帧。投影帧（∞，0,1）通常称为p1k的规范帧。

Homogeneous coordinates are also very useful to handle hyperplanes in terms of equations. If (ai)1≤i≤n+2 is a projective frame for P(E) associated with a basis (u1,...,un+1) for E, a nonnull linear form f is determined by n + 1 scalars α1,...,αn+1 (not all null), and a point x ∈ P(E) of homogeneous coordinates (x1,...,xn+1) belongs to the projective hyperplane P(H) of equation f iff  
齐次坐标对于处理方程中的超平面也非常有用。如果（a i）1≤i≤n+2是与e的基（u1，…，un+1）相关联的p（e）的投影框架，则非零线性形式f由n+1标量α1，…，αn+1（并非全部为零）确定，且齐次坐标（x1，…，xn+1）的点x∈p（e）属于方程f的投影超平面p（h）敌我识别

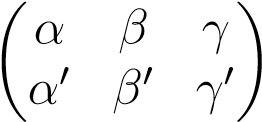
α1x1 + ··· + αn+1xn+1 = 0.  
α1X1+····+αN+1XN+1=0.

In particular, if P(E) is a projective plane, a line is defined by an equation of the form αx + βy + γz = 0. If P(E) is a projective space, a plane is defined by an equation of the form αx + βy + γz + δw = 0.  
特别地，如果p（e）是一个投影平面，直线由αx+βy+γz=0形式的方程定义。如果p（e）是投影空间，平面由αx+βy+γz+δw=0形式的方程定义。

As an application, let us find the coordinates of the intersection point of two distinct lines in a projective plane P(E) (with respect to some projective frame (a1,a2,a3,a4)). If D and D0 are two lines of equations  
作为一个应用，让我们找出投影平面p（e）中两条不同直线的交点坐标（相对于某个投影框架（a1、a2、a3、a4））。如果d和d0是两行方程

αx + βy + γz = 0 and α0x + β0y + γ0z = 0, (∗)  
αx+βy+γz=0和αx+β0y+γ0z=0，（）

then D and D0 are distinct lines iff the matrix  
然后d和d0是矩阵上的不同线



has rank 2. We claim that the intersection Q of the lines D and D0 has homogeneous coordinates  
排名2。我们认为d和d0线的交点q具有齐次坐标。

(βγ0 − β0γ: γα0 − γ0α: αβ0 − α0β); (†)  
（βγ0−β0γ：γα0−γ0α：αβ0−α0β）；（†）

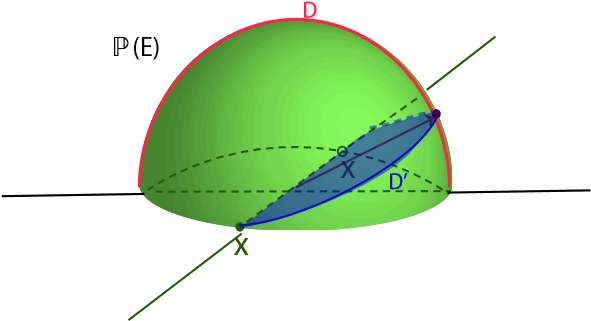
in other words, it is the projective point corresponding to the cross-product  
换句话说，它是与叉积相对应的投影点。

,  
，

as illustrated in Figure 25.8.  
如图25.8所示。

Indeed, the homogeneous coordinates of the intersection Q of D and D0 must satisfy simultaneously the two equations (∗), and since the two determinants  
实际上，d和d0交点q的齐次坐标必须同时满足这两个方程（），因为这两个行列式

and   
和



D

n

D

n

D

n

D’

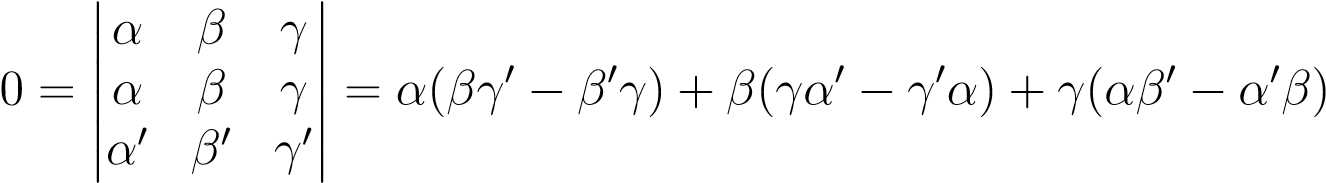
n

D’

x

Figure 25.8: The intersection of two projective lines in the projective plane P(E) is the cross product of the normals for the two corresponding planes in R3.  
图25.8：投影平面p（e）中两条投影线的交点是r3中两个对应平面的法线的交叉积。

are zero because they have two equal rows, and since by expanding these determinants with respect to their first row using the Laplace expansion formula we get  
是零，因为它们有两个相等的行，并且通过使用拉普拉斯展开式对这些行列式的第一行展开，我们得到



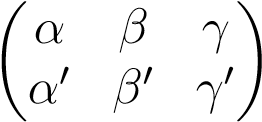
and  
和

,  
，

which confirms that the point  
这证实了这一点

Q = (βγ0 − β0γ: γα0 − γ0α: αβ0 − α0β)  
Q=（βγ0−β0γ：γα0−γ0α：αβ0−α0β）

satisfies both equations in (∗), and thus belongs to both lines D and D0. Since the matrix  
满足（）中的两个方程，因此属于第d行和第d0行。因为矩阵



has rank 2, at least one of the coordinates of Q is nonzero, so Q is indeed a point in the projective plane, and it is the intersection of the lines D and D0.  
有秩2，q的坐标中至少有一个是非零的，所以q确实是射影平面中的一个点，它是d和d0线的交点。

The result that we just proved yields the following criterion for three lines D,D0,D00 in a projective plane to pass through a common point (to be concurrent). In a projective plane, three lines D,D0,D00 of equations  
我们刚刚证明的结果给出了射影平面上三条线d，d0，d00通过一个公共点（要同时）的下列准则。在射影平面上，方程的三行d、d0、d00

αx + βy + γz = 0 α0x + β0y + γ0z = 0 α00x + β00y + γ00z = 0  
αx+βy+γz=0αx+β0y+γ0z=0α00x+β00y+γ00z=0

are concurrent iff  
是同时的iff

.  
.

We can also find the equation of the unique line D = hP,P 0i passing through two distinct points P = (u: v: w) and P 0 = (u0 : v0 : w0) of a projective plane. This line is given by the equation  
我们还可以找到唯一线d=hp，p 0i通过投影平面的两个不同点p=（u:v:w）和p 0=（u0:v0:w0）的方程。这条线由公式给出

, (††)  
，（††）

and since  
从那以后

has rank 2 because P =6 P 0, at least one of the coordinates of the equation (††) is nonzero. Observe that the coefficients of the equation (††) correspond to the cross-product  
排名为2，因为p=6 p 0，方程式（††）的至少一个坐标为非零。请注意，方程式（††）的系数与叉积相对应。

.  
.

The equation of the line D = hP,P 0i must be satisfied by the homogeneous coordinates of the points P and P 0. Equation (††) can be written as  
线d=hp，p 0i的方程必须由点p和p 0的齐次坐标来满足。方程式（††）可写为

,  
，

and a reasoning as in the case of the intersection of lines shows that the equation of the line passing through P and P 0 is given by equation (††).  
在直线交叉的情况下的推理表明，通过P和P 0的直线方程由方程（††）给出。

Then, in a projective plane, three points P = (u: v: w), P 0 = (u0 : v0 : w0) and P 00 = (u00 : v00 : w00) belong to a common line (are collinear) iff  
然后，在投影平面中，三个点p=（u:v:w），p 0=（u0:v0:w0）和p 00=（u00:v00:w00）属于公共线（共线）。

.  
.

More generally, in a projective space P(E) of dimension n ≥ 2, if n points P1,...,Pn are projectively independent and if Pi has homogeneous coordinates ( ) (with respect to some projective frame (a1,...,an+2)), then the equation of the unique hyperplane  
一般来说，在尺寸n≥2的射影空间p（e）中，如果n点p1，…，pn是射影独立的，如果pi有齐次坐标（）（相对于某些射影框架（a1，…，an+2）），那么唯一超平面的方程

H containing P1,...,Pn is given by the equation  
含p1，…，pn的h由公式给出。

.  
.

We also have the following proposition giving another characterization of projective frames.  
我们也有下面的命题，给出了射影框架的另一个特征。

Proposition 25.3. A family (ai)1≤i≤n+2 of n+2 points is a projective frame of P(E) iff for every i, 1 ≤ i ≤ n + 2, the subfamily (aj)j6=i is projectively independent.  
提案25.3.n+2点的族（a i）1≤i≤n+2是p（e）iff的投影框架，对于每个i，1≤i≤n+2，子族（aj）j6=i是投影独立的。

Proof. We leave as an (easy) exercise the fact that if (ai)1≤i≤n+2 is a projective frame, then each subfamily (aj)j=6 i is projectively independent. Conversely, pick some ui ∈ E −{0} such that ai = p(ui), 1 ≤ i ≤ n + 2. Since (aj)j6=n+2 is projectively independent, (u1,...,un+1) is a basis of E. Thus, we must have  
证据。作为一个简单的练习，如果（a i）1≤i≤n+2是一个投影帧，那么每个子族（a j）j=6i都是投影独立的。相反，选择一些ui∈e−0这样ai=p（ui），1≤i≤n+2。因为（aj）j6=n+2是投影独立的，（u1，…，un+1）是e的基础，所以我们必须

un+2 = λ1u1 + ··· + λn+1un+1,  
un+2=λ1u1+····+λn+1un+1，

for some λi ∈ K. However, since for every i, 1 ≤ i ≤ n+1, the family (aj)j=6i is projectively independent, we must have λi = 06 , and thus (λ1u1,...,λn+1un+1) is also a basis of E, and since  
对于某些λi∈k，但是，由于对于每个i，1≤i≤n+1，族（a j）j=6i是投影独立的，我们必须有λi=06，因此（λ1u1，…，λn+1un+1）也是e的基础，并且

un+2 = λ1u1 + ··· + λn+1un+1,  
un+2=λ1u1+····+λn+1un+1，

it induces the projective frame (ai)1≤i≤n+2.   
它诱导投影框（ai）1≤i≤n+2。

Figure 25.9 shows a projective frame (a,b,c,d) in a projective plane. With respect to this projective frame, the points a,b,c,d have homogeneous coordinates (1,0,0), (0,1,0), (0,0,1), and (1,1,1). Let a0 be the intersection of hd,ai and hb,ci, b0 be the intersection of hd,bi and ha,ci, and c0 be the intersection of hd,ci and ha,bi. Then the points a0,b0,c0 have homogeneous coordinates (0,1,1), (1,0,1), and (1,1,0). The diagram formed by the line segments ha,c0i, ha,b0i, hb,b0i, hc,c0i, ha,di, and hb,ci is sometimes called a M¨obius net; see Hilbert and Cohn-Vossen [90] (Chapter III, §15, page 96).  
图25.9显示了射影平面中的射影框架（A、B、C、D）。关于这个射影框架，点A、B、C、D具有齐次坐标（1,0,0）、（0,1,0）、（0,0,1）和（1,1,1）。设a0为hd、ai和hb的交点，ci、b0为hd、bi和ha、ci和c0的交点，hd、ci和ha、bi的交点。然后点a0，b0，c0具有齐次坐标（0,1,1），（1,0,1）和（1,1,0）。由线段ha、c0i、ha、b0i、hb、b0i、hc、c0i、ha、di和hb、ci组成的图有时称为m-obius网；见Hilbert和Cohn Vossen[90]（第三章，第15节，第96页）。

Recall that the equation of a line (a hyperplane in a projective plane) in terms of homogeneous coordinates with respect to the projective frame (a,b,c,d) is given by a homogeneous equation of the form  
回想一下，关于射影框架（a，b，c，d），直线（射影平面中的超平面）的齐次坐标方程由形式的齐次方程给出。

αx + βy + γz = 0,  
αx+βy+γz=0，

where α,β,γ are not all zero. It is easily verified that the equations of the lines ha,bi, ha,ci, hb,ci, are z = 0, y = 0, and x = 0, and the equations of the lines ha,di, hb,di, and hc,di,  
其中α，β，γ不都是零。可以很容易地证明线Ha、Bi、Ha、Ci、Hb、Ci的方程为z=0、y=0和x=0，线Ha、Di、Hb、Di和hc、Di的方程为：

*b*

*c*

*d*

*b*

*c*

*a*

*g*

*a*

z = 0

(-1,1

0)

,

(0

0)

,1,

0)

,0,

(1

‘

,1,

0)

(1

y

=

z

1)

(1

,1,

,

,1,

1)

(0

(0

,0,

1)

y

=

0

x

=

y

(1

,0,

1)

x

=

0

x

=

z

e

1)

(0

,-1,

x

=

y

+

z

z

=

x

+

y

f

(1

0,-1)

,

y

=

x

+

z

x

+

y

+

z

=

0

Figure 25.9: A projective frame (a,b,c,d).  
图25.9：投影框架（A、B、C、D）。

are y = z, x = z, and x = y. The equations of the lines ha0,b0i, ha0,c0i, hb0,c0i are z = x + y, y = x + z, and x = y + z.  
是y=z，x=z，x=y。线的方程式ha0，b0i，ha0，c0i，hb0，c0i是z=x+y，y=x+z，x=y+z。

If we let e be the intersection of hb,ci and hb0,c0i, f be the intersection of ha,ci and ha0,c0i,  
如果e是hb，ci和hb0的交点，c0i，f是ha，ci和ha0，c0i的交点，

− − h  
-小时

handcoordinates (0which correspond to the homogeneous coordinates (0b,cigisbe the intersection ofx = 0,and the equation of the line1,1), (1,0,ha,b1)i, andand h(a−0,b1,0i1, then it easily seen that,0)b0,c. For example, since the equation of the line0i is ,x−=1,y1)+forz, fore. e,f,gx = 0have homogeneous, we get z = −y,  
手坐标（0对应于齐次坐标（0b，cigisbe，x=0的交点，1,1），（1,0，h a，b1）i，and h（a−0，b1,0i 1，那么很容易看出，0）b0，c。例如，由于0的线方程是，x−=1，y1）+forz，fore。e，f，gx=0具有同质性，我们得到z=−y，

The coordinates of the points e,f,g satisfy the equation x+y +z = 0, which shows that they are collinear.  
点E、F、G的坐标满足方程x+y+z=0，表明它们共线。

As pointed out in Coxeter [45] (Proposition 2.41), this is a special case of the projective version of Desargues’s theorem (Proposition 25.7) applied to the triangles (a,b,c) and  
如coxeter[45]中所指出的（命题2.41），这是德沙格定理（命题25.7）的射影版本应用于三角形（a，b，c）和

(pointa0,b0,cd0. The line containing the points). Indeed, by construction, the linese,f,gha,ais called the0i, hb,b0i, andpolar line (or fundamental line)hc,c0i intersect in the common of d with respect to the triangle (a,b,c) (see Pedoe [132]). The diagram also shows the intersection g of ha,bi and ha0,b0i.  
（点a0，b0，cd0.包含点的线）。实际上，通过构造，线se、f、gha、ais称为0i、hb、b0i和极线（或基本线）hc、c0i与三角形（a、b、c）在d的公共部分相交（见pedoe[132]）。图中还显示了ha、bi和ha0、b0i的交叉点g。

The projective space of circles provides a nice illustration of homogeneous coordinates.  
圆的射影空间为均匀坐标提供了一个很好的说明。

Let E be the vector space (over R) consisting of all homogeneous polynomials of degree 2 in x,y,z of the form  
设e为向量空间（在r上），由x，y，z形式中2次的所有齐次多项式组成。

ax2 + ay2 + bxz + cyz + dz2  
轴2+AY2+BXZ+CyZ+DZ2

(plus the null polynomial). The projective space P(E) consists of all equivalence classes  
（加上零多项式）。射影空间p（e）由所有等价类组成。

[P]∼ = {λP | λ = 06 },  
[p]λpλ=06，

where P(x,y,z) is a nonnull homogeneous polynomial in E. We want to give a geometric interpretation of the points of the projective space P(E). In order to do so, pick some projective frame (a1,a2,a3,a4) for the projective plane RP2, and associate to every [P] ∈ P(E) the subset of RP2 known as its its zero locus (or zero set, or variety) V ([P]), and defined such that  
其中p（x，y，z）是e中的一个非零齐次多项式，我们想给出射影空间p（e）点的几何解释。为此，选取射影平面rp2的一些射影帧（a1，a2，a3，a4），并将rp2的子集（称为其零轨迹（或零集或变种）v（[p]）与每个[p]∈p（e）相关联，并定义如下：

V ([P]) = {a ∈ RP2 | P(x,y,z) = 0},  
v（[p]）=a∈rp2 p（x，y，z）=0，

where (x,y,z) are homogeneous coordinates for a.  
其中（x，y，z）是a的齐次坐标。

As explained earlier, we also use the simpler notation  
如前所述，我们还使用更简单的符号

V ([P]) = {(x,y,z) ∈ RP2 | P(x,y,z) = 0}.  
v（[p]）=（x，y，z）∈rp2 p（x，y，z）=0。

Actually, in order for V ([P]) to make sense, we have to check that V ([P]) does not depend on the representative chosen in the equivalence class [P] = {λP | λ = 06 }. This is because  
实际上，为了使v（[p]）有意义，我们必须检查v（[p]）不依赖于在等价类[p]=λpλ=06中选择的代表。这是因为

P(x,y,z) = 0 iff λP(x,y,z) = 0 when λ = 06 .  
当λ=06时，p（x，y，z）=0 iffλp（x，y，z）=0。

For simplicity of notation, we also denote V ([P]) by V (P). We also have to check that if  
为了简化表示法，我们还用v（p）表示v（[p]）。我们还要检查一下如果

(λx,λy,λz) are other homogeneous coordinates for a ∈ RP2, where λ = 06 , then  
（λx，λy，λz）是a∈rp2的其他齐次坐标，其中λ=06，则

P(x,y,z) = 0 iff P(λx,λy,λz) = 0.  
p（x，y，z）=0 iff p（λx，λy，λz）=0。

However, since P(x,y,z) is homogeneous of degree 2, we have  
然而，由于p（x，y，z）是2级的齐次，我们有

P(λx,λy,λz) = λ2P(x,y,z),  
p（λx，λy，λz）=λ2p（x，y，z）

and since λ = 06 ,  
既然λ=06，

P(x,y,z) = 0 iff λ2P(x,y,z) = 0.  
p（x，y，z）=0 iffλ2p（x，y，z）=0。

The above argument applies to any homogeneous polynomial P(x1,...,xn) in n variables of any degree m, since  
上述论点适用于任意m阶n变量中的任何齐次多项式p（x1，…，xn），因为

P(λx1,...,λxn) = λmP(x1,...,xn).  
p（λx1，…，λxn）=λmp（x1，…，xn）。

Thus, we can associate to every [P] ∈ P(E) the curve V (P) in RP2. One might wonder why we are considering only homogeneous polynomials of degree 2, and not arbitrary polynomials of degree 2? The first reason is that the polynomials in x,y,z of degree 2 do not form a vector space. For example, if P = x2 + x and Q = −x2 + y, the polynomial P + Q = x + y is not of degree 2. We could consider the set of polynomials of degree ≤ 2, which is a vector space, but now the problem is that V (P) is not necessarily well defined!.  
因此，我们可以将rp2中的曲线v（p）与每一个[p]∈p（e）联系起来。我们为什么只考虑2阶的齐次多项式，而不考虑2阶的任意多项式？第一个原因是2阶的x，y，z多项式不形成向量空间。例如，如果p=x2+x且q=−x2+y，则多项式p+q=x+y不属于2阶。我们可以考虑次数≤2的多项式集，这是一个向量空间，但现在的问题是v（p）不一定定义得很好！.

For example, if P(x,y,z) = −x2 + 1, we have  
例如，如果p（x，y，z）=-x2+1，我们有

P(1,0,0) = 0 and P(2,0,0) = −3,  
P（1,0,0）=0和P（2,0,0）=3，

and yet (2,0,0) = 2(1,0,0), so that P(x,y,z) takes different values depending on the representative chosen in the equivalence class [1,0,0]. Thus, we are led to restrict ourselves to homogeneous polynomials. Actually, this is usually an advantage more than a disadvantage, because homogeneous polynomials tend to be well behaved.  
然而（2,0,0）=2（1,0,0），因此p（x，y，z）根据在等价类[1,0,0]中选择的代表性取不同的值。因此，我们只能局限于齐次多项式。实际上，这通常是一个优点而不是缺点，因为齐次多项式往往表现得很好。

What are the curves V (P)? One way to “see” such curves is to go back to the hyperplane model of RP2 in terms of the plane H of equation z = 1 in R3. Then the trace of V (P) on H is the circle of equation  
什么是曲线v（p）？“看到”这些曲线的一种方法是根据r3中方程式z=1的平面h回到rp2的超平面模型。那么h上v（p）的迹线就是方程的圆。

ax2 + ay2 + bx + cy + d = 0.  
ax2+ay2+bx+cy+d=0.

Thus, we may think of P(E) as a projective space of circles. However, there are some problems. For example, V (P) may be empty! This happens, for instance, for P(x,y,z) = x2 + y2 + z2, since the equation  
因此，我们可以把p（e）看作是圆的投影空间。但也存在一些问题。例如，v（p）可能是空的！例如，当p（x，y，z）=x2+y2+z2时，就会出现这种情况，因为方程

x2 + y2 + z2 = 0  
x2+y2+z2=0

has only the trivial solution (0,0,0), which does not correspond to any point in RP2. Indeed, only nonnull vectors in R3 yield points in RP2. It is also possible that V (P) is reduced to a single point, for instance when P(x,y,z) = x2 + y2, since the only homogeneous solution of  
只有平凡解（0,0,0），它与rp2中的任何点都不对应。实际上，在r3中只有非空向量在rp2中产生点。也有可能将v（p）简化为一个点，例如当p（x，y，z）=x2+y2时，因为

x2 + y2 = 0  
x2+y2=0

is (0,0,1). Also, note that the map  
是（0,0,1）。另外，注意地图

[P] 7→ V (P)  
[P]7→V（P）

is not injective. For instance, P = x2 + y2 and Q = x2 + 2y2 define the same degenerate circle reduced to the point (0,0,1). We also accept as circles the union of two lines, as in the case  
不是注射剂。例如，p=x2+y2和q=x2+2y2定义了同一退化圆，并将其简化为点（0,0,1）。我们也接受两条线的并线作为圆，就像在这个例子中那样。

(bx + cy + dz)z = 0,  
（bx+cy+dz）z=0，

where a = 0, and even a double line, as in the case  
其中a=0，甚至是双行，例如

z2 = 0,  
z2=0，

where a = b = c = 0.  
其中a=b=c=0。

A clean way to resolve most of these problems is to switch to homogeneous polynomials over the complex field C and to consider curves in CP2. This is what is done in algebraic geometry (see Fulton [67] or Harris [86]). If P(x,y,z) is a homogeneous polynomial over C of degree 2 (plus the null polynomial), it is easy to show that V (P) is always nonempty, and in fact infinite. It can also be shown that V (P) = V (Q) implies that Q = λP for some λ ∈ C, with λ = 0 (6 see Samuel [138], Section 1.6, Theorem 10). Another advantage of switching to the complex field C is that the theory of intersection is cleaner. Thus, any two circles that do not contain a common line always intersect in four points, some of which might be multiple points (as in the case of tangent circles). This may seem surprising, since in the real plane, two circles intersect in at most two points. Where are the other two points? They turn out to be the points (1,i,0) and (1,−i,0), as one can immediately verify. We can think of them as complex points at infinity! Not only are they at infinity, but they are not real. No wonder we cannot see them! We will come back to these points, called the circular points, in Section 25.14.  
解决这些问题的一个干净方法是在复场C上切换到齐次多项式，并考虑CP2中的曲线。这就是在代数几何中所做的（见Fulton[67]或Harris[86]）。如果p（x，y，z）是2次C上的齐次多项式（加上零多项式），很容易证明v（p）总是非空的，实际上是无穷大的。也可以证明，v（p）=v（q）意味着对于某些λ∈c，q=λp，其中λ=0（6见Samuel[138]，第1.6节，定理10）。切换到复场C的另一个优点是交集理论更清晰。因此，任何不包含公共线的两个圆总是在四个点上相交，其中一些点可能是多个点（如切线圆）。这可能看起来很奇怪，因为在实际平面上，两个圆最多相交两个点。其他两点在哪里？它们被证明是点（1，i，0）和（1，−i，0），正如人们可以立即验证的那样。我们可以把它们看作无穷远处的复杂点！它们不仅是无穷大的，而且不是真实的。难怪我们看不见他们！我们将在第25.14节中回到这些点，即圆形点。

Going back to the vector space E of circles over R, it is worth saying that it can be shown that if V (P) = V (Q) contains at least two points (in which case, V (P) is actually infinite), then Q = λP for some λ ∈ R with λ = 0 (6 see Tisseron [170], Theorem 3.6.1 and Theorem 4.7). Thus, even over R, the mapping  
回到r上的圆的向量空间e，值得说明的是，如果v（p）=v（q）至少包含两个点（在这种情况下，v（p）实际上是无限的），那么对于某些λ∈r，q=λp，其中λ=0（6见Tisseron[170]、定理3.6.1和定理4.7）。因此，即使在R上，映射

[P] 7→ V (P)  
[P]7→V（P）

is injective whenever V (P) is neither empty nor reduced to a single point. Note that the projective space P(E) of circles has dimension 3. In fact, it is easy to show that three distinct points that are not collinear determine a unique circle (see Samuel [138], Section 1.6).  
当v（p）既不是空的，也不是减少到一个点时，是注射的。注意圆的射影空间p（e）有维3。事实上，很容易证明三个不共线的不同点决定了一个唯一的圆（见塞缪尔[138]第1.6节）。

In a similar vein, we can define the projective space of conics P(E) where E is the vector space (over R) consisting of all homogeneous polynomials of degree 2 in x,y,z,  
在类似的纹理中，我们可以定义二次曲线p（e）的投影空间，其中e是向量空间（在r上），它由x、y、z中2阶的所有齐次多项式组成。

ax2 + by2 + cxy + dxz + eyz + fz2  
ax2+by2+cxy+dxz+eyz+fz2

(plus the null polynomial). The curves V (P) are indeed conics, perhaps degenerate. To see this, we can use the hyperplane model of RP2. The trace of V (P) on the plane of equation z = 1 is the conic of equation  
（加上零多项式）。曲线v（p）确实是二次曲线，可能是退化的。为了看到这一点，我们可以使用RP2的超平面模型。方程z=1平面上v（p）的迹线是方程的二次曲线。

ax2 + by2 + cxy + dx + ey + f = 0.  
ax2+by2+cxy+dx+ey+f=0.

Another way to see that V (P) is a conic is to observe that in R3,  
另一种观察v（p）是二次曲线的方法是观察r3中，

ax2 + by2 + cxy + dxz + eyz + fz2 = 0  
ax2+by2+cxy+dxz+eyz+fz2=0

defines a cone with vertex (0,0,0), and since its section by the plane z = 1 is a conic, all of its sections by planes are conics. See Figure 25.10 for schematic illustration of a projective conic embedded in RP2.  
定义一个顶点为（0,0,0）的圆锥体，由于其平面截面z=1是一个圆锥体，因此其所有平面截面都是圆锥体。有关嵌入在RP2中的投影圆锥图的示意图，请参见图25.10。

The mapping  
地图

[P] 7→ V (P)  
[P]7→V（P）

is still injective when E is defined over the ground field C (Samuel [138], Section 1.6, Theorem 10), or if V (P) has at least two points when E is defined over R (Tisseron [170], Theorem 3.6.1 and Theorem 4.7). Note that the projective space P(E) of conics has dimension 5. In fact, it can be shown that five distinct points, no four of which are collinear, determine a  
当e在地磁场c上定义时（塞缪尔[138]，第1.6节，定理10），或者当e在r上定义时，如果v（p）至少有两个点（Tisseron[170]，定理3.6.1和定理4.7），则仍然是内射的。注意二次曲线的射影空间p（e）的尺寸为5。事实上，可以证明五个不同的点，其中没有四个共线，决定了

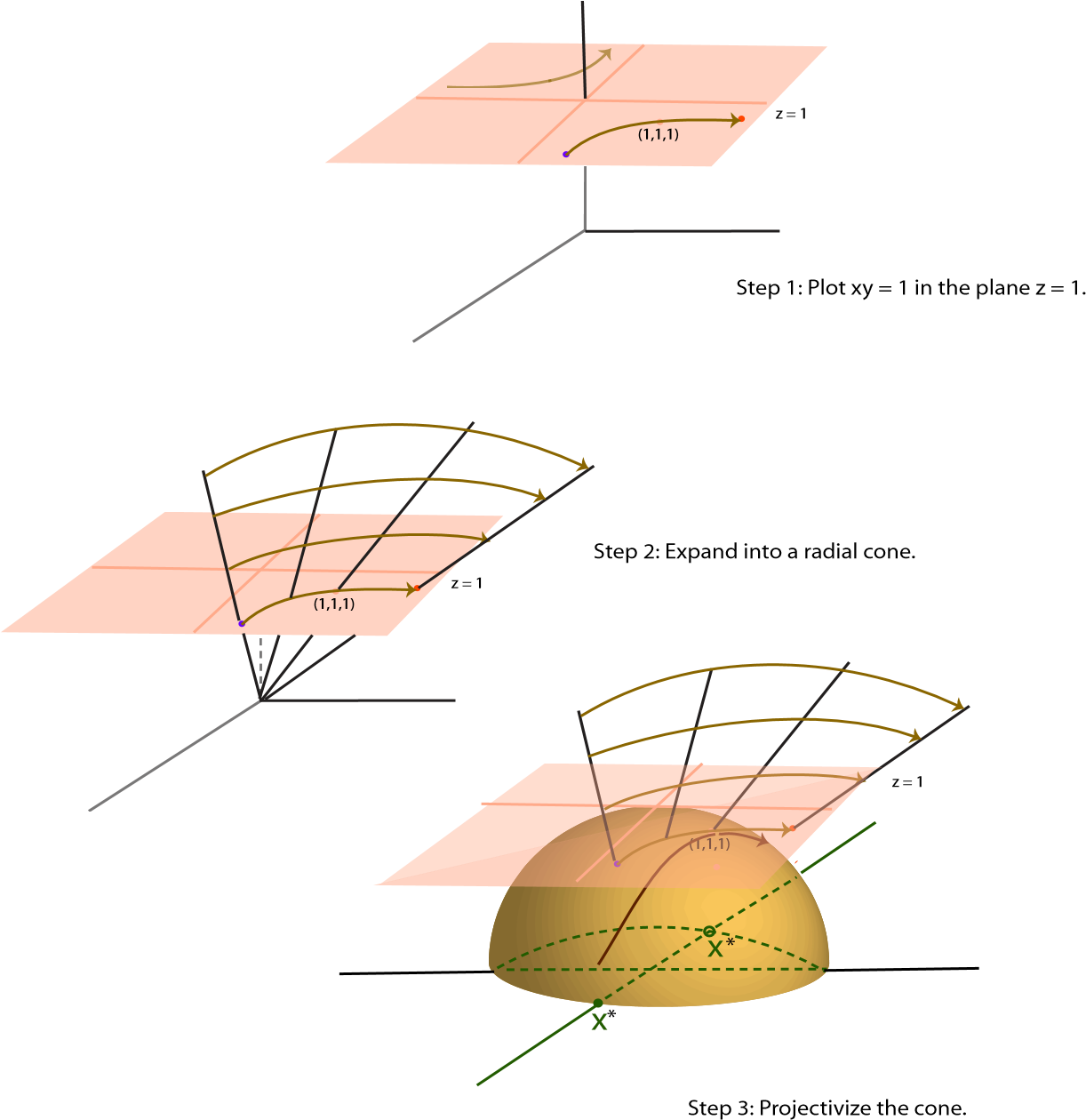


Figure 25.10: A three step process for constructing V (P) where P is the homogenous conic xy = z. In Step 2, we convert to homogenous coordinates via the transformation x → x/z, y → y/z.  
图25.10：构建v（p）的三步过程，其中p是均匀二次曲线x y=z。在步骤2中，我们通过变换x→x/z，y→y/z转换为均匀坐标。

unique conic (among many sources, see Samuel [138], Section 1.7, Theorem 17, or Coxeter [45], Theorem 6.56, where a geometric construction is given in Section 6.6).  
唯一二次曲线（在许多来源中，见Samuel[138]，第1.7节，定理17，或Coxeter[45]，定理6.56，其中几何结构在第6.6节中给出）。

In fact, if we pick a projective frame (a1,a2,a3,a4) in CP2 (or RP2), and if the five points p1,p2,p3,p4,p5 have homogeneous coordinates pi = (xi,yi,zi) for i = 1,...,5 and (x,y,z) are variables, then it is an easy exercise to show that the equation of the unique conic C passing through the points p1,p2,p3,p4,p5 is given by  
事实上，如果我们在CP2（或RP2）中选择一个射影帧（A1，A2，A3，A4），并且如果五点P1、P2、P3、P4、P5具有I＝1、…、5和（x，y，z）的齐次坐标PI=（Xi，Yi，Zi）是变量，那么这是一个简单的练习来证明唯一的圆锥曲线通过这些点的方程。p1，p2，p3，p4，p5由下式给出

.  
.

The polynomial obtained by expanding the above determinant according to the first row is a homogeneous polynomial of degree 2 in the variables x,y,z, and it is not the zero polynomial because the 5×6 matrix obtained by deleting the first row in the matrix of the determinant has rank 5. Indeed, this is the matrix of the linear system determining the six coefficients of the conic passign through p1,p2,p3,p4,p5 (up to a scalar), and since this conic is unique, this matrix must have rank 5.  
根据第一行展开上述行列式得到的多项式是变量x、y、z中二次齐次多项式，且不是零多项式，因为通过删除行列式矩阵中第一行得到的5×6矩阵具有秩5。实际上，这是线性系统的矩阵，通过p1、p2、p3、p4、p5（最高为一个标量）确定二次曲线通过的六个系数，由于这个二次曲线是唯一的，所以这个矩阵必须具有秩5。

It is also interesting to see what are lines in the space of circles or in the space of conics. In both cases we get pencils (of circles and conics, respectively). For more details, see Samuel [138], Sidler [156], Tisseron [170], Lehmann and Bkouche [112], Pedoe [132], Coxeter [45, 46], and Veblen and Young [177, 178].  
同样有趣的是，在圆的空间或圆锥曲线的空间中，线条是什么。在这两种情况下，我们都会得到铅笔（分别是圆和圆锥曲线）。有关详细信息，请参阅Samuel[138]、Sidler[156]、Tisseron[170]、Lehmann和Bkouche[112]、Pedoe[132]、Coxeter[45、46]和Veblen and Young[177、178]。

The generalization of the space of projective conics is the space of projective quadrics P(E), where E is the vector space (over a field K, typically K = R or K = C) consisting of all homogeneous polynomials P(x1,...,xN+1) of degree 2 in the variables x1,...,xN+1, with N ≥ 3 (plus the null polynomial). The zero locus V (P) of P is defined just as before as  
射影二次曲线的空间的推广是射影四次曲线的空间p（e），其中e是向量空间（在场k上，通常k=r或k=c），由变量x1，…，xn+1中2阶的所有齐次多项式p（x1，…，xn+1）组成，n≥3（加上零多项式MIAL）。p的零轨迹v（p）的定义与之前一样

V (P) = {(x1 : ··· : xN+1) ∈ PNK | P(x1,...,xN+1) = 0}.  
v（p）=（x1：···：xn+1）∈pnk p（x1，…，xn+1）=0。

If the field K is algebraically closed, in particular if K = C, then V (P) = V (Q) implies that there is some nonzero λ ∈ K such that Q = λP; see Berger [12] (Chapter 14, Theorem  
如果场k是代数闭合的，特别是如果k=c，那么v（p）=v（q）意味着存在一些非零的λ∈k，这样q=λp；参见Berger[12]（第14章，定理

14.1.6.2).  
14.1.6.2条）。

Another situation where the map [P] 7→ V (P) is injective involves the notion of simple (or regular) point of a quadric. For any a = (a1 : ··· : aN+1) ∈ PNK, let Pxi(a) be the partial derivative of P at a given by  
图[p]7→v（p）是内射的另一种情况涉及二次曲面的简单（或规则）点的概念。对于任意a=（a1：····：an+1）∈pnk，让pxi（a）是p在给定

.  
.

Strictly speaking, Pxi(a) depends on the representative (a1,...,aN+1) ∈ KN+1 chosen for the point a, but since P is homogeneous of degree 2, for any nonzero λ ∈ K,  
严格地说，pxi（a）依赖于a点所选的代表（a1，…，an+1）∈kn+1，但由于p是2阶的齐次，对于任何非零的λ∈k，

.  
.

Thus Pxi(a) is defined up to a nonzero scalar. In particular, whether or not Pxi(a) = 0 depends only the point a = (a1 : ··· : aN+1) ∈ PNK. Then the point a ∈ V (P) is said to be simple (or regular) if  
因此，pxi（a）被定义为非零标量。特别是，PxI（a）=0是否仅取决于点A=（a1：····：an+1）∈PNK。那么点a∈v（p）被称为简单（或规则）如果

Pxi(a) = 06 for some i, 1 ≤ i ≤ N + 1.  
pxi（a）=06，对于某些i，1≤i≤n+1。

Otherwise, if Px1(a) = ··· = PxN+1(a) = 0, we say that a ∈ V (P) is a singular point. If a ∈ V (P) is a regular point, then the tangent hyperplane TaV (P) to V (P) at a is the hyperplane given by the equation  
否则，如果Px1（a）=······=Pxn+1（a）=0，我们就说a∈v（p）是一个奇异点。如果a∈v（p）是一个正则点，那么a处的切线超平面tav（p）到v（p）是由方程给出的超平面。

Px1(a)x1 + ··· + PxN+1(a)xN+1 = 0.  
px1（a）x1+·····+pxn+1（a）xn+1=0。

It can be shown that if the field K is not the field F2 = {0,1} and if the quadric V (P) contains some regular point, then V (P) = V (Q) implies that there is some nonzero λ ∈ K such that Q = λP; see Samuel [138] (Chapter 3, Theorem 46).  
可以证明，如果场k不是场f2=0,1，如果二次v（p）包含一些正则点，那么v（p）=v（q）意味着存在一些非零的λ∈k，这样q=λp；参见Samuel[138]（第3章，定理46）。

Quadrics, projective, affine, and Euclidean, have been thoroughly investigated. Among many sources, the reader is referred to Berger [11], Samuel [138], Tisseron [170], Fresnel [66], and Vienne [179].  
四次方、射影、仿射和欧几里德已经被彻底研究过。在许多资料中，读者被称为伯杰[11]、塞缪尔[138]、蒂塞隆[170]、菲涅尔[66]和维也纳[179]。

We could also investigate algebraic plane curves of any degree m, by letting E be the vector space of homogeneous polynomials of degree m in x,y,z (plus the null polynomial). The zero locus V (P) of P is defined just as before as  
我们也可以研究任意m阶的代数平面曲线，方法是让e是m阶在x，y，z（加上零多项式）中的齐次多项式的向量空间。p的零轨迹v（p）的定义与之前一样

V (P) = {(x: y: z) ∈ RP2 | P(x,y,z) = 0}.  
v（p）=（x:y:z）∈rp2 p（x，y，z）=0。

Observe that when m = 1, since homogeneous polynomials of degree 1 are linear forms, we are back to the case where E = (R3)∗, the dual space of R3, and P(E) can be identified with the set of lines in RP2. But when m ≥ 3, things are even worse regarding the injectivity of the map [P] 7→ V (P). For instance, both P = xy2 and Q = x2y define the same union of two lines. It is necessary to consider irreducible curves, i.e., curves that are defined by irreducible polynomials, and to work over the field C of complex numbers (recall that a polynomial P is irreducible if it cannot be written as the product P = Q1Q2 of two polynomials Q1,Q2 of degree ≥ 1). We refer the reader to Fischer’s book for a beautiful (and very clear) introduction to algebraic curves [63]. The next step is Fulton [67].  
观察到当m=1时，由于阶1的齐次多项式是线性形式，我们回到E=（r3）的情形，r3和p（e）的对偶空间可以用rp2中的一组线来标识。但当m≥3时，关于图的注入率[p]7→v（p），情况更糟。例如，p=xy2和q=x2y都定义了两条线的同一个联合。有必要考虑不可约曲线，即由不可约多项式定义的曲线，并对复数的域C进行研究（如果不能将多项式p写成两个多项式q1，q2的乘积p=q1q2，则多项式p是不可约的）。我们请读者参考费舍尔的书，了解代数曲线的美丽（非常清晰）介绍[63]。下一步是富尔顿[67]。

We can also investigate algebraic surfaces in RP3 (or CP3), by letting E be the vector space of homogeneous polynomials of degree m in four variables x,y,z,t (plus the null polynomial). We can also consider the zero locus of a set of equations  
我们也可以研究rp3（或cp3）中的代数曲面，让e是四个变量x、y、z、t（加上零多项式）中m次齐次多项式的向量空间。我们也可以考虑一组方程的零轨迹。

E = {P1 = 0, P2 = 0, ..., Pn = 0},  
E=p1=0，p2=0，…，pn=0，

where P1,...,Pn are homogeneous polynomials of degree m in x,y,z,t, defined as  
式中，p1，…，pn是x，y，z，t中m阶的齐次多项式，定义为

V (E) = {(x: y: z: t) ∈ RP3 | Pi(x,y,z,t) = 0, 1 ≤ i ≤ n}.  
v（e）=（x:y:z:t）∈rp3 pi（x，y，z，t）=0，1≤i≤n。

This way, we can also deal with space curves.  
这样，我们也可以处理空间曲线。

Finally, we can consider homogeneous polynomials P(x1,...,xN+1) in N + 1 variables and of degree m (plus the null polynomial), and study the subsets of RPN or CPN (or more generally of PNK, for an arbitrary field K), defined as the zero locus of a set of equations  
最后，我们可以考虑n+1变量中的齐次多项式p（x1，…，xn+1）和m阶的齐次多项式（加上零多项式），并研究rpn或cpn的子集（或更一般的p n k，对于任意场k），定义为一组方程的零轨迹。

E = {P1 = 0, P2 = 0, ..., Pn = 0},  
E=p1=0，p2=0，…，pn=0，

where P1,...,Pn are homogeneous polynomials of degree m in the variables x1, ..., xN+1. For example, it turns out that the set of lines in RP3 forms a surface of degree 2 in RP5 (the Klein quadric). However, all this would really take us too far into algebraic geometry, and we simply refer the interested reader to Hulek [94], Fulton [67], and Harris [86].  
式中，p1，…，pn是变量x1，…，xn+1中m阶的齐次多项式。例如，结果表明，rp3中的一组线在rp5（klein二次曲面）中形成一个2度曲面。然而，所有这一切真的会把我们带到代数几何中去太远，我们只是把感兴趣的读者引向Hulek[94]、Fulton[67]和Harris[86]。

We now consider projective maps.  
我们现在考虑投影地图。

## 25.5 Projective Maps 25.5投影图

Given two nontrivial vector spaces E and F and a linear map f : E → F, observe that for every u,v ∈ (E − Kerf), if v = λu for some λ ∈ K − {0}, then f(v) = λf(u), and thus f restricted to (E −Kerf) induces a function P(f): (P(E)−P(Kerf)) → P(F) defined such that  
给定两个非平凡向量空间e和f和一个线性映射f:e→f，观察每个u，v∈（e−kerf），如果v=λu对于一些λ∈k−0，那么f（v）=λf（u），因此f限制为（e−kerf）诱导一个函数p（f）：（p（e）−p（kerf））→p（f）定义如下：

P(f)([u]∼) = [f(u)]∼,  
p（f）（[u]）=[f（u）]，

as in the following commutative diagram:  
如下图所示：

f  
f

E − Kerf / F − {0}  
E−切口/F−0

p p  
磷脂酶P

P(E)P(Kerf) / P(F)  
P（E）P（切口）/P（F）

P(f)  
P（F）

When f is injective, i.e., when Kerf = {0}, then P(f): P(E) → P(F) is indeed a welldefined function. The above discussion motivates the following definition.  
当f是注射剂时，即当kerf=0，那么p（f）：p（e）→p（f）确实是一个定义良好的函数。上述讨论激发了以下定义。

Definition 25.5. Given two nontrivial vector spaces E and F, any linear map f : E → F induces a partial map P(f): P(E) → P(F) called a projective map, such that if Kerf = {u ∈ E | f(u) = 0} is the kernel of f, then P(f): (P(E)−P(Kerf)) → P(F) is a total map defined such that  
定义25.5.给定两个非平凡向量空间e和f，任何线性映射f:e→f都会产生一个称为投影映射的偏映射p（f）：p（e）→p（f），如果kerf=u∈e f（u）=0是f的核，那么p（f）：（p（e）−p（kerf））→p（f）是一个定义为

P(f)([u]∼) = [f(u)]∼,  
p（f）（[u]）=[f（u）]，

as in the following commutative diagram:  
如下图所示：

f  
f

E − Kerf / F − {0}  
E−切口/F−0

p p  
磷脂酶P

P(E)P(Kerf) / P(F)  
P（E）P（切口）/P（F）

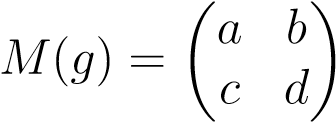
P(f)  
P（F）

If f is injective, i.e., when Kerf = {0}, then P(f): P(E) → P(F) is a total function called  
如果f是内射的，即当kerf=0时，那么p（f）：p（e）→p（f）是一个称为

a projective transformation, and when f is bijective, we call P(f) a projectivity, or projective isomorphism, or homography. The set of projectivities P(f): P(E) → P(E) is a group called the projective (linear) group, and is denoted by PGL(E).  
一个射影变换，当f是双射影时，我们称p（f）为射影性，或射影同构，或同形。射影率p（f）：p（e）→p（e）是一个称为射影（线性）群的群，用pgl（e）表示。

 One should realize that if a linear map f : E → F is not injective, then the projective map P(f): P(E) → P(F) is only a partial map, i.e., it is undefined on P(Kerf). In particular, if f : E → F is the null map (i.e., Kerf = E), the domain of P(f) is empty and P(f) is the partial function undefined everywhere. We might want to require in Definition 25.5 that f not be the null map to avoid this degenerate case. Projective maps are often defined only when they are induced by bijective linear maps.  
我们应该认识到，如果线性映射f:e→f不是内射的，那么投影映射p（f）：p（e）→p（f）只是局部映射，即p（kerf）上没有定义。特别是，如果f:e→f是空映射（即kerf=e），则p（f）的域为空，p（f）是处处未定义的部分函数。我们可能希望在定义25.5中要求f不是空映射以避免这种退化情况。射影映射通常只有在由双射影线性映射诱导时才被定义。

We take a closer look at the projectivities of the projective line P1K, since they play a role in the “change of parameters” for projective curves. A projectivity f : P1K → P1K is induced by some bijective linear map g: K2 → K2 given by some invertible matrix  
我们更仔细地看一下投影线p1k的投影率，因为它们在投影曲线的“参数变化”中起着作用。由一些可逆矩阵给出的双射线性映射g:k2→k2，得到了f:p1k→p1k的投影性。



with ad−bc = 06 . Since the projective line P1K is isomorphic to K ∪{∞}, it is easily verified that f is defined as follows:  
当ad−bc=06时。由于投影线p1k与k∞同构，很容易证明f的定义如下：

,  
，

,  
，

;  
；

From Section 25.4, we know that the points not at infinity are repesented by vectors of the form (z,1) where z ∈ K and that ∞ is represented by (1,0). First, assume c = 06 . Since  
从第25.4节中，我们知道不在无穷远处的点由形式（z，1）的向量表示，其中z∈k和∞由（1,0）表示。首先，假设c=06。自从

,  
，

if cz + d = 06 , that is, z =6 −d/c, then  
如果c z+d=06，即z=6−d/c，则

,  
，

so z is is mapped to (az + d)/cz + d). If cz + d = 0, then  
所以z映射到（az+d）/cz+d。如果cz+d=0，则

(az + d,0) ∼ (1,0) = ∞,  
（az+d，0）（1,0）=∞，

so −d/c is mapped to ∞. We also have  
因此−d/c映射到∞。我们也有

,  
，

and since c = 06 we have  
从C=06开始

(a,c) ∼ (a/c,1),  
（a，c）（a/c，1）、

so ∞ is mapped to a/c. The case where c = 0 is handled similarly.  
因此∞映射到A/C。C=0的情况处理类似。

If K = R or K = C, note that a/c is the limit of (az + b)/(cz + d), as z approaches infinity, and the limit of (az + b)/(cz + d) as z approaches −d/c is ∞ (when c 6= 0). Projections between hyperplanes form an important example of projectivities.  
如果k=r或k=c，注意a/c是（a z+b）/（cz+d）的极限，当z接近无穷大时，（az+b）/（cz+d）的极限是∞（当c 6=0时）。超平面之间的投影是投影的一个重要例子。

Definition 25.6. Given a projective space P(E), for any two distinct hyperplanes P(H) and P(H0), for any point c ∈ P(E) neither in P(H) nor in P(H0), the projection (or perspectivity) of center c between P(H) and P(H0) is the map f : P(H) → P(H0) defined such that for every a ∈ P(H), the point f(a) is the intersection of the line hc,ai through c and a with P(H0).  
定义25.6.给定一个投影空间p（e），对于任意两个不同的超平面p（h）和p（h0），对于任何点c∈p（e），无论是在p（h）还是在p（h0）中，中心c在p（h）和p（h0）之间的投影（或透视性）是映射f:p（h）→p（h0），定义为对于每一个a∈p（h），点f（a）是int。Hc、Ai至C和A线与P（H0）的截面。

Let us verify that f is well–defined and a bijective projective transformation. Since the hyperplanes P(H) and P(H0) are distinct, the hyperplanes H and H0 in E are distinct, and since c is neither in P(H) nor in P(H0), letting c = p(u) for some nonnull vector u ∈ E, then u /∈ H and u /∈ H0, and thus E = H ⊕Ku = H0 ⊕Ku. If π: E → H0 is the linear map  
让我们验证f是定义良好的，并且是一个双射射影变换。由于超平面p（h）和p（h0）是不同的，所以e中的超平面h和h0是不同的，并且c既不在p（h）中也不在p（h0）中，因此对于一些非空向量u∈e，让c=p（u），那么u/∈h和u/∈h0，因此e=h ku=h0 ku。如果π：e→h0是线性映射

(projection onto H0 parallel to u) defined such that  
（投影到与u平行的h0上）定义为

π(w + λu) = w,  
π（w+λu）=w，

for all w ∈ H0 and all λ ∈ K, since E = H ⊕ Ku = H0 ⊕ Ku, the restriction g: H → H0 of π: E → H0 to H is a linear bijection between H and H0, and clearly f = P(g), which shows that f is a projectivity.  
对于所有w∈h0和所有λ∈k，由于e=h ku=h0 ku，π：e→h0到h的约束g:h→h0是h和h0之间的线性双射，显然f=p（g），这表明f是一个投影性。

Remark: Going back to the linear map π: E → H0 (projection onto H0 parallel to u), note that P(π): P(E) → P(H0) is also a projective map, but it is not injective, and thus only a partial map. More generally, given a direct sum E = V ⊕W, the projection π: E → V onto V parallel to W induces a projective map P(π): P(E) → P(V ), and given another direct sum E = U ⊕ W, the restriction of π to U induces a perspectivity f between P(U) and P(V ). Geometrically, f is defined as follows: Given any point a ∈ P(U), if hP(W),ai is the smallest projective subspace containing P(W) and a, the point f(a) is the intersection of hP(W),ai with P(V ).  
注：回到线性映射π：e→h0（投影到平行于u的h0上），注意p（π）：p（e）→p（h0）也是一个投影映射，但它不是内射的，因此只是一个局部映射。更一般地说，给定一个直和e=v\_w，投影π：e→v与w平行，在v上产生一个投影图p（π）：p（e）→p（v），再给定另一个直和e=u\_w，π对u的限制，在p（u）和p（v）之间产生一个透视f。几何上，F的定义如下：给定任意点a∈p（u），如果hp（w），ai是包含p（w）和a的最小投影子空间，点f（a）是hp（w），ai与p（v）的交集。

Figure 25.11 illustrates a projection f of center c between two projective lines ∆ and ∆0 (in the real projective plane).  
图25.11说明了中心C在两条投影线∆和∆0之间的投影F（在实际投影平面中）。

If we consider three distinct points d1,d2,d3 on ∆ and their images on ∆0 under the projection f, then ratios are not preserved, that is,  
如果我们考虑∆上的三个不同点d1、d2、d3及其在投影f下的∆0上的图像，那么比率就不会被保留，也就是说，

.  
.

However, if we consider four distinct points d1,d2,d3,d4 on ∆ and their images on ∆0 under the projection f, we will show later that we have the following preservation of the so-called “cross-ratio”  
但是，如果我们考虑∆上的四个不同点d1、d2、d3、d4，以及它们在投影f下的∆0上的图像，我们稍后将证明我们保留了所谓的“交叉比”。

.  
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Cross-ratios and projections play an important role in geometry (for some very elegant illustrations of this fact, see Sidler [156]).  
交叉比和投影在几何学中起着重要作用（有关这一事实的一些非常优雅的说明，请参见Sidler[156]）。

*c*

*d*

1

*d*

2

*d*

3

*d*

4

*d*

′

1

*d*

′

2

*d*

′

3

*d*

′

4

*D*

1

*D*

2

*D*

3

*D*

4

∆

∆

′

Figure 25.11: A projection of center c between two lines ∆ and ∆0.  
图25.11：中心C在两条直线∆和∆0之间的投影。

We now turn to the issue of determining when two linear maps f,g determine the same projective map, i.e., when P(f) = P(g). The following proposition gives us a complete answer.  
我们现在讨论的问题是确定两个线性映射f，g何时确定同一投影映射，即p（f）=p（g）。下面的建议给了我们一个完整的答案。

Proposition 25.4. Given two nontrivial vector spaces E and F, for any two linear maps f : E → F and g: E → F, we have P(f) = P(g) iff there is some scalar λ ∈ K − {0} such that g = λf.  
提案25.4.给定两个非平凡向量空间e和f，对于任意两个线性映射f:e→f和g:e→f，我们得到p（f）=p（g），如果有一些标量λ∈k−0，那么g=λf。

Proof. If g = λf, it is clear that P(f) = P(g). Conversely, in order to have P(f) = P(g), we must have Kerf = Kerg. If Kerf = Kerg = E, then f and g are both the null map, and this case is trivial. If E−Kerf =6 ∅, by taking a basis of Imf and some inverse image of this basis, we obtain a basis B of a subspace G of E such that E = Kerf ⊕G. If dim(G) = 1, the restriction of any linear map f : E → F to G is determined by some nonzero vector u ∈ E and some scalar λ ∈ K, and the proposition is obvious. Thus, assume that dim(G) ≥ 2. For any two distinct basis vectors u,v ∈ B, since P(f) = P(g), there must be some nonzero scalars λ(u), λ(v), and λ(u + v) such that  
证据。如果g=λf，很明显p（f）=p（g）。相反，为了使p（f）=p（g），我们必须有切口=切口。如果kerf=kerg=e，那么f和g都是空映射，这种情况很简单。当e−kerf=6∅时，利用imf的基和该基的一些逆映象，得到e的子空间g的基b，使e=kerf g，当dim（g）=1时，任何线性映射f:e→f到g的约束由一些非零向量u∈e和一些标量λ∈k确定，并给出了相应的证明。位置很明显。因此，假设dim（g）≥2。对于任意两个不同的基向量u，v∈b，因为p（f）=p（g），必须有一些非零标度λ（u），λ（v）和λ（u+v），以便

g(u) = λ(u)f(u), g(v) = λ(v)f(v), g(u + v) = λ(u + v)f(u + v).  
g（u）=λ（u）f（u），g（v）=λ（v）f（v），g（u+v）=λ（u+v）f（u+v）。

Since f and g are linear, we get  
因为f和g是线性的，我们得到

g(u) + g(v) = λ(u)f(u) + λ(v)f(v) = λ(u + v)(f(u) + f(v)),  
g（u）+g（v）=λ（u）f（u）+λ（v）f（v）=λ（u+v）（f（u）+f（v）），

that is,  
也就是说，

(λ(u + v) − λ(u))f(u) + (λ(u + v) − λ(v))f(v) = 0.  
（λ（u+v）-λ（u））f（u）+（λ（u+v）-λ（v））f（v）=0.

Since f is injective on G and u,v ∈ B ⊆ G are linearly independent, f(u) and f(v) are also linearly independent, and thus we have  
由于f在g和u上是内射的，v∈b g是线性无关的，f（u）和f（v）也是线性无关的，因此我们得到

λ(u + v) = λ(u) = λ(v).  
λ（u+v）=λ（u）=λ（v）。

Now we have shown that λ(u) = λ(v), for any two distinct basis vectors in B, which proves that λ(u) is independent of u ∈ G, and proves that g = λf.   
现在我们证明了对于B中任意两个不同的基向量，λ（u）=λ（v），证明了λ（u）独立于u∈g，并证明了g=λf。

Proposition 25.4 shows that the projective linear group PGL(E) is isomorphic to the quotient group of the linear group GL(E) modulo the subgroup K∗idE (where K∗ = K − {0}). Using projective frames, we prove the following useful result.  
命题25.4表明射影线性群pgl（e）同构于子群k ide（其中k=k−0）的线性群gl（e）模的商群。利用射影框架，我们证明了以下有用的结果。

Proposition 25.5. Given two nontrivial vector spaces E and F of the same dimension n + 1, for any two projective frames (ai)1≤i≤n+2 for P(E) and (bi)1≤i≤n+2 for P(F), there is a unique projectivity h: P(E) → P(F) such that h(ai) = bi for 1 ≤ i ≤ n + 2.  
提案25.5.给定两个非平凡向量空间e和f，对于任意两个投影帧（a i）1≤i≤n+2，对于p（e）和（bi）1≤i≤n+2，对于p（f），有一个唯一的投影度h:p（e）→p（f），使得h（ai）=bi，对于1≤i≤n+2。

Proof. Let (u1,...,un+1) be a basis of E associated with the projective frame (ai)1≤i≤n+2, and let (v1,...,vn+1) be a basis of F associated with the projective frame (bi)1≤i≤n+2. Since (u1,...,un+1) is a basis, there is a unique linear bijection g: E → F such that g(ui) = vi, for 1 ≤ i ≤ n+1. Clearly, h = P(g) is a projectivity such that h(ai) = bi, for 1 ≤ i ≤ n+2. Let h0 : P(E) → P(F) be any projectivity such that h0(ai) = bi, for 1 ≤ i ≤ n + 2. By definition, there is a linear isomorphism f : E → F such that h0 = P(f). Since h0(ai) = bi, for 1 ≤ i ≤ n + 2, we must have f(ui) = λivi, for some λi ∈ K − {0}, where 1 ≤ i ≤ n + 1, and f(u1 + ··· + un+1) = λ(v1 + ··· + vn+1),  
证据。设（u1，…，un+1）为与射影帧（a i）1≤i≤n+2相关联的e的基础，设（v1，…，vn+1）为与射影帧（bi）1≤i≤n+2相关联的f的基础。因为（u1，…，un+1）是一个基础，所以有一个唯一的线性双射g:e→f，这样g（ui）=vi，对于1≤i≤n+1。显然，h=p（g）是一个投射性，使得h（a i）=bi，对于1≤i≤n+2。设h0:p（e）→p（f）为任何投影性，使得h0（ai）=bi，对于1≤i≤n+2。根据定义，有一个线性同构f:e→f，这样h0=p（f）。由于h0（ai）=bi，对于1≤i≤n+2，我们必须有f（ui）=λivi，对于一些λi∈k−0，其中1≤i≤n+1，和f（u1+·······+un+1）=λ（v1+······+vn+1），

for some λ ∈ K − {0}. By linearity of f, we have  
对于某些λ∈k−0。根据f的线性，我们得到

λ1v1 + ··· + λn+1vn+1 = λv1 + ··· + λvn+1,  
λ1v1+····+λn+1vn+1=λv1+···+λvn+1，

and since (v1,...,vn+1) is a basis of F, we must have  
既然（v1，…，vn+1）是f的基础，我们必须

λ1 = ··· = λn+1 = λ.  
λ1=····=λn+1=λ。

This shows that f = λg, and thus that  
这表明f=λg，因此

h0 = P(f) = P(g) = h,  
h0=p（f）=p（g）=h，

and h is uniquely determined.   
H是唯一确定的。

 The above proposition and Proposition 25.4 are false if K is a skew field. Also, Proposition 25.5 fails if (bi)1≤i≤n+2 is not a projective frame, or if an+2 is dropped.  
如果k是斜场，则上述命题和25.4都是假的。另外，如果（bi）1≤i≤n+2不是投影帧，或者如果一个+2被丢弃，则命题25.5失败。

As a corollary of Proposition 25.5, given a projective space P(E), two distinct projective lines D and D0 in P(E), three distinct points a,b,c on D, and any three distinct points a0,b0,c0 on D0, there is a unique projectivity from D to D0, mapping a to a0, b to b0, and c to c0. This is because, as we mentioned earlier, any three distinct points on a line form a projective frame.  
作为25.5命题的一个推论，给定一个射影空间p（e），p（e）中的两条不同的射影线d和d0，d上的三个不同点a、b、c和d0上的任何三个不同点a0、b0、c0，有一个从d到d0的独特射影性，映射a到a0、b到b0和c到c0。这是因为，正如我们前面提到的，直线上的任何三个不同的点都形成了一个投影框架。

Remark: As in the affine case, there is “fundamental theorem of projective geometry.” For simplicity, we state this theorem assuming that vector spaces are over the field K = R. Given any two projective spaces P(E) and P(F) of the same dimension n ≥ 2, for any bijectivea,b,c to collinear functionpoints f(fa):,fP((bE),f) →(c)P, then(F), iff is a projectivity. For more general fields,f maps any three distinct collinear pointsf = P(g) for some  
注：在仿射的情况下，有“射影几何的基本定理”。为了简单起见，我们假设向量空间在k=r的域上。给定任意两个射影空间p（e）和p（f），同维n≥2，任意双射影a、b、c到共线func点F（f a）：，f p（（be），f）→（c）p，然后（f），iff是一个投射性。对于更一般的字段，f映射任意三个不同的共线点sf=p（g）

“semilinear” bijectiondistinct points) is often called ag: E → Fcollineation. A map such as. For Kf =(preserving collinearity of any threeR, collineations and projectivities coincide. For more details, see Samuel [138].  
“半线性”双射离散点）通常称为ag:e→f线性。像这样的地图。对于kf=（保持任意三者的共线性，共线性和射影率是一致的。有关更多详细信息，请参见Samuel[138]。

Before closing this section, we illustrate the power of Proposition 25.5 by proving two interesting results. We begin by characterizing perspectivities between lines.  
在结束本节之前，我们通过证明两个有趣的结果来说明25.5号提案的威力。我们首先描述线条之间的透视性。

Proposition 25.6. Given any two distinct lines D and D0 in the real projective plane RP2, a projectivity f : D → D0 is a perspectivity iff f(O) = O, where O is the intersection of D and D0.  
提案25.6.在实射影平面rp2中任意两条不同的线d和d0，射影率f:d→d0是透视率iff（o）=o，其中o是d和d0的交集。

Proof. If f : D → D0 is a perspectivity, then by the very definition of f, we have f(O) = O. Conversely, let f : D → D0 be a projectivity such that f(O) = O. Let a,b be any two distinct points on D also distinct from O, and let a0 = f(a) and b0 = f(b) on D0. Since f is a bijection and since a,b,O are pairwise distinct, a0 6= b0. Let c be the intersection of the lines ha,a0i and hb,b0i, which by the assumptions on a,b,O, cannot be on D or D0. Then we can define the perspectivity g: D → D0 of center c, and by the definition of c, we have  
证据。如果f:d→d0是一个透视性，那么根据f的定义，我们有f（o）=o。相反，让f:d→d0是一个投影性，这样f（o）=o。让a，b是d上的任意两个不同点，也不同于o，让a0=f（a）和b0=f（b）在d0上。因为f是双射，而a，b，o是成对的，所以a0 6=b0。假设c是线ha、a0i和hb、b0i的交点，根据a、b、o的假设，这些线不能在d或d0上。然后我们可以定义中心c的透视率g:d→d0，根据c的定义，我们得到

g(a) = a0, g(b) = b0, g(O) = O.  
g（a）=a0，g（b）=b0，g（o）=o。

See Figure 25.12. However, f agrees with g on O,a,b, and since (O,a,b) is a projective frame for D, by Proposition 25.5, we must have f = g.   
见图25.12。然而，f在o，a，b上与g一致，由于（o，a，b）是d的投影框架，根据命题25.5，我们必须有f=g。

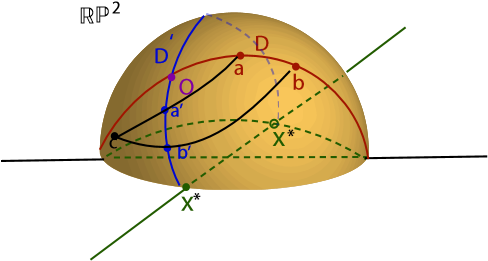
Using Proposition 25.6, we can give an elegant proof of a version of Desargues’s theorem (in the plane).  
利用25.6号命题，我们可以很好地证明（在平面上）德沙格定理的一个版本。

Proposition 25.7. (Desargues) Given two triangles (a,b,c) and (a0,b0,c0) in RP2, where the  
提案25.7.（desargues）在rp2中给出两个三角形（a，b，c）和（a0，b0，c0），其中

Apoints0 = hba,b,c,a0,c0i, B0,b0 =0, ch0a0are pairwise distinct and the lines,c0i, C0 = ha0,b0i are pairwise distinct, if the linesA = hb,ci, B = hha,aa,c0ii,, Chb,b=0ih, anda,bi,  
apoints0=hba，b，c，a0，c0i，b0，b0=0，ch0a0是成对的不同，而线，c0i，c0=ha0，b0i是成对的不同，如果线a=hb，ci，b=hha，aa，c0ii，chb，b=0ih，anda，bi，

hc,c0i intersect in a common point d distinct from a,b,c, a0,b0,c0, then the intersection points p = hb,ci∩hb0,c0i, q = ha,ci∩ha0,c0i, and r = ha,bi∩ha0,b0i belong to a common line distinct from A,B,C, A0,B0,C0.  
hc，c0i相交于与a，b，c，a0，b0，c0不同的公共点d，然后相交点p=hb，ci hb0，c0i，q=ha，ci ha0，c0i和r=ha，bi ha0，b0i属于与a，b，c，a0，b0，c0不同的公共线。

Proof.Ahc,c0,B0i0,CIn view of the assumptions on0. Let f : ha,a0i → hp. Letb,b0i hbe the perspectivity of center= ga,b,c◦f. Since both, a0,b0,c0, andf(dd, the point) = d and gr(dis on neither) = h0ia,anor0i, northe perspectivity of centerhb,b. It is also immediately shown that the line0i, the point p is on neither hb,b0i nor hc,c0i, and the pointhp,qi is distinct from the linesr andq is on neitherg: hb,bd0, we also havei → hha,aA,B,Cc,c0i be  
证明：AHC、C0、B0I0、CIN对假设0的看法。让f:ha，a0i→hp。Letb，b0i hbe中心的透视率=ga，b，c f。由于a0，b0，c0，and f（d d，该点）=d和gr（d is-on-neither）=h0ia，anor0i，center hb，b的北透视率。也立即显示线条0i，该点p既不在hb，b0i也不在hc，c0i上，并且该点hp，qi与m线路sr和q在附近：hb，bd0，我们也有i→hha，aa，b，cc，c0i



D

D

’

O

a

b

a’ = f(a)

b’ = f(b)

c

Figure 25.12: An illustration of the perspectivity construction of Proposition 25.6.  
图25.12：25.6提案的透视结构示意图。

h(d) = d. Thus by Proposition 25.6, the projectivity h: ha,a0i → hc,c0i is a perspectivity. Since  
H（d）=D。因此，根据命题25.6，投影性H:h a，a0i→hc，c0i是一个透视性。自从

h(a) = g(f(a)) = g(b) = c, h(a0) = g(f(a0)) = g(b0) = c0,  
h（a）=g（f（a））=g（b）=c，h（a0）=g（f（a0））=g（b0）=c0，

the intersection q of ha,ci and ha0,c0i is the center of the perspectivity h. Also note that the point m = ha,a0i∩hp,ri and its image h(m) are both on the line hp,ri, since r is the center of f and p is the center of g. Since h is a perspectivity of center q, the line hm,h(m)i = hp,ri passes through q, which proves the proposition.  
h a、ci和ha0、c0i的交叉点q是透视性h的中心。还要注意，点m=ha、a0i h p、r i及其图像h（m）都在线hp、ri上，因为r是f的中心，p是g的中心。因为h是中心q的透视性，所以线hm、h（m）i=hp，ri通过啊，这证明了这个命题。

Desargues’s theorem is illustrated in Figure 25.13. It can also be shown that every projectivity between two distinct lines is the composition of two perspectivities (not in a unique way). An elegant proof of Pappus’s theorem can also be given using perspectivities.  
德沙格定理如图25.13所示。也可以证明，两条不同线条之间的每一个投影都是两个透视图的组成部分（不是以独特的方式）。帕普斯定理的一个很好的证明也可以用透视法给出。

## 25.6 Finding a Homography Between Two Projective Frames 25.6在两个投影帧之间找到一个同形

In this section we present a method for finding the matrix (up to a scalar) of the unique homography (bijective projective transformation) mapping one projective frame to an other projective frame. This problem arises notably in computer vision in the context of image morphing.  
在本节中，我们提出了一种求唯一同形矩阵（双射射影变换）的方法，将一个射影帧映射到另一个射影帧。这一问题在图像变形背景下的计算机视觉中尤为突出。

We begin with the simple case of two nondegenerate quadrilatrerals ([p1],[p2],[p3],[p4]) and [(q1],[q2],[q3],[q4]) in RP2, that is, two projective frames, which means that (p1,p2,p3)  
我们从RP2中的两个非简并四分体（[p1]、[p2]、[p3]、[p4]）和[（q1]、[q2]、[q3]、[q4]）的简单情况开始，即两个投影帧，这意味着（p1、p2、p3）

*d*

*a*

*b*

*c*

*a*

′

*c*

′

*b*

′

*r*

*p*

*q*

Figure 25.13: Desargues’s theorem (projective version in the plane).  
图25.13：德沙格定理（平面中的投影形式）。

and (q1,q2,q3) are linearly independent, and that if we write  
和（q1，q2，q3）是线性无关的，如果我们写

p4 = α1p1 + α2p2 + α3p3  
P4=α1p1+α2p2+α3p3

and q4 = λ1q1 + λ2q2 + λ3q3,  
q4=λ1q1+λ2q2+λ3q3，

for some unique scalars α1,α2,α3 and λ1,λ2,λ3, then αi = 06 and λi = 06 for i = 1,2,3. The problem is to find the 3 × 3 matrix (up to a scalar) representing the unique homography h mapping [pi] to [qi] for i = 1,2,3,4.  
对于一些独特的量表α1、α2、α3和λ1、λ2、λ3，则αi=06和λi=06表示i=1、2、3。问题是找到3×3矩阵（达到一个标量），它表示i=1,2,3,4的唯一同形H映射[pi]到[qi]。

We will use the canonical basis E = (e1,e2,e3) of R3, with e1 = (1,0,0), e2 = (0,1,0), e3 = (0,0,1), and the bases P = (p1,p2,p3) and Q = (q1,q2,q3) of R3.  
我们将使用r3的规范基e=（e1，e2，e3），e1=（1，0，0），e2=（0，1，0），e3=（0，0，1），以及r3的基p=（p1，p2，p3）和q=（q1，q2，q3）。

As a first step, it is convenient to express (q1,q2,q3,q4) over the basis P = (p1,p2,p3), with q1 = (x1,y1,z1), q2 = (x2,y2,z2), q3 = (x3,y3,z3), q4 = (x4,y4,z4). Over the canonical basis  
作为第一步，可以方便地表示（q1、q2、q3、q4）基P=（p1、p2、p3），其中q1=（x1、y1、z1），q2=（x2、y2、z2），q3=（x3、y3、z3），q4=（x4、y4、z4）。超越规范基础

E, the points (p1,p2,p3,p4) are given by the coordinates),  
e，点（p1，p2，p3，p4）由坐标给出，

), and similarly, the points (q1,q2,q3,q4) are given by the  
，同样，点（q1、q2、q3、q4）由

coordinates  
协调

Proposition 25.8. 2With respect to the basis P = (p1,p2,p3), the matrix AP of the unique homography h of RP mapping the projective frame ([p1],[p2],[p3],[p4]) to the projective frame [(q1],[q2],[q3],[q4]) is given by  
提案25.8.2关于基P=（p1，p2，p3），将投影帧（[p1]、[p2]、[p3]、[p4]）映射到投影帧[（q1]、[q2]、[q3]、[q4]）的RP的唯一同形H的矩阵ap由下式给出。

.  
.

Proof. Let u1 = α1p1, u2 = α2p2, u3 = α3p3, and let v1 = λ1q1, v2 = λ2q2, v3 = λ3q3, so that  
证据。设u1=α1p1，u2=α2p2，u3=α3p3，v1=λ1q1，v2=λ2q2，v3=λ3q3，这样

p4 = u1 + u2 + u3  
P4=U1+U2+U3

and q4 = v1 + v2 + v3.  
q4=v1+v2+v3。

Because p1,p2,p3 are linearly independent and since αi = 06 for i = 1,2,3, the vectors (u1,u2,u2) are also linearly independent, so there is a unique linear map f : R3 → R3, such that  
因为p1，p2，p3是线性无关的，而且由于αi=06，i=1,2,3，向量（u1，u2，u2）也是线性无关的，所以有一个唯一的线性映射f:r3→r3，这样

f(ui) = vi i = 1,...,3,  
f（ui）=vi i=1，…，3，

and by linearity  
以及线性

f(p4) = f(u1 + u2 + u3) = f(u1) + f(u2) + f(u3) = v1 + v2 + v3 = q4.  
f（p4）=f（u1+u2+u3）=f（u1）+f（u2）+f（u3）=v1+v2+v3=q4。

With respect to the basis P = (p1,p2,p3), we have  
关于基础p=（p1，p2，p3），我们有

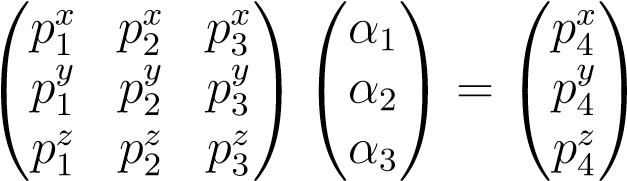
,  
，

so with respect to the basis P, the matrix of f is  
所以对于基P，f的矩阵是

,  
，

as claimed.   
如要求。

If we assume that we pick the coordinates of (p1,p2,p3,p4) and (q1,q2,q3,q4) with respect to the canonical basis E, then the coordinates α1,α2,α3 and λ1,λ2,λ3 are solutions of the systems  
假设我们选取（p1，p2，p3，p4）和（q1，q2，q3，q4）相对于标准基e的坐标，那么坐标α1，α2，α3和λ1，λ2，λ3是系统的解。



and  
和

,  
，

and the matrix AE of our linear map f with respect to the canonical basis is determined as follows.  
关于正则基的线性映射f的矩阵ae确定如下。

Proposition 25.9. With respect to the canonical basis2 E = (e1,e2,e3), the matrix AE of the unique homography h of RP mapping the projective frame ([p1],[p2],[p3],[p4]) to the projective frame [(q1],[q2],[q3],[q4]) is given by  
提案25.9.对于标准基2 e=（e1，e2，e3），将射影帧（[p1]、[p2]、[p3]、[p4]）映射到射影帧[（q1]、[q2]、[q3]、[q4]）的RP唯一同形H的矩阵ae由下式给出。

.  
.

Proof. Since f : R3 → R3 is the unique linear map given by  
证据。因为f:r3→r3是由

f(ui) = vi, i = 1,...,3,  
f（ui）=vi，i=1，…，3，

the map f : R3 → R3 is equal to the composition  
图f:r3→r3等于组成

f = fQ ◦ g,  
F=FQ G，

where g: R3 → R3 is the unique linear map given by  
其中，g:r3→r3是由

g(ui) = ei, i = 1,...,3,  
g（ui）=ei，i=1，…，3，

and fQ : R3 → R3 is the unique linear map given by  
而fq:r3→r3是由

fQ(ei) = vi, i = 1,...,3.  
fq（ei）=vi，i=1，…，3.

However, g: R3 → R3 is the inverse of the unique linear map fP : R3 → R3 given by  
然而，g:r3→r3是唯一线性映射fp:r3→r3的倒数，由

fP(ei) = ui, i = 1,...,3,  
fp（ei）=ui，i=1，…，3，

so f = fQ ◦ fP−1.  
所以f=fq fp−1。

The matrix BP representing fP over the canonical basis E is  
在标准基e上表示fp的矩阵bp是

,  
，

and similarly the matrix BQ representing fQ over E is  
同样的，表示FQ的Bq矩阵是

,  
，

and we have  
我们有

AE = BQBP−1.  
ae=BQBP−1。

Therefore, we have  
因此，我们有

,  
，

as claimed   
如声明

The above method generalizes immediately to any dimension (and any field K). If  
上述方法立即推广到任何维度（和任何字段k）。

([p1],...,[pn+1],[pn+2]) and [(q1],...,[qn+1],[qn+2]) are any two projective frames in a projective space P(E) where E is a K-vector space of dimension n + 1, then (p1,...,pn+1) is a basis of E denoted by P and (q1,...,qn+1) is a basis of E denoted Q, and we can write  
（[p1]、…、[p n+1]、[pn+2]）和[（q1]、…、[qn+1]、[qn+2]）是投影空间p（e）中任意两个投影帧，其中e是尺寸n+1的k矢量空间，则（p1，…，pn+1）是表示p的e的基，（q1，…，qn+1）是表示q的e的基，我们可以写

pn+2 = α1p1 + ··· + αn+1pn+1  
Pn+2=α1p1+····+αn+1pn+1

qn+2 = λ1q1 + ··· + λn+1qn+1  
qn+2=λ1q1+····+λn+1qn+1

for some unique αi,λi ∈ K such that αi = 06 and λi = 06 for i = 1,...,n + 1. If we assume that E = Kn+1, then the canonical basis is E = (e1,...,en+1). If we express the coordinates of the qj over the basis P by  
对于一些独特的αi，λi∈k，使得αi=06和λi=06，对于i=1，…，n+1。如果我们假设e=kn+1，那么规范基础是e=（e1，…，en+1）。如果我们把qj在p基上的坐标表示为

,  
，

then we have the following proposition.  
然后我们有下面的建议。

Proposition 25.10. With respect to the basis P = (p1,...,pn+1), the matrix AP of the unique homography h of P(E) where E is a K-vector space of dimension n + 1, mapping the projective frame ([p1],...,[pn+1],[pn+2]) to the projective frame [(q1],...,[qn+1],[qn+2]) is given by  
提案25.10。对于基P=（p1，…，p n+1），p（e）的唯一同形H的矩阵ap，其中e是尺寸n+1的k向量空间，将投影帧（[p1]、…，[pn+1]、[pn+2]）映射到投影帧[（q1]、…，[qn+1]、[qn+2]），由下式给出。

.  
.

If we express the coordinates of the vectors pi and qi over the canonical basis as pi = (p1i ,...,pni ,pni +1), qi = (qi1,...,qin,qin+1), i = 1,...,n + 2,  
如果我们把规范基上向量Pi和Qi的坐标表示为Pi=（p1i，…，pni，pni+1），Qi=（qi1，…，qi n，qin+1），i=1，…，n+2，

then we have the following result.  
然后我们得到以下结果。

Proposition 25.11. With respect to the canonical basis E = (e1,...,en+1), the matrix AE of the unique homography h of P(E) where E is a K-vector space of dimension n+1, mapping the projective frame ([p1],...,[pn+1],[pn+2]) to the projective frame [(q1],...,[qn+1],[qn+2]) is given by  
提案25.11.关于规范基e=（e1，…，e n+1），p（e）的唯一同形h的矩阵ae，其中e是尺寸n+1的k向量空间，将投影帧（[p1]、…，[pn+1]、[pn+2]）映射到投影帧[（q1]、…，[qn+1]、[qn+2]），由下式给出。

,  
，

where (α1,...,αn+1) and (λ1,...,λn+1) are the solutions of the systems  
其中（α1，…，αn+1）和（λ1，…，λn+1）是系统的解。

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| and 和 |  1 p1 1个P1  . .   .. …   γ   pn1 PN1   γ  pn1+1 pN1+ 1 | ... ... ……  ... …  ... … | p1n P1N  ... …  pnn pnn+1 PNN PNN+1 |  |
|  |  1 q1 1季度  . .   .. …   γ   q1n 问题1   γ  q1n+1 Q1N+ 1 | ... ... ……  ... …  ... … | qn1 QN1  ... …  qnn qnn+1 Qnn Qnn+1 | . . |

We now consider the special case where the points ([p1],[p2],[p3],[p4]) belong to the affine patch of RP2 corresponding to the plane H of equation z = 1. In this case, we may identify [pi] with pi, which has coordinates ( 1) with respect to the canonical basis (the pis are not points at infinity; points at infinity are of of form (x,y,0)). Then, the barycentric coordinates α1,α2,α3 of p4 are solutions of the systems  
我们现在考虑一个特殊情况，即点（[p1]、[p2]、[p3]、[p4]）属于方程z=1的平面h对应的rp2的仿射面片。在这种情况下，我们可以用π来标识[Pi]，它有关于规范基的坐标（Pi不是无穷大的点，无穷大的点是形式（x，y，0））。然后，P4的重心坐标α1，α2，α3是系统的解。

.  
.

By Proposition 25.9, we obtain the following result.  
根据25.9号提案，我们得出以下结果。

Proposition 25.12. With respect to the canonical basis E = (e1,e2,e3), the matrix AE of  
提案25.12.关于规范基e=（e1，e2，e3），矩阵ae

2 the unique homography h of RP mapping (p1,p2,p4,p4), points of the affine plane z = 1, to  
2.RP映射的唯一同形H（p1，p2，p4，p4），仿射平面的点z=1，to

[(q1],[q2],[q3], [q4]) is given by  
[（q1]、[q2]、[q3]、[q4]）由下式给出

.  
.

Observe that the above homography may map some of the affine points p1,p2,p3,p4 (which are not “points at infinity”) to arbitrary points in RP2, which may be points at infinity (in which case qiz = 0). The generalization to any dimension n ≥ 2 is immediate.  
观察上述同形图可以将仿射点p1、p2、p3、p4（不是“无穷远点”）映射到rp2中的任意点，该点可能是无穷远点（在这种情况下，qiz=0）。对任意维n≥2的推广是直接的。

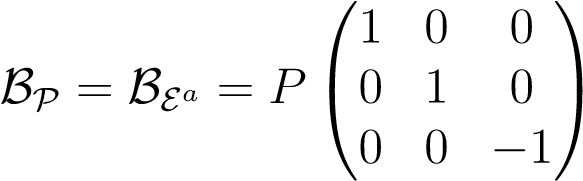
We define the basis), with 1), and  
我们定义了基础），用1）和

call it the affine canonical basis (of R2). We also define  
称之为仿射正则基（r2）。我们还定义

In the special case where (p1,p2,p3,p4) is the canonical square (), since  
在特殊情况下（p1，p2，p3，p4）是标准正方形（），因为

,  
，

we have α1 = 1,α2 = 1, and α3 = −1, so  
α1=1，α2=1，α3=-1，所以



where P is the change of basis matrix from the canonical basis E = (e1,e2,e3) to the affine  
式中，p是基矩阵从规范基e=（e1，e2，e3）到仿射的变化。

basis). We have  
基础）。我们有

|  |  |  |
| --- | --- | --- |
| and its inverse is 它的反方向是 |  | , ， |
| In this case, 在这种情况下， |  | . . |

,  
，

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | 0 0 −1 1 0 0  0 0−1 1 0 0 | |
|  | , ， |
| we obtain 我们得到 |  |  | , ， |

and since that is,  
既然是这样，

q1x q2x q3xλ1 0 0   
q1x q2x q3xλ10 0\_

AE = qq11yz qq22yz qq33yz 00 λ02 λ03.  
ae=qq11yz qq22yz qq33yz 00λ02λ03。

The generalization to any dimension n ≥ 2 is immediate.  
对任意维n≥2的推广是直接的。

Finally, we consider the special case where the points ([p1],[p2],[p3],[p4]) and the points [(q1],[q2],[q3],[q4]) belong to the affine patch of RP2 corresponding to the plane H of equation z = 1. In this case, we may also identify [qi] with qi, which has coordinates ( 1) with respect to the canonical basis. Then, the barycentric coordinates λ1,λ2,λ3 of q4 are solutions of the systems  
最后，我们考虑点（[p1]、[p2]、[p3]、[p4]）和点[（q1]、[q2]、[q3]、[q4]）属于方程z=1的平面h对应的rp2的仿射面片的特殊情况。在这种情况下，我们也可以用qi来标识[qi]，qi具有相对于规范基础的坐标（1）。然后，Q4的重心坐标λ1、λ2、λ3是系统的解。

.  
.

By Proposition 25.12 we obtain the following result.  
根据25.12号提案，我们得出以下结果。

Proposition 25.13. With respect to the canonical basis2 E = (e1,e2,e3), the matrix AE of the unique homography h of RP mapping (p1,p2,p4,p4) to (q1,q2,q3,q4), all points of the affine plane z = 1, is given by  
提案25.13.对于规范基2 e=（e1，e2，e3），RP映射（p1，p2，p4，p4）到（q1，q2，q3，q4）的唯一同形H的矩阵ae，仿射平面z=1的所有点由下式给出：

.  
.

If  
如果

,  
，

the transformed point of a point (x,y,1) in the affine plane z = 1,  
仿射平面z=1中点（x，y，1）的变换点，

,  
，

is not a point at infinity iff a31x + a32y + a33 = 06 , in which case it corresponds to the point in the affine plane z = 1 of coordinates  
不是无穷大的点iff a31x+a32y+a33=06，在这种情况下，它对应于坐标的仿射平面z=1中的点

.  
.

The generalization to any dimension n ≥ 2 is immediate.  
对任意维n≥2的推广是直接的。

Let us go back to the situation where the the points (p1,p2,p3,p4) and (q1,q2,q3,q4) are in the affine patch z = 1, and where the matrix of our linear map is expressed with respect to the basis P = (p1,p2,p3) and the coordinates of (q1,q2,q3,q4) are also expressed with respect to the basis P = (p1,p2,p3). In practical situations, for example in computer vision, it is important to find necessary and sufficient conditions for the unique projective transformation mapping (p1,p2,p3,p4) to (q1,q2,q3,q4) to be defined on the convex hull of the points p1,p2,p3,p4.  
让我们回到点（p1，p2，p3，p4）和（q1，q2，q3，q4）在仿射面片z=1中的情况，我们的线性映射的矩阵表示为基P=（p1，p2，p3），并且（q1，q2，q3，q4）的坐标也表示为基P=（p1，第2页，第3页）。在实际情况下，例如在计算机视觉中，重要的是要找到在点p1、p2、p3、p4的凸包上定义到（q1、q2、q3、q4）的唯一投影变换映射（p1、p2、p3、p4）的必要和充分条件。

Proposition 25.14. The unique projective transformation mapping (p1,p2,p3,p4) to (q1,q2, q3,q4) (all points in the affine plane H of equation z = 1) is defined on the convex hull of the points p1,p2,p3,p4 iff the scalars in each of the pairs (α1,λ1), (α2,λ2) and (α3,λ3), have the same sign.  
提案25.14.唯一射影变换映射（p1，p2，p3，p4）到（q1，q2，q3，q4）（方程式z=1的仿射平面h中的所有点）定义在点p1，p2，p3，p4的凸壳上，如果每个对（α1，λ1），（α2，λ2）和（α3，λ3）中的标量具有相同的符号。

Proof. With respect to the basis P, the equation of the plane H is  
证据。关于基P，平面h的方程是

x + y + z = 1,  
x+y+z=1，

so the image of p = (x,y,1 − x − y) under our linear map is  
所以我们的线性图下的p=（x，y，1−x−y）的图像是

.  
.

The above point is a point at infinity iff  
上面的点是无穷远处的点。

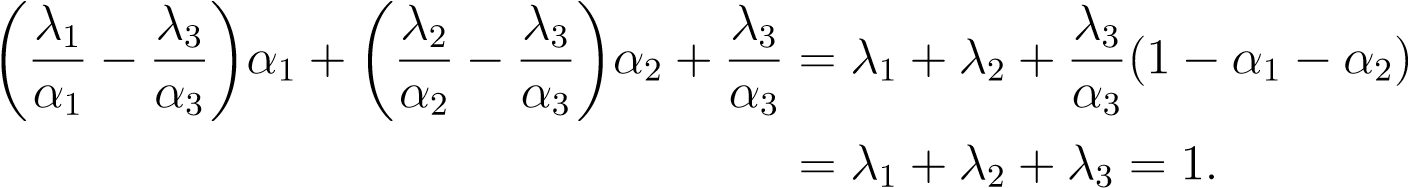
. (∗)  
.（）

The unique projective transformation mapping (p1,p2,p3,p4) to (q1,q2,q3,q4) is defined on the convex hull of the points p1,p2,p3,p4 iff all four points p1,p2,p3,p4 are strictly contained in one of the two open half spaces determined by the line of equation (∗), which means that the affine form in (∗) must have the same sign on these four points.  
唯一射影变换映射（p1，p2，p3，p4）到（q1，q2，q3，q4）是在点p1，p2，p3，p4的凸壳上定义的。如果所有四个点p1，p2，p3，p4都严格包含在方程（）所确定的两个半空间中的一个中，这意味着（）这四点上必须有相同的符号。

When we evaluate the affine form in (∗) on the four points p1,p2,p3,p4 using coordinates  
当我们用坐标对四个点p1，p2，p3，p4上（）的仿射形式进行评估时

(x,y,1 − x − y), w.r.t. the basis P = (p1,p2,p3),  
（x，y，1−x−y），w.r.t.基P=（p1，p2，p3）

1. for p1 = (1,0,0) we get λ1/α1,  
   对于p1=（1,0,0），我们得到λ1/α1，
2. for p2 = (0,1,0) we get λ2/α2,  
   对于p2=（0,1,0），我们得到λ2/α2，
3. for p3 = (0,0,1) we get λ3/α3,  
   对于p3=（0,0,1），我们得到λ3/α3，
4. and for p4 = (α1,α2,α3) we get  
   对于p4=（α1，α2，α3），我们得到



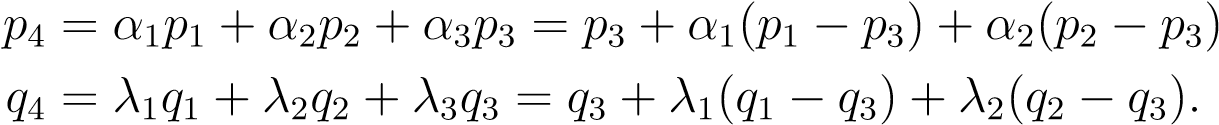
The fourth case shows that the sign of the affine form in (∗) is positive, and thus λ1/α1, λ2/α2,λ3/α3 > 0, which implies that the scalars in each of the pairs (α1,λ1), (α2,λ2) and (α3,λ3), must have the same sign.   
第四种情况表明（）中的仿射形式的符号为正，因此λ1/α1，λ2/α2，λ3/α3>0，这意味着每对（α1，λ1），（α2，λ2）和（α3，λ3）中的标量必须具有相同的符号。

The generalization to any dimension n ≥ 2 is immediate: the scalars in each pair (αi,λi) must have the same sign for i = 1,...,n + 2.  
对任何维数n≥2的推广是直接的：每对（αi，λi）中的标量对于i=1，…，n+2必须有相同的符号。

In dimension 2, since α3 = 1 − α1 − α2 and λ3 = 1 − λ1 − λ2, there are four cases to consider:  
在维度2中，由于α3=1−α1−α2和λ3=1−λ1−λ2，有四种情况需要考虑：

1. α1,λ1,α2,λ2 < 0. In this case, α3,λ3 > 1 so α3,λ3 also have the same sign.  
   α1，λ1，α2，λ2<0.在这种情况下，α3，λ3>1，因此α3，λ3也有相同的符号。
2. α1,λ1 < 0 and α2,λ2 > 0. In this case, since α3 = 1 − α1 − α2 and λ3 = 1 − λ1 − λ2, we must have either both α1 + α2 < 1 and λ1 + λ2 < 1, or both α1 + α2 > 1 and λ1 + λ2 > 1, in order for α3 and λ3 to have the same sign.  
   α1，λ1<0和α2，λ2>0。在这种情况下，由于α3=1-α1-α2和λ3=1-λ1-λ2，我们必须同时具有α1+α2<1和λ1+λ2<1，或者同时具有α1+α2>1和λ1+λ2>1，以便α3和λ3具有相同的符号。
3. α1,λ1 > 0 and α2,λ2 < 0. As in the previous case, since α3 = 1 − α1 − α2 and λ3 = 1 − λ1 − λ2, we must have either both α1 + α2 < 1 and λ1 + λ2 < 1, or both α1 + α2 > 1 and λ1 + λ2 > 1, in order for α3 and λ3 to have the same sign.  
   α1，λ1>0，α2，λ2<0.与前一种情况一样，由于α3=1-α1-α2和λ3=1-λ1-λ2，我们必须同时具有α1+α2<1和λ1+λ2<1，或者同时具有α1+α2>1和λ1+λ2>1，以便α3和λ3具有相同的符号。
4. α1,λ1,α2,λ2 > 0. As in the previous case, since α3 = 1−α1 −α2 and λ3 = 1−λ1 −λ2, we must have either both α1 + α2 < 1 and λ1 + λ2 < 1, or both α1 + α2 > 1 and λ1 + λ2 > 1, in order for α3 and λ3 to have the same sign.  
   α1，λ1，α2，λ2>0.在前面的例子中，由于α3=1−α1−α2和λ3=1−λ1−λ2，我们必须同时拥有α1+α2<1和λ1+λ2<1，或者同时拥有α1+α2>1和λ1+λ2>1，以便α3和λ3具有相同的符号。

Since α3 = 1 − α1 − α2 and λ3 = 1 − λ1 − λ2, we can write  
由于α3=1−α1−α2和λ3=1−λ1−λ2，我们可以写



In the affine frame (p3,(p1 − p3,p2 − p3)), points have coordinates (α1,α2), and in the affine frame (q3,(q1 − q3,q2 − q3)), points have coordinates (λ1,λ2). In the first affine frame, the line hp1,p2i is given by the equation α1 + α2 = 1, and in the second affine frame, the line hq1,q2i is given by the equation λ1 +λ2 = 1. The open half plane containing p3 and bounded by the line hp1,p2i corresponds to the points of coordinates (α1,α2) satisfying α1 + α2 < 1, and the other open half plane not containing p3 corresponds to the points of coordinates (α1,α2) satisfying α1 +α2 > 1. Similarly, the open half plane containing q3 and bounded by the line hq1,q2i corresponds to the points of coordinates (λ1,λ2) satisfying λ1 + λ2 < 1, and the other open half plane not containing q3 corresponds to the points of coordinates (λ1,λ2) satisfying λ1 + λ2 > 1.  
在仿射框架（p3，（p1−p3，p2−p3））中，点具有坐标（α1，α2），在仿射框架（q3，（q1−q3，q2−q3））中，点具有坐标（λ1，λ2）。在第一个仿射框中，线hp1，p2i由公式α1+α2=1给出，在第二个仿射框中，线hq1，q2i由公式λ1+λ2=1给出。含有p3且以线hp1、p2i为界的开半平面对应于满足α1+α2<1的坐标点（α1，α2），另一个不含p3的开半平面对应于满足α1+α2>1的坐标点（α1，α2）。同样，含有q3且以线hq1、q2i为界的开半平面对应于满足λ1+λ2<1的坐标点（λ1、λ2），另一个不包含q3的开半平面对应于满足λ1+λ2>1的坐标点（λ1、λ2）。

Then, the above conditions have the following interpretation in terms of regions in the affine plane z = 1:  
那么，上述条件对于仿射平面z=1中的区域有如下解释：

1. When α1 < 0 and α2 < 0, the point p4 lies in quadrant III (with respect to the affine frames (p3,(p1 − p3,p2 − p3))). Under the mapping f, the point q4 is also mapped to quadrant III (with respect to the affine frame (q3,(q1 −q3,q2 −q3))); see Figure 25.14.  
   当α1<0和α2<0时，点P4位于象限III（相对于仿射帧（p3，（p1-p3，p2-p3））。在映射F下，点Q4也映射到象限III（相对于仿射帧（q3，（q1-q3，q2-q3））；参见图25.14。

p

3

p

1

p

z= 1

z= 1

q

3

2

q

2

q

1

f

I

II

III

IV

I

II

III

IV

p

3

p

1

p

-

p

3

p

2

p

-

p

3

p

4

3

1

2

4

q

q

q

q

2

3

q

-

q

q

-

q

1

3

f

1

2

Figure 25.14: Case (1)  
图25.14：案例（1）

1. When α1,λ1 < 0 and α2,λ2 > 0, the points p4 and q4 belongs to quadrant II (with respect to the affine frames (p3,(p1 − p3,p2 − p3)) and (q3,(q1 − q3,q2 − q3))). Two possibilities occur. Either p4 belong to the open half space containing p3 and bounded by the line hp1,p2i and q4 belong to the open half space containing q3 and bounded by the line hq1,q2i, or p4 belong to the open half space not containing p3 and bounded by the line hp1,p2i and q4 belong to the open half space not containing q3 and bounded by the line hq1,q2i. The first possibility is illustrated by the top of Figure 25.15, while the second is illustrated by the bottom of Figure 25.15.  
   当α1、λ1<0和α2、λ2>0时，点p4和q4属于象限II（相对于仿射帧（p3，（p1−p3，p2−p3））和（q3，（q1−q3，q2−q3））。有两种可能性。p4属于包含p3的开放半空间，以hp1、p2i和q4线为界，属于包含q3的开放半空间，以hq1、q2i或p4线为界，属于不包含p3的开放半空间，以hp1、p2i和q4线为界，属于不包含co的开放半空间。第三行，以Hq1，q2i线为界。第一种可能性由图25.15的顶部说明，第二种可能性由图25.15的底部说明。
2. When α1,λ1 > 0 and α2,λ2 < 0, the points p4 and q4 belongs to quadrant IV (with respect to the affine frames (p3,(p1 − p3,p2 − p3)) and (q3,(q1 − q3,q2 − q3))). Two possibilities occur exactlty as in Case (2) depending on the position of p4 with respect to the line hp1,p2i and on the position of q4 with respect to the line hq1,q2i. The first possibility is illustrated by the top of Figure 25.16, while the second is illustrated by the bottom of Figure 25.16.  
   当α1，λ1>0和α2，λ2<0时，点P4和Q4属于象限IV（相对于仿射帧（P3，（P1−P3，P2−P3））和（Q3，（Q1−Q3，Q2−Q3））。两种可能性发生得很准确，如情况（2）所示，这取决于P4相对于线hp1，p2i得位置以及q4相对于线hq1，q2i得位置.图25.16顶部说明了第一种可能性，图25.16底部说明了第二种可能性.
3. When α1,λ1,α2,λ > 0 and α2,λ2 < 0, the points p4 and q4 belongs to quadrant I  
   当α1、λ1、α2、λ>0和α2、λ2<0时，点P4和Q4属于象限I。

p

3

p

1

p

-

p

3

p

2

p

-

p

3

p

4

3

1

2

4

q

q

q

q

3

q

-

q

q

-

q

1

3

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*2*

*1*

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p

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p

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3

p

2

p

-

p

3

*α*

*α*

*2*

*1*

p

4

3

1

2

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q

q

q

q

2

3

q

-

q

q

-

q

1

3

*λ*

*-*

*λ*

*1*

*2*

f

*+*

*=*

*1*

*+*

*=*

*1*

1

2

Figure 25.15: Case (2)  
图25.15：案例（2）

(with respect to the affine frames (p3,(p1 − p3,p2 − p3)) and (q3,(q1 − q3,q2 − q3))). Two possibilities occur exactlty as in Cases (2) and (3) depending on the position of p4 with respect to the line hp1,p2i and on the position of q4 with respect to the line hq1,q2i. The first possibility is illustrated by the top of Figure 25.17, while the second is illustrated by the bottom of Figure 25.17.  
（关于仿射帧（p3，（p1−p3，p2−p3））和（q3，（q1−q3，q2−q3））。根据p4相对于线hp1、p2i的位置和q4相对于线hq1、q2i的位置，有两种可能发生，如案例（2）和（3）所示。第一种可能出现在图25.17的顶部，第二种可能出现在图的底部。第25.17条。

Thus, if both (p1,p2,p3,p4) and (q1,q2,q3,q4) satisfy the conditions listed above, there is no point at infinity inside of the convex hull of the quadrangle (p1,p2,p3,p4).  
因此，如果（p1，p2，p3，p4）和（q1，q2，q3，q4）都满足上述条件，则四边形（p1，p2，p3，p4）的凸壳内部没有无穷大的点。

It remains to prove that the image of the convex hull of (p1,p2,p3,p4) is the convex hull of (q1,q2,q3,q4).  
还有待证明（p1，p2，p3，p4）的凸壳图像是（q1，q2，q3，q4）的凸壳图像。

Proposition 25.15. If both (p1,p2,p3,p4) and (q1,q2,q3,q4) satisfy the conditions of Proposition 25.14, then the image of the convex hull of (p1,p2,p3,p4) under the unique projective map mapping (p1,p2,p3,p4) to (q1,q2,q3,q4) is the convex hull of (q1,q2,q3,q4)  
提案25.15。如果（p1，p2，p3，p4）和（q1，q2，q3，q4）都满足命题25.14的条件，那么（p1，p2，p3，p4）在唯一投影映射（p1，p2，p3，p4）到（q1，q2，q3，q4）下的凸壳图像就是（q1，q2，q3，q4）的凸壳。

Proof. It suffices to show that the restriction of our projective transformation maps a line segment to the convex hull of the images of the endpoints of this segment. Thus, the problem  
证据。证明了射影变换的限制条件将直线段映射到该段端点图像的凸包。因此，问题是

p

3

p

1

p

-

p

3

p

2

p

-

p

3

p

4

3

1

2

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q

q

q

3

q

-

q

q

-

q

1

3

*α*

*-*

*α*

*2*

*1*

*λ*

*-*

*λ*

*1*

*2*

2

f

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*1*

+

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*1*

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1

2

p

3

p

1

p

-

p

3

p

2

p

-

p

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*α*

*2*

*1*

p

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q

q

2

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*λ*

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*2*

f

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+

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*1*

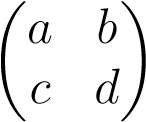
*+*

1

2

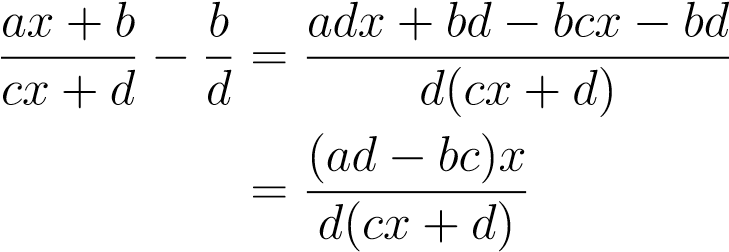
Figure 25.16: Case (3)  
图25.16：案例（3）

reduces to proving that if a projective transformation given by an invertible matrix  
减少到证明如果由可逆矩阵给出的射影变换

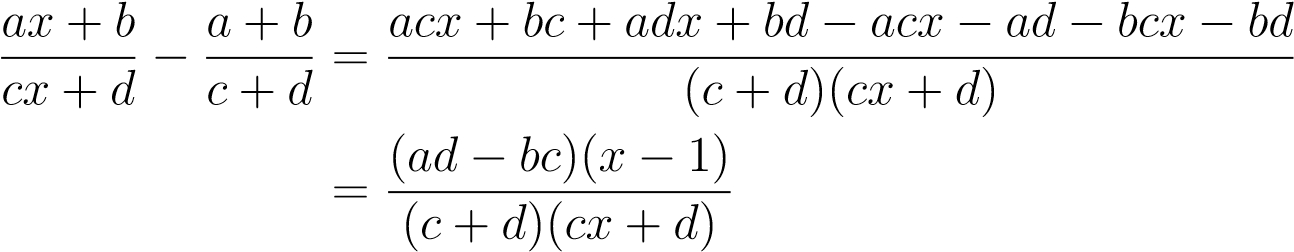


does not have points at infinity on the line segment in R2 corresponding to the points of coordinates (x,1) with 0 ≤ x ≤ 1, then the image of the line segment [(0,1),(1,1)] is the line segment [(b/d,1),((a + b)/(c + d),1)] (or [((a + b)/(c + d),1),(b/d,1)]).  
在r2的直线段上没有与0≤x≤1的坐标点（x，1）对应的无穷远点，则直线段的图像[（0,1），（1,1）]是直线段[（b/d，1），（（a+b）/（c+d），1）]（或[（（a+b）/（c+d），1），（b/d，1）]。

We have  
我们有



and  
和



25.7. AFFINE PATCHES  
25.7。仿射补丁

p

3

p

p

-

p

3

p

2

p

-

p

3

p

4

3

1

2

4

q

q

q

q

3

q

-

q

q

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*1*

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q

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q

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3

*λ*

*-*

*λ*

*1*

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f

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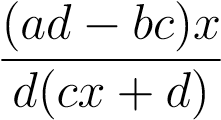
1

2

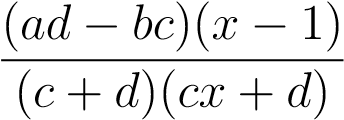
Figure 25.17: Case (4)  
图25.17：案例（4）

In order for our map to be defined for 0 ≤ x ≤ 1, cx + d must have a constant sign for  
为了使我们的地图定义为0≤x≤1，cx+d必须有一个常量符号

0 ≤ x ≤ 1, which means that d and c + d have the same sign. Then,  
0≤x≤1，表示d与c+d符号相同。然后，



and  
和



have opposite signs when 0 < x < 1, which means that the image of [0,1] is the interval [b/d,(a + b)/(c + d)] (or [(a + b)/(c + d),b/d]).   
当0<x<1时有相反的符号，这意味着[0,1]的图像是间隔[b/d，（a+b）/（c+d）]（或[（a+b）/（c+d），b/d]）。

We now consider the projective completion of an affine space. First, we introduce the notion of affine patch.  
我们现在考虑仿射空间的射影完备。首先，我们介绍仿射补丁的概念。

## 25.7 Affine Patches 25.7仿射补丁

Given an affine space E with associated vector space →−E, we can form the vector space Eb, the homogenized version of E, and then, the projective space P induced by Eb. This projective space, also denoted by Ee, has some very interesting properties. In fact, it satisfies a universal property, but before we can say what it is, we have to take a closer look at Ee.  
给定一个具有相关向量空间→−e的仿射空间e，我们可以形成向量空间eb，e的均匀化版本，然后由eb诱导的射影空间p。这个射影空间，也用ee表示，有一些非常有趣的性质。事实上，它满足一个普遍的属性，但在我们说出它是什么之前，我们必须仔细看看EE。

Since the vector space Eb is the disjoint union of elements of the form ha,λi, where a ∈ E and λ ∈ K − {0}, and elements of the form u ∈ →−E, observe that if ∼ is the equivalence relation on Eb used to define the projective space P , then the equivalence class [ha,λi]∼ of a weighted point contains the special representative a = ha,1i, and the equivalence class [u]∼ of a nonzero vector is just a point of the projective space P . Thus, there is a bijection  
由于向量空间eb是形式ha，λi的元素的不相交并，其中a∈e和λ∈k−0，以及形式u∈→−e的元素，观察如果是用于定义射影空间p的eb上的等价关系，那么一个权重的等价类[ha，λi]Ted点包含特殊的代表a=ha，1i，非零向量的等价类[u]只是射影空间p的点。因此，有一个双射

P   
磷

between P and the disjoint union , which allows us to view E as being embedded in P . The points of P E in P E will be called points at infinity, and the projective hyperplane P is called the hyperplane at infinity. We will also denote the point [u]∼ of P (where u = 0)6 by u∞.  
在P和不相交的联合之间，这允许我们将E视为嵌入在P中。p e中的点称为无穷远处的点，投影超平面p称为无穷远处的超平面。我们还将用u∞表示点[u]p（其中u=0）6。

Thus, we can think of as the projective completion of the affine space E obtained by adding points at infinity forming the hyperplane P . As we commented in Section 25.2 when we presented the hyperplane model of P(E), the notion of point at infinity is really an affine notion. But even if a vector space E doesn’t arise from the completion of an affine space, there is an affine structure on the complement of any hyperplane P(H) in the projective space P(E). In the case of Ee, the complement E of the projective hyperplane P is indeed an affine space. This is a general property that is needed in order to figure out the universal property of Ee.  
因此，我们可以把仿射空间e看作是在无穷远处加上点形成超平面p而得到的射影完备。正如我们在25.2节中所评论的，当我们提出p（e）的超平面模型时，无穷远点的概念实际上是一个仿射概念。但是，即使向量空间e不是仿射空间的完备形式，在射影空间p（e）中任何超平面p（h）的补上都存在仿射结构。在e e的情况下，射影超平面p的补e实际上是一个仿射空间。这是计算EE的通用属性所需的通用属性。

Proposition 25.16. Given a vector space E and a hyperplane H in E, the complement EH = P(E) − P(H) of the projective hyperplane P(H) in the projective space P(E) can be given an affine structure such that the associated vector space of EH is H. The affine structure on EH depends only on H, and under this affine structure, EH is isomorphic to an affine hyperplane in E.  
提案25.16。给定一个向量空间e和e中的超平面h，射影空间p（e）中射影超平面p（h）的补码e h=p（e）−p（h）可以给出一个仿射结构，使得e h的相关向量空间为h。e h上的仿射结构仅依赖于h，并且在这个仿射结构下。结构，eh同构于e中的仿射超平面。

Proof. Since H is a hyperplane in E, there is some w ∈ E−H such that E = Kw⊕H. Thus, every vector u in E−H can be written in a unique way as λw+h, where λ = 06 and h ∈ H. As a consequence, for every point [u] in EH, the equivalence class [u] contains a representative of the form w + λ−1h, with λ = 06 . Then we see that the map ϕ: (w + H) → EH, defined such that  
证据。由于h是e中的超平面，因此存在一些w∈e−h，e=kw h。因此，e−h中的每个向量u都可以用独特的方式写成λw+h，其中，λ=06和h∈h。因此，对于eh中的每个点[u]，等价类[u]都包含形式w+λ−1h，w的代表。ithλ=06.然后我们看到地图\_：（w+h）→eh，定义如下：

ϕ(w + h) = [w + h],  
⑨（w+h）=[w+h]，

is a bijection. In order to define an affine structure on EH, we define +: EH × H → EH as follows: For every point [w + h1] ∈ EH and every h2 ∈ H, we let  
是一个双射。为了在eh上定义一个仿射结构，我们定义了+：eh×h→eh如下：对于每一点[w+h1]∈eh和每一个h2∈h，我们让

[w + h1] + h2 = [w + h1 + h2].  
[w+h1]+h2=[w+h1+h2]。

The axioms of an affine space are immediately verified. Now, w + H is an affine hyperplane is E, and under the affine structure just given to EH, the map ϕ: (w+H) → EH is an affine  
仿射空间的公理立即得到验证。现在，w+h是仿射超平面，e是仿射结构，在刚刚给eh的仿射结构下，图（w+h）→eh是仿射

25.7. AFFINE PATCHES  
25.7。仿射补丁

map that is bijective. Thus, EH is isomorphic to the affine hyperplane w + H. If we had chosen a different vector w0 ∈ E −H such that E = Kw0 ⊕H, then EH would be isomorphic to the affine hyperplane w0 + H parallel to w + H. But these two hyperplanes are clearly isomorphic by translation, and thus the affine structure on EH depends only on H.   
双目标地图。因此，e h与仿射超平面w+h是同构的，如果我们选择一个不同的向量w0∈e−h，使e=kw0 h，那么eh将与平行于w+h的仿射超平面w0+h同构，但这两个超平面通过平移显然是同构的，因此仿射结构你对eh的依赖仅仅在于h。

An affine space of the form EH is called an affine patch on P(E). Proposition 25.16 allows us to view a projective space P(E) as the result of gluing some affine spaces together, at least when E is of finite dimension. For example, when E is of dimension 2, a hyperplane in E is just a line, and the complement of a point in the projective line P(E) can be viewed as an affine line. Thus, we can view P(E) as being covered by two affine lines glued together as illustrated by When K = R, this shows that topologically, the projective line RP1 is equivalent to a circle. See Figure 25.18. When E is of dimension 3, a hyperplane in E is  
形式eh的仿射空间称为p（e）上的仿射补丁。命题25.16允许我们看到投影空间p（e），至少当e是有限维的时候，是把一些仿射空间粘合在一起的结果。例如，当e是维数2时，e中的超平面只是一条线，投影线p（e）中点的补部可以看作是仿射线。因此，我们可以把p（e）看作是由两条粘在一起的仿射线所覆盖，如当k=r时所示，这表明在拓扑上，射影线rp1等于一个圆。见图25.18。当e为尺寸3时，e中的超平面为

y = 1

y = 0

Figure 25.18: The covering of RP1 by the affine lines y = 0 and y = 1.  
图25.18：用仿射线y=0和y=1覆盖rp1。

just a plane, and the complement of a projective line in the projective plane P(E) can be viewed as an affine plane. Thus, we can view P(E) as being covered by three affine planes glued together as illustrated by Figure 25.19.  
只要一个平面，射影平面p（e）中射影线的补部就可以看作是仿射平面。因此，我们可以将p（e）视为由三个粘合在一起的仿射平面覆盖，如图25.19所示。

However, even when K = R, it is much more difficult to come up with a geometric embedding of the projective plane RP2 in A3, and in fact, this is impossible! Nevertheless, there are some fascinating immersions of the projective space RP2 as 3D surfaces with selfintersection, one of which is known as the Boy surface. We urge our readers to consult the remarkable book by Hilbert and Cohn-Vossen [90] for drawings of the Boy surface, and more. One should also consult Fischer’s books [62, 61], where many beautiful models of surfaces are displayed, and the commentaries in Chapter 6 of [61] regarding models of RP2. More generally, when E is of dimension n+1, the projective space P(E) is covered by n+1 affine patches (hyperplanes) glued together. This idea is very fruitful, since it allows the treatment of projective spaces as manifolds, and it is essential in algebraic geometry.  
然而，即使当k=r时，要想在a3中嵌入射影平面rp2也要困难得多，事实上，这是不可能的！尽管如此，投影空间rp2还是有一些令人着迷的沉浸感，即自相交的三维表面，其中一个被称为“男孩表面”。我们敦促读者参考希尔伯特和科恩·沃森的这本非凡的书[90]来获取男孩表面的绘画，等等。我们还应该参考费舍尔的书[62，61]，其中展示了许多美丽的表面模型，以及[61]第6章中关于RP2模型的评论。通常，当e的尺寸为n+1时，投影空间p（e）被粘在一起的n+1仿射片（超平面）覆盖。这个想法是非常有成效的，因为它允许把射影空间作为流形来处理，并且它在代数几何中是必不可少的。

We can now go back to the projective completion Ee of an affine space E.  
我们现在可以回到仿射空间e的射影完备ee。

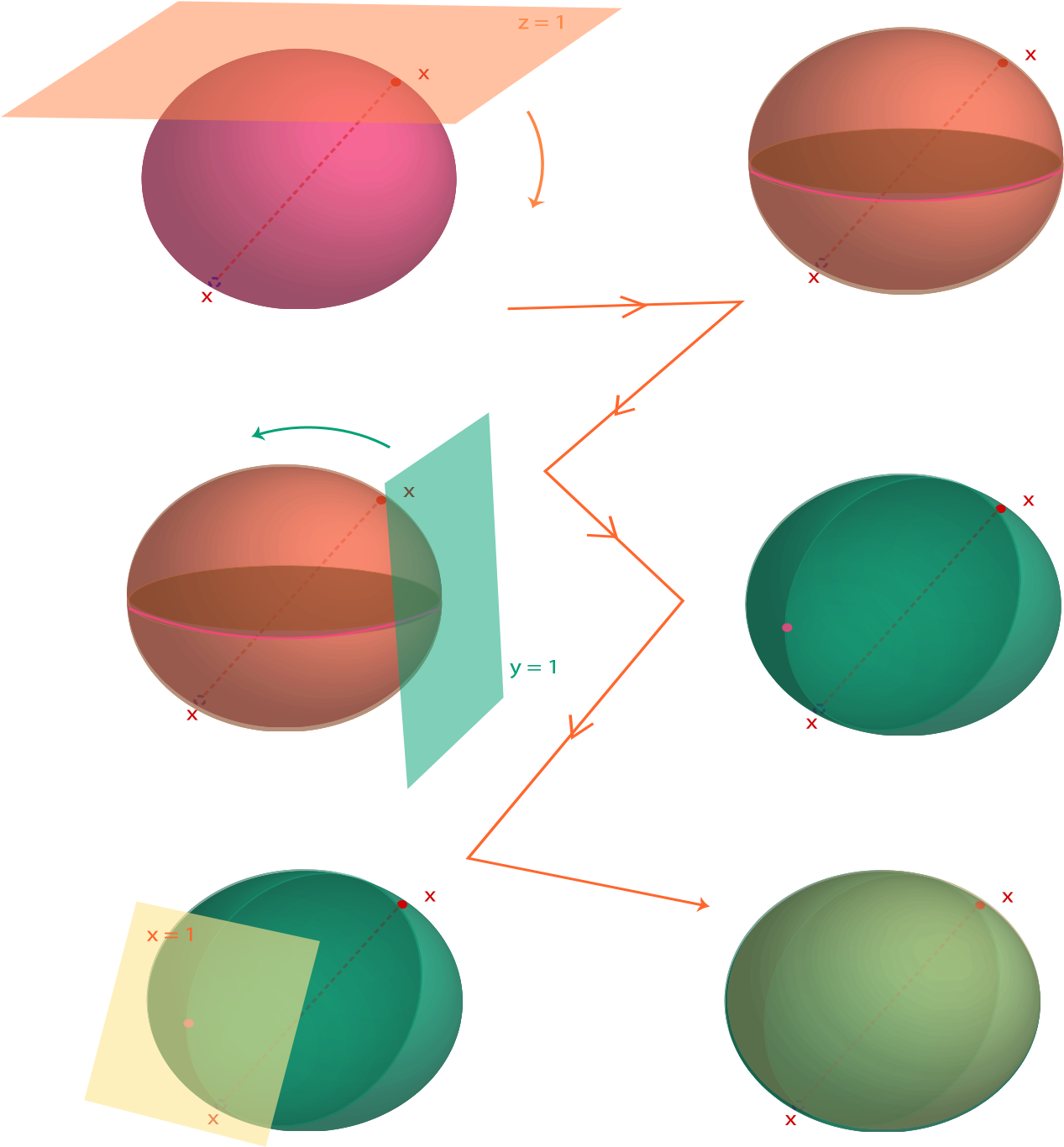


Figure 25.19: The covering of RP2 by the affine planes z = 1, x = 1, and y = 1. The plane z = 1 covers everything but the circle x2 + y2 = 1 in the xy-plane. The plane y = 1 covers that circle modulo the point (1,0,0), which is then covered by the plane x = 1.  
图25.19：仿射平面z=1、x=1和y=1覆盖了rp2。平面z=1覆盖了除xy平面中的圆x2+y2=1之外的所有内容。平面y=1覆盖该圆的模点（1,0,0），然后被平面x=1覆盖。

## 25.8 Projective Completion of an Affine Space 25.8仿射空间的射影完备

We begin by spelling out the universal property characterizing the projective completion of an affine space (E,→−E). Then, we prove thatwhere is the projective space obtained associated with the vector space E obtained from E by the hat construction from Chapter 24 is indeed a projective completion of (E,→−E).  
我们首先阐述了一个仿射空间（e，→−e）的射影完备的普遍性质。然后，我们证明，从第24章的hat构造得到的与从e得到的向量空间e相关的射影空间在哪里，实际上是（e，→−e）的射影完成。

Definition 25.7. Given any affine space E with associated vector space →−E, a projective completion of the affine space E with hyperplane at infinity P(H) is a triple hP(E),P(H),ii, where E is a vector space, H is a hyperplane in E, i: E → P(E) is an injective map such projective spacethat i(E) = EH andP(Fi) (is affine (wherewhere F is some vector space), every hyperplaneEH = P(E) − P(H) is an affine patch), and for everyH in F, and every map f : E → P(F) such that f(E) ⊆ FH and f is affine (where FH = P(F) − P(H) is an  
定义25.7.给定任意具有相关向量空间→−e的仿射空间e，在无穷大p（h）上具有超平面的仿射空间e的射影完备为三重hp（e），p（h），i i，其中e是向量空间，h是e中的超平面，即：e→p（e）是射影映射，即i（e）=eh和p（fi）（是仿射（其中f是一些向量空间），每个超平面=p（e）−p（h）是仿射补丁，对于f中的每一个平面，以及每个映射f:e→p（f），使得f（e）fh和f是仿射（其中fh=p（f）−p（h）是一个

25.8. PROJECTIVE COMPLETION OF AN AFFINE SPACE affine patch), there is a unique projective map) such that  
25.8。仿射空间仿射面片的射影完成），有一个独特的射影图）这样

f = fe◦ i and P   
F=铁I和P

(where →−i : →−E → H and →−f : →−E → H are the linear maps associated with the affine maps i: E → P(E) and f : E → P(F)), as in the following diagram:  
（其中→−i：→−e→h和→−f：→−e→h是与仿射图i:e→p（e）和f:e→p（f）相关联的线性图，如下图所示：

r  
R

E IIIIIiIIIIfI/IIEIHIII⊆IIIPI$ (E )fe⊇ryrPrrr(rHrr)rPro rrPr(r−→rir) rrP  
E IIIII IIIIIFI/I IIEIHIII IIIPI$（E）Fe Ryrprr（rhrr）Rpro rrp r（r−→Rir）rrp



FH ⊆ P(F) ⊇ P(H)  
f h p（f）p（h）

The points of P(E) in P(H) are called points at infinity, and the projective hyperplane P(H) is called the hyperplane at infinity. We will also denote the point [u]∼ of P(H) (where u = 0)6 by u∞. As usual, objects defined by a universal property are unique up to isomorphism. We leave the proof as an exercise.  
p（h）中p（e）的点称为无穷远点，投影超平面p（h）称为无穷远点。我们还将用u∞来表示p（h）（其中u=0）6的点[u]。通常，由一个普遍属性定义的对象在同构上是唯一的。我们把证据留作练习。

The importance of the notion of projective completion stems from the fact that every affine map f : E → F extends in a unique way to a projective map fe: P(E) → P(F), where hP(E),P(HE),iEi is a projective completion ofis a projective completion of F, provided that the restriction of f to P E agrees with P f , as illustrated in the following commutative diagram:  
射影完成概念的重要性源于每个仿射映射f:e→f都以独特的方式延伸到射影映射fe:p（e）→p（f），其中hp（e）、p（he）、iei是f的射影完成的射影完成，前提是f对p e ag的限制带有p f的Rees，如下图所示：

f  
f

E / F iEiF  
E/F IEIF公司

P(E) e / P(F).  
P（E）E/P（F）。

f  
f

We will now show that is the projective completion of E, where i: E → Ee is the injection of E into . For example, if E = A1K is an affine line, its projective completion Af1K is isomorphic to the projective line P(K2), and they both can be identified with A1K ∪ {∞}, the result of adding a point at infinity (∞) to A1K. In general, the projective completion AfmK of the affine space AmK is isomorphic to P(Km+1). Thus, is isomorphic to RPm, andis isomorphic to CPm.  
现在我们将证明这是e的投影完成，其中i:e→ee是e的注入。例如，如果e=a1k是仿射线，其射影完备af1k与射影线p（k2）是同构的，两者都可以用a1k∞来标识，即在无穷大（∞）处加一个点到a1k的结果。一般来说，仿射空间amk的射影完备afmk是isom。奥菲奇到P（km+1）。因此，与RPM同构，与CPM同构。

First, let us observe that if E is a vector space and H is a hyperplane in E, then the homogenization of the affine patch EH (the complement of the projective hyperplane P(H) in P(E)) is isomorphic to E. The proof is rather simple and uses the fact that there is an affine bijection between EH and the affine hyperplane w + H in E, where w ∈ E − H is any fixed vector. Choosing w as an origin in EH, we know that EcH = H +b Kw, and since E = H ⊕ Kw, it is obvious how to define a linear bijection between EcH = H +b Kw and E = H ⊕ Kw. As a consequence the projective spaces EfH and P(E) are isomorphic, i.e., there is a projectivity between them.  
首先，让我们观察一下，如果e是向量空间，h是e中的超平面，那么仿射面片eh（p（e）中投影超平面p（h）的补码）的同构化就是e的同构化，证明相当简单，并且使用了eh和t之间存在仿射双射的事实。他在e中仿射超平面w+h，其中w∈e−h是任何固定向量。选择w作为e h的原点，我们知道ech=h+b kw，由于e=h\_kw，如何定义ech=h+b kw和e=h\_kw之间的线性双射关系是显而易见的。因此，射影空间efh和p（e）是同构的，即它们之间存在射影性。

Proposition 25.17. Given any affine space E,, for every projective space P(F) (where F is some vector space), every hyperplane H in F, and every map f : E → P(F) such that f(E) ⊆ FH and f is affine (FH being viewed as an affine patch), there is a unique projective map fe: Ee → P(F) such that  
提案25.17。给定任意仿射空间e，，对于每个射影空间p（f）（其中f是一些向量空间），f中的每个超平面h，以及每个映射f:e→p（f），使得f（e）fh和f是仿射（fh被视为仿射补丁），有一个独特的射影映射fe:ee→p（f），这样

f = fe◦ i and P ,  
f=fe i和p，

(where →−i : →−E → →−E and →−f : →−E → H are the linear maps associated with the affine maps i: E → Ee and f : E → P(F)), as in the following diagram:  
（其中→−i：→−e→→−e和→−f：→−e→h是与仿射图i:e→ee和f:e→p（f）相关联的线性图，如下图所示：



E IIIIIiIIIIfI/IIEIHIII⊆IIIPI$ (E )fe⊇ryrPrrr(rHrr)rPro rrPr(r−→rir)rrrP  
E IIIII IIIIIFI/I IIEIHIII IIIPI$（E）Fe Ryrprr（rhrr）Rpro Rpr（r−→Rir）Rrrp



FH ⊆ P(F) ⊇ P(H)  
f h p（f）p（h）

Proof. The existence of feis a consequence of Proposition 24.6, where we observe thatis isomorphic to F. Just take the projective map P), where is the unique linear map extending f. It remains to prove its uniqueness.  
证据。fei的存在是24.6命题的一个结果，在这里我们观察到它与f同构。只要取射影映射p），这里是唯一的线性映射延伸f，它仍然是证明其唯一性的。

As explained in the proof of Proposition 25.16, the affine patch FH is affinely isomorphic to some affine hyperplane of the form w + H for some w ∈ F − H. If we pick any a ∈ E, since by hypothesis f(a⊕) ∈ FH, we may assume that→ w ∈ F −H is chosen so that∈ f(a∈) = [→− w], and we have F = Kw H. Since f : E FH is affine, for any a E and any u E, we have  
如命题25.16的证明所解释的那样，对于某些w∈f−h，仿射面片fh与形式为w+h的某些仿射超平面是仿射同构的。如果我们选取任何a∈e，因为根据假设f（a）∈fh，我们可以假定选择→w∈f−h，这样∈f（a∈）=[→−w]，一个d我们有f=kw h。因为f:e fh是仿射的，对于任何a e和任何u e，我们有

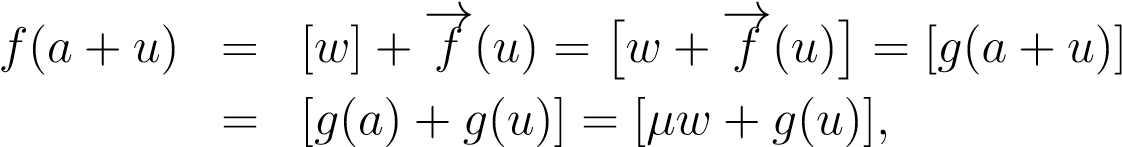
f(a + u) = f(a) + →−f (u) = w + →−f (u),  
F（A+U）=F（A）+→−F（U）=W+→−F（U）、

whereis a linear map, and where f(a) is viewed as the vector w.  
其中是线性图，其中f（a）被视为向量w。

Assume that) exists with the desired property. Then there is some linear map∈ g: Eb → F such that fe = P(g). Our goal is to prove that→− for some nonzero µ K. First, we prove that g vanishes on Ker f .  
假设）具有所需属性。然后有一些线性映射∈g:eb→f，这样fe=p（g）。我们的目标是证明→−对于一些非零μK。首先，我们证明G在Ker F上消失。

Since , we must have f(a) = [w] = [g(a)], and thus g(a) = µw, for some µ = 0.6  
因为，对于一些μ=0.6，我们必须有f（a）=[w]=[g（a）]，因此g（a）=μw。

Also, for every u ∈ E,  
而且，对于每一个u∈e，



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and thus we must have  
因此我们必须

λ(u)w + λ(u)→−f (u) = µw + g(u), (∗1)  
λ（u）w+λ（u）→−f（u）=w+g（u），（1）

for some λ(u) 6= 0.  
对于某些λ（u）6=0。

If Ker→−f = →−E, the linear map is the null map, and since we are requiring that the restriction of fe to P →−E be equal to P f , the linear map g must also be the null map on →−E. Thus, fe is unique, and the restriction of fe to P is the partial map undefined everywhere.  
如果ker→−f=→−e，则线性映射为零映射，由于我们要求fe对p→−e的限制等于p f，因此线性映射g也必须是→−e上的零映射，因此fe是唯一的，fe对p的限制是到处未定义的部分映射。

If →−E − Ker→−f 6= ∅, by taking a basis of Im→−f →−Eand some inverse image of this basis, we= Ker→−f ⊕ →−G. Since →−E = Ker→−f where dimobtain a basis of a subspace1, for any x ∈ Ker f and any nonnull vectorsuch that y ∈ →−G, we have ⊕ →−G  
如果→−E−KER→−F 6=∅，通过取im→−F→−E的基和该基的一些逆像，我们=KER→−F→−G。因为→−E=KER→−F，其中dim获得子空间1的基，对于任何x∈KER F和任何非空矢量，例如y∈→−G，我们有→−G

|  |  |  |
| --- | --- | --- |
| λ(x)w λ（x）w | = = | µw + g(x), 礹w+g（x）、 |
| λ(y)w + λ(y)→−f (y) λ（y）w+λ（y）→−f（y） | = = | µw + g(y), 礹w+g（y）、 |

and  
和

λ(x + y)w + λ(x + y)→−f (x + y) = µw + g(x + y),  
λ（x+y）w+λ（x+y）→−f（x+y）＝μw+g（x+y），

which by linearity yields  
通过线性关系得出

(λ(x + y) − λ(x) − λ(y) + µ)w + (λ(x + y) − λ(y))→−f (y) = 0.  
（λ（x+y）−λ（x）−λ（y）+μ）w+（λ（x+y）−λ（y））→−f（y）=0.

Since F = Kw ⊕ H and →−f : →−E → H, we must have λ(x + y) = λ(y) and λ(x) = µ. Then the equation  
由于f=kw h和→−f：→−e→h，我们必须得到λ（x+y）=λ（y）和λ（x）=µ。那么方程

λ(x)w = µw + g(x)  
λ（x）w=μw+g（x）

yields µw = µw + g(x), shows that g vanishes on Ker→−f . If dim = 1 then by (∗1), for any y ∈ →−G we have  
得到μw=μw+g（x），表明g在Ker→−f上消失。如果dim=1，那么（1），对于任何y∈→−g，我们有

λ(y)w + λ(y)→−f (y) = µw + g(y),  
λ（y）w+λ（y）→−f（y）=μw+g（y），

and for any ν 6= 0 we have  
对于任何v 6=0，我们有

λ(νy)w + λ(νy)→−f (νy) = µw + g(νy),  
λ（νy）w+λ（νy）→−f（νy）=μw+g（νy），

which by linearity yields  
通过线性关系得出

(λ(νy) − νλ(y) − µ + νµ)w + (νλ(νy) − νλ(y))→−f (y) = 0.  
（λ（νy）−νλ（y）−祄+ν祄）w+（νλ（νy）−νλ（y））→−f（y）=0.

Since F = Kw ⊕ µH)(1, →−f−:ν→−E) = 0.→ H, and ν = 06 , we must have λ(νy) = λ(y). Then we must also have (λ(y) −  
由于f=kw祫h）（1，→−f−：ν→−e）=0.→h，且ν=06，我们必须有λ（νy）=λ（y）。那么我们还必须有（λ（y）−

If K = {0,1}, since the only nonzero scalar is 1, it is immediate that6 ∈ →− g(y) = →−f (y), and we are done. Otherwise, for ν = 0,1, we get λ(y) = µ for all y G. Then equation  
如果k=0,1，因为唯一的非零标量是1，那么6∈→−g（y）=→−f（y）是直接的，我们就完成了。否则，对于ν=0,1，我们得到所有y g的λ（y）=μ，然后方程

λ(y)w + λ(y)→−f (y) = µw + g(y)  
λ（y）w+λ（y）→−f（y）=μw+g（y）

yields g = µ→−f on G, and since g vanishes on Ker→−f we get g = µ→−f on →−E and the restriction of →− is equal to P. But now, by Proposition 24.6 and since is isomorphic to F, the linear map fbis completely determined by  
在g上产生g=μ→−f，由于g在ker→−f上消失，我们在→−e上得到g=μ→−f，并且→−的限制等于p。但是现在，根据命题24.6，由于与f同构，线性图fbi完全由

,  
，

and g is completely determined by  
G完全由

.  
.

Thus, we have.  
因此，我们有。

Otherwise, if dim 2, then for any two distinct basis vectors u and v in B,  
否则，如果dim 2，那么对于b中任意两个不同的基向量u和v，

|  |  |  |
| --- | --- | --- |
| λ(u)w + λ(u)→−f (u) λ（u）w+λ（u）→−f（u） | = = | µw + g(u), 礹w+g（u）、 |
| λ(v)w + λ(v)→−f (v) λ（v）w+λ（v）→−f（v） | = = | µw + g(v), 礹w+g（v）、 |

and  
和

λ(u + v)w + λ(u + v)→−f (u + v) = µw + g(u + v),  
λ（u+v）w+λ（u+v）→−f（u+v）=礹w+g（u+v），

and by linearity, we get  
通过线性，我们得到

(λ(u + v) − λ(u) − λ(v) + µ)w + (λ(u + v) − λ(u))→−f (u) + (λ(u + v) − λ(v))→−f (v) = 0.  
（λ（u+v）-λ（u）-λ（v）+μw+（λ（u+v）-λ（u））→f（u）+（λ（u+v）-λ（v））→f（v）=0.

Since F = Kw ⊕H, →−f : →−E → H, and →−f (u) and →−f (v) are linearly independent (because →−f in injective on →−G), we must have  
由于f=kw h，→−f：→−e→h和→−f（u）和→−f（v）是线性独立的（因为→−f在→−g上的注入中），我们必须

λ(u + v) = λ(u) = λ(v) = µ,  
λ（u+v）=λ（u）=λ（v）=μ，

which implies that g = µ→−f on →−E, and the restriction of fe = P(g) to P is equal to P . As in the previous case, g is completely determined by  
这意味着在→−e上g=祆→−f，并且fe=p（g）到p的限制等于p。与前一种情况一样，G完全由

.  
.

Again, we have g = µfb, and thus feis unique.   
同样，我们有g=μfb，因此feis是唯一的。

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 The requirement that the restriction of fe = P(g) to P be equal to P is necessary for the uniqueness of fe. The problem comes up when f is a constant map. Indeed, if f is the constant map defined such that f(a) = [w] for some fixed vector w ∈ F, it can be shown that any linear map g: Eb → F defined such that→− → g(a) = µw and g(u) =◦ ϕ(u)w for all u ∈ →−E, for some µ = 06 , and some linear form ϕ: E F satisfies f = P(g) i.  
为了使铁的唯一性，必须限制铁=P（g）到P等于P。当f是一个常数映射时，问题就出现了。事实上，如果f是定义为f（a）=[w]的常数映射，对于某些固定向量w∈f，可以证明任何线性映射g:eb→f都定义为→−→g（a）=μw和g（u）=（u）w，对于所有u∈→−e，对于某些μ=06，和一些线性形式：e f满足f=p（g）i。

Proposition 25.17 shows that is the projective completion of the affine space E.  
命题25.17表明，这是仿射空间e的射影完备。

The projective completion Ee of an affine space E is a very handy place in which to do geometry in, mainly because the following facts can be easily established.  
仿射空间e的射影完备ee是一个非常方便的几何处理的地方，主要是因为下列事实可以很容易地建立起来。

There is a bijection between affine subspaces of E and projective subspaces of Ee not contained in P . Two affine subspaces of E are parallel iff the corresponding projective subspaces of Ee have the same intersection with the hyperplane at infinity P. There is also a bijection between affine maps from E to F and projective maps from Ee to Fe mapping the hyperplane at infinity P into the hyperplane at infinity P →−F . In the projective plane, two distinct lines intersect in a single point (possibly at infinity, when the lines are parallel). In the projective space, two distinct planes intersect in a single line (possibly at infinity, when the planes are parallel). In the projective space, a plane and a line not contained in that plane intersect in a single point (possibly at infinity, when the plane and the line are parallel).  
e的仿射子空间与p中不包含的ee的投影子空间之间存在双射。e的两个仿射子空间是平行的，如果e的相应投影子空间在无穷大p处与超平面有相同的交集，则e到f的仿射映射与e到fe的投影映射之间也存在双射，将无穷大p处的超平面映射到超平面。在无穷大P→−F。在射影平面中，两条不同的直线相交于一个点（当直线平行时，可能在无穷远处）。在射影空间中，两个不同的平面在一条直线上相交（当平面平行时，可能在无穷远处）。在射影空间中，一个平面和一条不包含在该平面中的线相交于一个点（当平面和线平行时，可能在无穷远处）。

## 25.9 Making Good Use of Hyperplanes at Infinity 25.9充分利用无限远超平面

Given a vector space E and a hyperplane H in E, we have already observed that the projective spacesand P(E) are isomorphic. Thus, P(H) can be viewed as the hyperplane at infinity in P(E), and the considerations applying to the projective completion of an affine space apply to the affine patch EH on P(E). This fact yields a powerful and elegant method for proving theorems in projective geometry. The general schema is to choose some projective hyperplane P(H) in P(E), view it as the “hyperplane at infinity,” then prove an affine version of the desired result in the affine patch EH (the complement of P(H) in P(E), which has an affine structure), and then transfer this result back to the projective space P(E). This technique is often called “sending objects to infinity.” We refer the reader to geometry textbooks for a comprehensive development of these ideas (for example, Berger [11, 12], Samuel [138], Sidler [156], Tisseron [170], or Pedoe [132]), but we cannot resist presenting the projective versions of the theorems of Pappus and Desargues. Indeed, the method of sending points to infinity provides some strikingly elegant proofs. We begin with Pappus’s theorem, illustrated in Figure 25.20.  
给定一个向量空间e和e中的超平面h，我们已经观察到射影空间和p（e）是同构的。因此，p（h）在p（e）中可视为无穷远的超平面，应用于仿射空间射影完备的考虑也适用于p（e）上的仿射面片eh。这一事实为证明射影几何中的定理提供了一种强大而优雅的方法。一般的模式是在p（e）中选择一些投影超平面p（h），将其视为“无穷远的超平面”，然后在仿射补丁eh（p（e）中p（h）的补码，它具有仿射结构）中证明所需结果的仿射版本，然后将该结果转移回射影空间P（E）。这种技术通常被称为“将物体发送到无限远的地方”。我们把读者引向几何教科书来全面发展这些思想（例如，伯杰[11，12]、塞缪尔[138]、西德勒[156]、蒂塞隆[170]或皮多[132]），但我们不能拒绝呈现投射的诗句。关于帕普斯定理和德沙格定理。实际上，将点发送到无穷大的方法提供了一些引人注目的优雅证明。我们从Pappus定理开始，如图25.20所示。

Proposition 25.18. (Pappus) Given any projective plane P(E) and any two distinct lines D and D0, for any distinct points a,b,c,a0,b0,c0, with a,b,c on D and a0,b0,c0 on D0, if  
提案25.18。（pappus）给定任何投影平面p（e）和任何两条不同的线d和d0，对于任何不同的点a、b、c、a0、b0、c0，其中a、b、c在d上，a0、b0、c0在d0上，如果

*a*

*b*

*c*

*a*

′

*b*

′

*c*

′

*r*

*q*

*p*

Figure 25.20: Pappus’s theorem (projective version).  
图25.20:Pappus定理（投影版本）。

a,b,c,a0,b0,c0 are distinct from the intersection of D and D0, then the intersection points p = hb,c0i ∩ hb0,ci, q = ha,c0i ∩ ha0,ci, and r = ha,b0i ∩ ha0,bi are collinear.  
a、b、c、a0、b0、c0与d和d0的交点不同，那么交点p=hb、c0i hb0、ci、q=ha、c0i ha0、ci和r=ha、b0i ha0、bi是共线的。

Proof. First, since any two lines in a projective plane intersect in a single point, the points  
证据。首先，由于射影平面上的任意两条直线相交于一个点，因此

hp,q,rXa0,b=iPare well defined. Choose ∆ =are parallel, and similarly(E) − ∆. Since ha,b0i andhb,chap,r0,bii intersect at a point at infinityas the line at infinity, and consider the affine planeb0,ci are parallel. Thus, by the affine version ofr on ∆, ha,b0i and h 0i and h  
hp，q，rxa0，b=i定义良好。选择∆=平行，同样选择（e）−∆。由于ha、b0i和hb、chap、r0、bii在无穷远处与直线相交，因此认为仿射平面b0、ci是平行的。因此，通过∆、ha、b0i和h 0i和h上的r的仿射形式

Pappus’s theorem (Proposition 23.12), the lines ha,c0i andp,rhai, which means that0,ci are parallel, which meansp,q,r are that their intersection q is on the line at infinity ∆ = h collinear.   
Pappus定理（命题23.12），线ha，c0i和p，rhai，这意味着0，ci是平行的，这意味着sp，q，r是它们的交叉点q在无穷大的直线上∆=h共线。

By working in the projective completion of an affine plane, we can obtain an improved version of Pappus’s theorem for affine planes. The reader will have to figure out how to deal with the special cases where some of p,q,r go to infinity.  
通过研究仿射平面的射影完备，我们可以得到仿射平面的帕普斯定理的一个改进版本。读者将不得不弄清楚如何处理一些特殊情况，其中p，q，r中的一些是无穷大的。

Now, we prove a projective version of Desargues’s theorem slightly more general than that given in Proposition 25.7. It is interesting that the proof is radically different, depending on the dimension of the projective space P(E). This is not surprising. In axiomatic presentations of projective plane geometry, Desargues’s theorem is independent of the other axioms. Desargues’s theorem is illustrated in Figure 25.21.  
现在，我们证明了德沙格定理的一个射影形式，比25.7命题中给出的稍微更普遍一些。有趣的是，根据投影空间p（e）的尺寸，证明是完全不同的。这并不奇怪。在射影平面几何的公理表示中，德沙格定理独立于其他公理。德沙格定理如图25.21所示。

Proposition 25.19. (Desargues) Let P(E) be a projective space. Given two triangles (a,b,c) and (a0,b0,c0), where the points a,b,c,a0,b0,c0 are pairwise distinct and the lines A = hb,ci, B = ha,ci, C = ha,bi, A0 = hb0,c0i, B0 = ha0,c0i, C0 = ha0,b0i are pairwise distinct, if the  
提案25.19。（德沙格）让p（e）是一个投影空间。给定两个三角形（a，b，c）和（a0，b0，c0），其中点a，b，c，a0，b0，c0是成对不同的，线a=hb，ci，b=ha，ci，c=ha，bi，a0=hb0，c0i，b0=ha0，c0i，c0=ha0，b0i是成对不同的，如果

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lines ha,a0i, hb,b0i, and hc,c0i intersect in a common point d distinct from a,b,c, a0,b0,c0, then the intersection points p = hb,ci ∩ hb0,c0i, q = ha,ci ∩ ha0,c0i, and r = ha,bi ∩ ha0,b0i belong to a common line distinct from A,B,C, A0,B0,C0.  
线ha、a0i、hb、b0i和hc、c0i相交于与a、b、c、a0、b0、c0不同的公共点d，然后交点p=hb、ci hb0、c0i、q=ha、ci ha0、c0i和r=ha、bi ha0、b0i属于与a、b、c、a0、b0、c0不同的公共线。

Proof.A0,B0,CFirst, it is immediately shown that the line0. Let us assume that P(E) has dimension nh≥p,q3. If the seven pointsi is distinct from the linesd,a,b,c,aA,B,C0,b0,c,0 generate a projective subspace of dimension 3, then by Proposition 25.1, the intersection of the two planes ha,b,ci and ha0,b0,c0i is a line, and thus p,q,r are collinear.  
证明.a0，b0，cfirst，它立即显示第0行。假设p（e）的维数nh≥p，q3。如果七个点si不同于线sd，a，b，c，aa，b，c0，b0，c，0生成一个维度3的投影子空间，那么根据命题25.1，两个平面ha，b，ci和ha0，b0，c0i的交点是一条线，因此p，q，r是共线的。

If P(E) has dimension n = 2 or the seven points d,a,b,c,a0,b0,c0 generate a projective subspace of dimension 2, we use the following argument. In the projective plane X generated by the seven points d,a,b,c,a0,b0,c0, choose the projective line ∆ = hp,ri as the line at infinity. Then in the affine plane Y = X −∆, the linesa,a0i, hhb,bb,c0ii, andand hhbc,c0,c00ii are either parallel orare parallel, and the lines ha,bi and ha0,b0i are parallel, and the lines h concurrent. Then by the converse of the affine version of Desargues’s theorem (Proposition  
如果p（e）的维数n=2，或者七点d、a、b、c、a0、b0、c0生成维数2的投影子空间，我们使用以下参数。在由七个点d、a、b、c、a0、b0、c0生成的投影平面x中，选择投影线∆=hp，ri作为无穷大处的线。然后在仿射平面y=x−∆中，直线a、a0i、hhb、bb、c0ii和hhbc、c0、c0ii是平行的或是平行的，直线ha、bi和ha0、b0i是平行的，直线h是平行的。然后通过德沙格定理（命题）的仿射形式的逆

23.13)to the line at infinity ∆ =, the lines ha,ci and hhp,ra0,ci0, and thus thati are parallel, which means that their intersectionp,q,r are collinear. q belongs  
23.13）对于无穷大∆处的线，线ha、ci和hhp、ra0、ci0，因此i是平行的，这意味着它们的相交p、q、r是共线的。q属于

*d*

*a*

*b*

*c*

*a*

′

*c*

′

*b*

′

*r*

*p*

*q*

Figure 25.21: Desargues’s theorem (projective version).  
图25.21：德沙格定理（射影版）。

The converse of Desargues’s theorem also holds. Using the projective completion of an affine space, it is easy to state an improved affine version of Desargues’s theorem. The reader will have to figure out how to deal with the case where some of the points p,q,r go to infinity. It can also be shown that Pappus’s theorem implies Desargues’s theorem. Many results of projective or affine geometry can be obtained using the method of “sending points to infinity.”  
德沙格定理的逆命题也成立。利用仿射空间的射影完备，可以很容易地描述一个改进的德沙格定理的仿射形式。读者必须弄清楚如何处理点P，Q，R到无穷大的情况。也可以证明Pappus定理隐含了Desargues定理。射影几何或仿射几何的许多结果可以用“发送点到无穷大”的方法得到。

We now discuss briefly the notion of cross-ratio, since it is a major concept of projective geometry.  
我们现在简单地讨论交叉比的概念，因为它是射影几何的一个主要概念。

## 25.10 The Cross-Ratio 25.10交叉比

Recall that affine maps preserve the ratio of three collinear points. In general, projective maps do not preserve the ratio of three collinear points. However, bijective projective maps preserve the “ratio of ratios” of any four collinear points (three of which are distinct). Such ratios are called cross-ratios (in French, “birapport”). There are several ways of introducing cross-ratios, but since we already have Proposition 25.5 at our disposal, we can circumvent some of the tedious calculations needed if other approaches are chosen.  
回想一下，仿射映射保持三个共线点的比率。一般来说，投影图不保留三个共线点的比例。然而，双射射影映射保留了任何四个共线点（其中三个点是不同的）的“比率”。这种比率称为交叉比率（法语称为“birapport”）。有几种引入交叉比的方法，但是由于我们已经有了25.5号提案，如果选择了其他方法，我们可以绕过一些繁琐的计算。

Given a field K, say K = R, recall that the projective line P1K consists of all equivalence classes [x,y] of pairs (x,y) ∈ K2 such that (x,y) = (06 ,0), under the equivalence relation ∼ defined such that  
给定一个场k，假设k=r，回想一下，射影线p1k由对（x，y）的所有等价类[x，y]组成∈k2，这样（x，y）=（06，0），在等价关系下定义如下：

(x1,y1) ∼ (x2,y2) iff x2 = λx1 and y2 = λy1,  
（x1，y1）（x2，y2）iff x2=λx1和y2=λy1，

for some λ ∈ K−{0}. Letting ∞ = [1,0], the projective line P1K is in bijection with K∪{∞}. Furthermore, letting 0 = [0,1] and 1 = [1,1], the triple (∞,0,1) forms a projective frame for P1K. Using this projective frame and Proposition 25.5, we define the cross-ratio of four collinear points as follows.  
对于某些λ∈k−0。设∞=[1,0]，投影线p1k为双射，k∞。另外，假设0=[0,1]和1=[1,1]，三重（∞，0,1）构成了p1k的投影框架，利用这个投影框架和命题25.5，我们定义了四个共线点的交叉比如下。

Definition 25.8. Given a projective line ∆ = P(D) over a field K, for any sequence (a,b,c,d) of four points in ∆, where a,b,c are distinct (i.e., (a,b,c) is a projective frame), the cross-ratio [a,b,c,d] is defined as the element h(d) ∈ P1K, where h: ∆ → P1K is the unique projectivity such that h(a) = ∞, h(b) = 0, and h(c) = 1 (which exists by Proposition 25.5, since (a,b,c) is a projective frame for ∆ and (∞,0,1) is a projective frame for P1K). For any projective space P(E) (of dimension ≥ 2) over a field K and any sequence (a,b,c,d) of four collinear points in P(E), where a,b,c are distinct, the cross-ratio [a,b,c,d] is defined using the projective line ∆ that the points a,b,c,d define. For any affine space E and any sequence (a,b,c,d) of four collinear points in E, where a,b,c are distinct, the cross-ratio [a,b,c,d] is defined by considering E as embedded in Ee.  
定义25.8.给定一条在k域上的投影线∆=p（d），对于∆中四个点的任意序列（a、b、c、d），其中a、b、c是不同的（即（a、b、c）是投影帧），交叉比[a、b、c、d]定义为元素h（d）∈p1k，其中h：∆→p1k是唯一的投影性，因此h（a）=∞，h（b）=0，h（c）=1（通过命题25.5存在），因为（a，b，c）是∆的投影框，而（∞，0,1）是p1k的投影框。对于场k上的任何投影空间p（e）（尺寸≥2）和p（e）中四个共线点的任何序列（a、b、c、d），其中a、b、c是不同的，利用点a、b、c、d定义的投影线∆定义交叉比[a、b、c、d]。对于任意仿射空间e和e中四个共线点的任何序列（a，b，c，d），其中a，b，c是不同的，交叉比[a，b，c，d]是通过考虑e嵌入ee来定义的。

It should be noted that the definition of the cross-ratio [a,b,c,d] depends on the order of the points. Thus, there could be 24 = 4! different possible values depending on the permutation of {a,b,c,d}. In fact, there are at most 6 distinct values. Also, note that [a,b,c,d] = ∞ iff d = a, [a,b,c,d] = 0 iff d = b, and [a,b,c,d] = 1 iff d = c. Thus, [a,b,c,d] ∈ K − {0,1} iff d /∈ {a,b,c}.  
应注意的是，交叉比[A、B、C、D]的定义取决于各点的顺序。因此，可能有24=4！不同的可能值取决于a、b、c、d的排列。实际上，最多有6个不同的值。另外，请注意，[a，b，c，d]=∞iff d=a，[a，b，c，d]=0 iff d=b，and[a，b，c，d]=1 iff d=c。因此，[a，b，c，d]∈k−0,1 iff d/∈a，b，c。

25.10. THE CROSS-RATIO  
25.10条。交叉比

The following proposition is almost obvious, but very important. It shows that projectivities between projective lines are characterized by the preservation of the cross-ratio of any four points (three of which are distinct).  
下面的命题几乎是显而易见的，但非常重要。结果表明，射影线之间的射影性具有保持任意四个点（其中三个点是不同的）的交叉比的特征。

Proposition 25.20. Given any two projective lines ∆ and ∆0, for any sequence (a,b,c,d) of points in ∆ and any sequence (a0,b0,c0,d0) of points in ∆0, if a,b,c are distinct and a0,b0,c0 are distinct, there is a unique projectivity f : ∆ → ∆0 such that f(a) = a0, f(b) = b0, f(c) = c0, and f(d) = d0 iff [a,b,c,d] = [a0,b0,c0,d0].  
提案25.20。给定任意两条射影线∆和∆0，对于∆0点的任意序列（a，b，c，d）和∆0点的任意序列（a0，b0，c0，d0），如果a，b，c是不同的，a0，b0，c0是不同的，则有一个唯一的射影率f：∆→∆0，这样f（a）=a0，f（b）=b0，f（c）=c0，f（d）=d0 iff[a，b，c，d]=[a0，b0，c0，d0]。

Proof. First, assume that f : ∆ → ∆0 is a projectivity such that f(a) = a0, f(b) = b0, f(c) = c0, and f(d) = d0. Let h: ∆ → P1K be the unique projectivity such that h(a) = ∞, h(b) = 0, and h(c) = 1, and let h0 : ∆0 → PK1 be the unique projectivity such that h0(a0) = ∞, h0(b0) = 0, and h0(c0) = 1. By definition, [a,b,c,d] = h(d) and [a0,b0,c0,d0] = h0(d0). However, h0 ◦f : ∆ → P1K is a projectivity such that (h0 ◦f)(a) = ∞, (h0 ◦f)(b) = 0, and (h0 ◦f)(c) = 1, and by the uniqueness of h, we get h = h0 ◦ f. But then, [a,b,c,d] = h(d) = h0(f(d)) = h0(d0) = [a0,b0,c0,d0].  
证据。首先，假设f：∆→∆0是一个投影性，这样f（a）=a0，f（b）=b0，f（c）=c0，f（d）=d0。设h：∆→p1k为唯一射影率，使h（a）=∞，h（b）=0，h（c）=1，设h0：∆0→pk1为唯一射影率，使h0（a0）=∞，h0（b0）=0，h0（c0）=1。根据定义，[a，b，c，d]=h（d）和[a0，b0，c0，d0]=h0（d0）。然而，h0 f：∆→p1k是一个投影性，因此（h0 f）（a）=∞，（h0 f）（b）=0和（h0 f）（c）=1，并且通过h的唯一性，我们得到h=h0 f。但是，然后，[a，b，c，d]=h（d）=h0（f（d））=h0（d0）=[a0，b0，c0，d0]。

Conversely, assume that [a,b,c,d] = [a0,b0,c0,d0]. Since (a,b,c) and (a0, b0, c0) are projective frames, by Proposition 25.5, there is a unique projectivity g: ∆ → ∆0 such that g(a) = a0, g(b) = b0, and g(c) = c0. Now, h0 ◦ g: ∆ → P1K is a projectivity such that (h0 ◦ g)(a) = ∞, (h0 ◦ g)(b) = 0, and (h0 ◦ g)(c) = 1, and thus, h = h0 ◦ g. However, h0(d0) = [a0,b0,c0,d0] = [a,b,c,d] = h(d) = h0(g(d)), and since h0 is injective, we get d0 = g(d).   
相反，假设[a，b，c，d]=[a0，b0，c0，d0]。由于（a，b，c）和（a0，b0，c0）是射影帧，根据命题25.5，有一个唯一的射影度g：∆→∆0，这样g（a）=a0，g（b）=b0，g（c）=c0。现在，h0 g：∆→p1k是一个投影性，这样（h0 g）（a）=∞，（h0 g）（b）=0，和（h0 g）（c）=1，因此，h=h0 g。然而，h0（d0）=[a0，b0，c0，d0]=[a，b，c，d]=h（d）=h0（g（d）），由于h0是注射剂，我们得到d0=g（d）。

As a corollary of Proposition 25.20, given any three distinct points a,b,c on a projective line ∆, for every λ ∈ P1K there is a unique point d ∈ ∆ such that [a,b,c,d] = λ.  
作为命题25.20的一个推论，给定射影线∆上任意三个不同的点a、b、c，对于每一个λ∈p1k，都有一个唯一的点d∈∆使得[a、b、c、d]=λ。

In order to compute explicitly the cross-ratio, we show the following easy proposition.  
为了明确计算交叉比，我们给出了以下简单的命题。

Proposition 25.21. Given any projective line ∆ = P(D), for any three distinct points a,b,c in ∆, if a = p(u), b = p(v), and c = p(u + v), where (u,v) is a basis of D, and for any  
提案25.21。给定任何投影线∆=p（d），对于∆中的任何三个不同点a、b、c，如果a=p（u），b=p（v）和c=p（u+v），其中（u，v）是d的基，并且对于任何

[λ,µ]∼ ∈ P1K and any point d ∈ ∆, we have  
[λ，μ]p1k和任意点d∆，我们有

d = p(λu + µv) iff [a,b,c,d] = [λ,µ]∼.  
d=p（λu+μv）iff[a，b，c，d]=[λ，μ]。

Proof. If (e1,e2) is the basis of K2 such that e1 = (1,0) and e2 = (0,1), it is obvious that p(e1) = ∞, p(e2) = 0, and p(e1 + e2) = 1. Let f : D → K2 be the bijective linear map such that f(u) = e1 and f(v) = e2. Then f(u + v) = e1 + e2, and thus f induces the unique projectivity P(f): P(D) → P1K such that P(f)(a) = ∞, P(f)(b) = 0, and P(f)(c) = 1.  
证据。如果（e1，e2）是k2的基础，使得e1=（1,0）和e2=（0,1），很明显p（e1）=∞，p（e2）=0，和p（e1+e2）=1。设f:d→k2为双射线性映射，使f（u）=e1，f（v）=e2。然后f（u+v）=e1+e2，因此f诱导独特的投射性p（f）：p（d）→p1k，使p（f）（a）=∞，p（f）（b）=0和p（f）（c）=1。

Then  
然后

P(f)(p(λu + µv)) = [f(λu + µv)]∼ = [λe1 + µe2]∼ = [λ,µ]∼,  
p（f）（p（λu+μv））=[f（λu+μv）]至=[λe1+μe2]至=[λ，μ]至，

that is,  
也就是说，

d = p(λu + µv) iff [a,b,c,d] = [λ,µ]∼,  
d=p（λu+μv）iff[a，b，c，d]=[λ，μ]，

as claimed.   
如要求。

We can now compute the cross-ratio explicitly for any given basis (u,v) of D. Assume that a,b,c,d have homogeneous coordinates [λ1,µ1], [λ2,µ2], [λ3,µ3], and [λ4,µ4] over the projective frame induced by (u,v). Letting wi = λiu + µiv, we have a = p(w1), b = p(w2), c = p(w3), and d = p(w4). Since a and b are distinct, w1 and w2 are linearly independent, and we can write w3 = αw1 + βw2 and w4 = γw1 + δw2, which can also be written as  
我们现在可以显式计算d的任何给定基（u，v）的交叉比。假设a、b、c、d在（u，v）诱导的投影帧上具有均匀坐标[λ1，μ1]、[λ2，μ2]、[λ3，μ3]和[λ4，μ4]。假设wi=λiu+μiv，我们得到a=p（w1），b=p（w2），c=p（w3），d=p（w4）。由于a和b是不同的，w1和w2是线性无关的，我们可以写w3=αw1+βw2和w4=γw1+δw2，也可以写为

,  
，

and by Proposition 25.21, [. However, since w1 and w2 are linearly independent, it is possible to solve for α,β,γ,δ in terms of the homogeneous coordinates, obtaining expressions involving determinants:  
根据25.21号提案，[然而，由于w1和w2是线性无关的，因此可以用齐次坐标解α、β、γ、δ，得到涉及行列式的表达式：

,  
，

and thus, assuming that d =6 a, we get  
因此，假设d=6a，我们得到

.  
.

When d = a, we have [a,b,c,d] = ∞. In particular, if ∆ is the projective completion of an affine line D, then µi = 1, and we get  
当d=a时，我们得到[a，b，c，d]=∞。特别地，如果∆是仿射线d的投影完成，那么μi=1，我们得到

.  
.

When d = ∞, we get  
当d=∞时，我们得到

,  
，

which is just the usual ratio (although we defined it earlier as −ratio(a,c,b)).  
这只是通常的比率（尽管我们之前将其定义为−比率（a，c，b））。

We briefly mention some of the properties of the cross-ratio. For example, the crossratio [a,b,c,d] is invariant if any two elements and the complementary two elements are transposed, and letting 0−1 = ∞ and ∞−1 = 0, we have  
我们简单地提到了交叉比的一些性质。例如，如果任意两个元素和互补的两个元素被转置，那么交叉比[a，b，c，d]是不变的，并且让0−1=∞和∞−1=0，我们得到

[a,b,c,d] = [b,a,c,d]−1 = [a,b,d,c]−1  
[A，B，C，D]=[B，A，C，D]-1=[A，B，D，C]-1

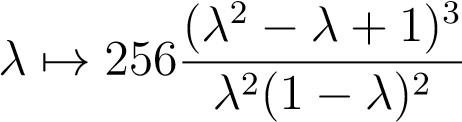
and  
和

[a,b,c,d] = 1 − [a,c,b,d].  
【A、B、C、D】=1−【A、C、B、D】。

25.10. THE CROSS-RATIO  
25.10条。交叉比

Since the permutations of {a,b,c,d} are generated by the above transpositions, the cross-λ = [a,b,c,d], if λ ∈ {∞,0,1}, then any permutation ratio takes at most six values. Letting of {a,b,c,d} yields a cross-ratio in {∞,0,1}, and if λ /∈ {∞,0,1}, then there are at most the six values λ, .  
由于a、b、c、d的置换是由上述置换产生的，因此交叉λ=[a、b、c、d]，如果λ∈∞，0,1，则任何置换比最多取6个值。放开a，b，c，d得出∞，0,1中的交叉比，如果λ/∞，0,1，则最多有六个值。

It can be shown that the function  
可以看出，函数



takes a constant value on the six values listed above.  
对上面列出的六个值取一个常量值。

We also define when four points form a harmonic division. For this, we need to assume that K is not of characteristic 2.  
我们还定义了当四个点形成一个调和除法。为此，我们需要假设k不是特征2。

Definition 25.9. Given a projective line ∆, we say that a sequence of four collinear points  
定义25.9.给定一条射影线∆，我们称为四个共线点的序列

([a,b,c,da,b,c,d] =) in ∆ (where−1, we also say thata,b,c are distinct) forms ac and d are harmonic conjugatesharmonic divisionofif [aa,b,c,dand b.] = −1. When  
（[a，b，c，da，b，c，d]=）在∆中（其中−1，我们也说a，b，c是不同的）形式ac和d是谐波共轭的sharmonic分型of[aa，b，c，d and b.]=−1。什么时候？

If a,b,c are distinct collinear points in some affine space, from  
如果a，b，c是某些仿射空间中不同的共线点，从

,  
，

we note that c is the midpoint of (a,b) iff [a,b,c,∞] = −1a,b,c,d, that is, if) on the real line, where(a,b,c,∞) forms a harmonic division. Figure 25.22 shows a harmonic division ( the coordinates of (a,b,c,d) are (−2,2,1,4).  
我们注意到，c是（a，b）iff[a，b，c，∞]=-1a，b，c，d的中点，也就是说，如果）在实线上，其中（a，b，c，∞）形成一个调和除法。图25.22显示了谐波划分（a、b、c、d）的坐标为（−2、2、1、4）。

a c b d  
A C B D

Figure 25.22: Four points forming a harmonic division.  
图25.22：构成谐波分区的四个点。

If ∆ = P1K and a,b,c,d are all distinct from ∞, then we see immediately from the formula  
如果∆=p1k和a、b、c、d都与∞不同，那么我们可以立即从公式中看到

that [a,b,c,d] = −1 iff .  
即[a，b，c，d]=-1 iff。

We also check immediately that [a,b,c,∞] = −1 iff  
我们还立即检查[a，b，c，∞]=−1 iff

a + b = 2c.  
A+B=2c。

There is a nice geometric interpretation of harmonic divisions in terms of quadrangles (or complete quadrilaterals). Consider the quadrangle (projective frame) (a,b,c,d) in a hprojective plane, and letd,bi and ha,ci, and c0 be the intersection ofa0 be the intersection ofhd,cihandd,aiha,bandi. If we lethb,ci, b0 be the intersection ofg be the intersection  
有一个很好的几何解释谐波划分的四角（或完全四边形）。在hprojective平面中考虑四边形（投影框）（a、b、c、d），Letd、bi和ha、ci和c0是hd、cihandd、aiha、bandi的交点。如果我们遗忘了，ci，b0是g的交集，是g的交集。

of ha,bi and ha0,b0i, then it is an interesting exercise to show that (a,c,b,d) as a projective frame and to computea,b,g,c0) is a harmonic division. One way to prove this is to pick (  
关于ha，bi和ha0，b0i，那么证明（a，c，b，d）作为一个投影框架和计算，b，g，c0）是一个调和除法是一个有趣的练习。证明这一点的一种方法是（

the coordinates of,b,g,c0], which is computed using the above formula. Another way is to send some wella0,b0,c0, and g. Then because ha,ci is the line at infinity, [a,b,g,c0] =  
b，g，c0]的坐标，用上述公式计算。另一种方法是发送一些wella0、b0、c0和g。然后，因为ha、ci是无穷远的直线，[a、b、g、c0]。=

[chosen points to infinity; see Berger [11] (Chapter 6, Section 6.4).∞  
[选择的点指向无穷大；见Berger[11]（第6章，第6.4节）。∞

*a*

*b*

*c*

*d*

*b*

′

*c*

′

*a*

′

*g*

Figure 25.23: A quadrangle, and harmonic divisions.  
图25.23：四边形和谐波分区。

In fact, it can be shown that the following quadruples of lines induce harmonic divihsions: (ha,dc,aii,,hhba00,a,b00ii,)hond,bhic,d,hbi0,c; see Figure 25.23. For more on harmonic divisions, the inter-0i) on ha,bi, (hb,ai,hc0,a0i, hd,ci,hc0,b0i) on ha,ci, and (hb,ci, a0,c0i,h  
事实上，可以看出，下列四重线引起谐波分裂：（ha，d c，aii，，hhba00，a，b00ii，）hond，bhic，d，hbi0，c；见图25.23。关于谐波划分的更多信息，Ha、Bi（hb、ai、hc0、a0i、hd、ci、hc0、b0i）和Ha、ci和（hb、ci、a0、c0i、h）上的inter-0i

ested reader should consult any text on projective geometry (for example, Berger [11, 12], Samuel [138], Sidler [156], Tisseron [170], or Pedoe [132]).  
尊敬的读者应该参考任何有关射影几何的文本（例如，Berger[11，12]、Samuel[138]、Sidler[156]、Tisseron[170]或Pedoe[132]）。

## 25.11 Fixed Points of Homographies and Homologies; Homographies of RP1 and RP2 25.11同系物和同系物的固定点；RP1和RP2的同系物

PLet(EP)(be homography (or projectivity) ofE) be a projective space where E is a vector space over some fieldP(E) where h is given by the linear isomorphismK, and let h: P(E) → f : E → E so that h = P(f). Observe that if u ∈ E is an eigenvector of f for some eigenvalue  
Plet（e p）（e的同构（或射影性）是一个射影空间，其中e是某个场p（e）上的向量空间，其中h由线性同构mk给出，并让h:p（e）→f:e→e使h=p（f）。观察，如果u∈e是某个特征值f的特征向量

λ ∈ K, then h([u]) = [f(u)] = [λu] = [u]  
λ∈k，则h（[u]）=[f（u）]=[λu]=[u]

since λ = 06 because f is an isomorphism, which means that the point [u] ∈ P(E) is a fixed pointh of h. In other words, eigenvectors of f induce fixed points of h = P(f).  
因为λ=06，因为f是同构的，这意味着点[u]∈p（e）是h的不动点，换句话说，f的特征向量诱导h=p（f）的不动点。

Consequently, it makes sense to try to classify homographies in terms of their fixed points. Of course this depends on the field K. If K is algebraically closed, for instance K = C, then all the eigenvalues of f belong to K, and we can use the Jordan form of a matrix representing f. If K = R, which is of particular interest to us, then we can use the real Jordan form, and we can obtain a compete classification for E = R2 and E = R3. We will also see that special kinds of homographies that leave every point of some projective hyperplane P(H) fixed, called homologies, play a special role.  
因此，试着根据不动点对同素词进行分类是有意义的。当然，这取决于场k。如果k是代数闭合的，例如k=c，那么f的所有特征值都属于k，我们可以使用表示f的矩阵的乔丹形式。如果k=r，这对我们特别有意义，那么我们可以使用真实的乔丹形式，我们可以得到c。e=r2和e=r3的综合分类。我们还将看到，使某些投影超平面p（h）的每个点保持不变的特殊类同构，称为同构，起着特殊的作用。

We begin with the classification of the homographies of the real projective line RP1. Since a homography h of RP1 is represented by a real invertible 2 × 2 matrix  
我们从实射影线rp1的同构图分类开始。因为Rp1的同构式h由一个实可逆的2×2矩阵表示。

,  
，

and since A either 0, 1, or 2, real eigenvalues, the homography h has 0, 1, or 2 fixed points.  
因为0，1，或2，实特征值，所以同形H有0，1，或2个不动点。

Definition 25.10. A homography of the real projective line RP1 not equal to the identity is elliptic if is has no fixed point, parabolic if it has a single fixed point, or hyperbolic if it has two fixed points.  
定义25.10.不等于恒等式的实射影线rp1的同形是椭圆的，如果没有不动点，则是抛物线；如果有单不动点，则是双曲的，如果有两个不动点。

1. Elliptic homographies. In this case, (a + d)2 − 4(ad − bc) < 0, so A has two distinct complex conjugate eigenvalues α ± iβ, and in C2, they correspond to two complex eigenvectors w1 = u + iv and w2 = u − iv, with u,v ∈ R2. Since  
   椭圆同形。在这种情况下，（a+d）2−4（ad−bc）<0，因此a有两个不同的复共轭特征值α±iβ，在c2中，它们对应于两个复特征向量w1=u+i v和w2=u−iv，其中u，v∈r2。自从

f(w1) = (α − iβ)w1  
F（w1）=（α−iβ）w1

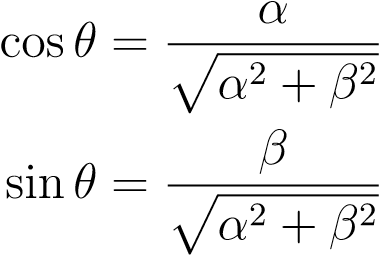
we obtain  
我们得到

f(u) + if(v) = αu + βv + i(−βu + αv),  
F（u）+如果（v）=αu+βv+i（−βu+αv），

which shows that in the basis (u,v), the homography h is represented by the matrix  
这表明在基（u，v）中，用矩阵表示同形H。

.  
.

If we let θ ∈ (0,2π) be the angle given by  
如果我们让θ∈（0,2π）为



and write then   
然后写

which corresponds to a similarity. Observe that h is an involution, that is, h2 = id iff θ = π/2.  
相当于相似性。观察h为对合，即h2=id iffθ=π/2。

1. Parabolic homographies. In this case, we must have (a + d)2 − 4(ad − bc) = 0. The matrix A is not diagonalizable and it has a Jordan form of the form  
   抛物线均形。在这种情况下，我们必须（a+d）2−4（ad−bc）=0。矩阵A不可对角化，它具有形式的约旦形式

.  
.

In the affine line y = 1, a parabolic homography behaves like the translation by 1/λ.  
在仿射线y=1中，抛物线同形表示为1/λ的平移。

1. Hyperbolic homographies. In this case, (a+d)2 −4(ad−bc) > 0, so A has two distinct nonzero reals eigenvalues λ and µ, and in a basis of eigenvectors it is represented by the diagonal matrix  
   双曲同形。在这种情况下，（a+d）2−4（ad−bc）>0，因此a有两个不同的非零实特征值λ和礹，在特征向量的基础上，它由对角矩阵表示。

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If P and Q are the distinct fixed points of the the homography h, it is not hard to show that for every M ∈ RP1 such that M =6 P,Q, we have  
如果p和q是同形H的不同不动点，则不难证明对于每一个m∈rp1，m=6p，q，我们有

[P,Q,M,h(M)] = k  
[P，Q，M，H（M）]=K

where k = λ/µ. For example, see Sidler [156] (Chapter 3, Proposition 3.3.1), and Berger [11] (Lemma 6.6.3). It can also be shown that h is an involution (h2 = id) with two distinct fixed points P and Q iff a + d = 0 iff k = −1 in the above equation; see Sidler [156] (Chapter 3, Proposition 3.3.2), and Samuel [138] (Section 2.4).  
其中k=λ/μ。例如，参见Sidler[156]（第3章，提案3.3.1）和Berger[11]（Lemma 6.6.3）。也可以证明，h是一个对合（h2=id），在上述方程中有两个不同的不动点p和q iff a+d=0 iff k=−1；见Sidler[156]（第3章，命题3.3.2）和Samuel[138]（第2.4节）。

We now classify the homographies of RP2. Since the characteristic polynomial of a 3×3 real matrix A has degree 3 and since every real polynomial of degree 3 has at least one real zero, A has some real eigenvalue. Since C is algebraically closed, every complex polynomial of degree 3 has three zeros (counted with multiplicity), in which case, all three eigenvalues of a 3 × 3 complex matrix A belong to C. Thus we have the following useful fact.  
我们现在对rp2的同构图进行分类。因为3×3实矩阵A的特征多项式有3次，而且3次的每一实多项式至少有一个实零，所以A有一些实特征值。因为C是代数闭的，所以三次的每一个复多项式都有三个零（用重数计数），在这种情况下，一个3×3复矩阵A的三个特征值都属于C，因此我们有以下有用的事实。

Proposition 25.22. Every homography of the real projective plane RP2 or of the complex projective plane CP2 has at least one fixed point.  
提案25.22.实射影平面rp2或复射影平面cp2的每一个同形都至少有一个固定点。

Here is the classification of the homographies of RP2 based on the real Jordan form of a 3×3 matrix. Most details are left as exercises. We denote by Γ the 3×3 matrix representing the real Jordan form of the matrix of the linear map representing the homography h.  
这里是基于3×3矩阵的实约但形式的RP2同构图的分类。大部分细节留作练习。我们用\_表示3×3矩阵，表示表示表示同形H的线性映射矩阵的实乔丹形式。

1. Three real eigenvalues α,β,γ. The matrix Γ has the form  
   三个实特征值α，β，γ。矩阵\_的形式

,  
，

with α,β,γ ∈ R nonzero and all distinct. As illustrated in Figure 25.24, the homography h has three fixed points P,Q,R, forming a triangle. The sides (lines) of this triangle are invariant under h. The restriction of h to each of these sides is hyperbolic.  
α、β、γ∈R非零且全部不同。如图25.24所示，同形H有三个固定点P、Q、R，形成三角形。这个三角形的边（线）在h下是不变的。h对这些边的限制是双曲线的。

z = 1

(0

,0,

0)

P

P

Q

Q

R

R

Figure 25.24: Case (I): The left figure is the hyperplane representation of RP2 and a homography with fixed points P,Q,R. The purple (linear) hyperplane maps to itself in a manner which is not the identity.  
图25.24：案例（i）：左边的图是rp2的超平面表示和不动点p、q、r的同形图。紫色（线性）超平面以非同一性的方式映射到自身。

1. One real eigenvalue α and two complex conjugate eigenvalues. Then Γ has the form  
   一个实特征值α和两个复共轭特征值。那么\_有表格了

,  
，

with α,γ ∈ R nonzero. The homography h, which is illustrated in Figure 25.25, has one fixed point P, and a line ∆ invariant under h and not containing P. The restriction of h to ∆ is elliptic.  
α，γ∈R非零。图25.25所示的同形图h有一个不动点p，h下有一条不含p的直线∆不变量。h对∆的限制是椭圆的。

P

z = 1

(0

,0,

0)

P

A

A

B

B

C

C

D

D

∆

∆

Figure 25.25: Case (II): The left figure is the hyperplane representation of RP2 and a homography with fixed point P and invariant line ∆. The purple (linear) hyperplane maps to itself under a rotation and rescaling.  
图25.25：情况（ii）：左图是rp2的超平面表示和具有固定点p和不变线∆的同形图。紫色（线性）超平面在旋转和重新缩放下映射到自身。

1. Two real eigenvalues α,β. The matrix Γ has the form  
   两个实特征值α，β。矩阵\_的形式

,  
，

with α,β ∈ R nonzero and distinct. The homography h, as illustrated in Figure 25.26, has one fixed point P, and a line ∆ invariant under h and not containing P. The restriction of h to ∆ is the identity. Every line through P is invariant under h and the restriction of h to this line is hyperbolic.  
α，β∈R非零且不同。如图25.26所示，同形图h有一个不动点p，h下的直线∆不变，不含p。h对∆的限制是恒等式。通过p的每一条线在h下都是不变的，h对这条线的限制是双曲线的。

1. One real eigenvalue α. The matrix Γ has the form  
   一个实特征值α。矩阵\_的形式

,  
，

with α ∈ R nonzero. As illustrated by Figure 25.27, the homography h has one fixed point P, and a line ∆ invariant under h containing P. The restriction of h to ∆ is the identity. Every line through P is invariant under h and the restriction of h to this line is parabolic.  
α∈R非零。如图25.27所示，同形图H有一个不动点P，H下的直线∆不变，包含P。H对∆的限制是同一性。通过p的每一条线在h下都是不变的，h对这条线的约束是抛物线的。

P

∆

z = 1

(0

,0,

0)

P

∆

Figure 25.26: Case (III): The left figure is the hyperplane representation of RP2 and a homography with fixed point P and invariant line ∆. The purple (linear) hyperplane maps to itself under rescaling; as such the restriction of the homography to ∆ is the identity. The green (linear) hyperplane also is invariant under the homography, but the invariance is not given by the identity map.  
图25.26：情况（iii）：左图是rp2的超平面表示和具有固定点p和不变线∆的同形图。紫色（线性）超平面在重新缩放下映射到自身；因此，同形性对∆的限制是同一性。绿色（线性）超平面在同形下也是不变的，但其不变性不是由同一映射给出的。

P

∆

z = 1

(0

,0,

0)

P

∆

Figure 25.27: Case (IV): The left figure is the hyperplane representation of RP2 and a homography with fixed point P and invariant line ∆ containing P. The purple (linear) hyperplane maps to itself under rescaling; as such the restriction of the homography to ∆ is the identity. The green (linear) hyperplane also is invariant under the homography, but the invariance is not given by the identity map.  
图25.27：案例（iv）：左边的图是rp2的超平面表示和一个具有固定点p和含有p的不变线∆的同形图。紫色（线性）超平面在重新缩放下映射到自身；因此，同形图对∆的限制是同一性。绿色（线性）超平面在同形下也是不变的，但其不变性不是由同一映射给出的。

1. Two real eigenvalues α,β. The matrix Γ has the form  
   两个实特征值α，β。矩阵\_的形式

,  
，

with α,β ∈ R nonzero and distinct. The homography h, which is illustrated in Figure 25.28, has two fixed points P and Q. The line hP,Qi is invariant under h, and there is is another line ∆ through Q invariant under h. The restriction of h to ∆ is parabolic, and the restriction of h to hP,Qi is hyperbolic.  
α，β∈R非零且不同。图25.28所示的同形图h有两个不动点p和q。线hp，qi在h下不变，还有一条线∆到q在h下不变。h对∆的限制是抛物线的，h对hp，qi的限制是双曲线的。

P

∆

z = 1

(0

,0,

0)

P

∆

Q

Q

Figure 25.28: Case (V): The left figure is the hyperplane representation of RP2 and a homography with fixed points P,Q and invariant line ∆. Both the purple and green (linear) hyperplanes are invariant under the homography, but the invariance is not given by the identity map.  
图25.28：情况（v）：左图是rp2的超平面表示和具有固定点p、q和不变线∆的同形图。紫色和绿色（线性）超平面在同形下都是不变的，但其不变性不是由同一映射给出的。

1. One real eigenvalue α. The matrix Γ has the form  
   一个实特征值α。矩阵\_的形式

,  
，

with α ∈ R nonzero. The homography h, which is illustrated in Figure 25.29, has one fixed point P, and a line ∆ invariant under h containing P. The restriction of h to ∆ is parabolic.  
α∈R非零。图25.29所示的同形图H有一个固定点P，在H下有一条直线∆不变量，其中包含P。H对∆的限制是抛物线的。

For the classification of the homographies of CP2, Case (II) becomes Case (I).  
对于CP2的同形图分类，案例（i i）变为案例（i）。

P

∆

z = 1

(0

,0,

0)

P

∆

Figure 25.29: Case (VI): The left figure is the hyperplane representation of RP2 and a homography with fixed point P and invariant line ∆. The purple (linear) hyperplane maps to itself in a manner which is not the identity.  
图25.29：案例（vi）：左图是rp2的超平面表示和具有固定点p和不变线∆的同形图。紫色（线性）超平面以非同一性的方式映射到自身。

Observe that in Cases (III) and (IV), the homography h has a line ∆ of fixed points, as well as a fixed point P. In Case (III), P /∈P ∆, and in Case (IV),is called the center and the line ∆ is calledP ∈ ∆. This kind of homography is called a homology. The point the axis (or base). Some authors only use the term homology when P /∈ ∆, and when P ∈ ∆, they use the term elation. When P ∈ ∆, other authors use the termO (instead of P). projective transvection, which we prefer. The center is usually denoted by  
观察在情况（iii）和（iv）中，同形图h有一条不动点∆线和一条不动点p。在情况（iii）中，p/∈p∆，在情况（iv）中，称为中心，而线∆称为dp∈∆。这种同形被称为同源。轴（或基准）的点。有些作者只在p/∈∆时使用术语同调，而当p∈∆时使用术语关联。当p∈∆时，其他作者使用termo（而不是p）。投影变换，我们更喜欢。中心通常用

One of the nice features of homologies (and projective transvections) is that there is a nice geometric construction of the image h(M) of a point M in terms of the center O, the axis ∆, and any pair (A,A0) where A0 = h(A), A 6= O, and A /∈ ∆.  
同系物（和射影变换）的一个很好的特征是，一个点m的图像h（m）有一个很好的几何结构，关于中心o，轴∆，以及任何一对（a，a0），其中a0=h（a），a 6=o，和a/∈∆。

throughThis construction is possible because for any pointO. This can be proved using Desargues’ Theorem; for example, see Silder [156]M =6 O, the line hM,h(M)i passes  
通过这个构造是可能的，因为对于任何点。这可以用Desargues定理来证明；例如，参见Silder[156]m=6 o，线hm，h（m）i通过

(Chapter 4, Section 4.2). We will prove this property for a generalization of homologies to any projective space P(E), where E is a vector space of any finite dimension.  
（第4章第4.2节）。我们将证明这一性质，以推广到射影空间p（e）的同系物，其中e是任何有限维的向量空间。

linelineFor the construction, first assume thathhA,Mi intersects ∆ in some point I. SinceM =6 OI is not on the line∈ ∆, it is fixed byA,Ii, its imagehA,Ah, so the image of the0iM. In this case, the0 = h(M) is on  
线条对于施工，首先假设hha，mi与∆相交于某一点i。sincem=6 oi不在∈∆线上，它由a，ii，它的图像ha，ah固定，因此图像为0im。在这种情况下，0=h（m）开启

A,Ii is the line hA0,Ii, and since M is on the line h the line hA0,Ii. But M0 = h(M) is also on the line hO,Mi, which implies that M0 = h(M) is the intersection point of the lines hA0,Ii and hO,Mi; see Figure 25.30.  
A，II是线HA0，II，因为M在H线上，所以是线HA0，II。但m0=h（m）也在ho，mi线上，这意味着m0=h（m）是ha0，ii和ho，mi线的交点；见图25.30。

If M 6= O is on the line hA,A0i, then we use the construction of the image B0 of some pointintersectionB 6= OJand not onof hM,Bi and ∆, and thenhA,A0i as before, and then repeat the construction by finding theM0 = h(M) is the intersection point of hB0,Ji and hA,A0i; see Figure 25.31.  
如果m 6=o在ha，a0i线上，那么我们使用一些点相交b 6=oj的图像b0的构造，而不是hm，bi和∆，然后像以前一样使用ha，a0i，然后重复构造，找到them0=h（m）是hb0，ji和ha，a0i的交点；见图25.31。

O

A

A’

M

∆

O

A

A’

M

∆

I

O

A

A’

M

∆

I

O

A

A’

M

∆

I

M’

Set up

Step 1

Step 2

Step 3

Figure 25.30: The three step process for determining the homology point h(M) = M0 when  
图25.30：确定同源点h（m）=m0的三步过程

M is not on the line hA,A0i. Step 1 finds the intersection between the extension ofA0,Ii. Step 3 extends hO,M0i and determines its intersectionhA,Mi and ∆. Step 2 forms the line h with hA0,Ii. The intersection point is M0.  
m不在ha，a0i线上。步骤1找到0，ii延伸段之间的交点。步骤3扩展ho、m0i并确定其相交ha、mi和∆。第2步用HA0、II组成H线。交叉点是m0。

O

A

A’

M

∆

O

A

A’

M

∆

I

Set up

Step 1

Step 2

Step 3

B

B

O

A

A’

M

∆

I

B

O

A

A’

M

∆

I

B

B’

O

A

A’

M

∆

I

B

B’

J

O

A

A’

M

∆

I

B

B’

J

M’

Step 4

Step 5

Figure 25.31: The five step process for determining the homology point h(M) = M0 when intersection betweenM is on the line hA,AhM,B0i. Steps 1 through 3 determine the linei and ∆, namely J. Step 5 forms the linehB,BhJ,B0i. Step 4 finds the0i and intersects it with hA,A0i. The intersection point is M0.  
图25.31：当直线ha、ahm、b0i相交时，确定同源点h（m）=m0的五步过程。步骤1至3确定直线i和∆，即j。步骤5形成直线hb、bhj、b0i。步骤4找到0i并与ha、a0i相交。交叉点iS M0。

The above construction also works if O ∈ ∆; see Figures 25.32 and 25.33.  
如果o∈∆，上述结构也可以工作；见图25.32和25.33。

O

A

A’

M

∆

Set up

Step 1

Step 2

O

A

A’

M

∆

I

O

A

A’

M

∆

I

O

A

A’

M

∆

I

M’

Step 3

Figure 25.32: The three step process for determining the elation point h(M) = M0 when M is not on the line hA,A0i. Step 1 finds the intersection between the extension of hA,Mi and ∆. Step 2 forms the line hA0,Ii. Step 3 extends hA0Ii and determines its intersection with hO,Mi. The intersection point is M0.  
图25.32：当m不在ha，a0i线上时，确定相关点h（m）=m0的三步过程。步骤1找到ha，mi和∆的延伸段之间的交叉点。步骤2形成线HA0，II。步骤3扩展HA0II并确定其与HO、MI的交叉点。交叉点是m0。

Another useful property of homologies (here, O /∈ ∆) is that for any line d passing through the center O, if I is the intersection point of the line d and ∆, then for any M ∈ d distinct from O and not on ∆ and its image M0, the cross-ratio [O,I,M,M0] is independent of d. If [O,I,M,M0] = −1 for all M =6 O, we say that h is a harmonic homology. It can be shown that a homography h is a harmonic homology iff h is an involution (h2 = id); see Silder [156] (Chapter 4, Section 4.4). It can also be shown that any homography of RP2 can be expressed as the composition of two homologies; see Silder [156] (Chapter 4, Section 4.5).  
同系物的另一个有用性质（这里，o/∈∆）是对于任何通过圆心o的d线，如果i是d线和∆线的交点，那么对于任何与o不同的m∈d，而不是在∆及其图像m0上，交叉比[o，i，m，m0]与d无关。如果[o，i，m，m0]。=-1对于所有m=6o，我们说h是调和同调。可以证明，同形H是调和同调，如果H是对合（h2=id）；见Silder[156]（第4章，第4.4节）。也可以证明，rp2的任何同形性可以表示为两个同源性的组成；见silder[156]（第4章，第4.5节）。

We now consider the generalization of the notion of homology (and projective transvection) to any projective space P(E), where E is a vector space of any finite dimension over a field K. We need to review a few concepts from Section 7.15.  
我们现在考虑同调概念（和射影变换）到任何射影空间p（e）的推广，其中e是K域上任何有限维的向量空间。我们需要回顾7.15节中的一些概念。

Let E be a vector space and let H be a hyperplane in E. Recall from Definition 7.6 that for any nonzero vector u ∈ E such that u 6∈ H, and any scalar α = 06 ,1, a linear map f : E → E such that f(x) = x for all x ∈ H and f(x) = αx for every x ∈ D = Ku is called a dilatation of hyperplane H, direction D, and scale factor α. See Figure 25.34.  
设e为向量空间，设h为e中的超平面。从定义7.6中回忆，对于任何非零向量u∈e，使u 6∈h，以及任何标量α=06，1，线性映射f:e→e，使f（x）=x代表所有x∈h和f（x）=αx代表每个x∈d=ku，称为超平面的扩张。e h、方向d和比例因子α。见图25.34。

From Definition 7.7, for any nonzero nonlinear form ϕ ∈ E∗ defining H (which means that H = Ker(ϕ)) and any nonzero vector u ∈ H, the linear map τϕ,u given by  
根据定义7.7，对于任意非零非线性形式，定义h（即h=ker（\_））和任意非零向量u h，线性映射τ\_，u由下式给出

τϕ,u(x) = x + ϕ(x)u, ϕ(u) = 0,  
τ，u（x）=x+（x）u，（u）=0，

for all x ∈ E is called a transvection of hyperplane H and direction u. See Figure 25.35.  
对于所有x∈e，称为超平面h和方向u的矢量变换。见图25.35。

O

A

A’

M

∆

B

O

A

A

’

M

∆

B

I

Set up

O

A

A’

M

∆

B

I

O

A

A’

M

∆

B

I

B’

Step 1

Step 2

Step 3

J

O

A

A’

M

∆

B

I

B’

J

O

A

A’

M

∆

B

I

B’

M’

Step 4

Step 5

Figure 25.33: The five step process for determining elation point h(M) = M0 when M is on  
图25.33：M开启时确定关联点h（m）=m0的五步过程

The intersection point isthe linebetweenhhA,AM,B0ii. Steps 1 through 3 determine the lineand ∆, namelyM0. J. Step 5 forms the linehB,BhJ,B00ii. Step 4 finds the intersectionand intersects it with hA,A0i.  
交叉点是NHHA、AM、B0II之间的直线。步骤1到3确定直线和∆，名称lYm0.j。步骤5形成直线hb，bhj，b00ii。步骤4找到交叉点并将其与ha，a0i交叉。

0)

,0,

(0

u

H

v

(0

,0,

0)

u

H

h

α

u

f(v)

Figure 25.34: A dilation f of the xy-plane in direction u = (1,1,1). Every vector v not in the xy-plane determines a rose-colored plane through u, and the image f(v) is an element of this rose hyperplane since it is stretched in the u direction.  
图25.34:u方向上xy平面的膨胀f=（1,1,1）。不在xy平面中的每个向量v都通过u确定一个玫瑰色平面，而图像f（v）是这个玫瑰色超平面的一个元素，因为它是在u方向拉伸的。

0)

,0,

(0

H

u

x

,0,

(0

0)

H

u

x

f

(

x

)

Figure 25.35: A transvection τϕ,u of the xy-plane in direction u = (0,1,0), where ϕ(x,y,z) = z. Every vector x not in the xy-plane determines a light-blue plane through x and u. The image f(x) stays in the light-blue hyperplane since it is ”stretched“ in the u direction by a factor of ϕ(x,y,z).  
图25.35：x y平面在u=（0,1,0）方向上的矢量τ，u，式中，（x，y，z）=z。不在xy平面上的每个矢量x通过x和u确定浅蓝色平面。图像f（x）保持在浅蓝色超平面中，因为它在u方向上被“拉伸”了一个因数\_（x，y，z）。

Proposition 25.23, which we repeat here for the convenience of the reader, characterizes the linear isomorphisms f =6 id that leave every point in the hyperplane H fixed.  
为了方便读者，我们在这里重复这个命题25.23，它描述了线性同构f=6 id，使超平面h中的每个点保持不变。

Proposition 25.23. Let f : E → E be a bijective linear map of a finite-dimensional vector space E and assume that f =6 id and that f(x) = x for all x ∈ H, where H is some hyperplane in E. If det(f) = 1, then f is a transvection of hyperplane H; otherwise, f is a dilatation of hyperplane H. In either case, the vector u is uniquely defined up to a nonzero scalar.  
提案25.23。设f:e→e为有限维向量空间e的双射线性映射，假设f=6 id，且f（x）=x表示所有x∈h，其中h是e中的某个超平面。如果det（f）=1，则f是超平面h的一个超矢量化；否则，f是超平面h的一个扩张。在任何情况下，vecTor U是唯一定义为非零标量的。

Proof. Only the last part was not proved in Proposition 7.23, Since f is bijective and the identity on H, the linear map f − id has kernel exactly H. Since H is a hyperplane in E, the image of f −id has dimension 1, and since u belong to this image, it is uniquely defined up to a nonzero scalar.   
证据。只有最后一部分没有在命题7.23中得到证明，因为f是双射的，而h上的恒等式，线性映射f−id的核正好是h。由于h是e中的超平面，f−id的图像有维数1，并且由于u属于这个图像，所以它被唯一地定义为非零标量。

The proof of Proposition 7.23 shows that if dim(E) = n + 1 and if f is a dilatation of hyperplane H, direction D = Ku, and scale α, then 1 is an eigenvalue of f with multiplicity n and α = 06 ,1 is an eigenvalue of f with multiplicity 1; the vector u is an eigenvector for α, and f is diagonalizable. If f is a transvection of hyperplane H and direction u, then 1 is the only eigenvalue of f, and it has multiplicity n; the vector u is an eigenvector for 1, and f is not diagonalizable.  
命题7.23的证明表明，如果dim（e）=n+1，如果f是超平面h的扩张，方向d=ku，尺度α，那么1是f的特征值，具有多重性n和α=06，1是f的特征值，具有多重性1；向量u是α的特征向量，f是对角的。竹叶提取物。如果f是超平面h和方向u的矢量，那么1是f的唯一特征值，它具有多重性n；向量u是1的特征向量，f不可对角化。

A homology is the projective version of the type of maps involved in Proposition 25.23.  
同调是25.23号命题所涉及的映射类型的投影版本。

Definition 25.11. For any vector space E and any hyperplane H in E, a homography h: P(E) → P(E) is a homology of axis (or base) P(H) if h(P) = P for all P ∈ P(H). In other words, the restriction of h to P(H) is the identity. More explicitly, if h = P(f) for some linear isomorphism f : E → E, we have P(f)(P) = P for all points P = [u] ∈ P(H).  
定义25.11.对于任意向量空间e和e中的任何超平面h，如果h（p）=p，则同构h:p（e）→p（e）是轴（或基）p（h）的同构，如果h（p）=p表示所有p∈p（h）。换句话说，H到P（H）的限制就是身份。更明确地说，如果对于某些线性同构f:e→e，h=p（f），我们得到p（f）（p）=p，表示所有点p=[u]∈p（h）。

Using Proposition 25.23 we obtain the following characterization of homologies. Write dim(E) = n + 1.  
利用命题25.23，我们得到了以下同系物的特征。写下dim（e）=n+1。

Proposition 25.24. If h: P(E) → P(E) is a homology of axis P(H) and if h =6 id, then for any linear isomorphism f : E → E such that h = P(f), the following properties hold:  
提案25.24.如果h:p（e）→p（e）是p（h）轴的同系物，如果h=6 id，那么对于任何线性同构f:e→e，如果h=p（f），以下属性保持不变：

1. Either f is a dilatation of hyperplane H and of direction u for some nonzero u ∈ E−H uniquely defined up to a scalar;  
   f是超平面h的扩张，对于某个非零u∈e−h，它的方向u是唯一定义为一个标量的；
2. Or f is a transvection of hyperplane H and direction u for some nonzero u ∈ H uniquely defined up to a scalar.  
   或f是超平面h和方向u的矢量化，对于某个非零u∈h，其唯一定义为一个标量。

In both cases, O = [u] ∈ P(E) is a fixed point of h, and h has no other fixed points besides O and points in P(H). In Case (1), O /∈ P(H), whereas in Case (2), O ∈ P(H). Furthermore, for any point M ∈ P(E), if M =6 O and if M /∈ P(H), then the line hM,h(M)i passes through O. If dim(E) ≥ 3, the point O is the only point satisfying the above property.  
在这两种情况下，O=[U]∈P（E）是H的不动点，H除了O和P（H）中的点外，没有其他不动点。在情况（1）中，o/∈p（h），而在情况（2）中，o∈p（h）。另外，对于任意点m∈p（e），如果m=6o，如果m/∈p（h），则线hm，h（m）i通过o，如果dim（e）≥3，则点o是满足上述性质的唯一点。

Proof. Since the restriction of h = P(f) to P(H) is the identity, and since P(f) = P(idH), by Proposition 25.4 we have f = λidH on H for some nonzero scalar λ ∈ K. Then g = λ−1f is the identity on H, so by Proposition 25.23 we obtain (1) and (2).  
证据。由于h=p（f）对p（h）的约束是恒等式，并且p（f）=p（idh），根据命题25.4，对于一些非零标量λ∈k，我们在h上有f=λidh，那么g=λ−1f是h上的恒等式，因此根据命题25.23，我们得到（1）和（2）。

In Case (1), we have g(u) = αu, so P(g)([u]) = P(f)([u]) = [u]. In Case (2), g(u) = u, so again P(g)([u]) = P(f)([u]) = [u]. Therefore, O = [u] is a fixed point of P(f). In Case (1), the eigenvalues of f are 1 with multiplicity n and α with multiplicity 1. If Q = [v] =6 O was a fixed point of h not in P(H), then v would be an eigenvector corresponding to a nonzero eigenvalue λ of f with λ = 16 ,α, and then f would have n + 2 eigenvalues (counted with multiplicty), which is absurd. In Case (2), the only eigenvalue of f is 1, with multiplicity n, so f not diagonalizable, and as above, a vector v such that Q = [v] is a fixed point of h not in P(H) would be an eigenvector corresponding to a nonzero eigenvalue λ = 16 of f, so f would be diagonalizable, a contradiction.  
在例（1）中，我们有g（u）=αu，所以p（g）（[u]）=p（f）（[u]）=u。在第（2）种情况下，g（u）=u，因此p（g）（[u]）=p（f）（[u]）=u。因此，o=[u]是p（f）的固定点。在例（1）中，f的特征值为1，具有多重性n，α具有多重性1。如果q=[v]=6o是h的不动点，而不是p（h），那么v是与f的非零特征值λ相对应的特征向量，其中λ=16，α，那么f将具有n+2特征值（用乘法计数），这是荒谬的。在第（2）种情况下，f的唯一特征值是1，具有多重性n，因此f不可对角化，如上所述，q=[v]是h的固定点而不在p（h）中的向量v将是对应于f的非零特征值λ=16的特征向量，因此f将是可对角化的，这是一个矛盾。

Since in Case (1), for any x =6 u and x /∈ H we have x = λu + h for some unique h ∈ H and some unique λ = 06 , so g(x) = g(λu) + g(h) = λαu + h = λu + h + (λα − λ)u = x + λ(α − 1)u,  
因为在案例（1）中，对于任何x=6 u和x/∈h，对于一些唯一的h∈h和一些唯一的λ=06，我们有x=λu+h，因此g（x）=g（λu）+g（h）=λαu+h=λu+h+（λα−λ）u=x+λ（α−1）u，

which shows that O,[x] and P(g)([x]) = P(f)([x]) are collinear. In Case (2), for any x =6 u and x /∈ H we have  
这表明O、[X]和P（G）（[X]）=P（F）（[X]）是共线的。在案例（2）中，对于任何x=6u和x/∈h，我们有

g(x) = x + ϕ(x)u,  
g（x）=x+（x）u，

which also shows that O,[x] and P(g)([x]) = P(f)([x]) are collinear. The last property is left as an exercise (see Vienne [179], Chapter 4, Proposition 7).   
这也表明O、[X]和P（G）（[X]）=P（F）（[X]）是共线的。最后一项财产留作练习（见维也纳[179]，第4章，提案7）。

Proposition 25.24 suggests the following definition.  
提案25.24提出了以下定义。

Definition 25.12. Let h: P(E) → P(E) be a homology of axis P(H) with h =6 id, where h = P(f) for some linear isomorphism f : E → E. The fixed point O = [u] associated with the vector u involved in the definition of f, which is unique up to a scalar, is called the center of h. If O ∈ P(H), then h is called a projective transvection (or elation).  
定义25.12.设h:p（e）→p（e）是p（h）轴与h=6 id的同系物，其中h=p（f）是某些线性同构f:e→e的同系物。与f定义中涉及的向量u相关的不动点o=[u]被称为h的中心。如果o∈p（h），则h被称为pr。目标转移（或兴高采烈）。

The same geometric construction that we used in the case of the projective plane shows that a homology is determined by its center O, its axis P(H), and a pair of points A and A0 = h(A), with A =6 O and A /∈ P(H). As a kind of converse, we have the following proposition which is easily shown; see Vienne [179] (Chapter IV, Proposition 8).  
我们在射影平面的情况下使用的相同几何结构表明，同调是由它的中心o、轴p（h）和一对点a和a0=h（a）确定的，a=6o和a/∈p（h）。作为一种反义词，我们有以下易于展示的命题；见维也纳[179]（第四章，命题8）。

Proposition 25.25. Let P(H) be a hyperplane of P(E) and let O ∈ P(E) be a point. For any pair of distinct points (A,A0) such that O,A,A0 are collinear and A,A0 ∈/ P(H)∪{O}, there is a unique homology h: P(E) → P(E) of centrer O and axis P(H) such that h(A) = A0.  
提案25.25。设p（h）为p（e）的超平面，设o∈p（e）为点。对于任何一对不同的点（a，a0），如o，a，a0是共线，a，a0∈/p（h）o，中心的h:p（e）→p（e）和轴p（h）有一个唯一的同源性h（a）=a0。

Remark: From the proof of Proposition 7.23, since every dilatation can be represented by a matrix of the form   α 0 ··· 0  
注：从命题7.23的证明来看，由于每一次扩张都可以用α0····0形式的矩阵表示。

0 1 0  
0 1 0\_

... ... ...,  
………………，

  
γ

0 0 ··· 1  
0 0···1

we see that by choosing the hyperplane at infinity to be x1 = 0, on the affine hyperplane x1 = 1, a homology becomes a central magnification by α−1. Similarly, since every transvection  
我们看到，通过选择无限远的超平面为x1=0，在仿射超平面x1=1上，同源性通过α-1变成中心放大。同样地，因为每一次交通

|  |  |  |
| --- | --- | --- |
| can be represented by a matrix of the form 可以用形式的矩阵表示 |  |  |
|  γ  1 0 1 0  α 1 α1  ... …   γ   γ  0 0 0 0 | ··· ·········  ... ··· …········· | 0 0℃  0 0℃  ..., ……  1 一 |

we see that by choosing the hyperplane at infinity to be x1 = 0, on the affine hyperplane x1 = 1, an elation becomes a translation.  
我们看到，通过选择无限远的超平面为x1=0，在仿射超平面x1=1上，一种兴奋变成了一种翻译。

Theorem 7.26 immediately yields the following result showing that the group of homographies PGL(E) is generated by the homologies.  
定理7.26立即得出以下结果，表明同素异形群pgl（e）是由同系物生成的。

Theorem 25.26. Let E be any finite-dimensional vector space over a field K of characteristic not equal to 2. Then, the group of homographies PGL(E) is generated by the homologies.  
定理25.26。设e为特征不等于2的场k上的任何有限维向量空间。然后，由同系物生成一组同系物pgl（e）。

When E = R3, we saw earlier that the involutions of RP2 have a nice structure. In particular, if an involution has two fixed points, then it is a harmonic homology.  
当e=r3时，我们在前面看到rp2的对合有一个很好的结构。特别是，如果一个对合有两个不动点，那么它就是一个调和同调。

If dim(E) ≥ 4, it is harder to characterize the involutions of P(E), but it is possible. The case where the linear isomorphism f : E → E defining the involutive homography h = P(f) has no eigenvalue in the field K is quite different from the case where f has some eigenvalue in K. In the first case, h has no fixed point. It turns out that this implies that dim(E) is even and there is a simple description of the matrices representing an involution. If h has some fixed point, then f is an involution of E, so it has the eigenvalues +1 and −1, and E is the direct sum of the corresponding eigenspaces E1 and E−1. Then h can be described in terms of P(E1) and P(E−1). For details, we refer the reader to Vienne [179] (Chapter IV, Propositions 11 and 12).  
如果dim（e）≥4，则很难描述p（e）的对合，但这是可能的。线性同构f:e→e定义对合同构h=p（f）在k域中没有特征值的情况与f在k域中有一些特征值的情况有很大不同，在第一种情况下，h没有固定点。结果表明，这意味着dim（e）是偶数，并且有一个表示对合的矩阵的简单描述。如果h有一个不动点，那么f是e的对合，因此它有特征值+1和−1，e是相应特征值e1和e−1的直接和。然后h可以用p（e1）和p（e-1）来描述。有关详细信息，我们请读者参考维也纳[179]（第四章，命题11和12）。

## 25.12 Duality in Projective Geometry 25.12射影几何中的对偶性

We now consider duality in projective geometry. Given a vector space E of finite dimension nE+1∗ is isomorphic to, recall that itsEdual space. We also have a canonical isomorphism betweenE∗ is the vector space of all linear formsEfand its bidual: E → K and thatE∗∗, which allows us to identify E and E∗∗.  
我们现在考虑射影几何中的对偶性。给定一个有限维ne+1的向量空间e是同构的，回想一下它的程序空间。我们也有一个典型的同构，在之间是所有线性形式的向量空间，它的双：e→k和e，这使得我们能够识别e和e。

Let H(E) denote the set of hyperplanes in P(E). In Section 25.3 we observed that the map  
设h（e）表示p（e）中的超平面集。在第25.3节中，我们观察到地图

p(f) 7→ P(Kerf)  
P（F）7→P（切口）

is a bijection between P(E∗) and H(E), in which the equivalence classP(Kerfp)(. Using the abovef) = {λf | λ = 06 } bijection betweenof a nonnull linear formP(E∗)fand∈ EH∗(Eis mapped to the hyperplane), a projective subspace(E), namely the familyP(U) of P(E∗) (where U is a  
是p（e）和h（e）之间的双射，其中等价类p（kerfp）（。使用上述f）=λfλ=06非空线性形式p（e）fand∈eh（e is映射到超平面）之间的双射，一个投影子空间（e），即p（e）的族yp（u）（其中u是a

subspace of E∗) can be identified with a subset of H  
e）的子空间可以用h的子集来标识。

{P(H) | H = Kerf, f ∈ U − {0}}  
\_p（h）h=切口，f∈u−0

Uconsisting of the projective hyperplanes in. Such subsets of H(E) are called linear systems (of hyperplanes)H(E) corresponding to nonnull linear forms in.  
u射影超平面的存在。这种H（E）的子集称为线性系统（超平面）H（E），对应于中的非零线性形式。

and linear systems as projective subspaces ofThe bijection between P(E∗) and H(E) allows us to viewH(E). In the projective spaceH(E) as a projective space,H(E), a point is a hyperplane in P(E)! The duality between subspaces of E and subspaces of E∗ (reviewed below) and the fact that there is a bijection betweenduality between the set of projective subspaces of P(EP)(Eand the set of linear systems in∗) and H(E) yields a powerful  
线性系统作为p（e）和h（e）之间双射的投影子空间，使我们可以看到h（e）。在投影空间h（e）中，作为投影空间h（e），点是p（e）中的超平面！e的子空间与e的子空间之间的对偶性（见下文），以及p（ep）（和中的线性系统集）和h（e）的射影子空间集之间存在一个双射影的事实，产生了一个强大的

H(E) (or equivalently, the set of projective subspaces of P(E∗)).  
h（e）（或相当于p（e）的射影子空间集）。

The idea of duality in projective geometry goes back to Gergonne and Poncelet, in the early nineteenth century. However, Poncelet had a more restricted type of duality in mind (polarity with respect to a conic or a quadric), whereas Gergonne had the more general idea of the duality between points and lines (or points and planes). This more general duality arises from a specific pairing between E and E∗ (a nonsingular bilinear form). Here we consider the pairing h−,−i: E∗ × E → K, defined such that  
射影几何学中的对偶概念可以追溯到19世纪初的格冈涅和庞塞莱。然而，Poncelet在头脑中有一种更为有限的二元性（关于二次曲线或二次曲线的极性），而Gergonne对点和线（或点和平面）之间的二元性有更一般的概念。这种更普遍的二元性来自e和e（一种非奇异双线性形式）之间的特定配对。这里我们考虑配对h−，−i:e×e→k，定义如下：

hf,vi = f(v),  
hf，vi=f（v）

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for all f ∈ E∗ and all v ∈ E. Recall that given a subset V of E (respectively a subset U of  
对于所有f e和所有v e，回想一下给定的e的子集v（分别是

E∗), the orthogonal V 0 of V is the subspace of E∗ defined such that  
e），v的正交v 0是e的子空间，定义如下：

V 0 = {f ∈ E∗ | hf,vi = 0, for every v ∈ V },  
v 0=f∈e hf，vi=0，对于每个v∈v，

and that the orthogonal U0 of U is the subspace of E defined such that  
u的正交u0是e的子空间，定义为

U0 = {v ∈ E | hf,vi = 0, for every  
u0=v∈e hf，vi=0，每

Then, by Theorem 10.1 (since E and E∗ have the same finite dimension n + 1), U = U00,  
然后，根据定理10.1（因为e和e具有相同的有限维n+1），u=u00，

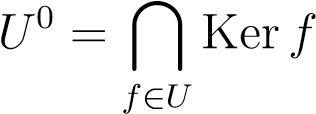
V = V 00, and the maps  
V=V 00，地图

V 7→ V 0 and U 7→ U0  
V 7→V 0和U 7→U0

are inverse bijections, where V is a subspace of E, and U is a subspace of E∗.  
是反双射，其中v是e的子空间，u是e的子空间。

These maps set up a duality between subspaces of E and subspaces of E∗. Furthermore, we know that U has dimension k iff U0 has dimension n+1−k, and similarly for V and V 0.  
这些地图在e的子空间和e的子空间之间建立了一个对偶性。此外，我们知道u的维数是k，如果u0的维数是n+1−k，同样的，v和v 0的维数也是。

Since a linear system P = P(U) of hyperplanes in H(E) corresponds to a subspace U of E∗, and since  
因为H（e）中超平面的线性系统p=p（u）对应于e的子空间u，并且



is the intersection of all the hyperplanes defined by nonnull linear forms in U, we can view a linear system P = P(U) = P(U00) in H(E) as the family of hyperplanes in P(E) containing P(U0).  
是由u中的非空线性形式定义的所有超平面的交集，我们可以将h（e）中的线性系统p=p（u）=p（u00）视为p（e）中包含p（u0）的超平面族。

In view of the identification of P(E∗) with the set H(E) of hyperplanes in P(E), by passing to projective spaces, the above bijection between the set of subspaces of E and the set of subspaces of E∗ yields a bijection between the set of projective subspaces of P(E) and the set of linear systems in H(E) (or equivalently, the set of projective subspaces of P(E∗)) called duality. Recall that a point of H(E) is a hyperplane in P(E).  
鉴于p（e）中超平面集h（e）对p（e）的识别，通过传递到射影空间，e的子空间集与e的子空间集之间的上述双射产生p（e）的射影子空间集与线性系统集之间的双射。在h（e）中（或等价地，p（e）的射影子空间集称为对偶。回想一下H（e）点是P（e）中的超平面。

More specifically, assuming that E has dimension n + 1, so that P(E) has dimension n, if Q = P(V ) is any projective subspace of P(E) (where V is any subspace of E) and if P = P(U) is any linear system in H(E) (where U is any subspace of E∗), we get a subspace  
更具体地说，假设e的维数为n+1，那么p（e）的维数为n，如果q=p（v）是p（e）的任何投影子空间（其中v是e的任何子空间），如果p=p（u）是h（e）中的任何线性系统（其中u是e的任何子空间），我们得到一个子空间。

Q0 of H(E) defined by  
h（e）的q0由定义

Q0 = {P(H) | Q ⊆ P(H), P(H) a hyperplane in H(E)},  
q0=p（h）q p（h），p（h）h（e）中的超平面，

and a subspace P0 of P(E) defined by  
p（e）的子空间p0定义为

P0 = \{P(H) | P(H) ∈ P, P(H) a hyperplane in H(E)}.  
p0=p（h）p（h）∈p，p（h）H（e）中的超平面。

We have P = P00 and Q = Q00. Since Q0 is determined by P(V 0), if Q = P(V ) has dimension k (i.e., if V has dimension k + 1), then Q0 has dimension n − k − 1 (since V has dimension k+1 and dim(E) = n+1, then V 0 has dimension n+1−(k+1) = n−k). Thus,  
我们有p=p00和q=q00。因为q0由p（v 0）决定，如果q=p（v）的尺寸为k（即v的尺寸为k+1），则q0的尺寸为n−k−1（因为v的尺寸为k+1，dim（e）=n+1，则v 0的尺寸为n+1−（k+1）=n−k）。因此，

dim(Q) + dim(Q0) = n − 1,  
尺寸（Q）+尺寸（Q0）=N-1，

and similarly, dim(P) + dim(P0) = n − 1.  
同样，dim（p）+dim（p0）=n-1。

A linear system P = P(U) of hyperplanes in H(E) is called a pencil of hyperplanes if it corresponds to a projective line in P(E∗), which means that U is a subspace of dimension 2 of E∗. From dim(P) + dim(P0) = n − 1, a pencil of hyperplanes P is the family of hyperplanes in H(E) containing some projective subspace P(V ) of dimension n − 2 (where P(V ) is a projective subspace of P(E), and P(E) has dimension n). When n = 2, a pencil of hyperplanes in H(E), also called a pencil of lines, is the family of lines passing through a given point. When n = 3, a pencil of hyperplanes in H(E), also called a pencil of planes, is the family of planes passing through a given line.  
H（e）中超平面的线性系统p=p（u）如果对应于p（e）中的投影线，则称为超平面的铅笔，这意味着u是e的维2的子空间。从dim（p）+dim（p0）=n−1开始，超平面p的一支铅笔是h（e）中的超平面族，包含维度n−2的一些投影子空间p（v）（其中p（v）是p（e）的投影子空间，p（e）具有维度n）。当n=2时，h（e）中的超平面铅笔，也称为线笔，是穿过给定点的线族。当n=3时，h（e）中的超平面铅笔，也称为平面铅笔，是通过给定直线的平面族。

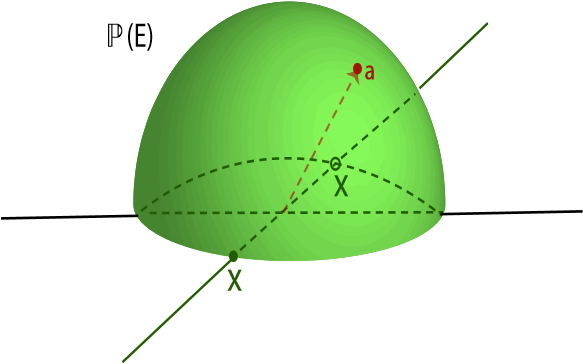
When n = 2, the above duality takes a rather simple form. In this case (of a projective plane P(E)), the duality is a bijection between points in P(E) and lines in P(E∗), represented by pencils of lines in H(E), with the following properties:  
当n=2时，上面的对偶形式相当简单。在这种情况下（投影平面p（e）），对偶是p（e）中点和p（e）中线之间的双射，用h（e）中线的铅笔表示，具有以下特性：

* A point a in P(E) maps to the line Da in P(E∗) represented by the pencil of lines in H(E) containing a, also denoted by a∗. See Figure 25.36.  
  p（e）中的点a映射到p（e）中的线da，由h（e）中的线笔表示，其中包含a，也用a表示。见图25.36。
* A line D in P(E) maps to the point pD in P(E∗) represented by the line D in H(E). See Figure 25.37.  
  p（e）中的线d映射到p（e）中的点pd，该点由h（e）中的线d表示。见图25.37。
* Two points a,b in P(E) map to lines Da,Db in P(E∗) represented by pencils of lines through a and b, and the intersection of Da and Db is the point pha,bi in P(E∗) corresponding to the line ha,bi belonging to both pencils. The point pha,bi is the image of the line ha,bi via duality. See Figure 25.38  
  p（e）中的两个点a、b映射到线da，db映射到线p（e），用铅笔通过a和b表示，da和db的交点是与线ha、bi对应的点pha、bi in p（e），bi属于这两个铅笔。点pha，bi是线ha，bi通过对偶的形象。见图25.38
* A line D in P(E) containing two points a,b maps to the intersection pD of the lines Da and Db in P(E∗) which are the images of a and b under duality. This is because a,b map to lines Da,Db in P(E∗) represented by pencils of lines through a and b, and the intersection of Da and Db is the point pD in P(E∗) corresponding to the line D = ha,bi belonging to both pencils. The point pD is the image of the line D = ha,bi under duality. Once again, see Figure 25.38.  
  p（e）中包含两个点a，b的线d映射到p（e）中da和db线的交点pd，这是二元性下a和b的图像。这是因为A，B映射到P（E）中的线d a，d b，用穿过A和B的线的铅笔表示，并且da和db的交点是P（E）中的点p d，对应于D=ha，bi，属于这两支铅笔。点pd是二重性下d=ha，bi线的图像。再次参见图25.38。
* If a ∈ D, where a is a point and D is a line in P(E), then pD ∈ Da in P(E∗). This is because under duality, a is mapped to the line Da in P(E∗) represented by the pencil of lines containing a, and D is mapped to the point pD ∈ P(E∗) represented by the line D through a in this pencil, so pD ∈ Da.  
  如果a d，其中a是点，d是p（e）中的线，那么pd da在p（e）中。这是因为在对偶性下，a被映射到p（e）中的线da，用含有a的线的铅笔表示，d被映射到点pd p（e），用该铅笔中的线d通过a表示，所以pd da。

The reader will discover that the dual of Desargues’s theorem is its converse. This is a nice way of getting the converse for free! We will not spoil the reader’s fun and let him discover the dual of Pappus’s theorem.  
读者会发现德沙格定理的对偶是它的逆命题。这是一个免费获得谈话的好方法！我们不会破坏读者的乐趣，让他发现帕普斯定理的对偶性。

In general, when n ≥ 2, the above duality is a bijection between points in P(E) and hyperplanes in P(E∗), which are represented by linear systems of dimension n − 1 in H(E), with the following properties:  
一般来说，当n≥2时，上述对偶是p（e）中点与p（e）中超平面之间的双射，用h（e）中n−1的线性系统表示，具有以下性质：

25.12. DUALITY IN PROJECTIVE GEOMETRY  
12月25日。射影几何中的对偶性





a

D

\*

a

D

b

\*

z = 1

*H*

(

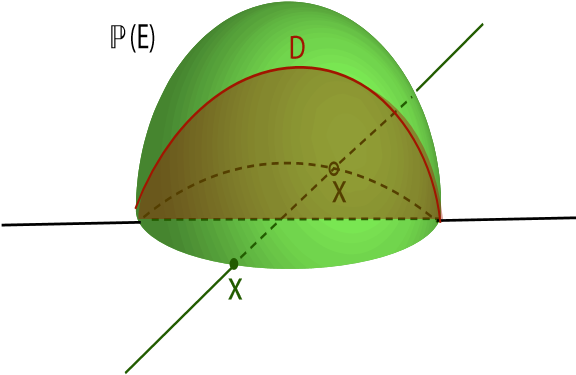
E

)

Figure 25.36: The duality between a point in P(E) and a line in P(E∗). The line in P(E∗) is also represented by the pencil of lines through a in H(E).  
图25.36：p（e）中的点与p（e）中的线之间的对偶性。p（e）中的线也由h（e）中的线笔表示。

* A point a in P(E) maps to the hyperplane Ha in P(E∗) (the linear system of hyperplanes in H(E) containing a, also denoted by a∗).  
  p（e）中的点a映射到p（e）中的超平面ha（h（e）中包含a的超平面线性系统，也用a表示）。
* A hyperplane H in P(E) maps to the point pH in P(E∗) (represented by the hyperplane H in H(E)).  
  p（e）中的超平面h映射到p（e）中的点ph（由h（e）中的超平面h表示）。

To conclude our quick tour of projective geometry, we establish a connection between the cross-ratio of hyperplanes in a pencil of hyperplanes with the cross-ratio of the intersection points of any line not contained in any hyperplane in this pencil with four hyperplanes in this pencil.  
为了结束我们对射影几何的快速浏览，我们建立了超平面铅笔中超平面的交叉比与铅笔中任何超平面中不包含的线的交叉点的交叉比与铅笔中四个超平面的交叉比之间的联系。



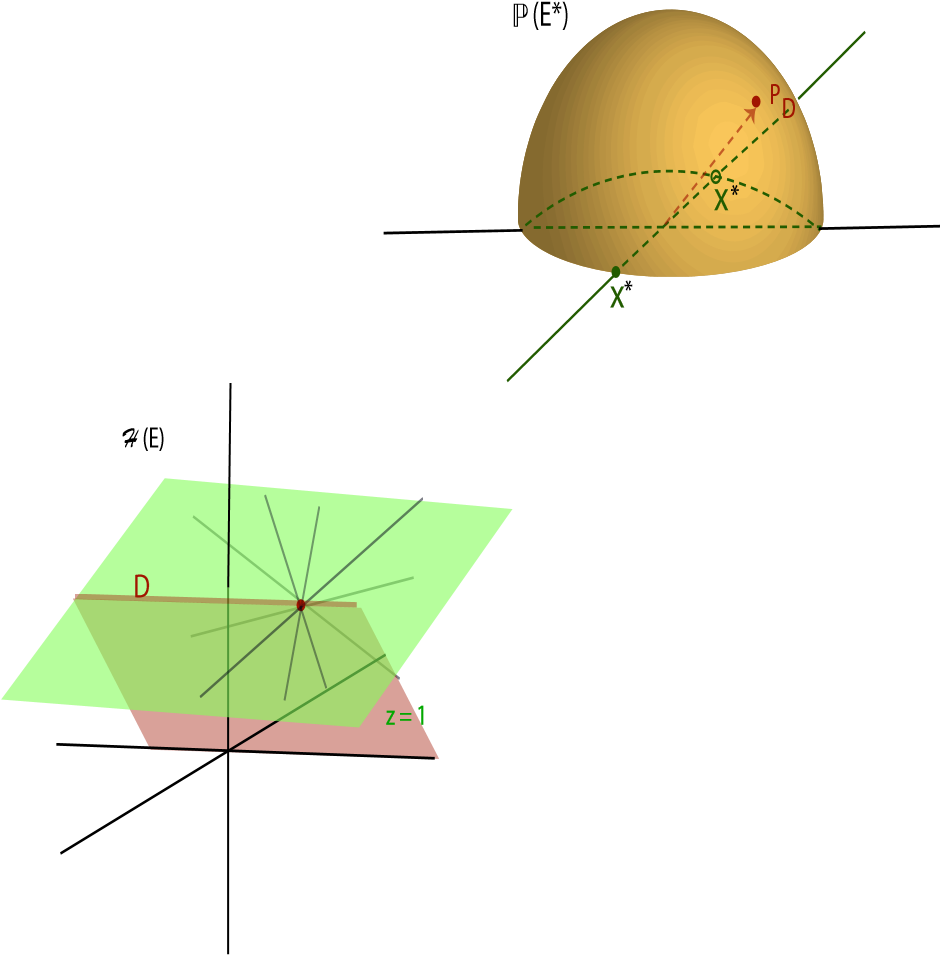


Figure 25.37: The duality between a line in P(E) and point in P(E∗). The point in P(E∗) is also represented by Line D in H(E).  
图25.37：p（e）中的一条线和p（e）中的点之间的对偶性。p（e）中的点也由h（e）中的d行表示。

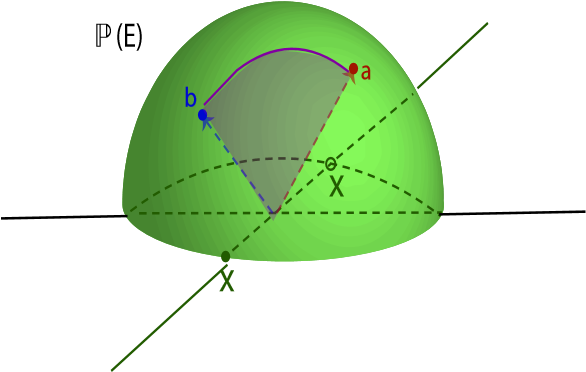
## 25.13 Cross-Ratios of Hyperplanes 25.13超平面的交叉比

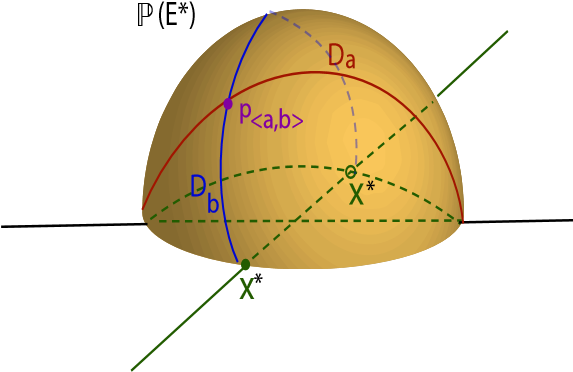
Given a pencil P = P(U) of hyperplanes in H(E), for any sequence (H1, H2, H3, H4) of hyperplanes in this pencil, if H1,H2,H3 are distinct, we define the cross-ratio [H1,H2,H3,H4] as the cross-ratio of the hyperplanes Hi considered as points on the projective line P in P(E∗). In particular, in a projective plane P(E), given any four concurrent lines D1, D2, D3, D4, where D1, D2, D3 are distinct, for any two distinct lines ∆ and ∆0 not passing through the common intersection c of the lines Di, letting di = ∆ ∩ Di, and d0i = ∆0 ∩ Di, note that the projection of center c from ∆ to ∆0 maps each di to d0i.  
给定H（e）中超平面的铅笔p=p（u），对于铅笔中任何超平面序列（h1、h2、h3、h4），如果h1、h2、h3是不同的，我们将交叉比[h1、h2、h3、h4]定义为被视为P（e）中投影线p上点的超平面的交叉比。特别是，在投影平面p（e）中，给定任意四条平行线d1、d2、d3、d4，其中d1、d2、d3是不同的，对于任何两条不通过直线di的公共交叉点c的不同直线∆和∆0，让di=∆di和d0i=∆0 di，注意投影从∆到∆0的F中心C将每个di映射到d0i。

Since such a projection is a projectivity, and since projectivities between lines preserve cross-ratios, we have  
因为这样的投影是投影，而且线之间的投影保持交叉比，所以我们有

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，

25.13. CROSS-RATIOS OF HYPERPLANES  
25.13条。超平面的交叉比





a

z = 1

*H*

(

E

)

b

D

<

a

,

b

>

p

Figure 25.38: The duality between a line through two points in P(E) and a point incident to two lines in P(E∗).  
图25.38：穿过p（e）中两点的线与入射到p（e）中两点的点之间的对偶性。

which means that the cross-ratio of the di is independent of the line ∆ (see Figure 25.39). In fact, this cross-ratio is equal to [D1,D2,D3,D4], as shown in the next proposition.  
这意味着di的交叉比与直线∆无关（见图25.39）。事实上，这个交叉比等于[d1，d2，d3，d4]，如下一个命题所示。

Proposition 25.27. Let P = P(U) be a pencil of hyperplanes in H(E), and let ∆ = P(D) be any projective line such that ∆ ∈/ H for all H ∈ P. The map h: P → ∆ defined such that h(H) = H ∩ ∆ for every hyperplane H ∈ P is a projectivity. Furthermore, for any sequence (H1,H2,H3,H4) of hyperplanes in the pencil P, if H1,H2,H3 are distinct and di = ∆ ∩ Hi, then [d1,d2,d3,d4] = [H1,H2,H3,H4].  
提案25.27。设p=p（u）为h（e）中超平面的一支铅笔，设∆=p（d）为任意一条射影线，使∆∈/h表示所有h∈p。图h:p→∆定义为每个超平面h∈p的h（h）=h∆为射影性。此外，对于铅笔p中的任何超平面序列（h1、h2、h3、h4），如果h1、h2、h3是不同的且di=∆hi，则[d1、d2、d3、d4]=[h1、h2、h3、h4]。

Proof. First, the map h: P → ∆ is well–defined, since in a projective space, every line ∆ = P(D) not contained in a hyperplane intersects this hyperplane in exactly one point. Since P = P(U) is a pencil of hyperplanes in H(E), U has dimension 2, and let ϕ and ψ be two nonnull linear forms in E∗ that constitute a basis of U, and let F = ϕ−1(0) and  
证据。首先，映射h:p→∆定义得很好，因为在投影空间中，不包含在超平面中的每一条线∆=p（d）都与该超平面在一个点上相交。由于p=p（u）是h（e）中超平面的一支铅笔，u的尺寸为2，并且让\_和ψ是e中构成u基础的两个非零线性形式，并且让f=\_−1（0）和

*c*

*d*

1

*d*

2

*d*

3

*d*

4

*d*

′

1

*d*

′

2

*d*

′

3

*d*

′

4

*D*

1

*D*

2

*D*

3

*D*

4

∆

∆

′

Figure 25.39: A pencil of lines and its cross-ratio with intersecting lines.  
图25.39：一支铅笔及其与相交线的交叉比。

G = ψ−1(0). Let a = P(F) ∩ ∆ and b = P(G) ∩ ∆. There are some vectors u,v ∈ D such that a = p(u) and b = p(v), and since ϕ and ψ are linearly independent, we have a =6 b, and we can choose ϕ and ψ such that ϕ(v) = −1 and ψ(u) = 1. Also, (u,v) is a basis of D.  
G=ψ−1（0）。设a=p（f）∆和b=p（g）∆。有一些向量u，v∈d，这样a=p（u）和b=p（v），并且由于ψ和ψ是线性独立的，所以我们有a=6b，我们可以选择ψ和ψ，从而使（v）=-1和ψ（u）=1。另外，（u，v）是d的基础。

Then a point p(αu+βv) on ∆ belongs to the hyperplane H = p(γϕ+δψ) of the pencil P iff  
那么∆上的点p（αu+βv）属于铅笔p iff的超平面h=p（γ\_+δψ）。

(γϕ + δψ)(αu + βv) = 0,  
（銄+δψ）（αu+βv）=0，

which, since ϕ(u) = 0, ψ(v) = 0, ϕ(v) = −1, and ψ(u) = 1, yields γβ = δα, which is equivalent to [α,β] = [γ,δ] in P(K2). But then the map h: P → ∆ is a projectivity. Letting di = ∆ ∩ Hi, since by Proposition 25.20 a projectivity of lines preserves the cross-ratio, we get [d1,d2,d3,d4] = [H1,H2,H3,H4].   
其中，由于\_（u）=0，ψ（v）=0，（v）=-1和ψ（u）=1，产生γβ=δα，相当于p（k2）中的[α，β]=[γ，δ]。但是图h:p→∆是一个投影。假设di=∆hi，因为根据命题25.20，线的投影保持了交叉比，我们得到[d1，d2，d3，d4]=[h1，h2，h3，h4]。

## 25.14 Complexification of a Real Projective Space 25.14真实投影空间的复杂性

Notions such as orthogonality, angles, and distance between points are not projective concepts. In order to define such notions, one needs an inner product on the underlying vector space. We say that such notions belong to Euclidean geometry. At first glance, the fact that some important Euclidean concepts are not covered by projective geometry seems a major drawback of projective geometry. Fortunately, geometers of the nineteenth century (including Laguerre, Monge, Poncelet, Chasles, von Staudt, Cayley, and Klein) found an astute way of recovering certain Euclidean notions such as angles and orthogonality (also circles) by embedding real projective spaces into complex projective spaces. In the next two sections we will give a brief account of this method. More details can be found in Berger [11, 12], Pedoe [132], Samuel [138], Coxeter [43, 44], Sidler [156], Tisseron [170], Lehmann and Bkouche [112], and, of course, Volume II of Veblen and Young [178].  
正交性、角度和点之间的距离等概念不是射影概念。为了定义这些概念，需要在底层向量空间上有一个内积。我们说这些概念属于欧几里得几何。乍一看，射影几何并没有涵盖一些重要的欧几里德概念，这似乎是射影几何的一个主要缺点。幸运的是，十九世纪的几何学家（包括拉盖尔、蒙格、庞塞莱、查理斯、冯·斯泰德、凯莱和克莱恩）发现了一种巧妙的方法，通过将真实的射影空间嵌入复杂的射影空间来恢复某些欧几里德概念，如角度和正交性（也包括圆）。锿。在接下来的两个部分中，我们将简要介绍这种方法。更多细节见Berger[11，12]、Pedoe[132]、Samuel[138]、Coxeter[43，44]、Sidler[156]、Tisseron[170]、Lehmann和Bkouche[112]，当然还有Veblen和Young的第二卷[178]。

25.14. COMPLEXIFICATION OF A REAL PROJECTIVE SPACE  
25.14条。实射影空间的复杂性

The first step is to embed a real vector space E into a complex vector space EC. A quick but somewhat bewildering way to do so is to define the complexification of E as the tensor product C ⊗ E. A more tangible way is to define the following structure.  
第一步是将实向量空间E嵌入到复向量空间EC中。一种快速但有些令人困惑的方法是将e的复杂性定义为张量积c e。一种更具体的方法是定义以下结构。

Definition 25.13. Given a real vector space E, let EC be the structure E × E under the addition operation  
定义25.13.给定一个实向量空间e，让ec为加法运算下的结构e×e。

(u1, u2) + (v1, v2) = (u1 + v1, u2 + v2),  
（u1，u2）+（v1，v2）=（u1+v1，u2+v2）

and let multiplication by a complex scalar z = x + iy be defined such that  
并定义乘以复数标量z=x+iy

(x + iy) · (u, v) = (xu − yv, yu + xv).  
（x+iy）·（u，v）=（xu−yv，yu+xv）。

It is easily shown that the structure EC is a complex vector space. It is also immediate that  
结果表明，结构EC是一个复杂的矢量空间。也很快

(0, v) = i(v, 0),  
（0，v）=i（v，0），

and thus, identifying E with the subspace of EC consisting of all vectors of the form (u, 0), we can write  
因此，用包含形式（u，0）所有向量的EC子空间来识别e，我们可以写

(u, v) = u + iv.  
（u，v）=u+iv。

Given a vector w = u + iv, its conjugate w is the vector w = u − iv. Then conjugation is a map from EC to itself that is an involution. If (e1,...,en) is any basis of E, then ((e1,0),...,(en,0)) is a basis of EC. We call such a basis a real basis.  
给定一个向量w=u+iv，它的共轭w是向量w=u−iv。那么共轭是从ec到自身的映射，这是一个对合。如果（e1，…，en）是e的任何基础，那么（（e1，0），…，（en，0））是ec的基础。我们称这种基础为真正的基础。

Given a linear map f : E → E, the map f can be extended to a linear map fC: EC → EC defined such that fC(u + iv) = f(u) + if(v).  
给定线性映射f:e→e，映射f可扩展为线性映射fc:ec→ec，定义为fc（u+iv）=f（u）+if（v）。

We define the complexification of P(E) as P). If E, is a real affine space, we define the complexified projective completion of E, E as P) and denote it by EeC. Then Ee is naturally embedded in EeC, and it is called the set of real points of EeC.  
我们将p（e）的复杂性定义为p）。如果e是一个实仿射空间，我们将e，e的复射影完备定义为p），并用eec表示。然后EE自然地嵌入到EEC中，被称为EEC的实数点集。

If E has dimension n+1 and (e1,...,en+1) is a basis of E, given any homogeneous polynomial P(x1,...,xn+1) over C of total degree m, because P is homogeneous, it is immediately verified that  
如果e的维数为n+1，且（e1，…，en+1）是e的基础，给定总度数m的c上的任何齐次多项式p（x1，…，xn+1），因为p是齐次的，立即验证

P(x1,...,xn+1) = 0  
P（x1，…，xn+1）=0

iff  
敌我识别

P(λx1,...,λxn+1) = 0,  
p（λx1，…，λxn+1）=0，

for any λ = 06 . Thus, we can define the hypersurface V (P) of equation P(x1,...,xn+1) = 0 as the subset of EeC consisting of all points of homogeneous coordinates (x1,...,xn+1) such that P(x1,...,xn+1) = 0. We say that the hypersurface V (P) of equation P(x1,...,xn+1) = 0 is real whenever P(x1,...,xn+1) = 0 implies that P(x1,...,xn+1) = 0.  
对于任何λ=06。因此，我们可以将方程p（x1，…，xn+1）=0的超曲面v（p）定义为由齐次坐标（x1，…，xn+1）的所有点组成的EEC的子集，这样p（x1，…，xn+1）=0。我们认为当p（x1，…，xn+1）=0表示p（x1，…，xn+1）=0时，方程p（x1，…，xn+1）=0的超曲面v（p）是实的。

 Note that a real hypersurface may have points other than real points, or no real points at all. For example,  
注意，真实的超曲面可能有实点以外的点，或者根本没有实点。例如，

x2 + y2 − z2 = 0  
x2+y2−z2=0

contains real and complex points such as (1,i,0) and (1,−i,0), and  
包含实点和复杂点，如（1、i、0）和（1、−i、0），以及

x2 + y2 + z2 = 0  
x2+y2+z2=0

contains only complex points. When m = 2 (where m is the total degree of P), a hypersurface is called a quadric, and when m = 2 and n = 2, a conic. When m = 1, a hypersurface is just a hyperplane.  
只包含复杂点。当m=2（其中m是p的总度数）时，超曲面称为二次曲面；当m=2和n=2时，称为二次曲面。当m=1时，超曲面只是一个超平面。

Given any homogeneous polynomial P(x1,...,xn+1) over R of total degree m, since viewed as a homogeneous polynomial over C defines a hypersurface V (P)C in EC, and also a hypersurface V (P) in P(E). It is clear that V (P) is naturally embedded in V (P)C, and V (P)C is called the complexification of V (P).  
给定总次数m的r上的任何齐次多项式p（x1，…，xn+1），因为被视为c上的齐次多项式，所以在ec中定义了超曲面v（p）c，在p（e）中定义了超曲面v（p）。很明显V（P）是自然嵌入V（P）C中的，V（P）C被称为V（P）的复杂性。

We now show how certain real quadrics without real points can be used to define orthogonality and angles.  
我们现在展示了如何使用没有实点的实数四次曲面来定义正交性和角度。

## 25.15 Similarity Structures on a Projective Space 25.15射影空间上的相似结构

We begin with a real Euclidean plane E,. We will show that the angle of two lines D1 and D2 can be expressed as a certain cross-ratio involving the lines D1, D2 and also two lines DI and DJ joining the intersection point D1 ∩D2 of D1 and D2 to two complex points at infinity I and J called the circular points. However, there is a slight problem, which is that we haven’t yet defined the angle of two lines! Recall that we define the (oriented) angle of two unit vectors u1, u2 as the equivalence class of pairs of unit vectors under the equivalence relation defined such that  
我们从一个真正的欧几里得平面E开始。我们将证明，两条直线d1和d2的夹角可以表示为一个特定的交叉比，涉及到直线d1、d2，以及两条直线di和dj，将d1和d2的交点d1 d2连接到无穷大i和j处的两个复点，称为圆点。但是，还有一个小问题，就是我们还没有定义两条线的角度！回想一下，我们把两个单位向量u1，u2的（定向）角定义为在等价关系下单位向量对的等价类，这样

hu1,u2i ≡ hu3,u4i  
hu1、u2i hu3、u4i

iff there is some rotation r such that r(u1) = u3 and r(u2) = u4. The set of (oriented) angles of vectors is a group isomorphic to the group SO(2) of plane rotations. If the Euclidean plane is oriented, the measure of the angle of two vectors is defined up to 2kπ (k ∈ Z). The angle of two vectors has a measure that is either θ or 2π − θ, where θ ∈ [0,2π[ , depending on the orientation of the plane. The problem with lines is that they are not oriented: A line is defined by a point a and a vector u, but also by a and −u. Given any two lines D1 and D2, if r is a rotation of angle θ such that r(D1) = D2, note that the rotation −r of angle θ +π also maps D1 onto D2. Thus, in order to define the (oriented) angle D\1D2 of two lines D1, D2, we define an equivalence relation on pairs of lines as follows:  
如果有旋转R，R（u1）=u3，R（u2）=u4。向量的（定向）角集是一个与平面旋转的SO（2）群同构的群。如果欧几里得平面是定向的，那么两个向量的角度的测量被定义为最高2 kπ（k∈z）。两个矢量的角度有一个θ或2π−θ的度量，其中θ∈[0,2π[，取决于平面的方向。线的问题在于它们没有定向：线由点A和向量U定义，也由A和−U定义。给定任意两条线d1和d2，如果r是角θ的旋转，因此r（d1）=d2，注意角θ+π的旋转−r也将d1映射到d2。因此，为了定义两条直线d1，d2的（定向）角d\1d2，我们定义了一个直线对上的等价关系，如下所示：

hD1,D2i ≡ hD3,D4i  
hd1、d2i hd3、d4i

if there is some rotation r such that r(D1) = D2 and r(D3) = D4.  
如果存在旋转R，使得R（d1）=d2和R（d3）=d4。

It can be verified that the set of (oriented) angles of lines is a group isomorphic to the quotient group SO(2)/{id,−id}, also denoted by PSO(2). In order to define the measure of the angle of two lines, the Euclidean plane E must be oriented. The measure of the angle D\1D2 of two lines is defined up to kπ (k ∈ Z). The angle of two lines has a measure that is either θ or π − θ, where θ ∈ [0,π[ , depending on the orientation of the plane. We now go back to the circular points.  
可以证明线的一组（定向）角是与商群so（2）/id、−id同构的群，也由pso（2）表示。为了定义两条直线的角度测量，欧几里得平面E必须定向。两条直线的角d \1d2的测量定义为Kπ（K∈Z）。两条直线的角度有一个θ或π−θ的度量，其中θ∈[0，π[，取决于平面的方向。我们现在回到圆点。

Let (a0,a1,a2,a3) be any projective frame for EeC such that (a0,a1) arises from an orthonormal basis (u1,u2) of →−E and the line at infinity H corresponds to z = 0 (where (x,y,z) are the homogeneous coordinates of a point w.r.t. (a0,a1,a2,a3)). Consider the points belonging to the intersection of the real conic Σ of equation  
设（a0，a1，a2，a3）为EEC的任何投影帧，使（a0，a1）从→−e的正交基（u1，u2）产生，无穷大h处的线对应于z=0（其中（x，y，z）是点W.R.T.（a0，a1，a2，a3）的齐次坐标）。考虑方程实二次曲线∑的交点

x2 + y2 − z2 = 0  
x2+y2−z2=0

with the line at infinity z = 0. For such points, x2 + y2 = 0 and z = 0, and since  
直线在无穷大z=0。对于这些点，x2+y2=0，z=0，并且

x2 + y2 = (y − ix)(y + ix),  
x2+y2=（y−ix）（y+ix）

we get exactly two points I and J of homogeneous coordinates (1,−i,0) and (1,i,0). The points I and J are called the circular points, or the absolute points, of EeC. They are complex points at infinity. Any line containing either I or J is called an isotropic line.  
我们得到两个齐次坐标点i和j（1、−i，0）和（1，i，0）。I点和J点被称为EEC的圆点或绝对点。它们是无穷远的复点。任何含有i或j的线都称为各向同性线。

What is remarkable about I and J is that they allow the definition of the angle of two lines in terms of a certain cross-ratio. Indeed, consider two distinct real lines D1 and D2 in E, and let DI and DJ be the isotropic lines joining D1 ∩D2 to I and J. We will compute the cross-ratio [D1,D2,DI,DJ]. For this, we simply have to compute the cross-ratio of the four points obtained by intersecting D1,D2,DI,DJ with any line not passing through D1 ∩ D2. By changing frame if necessary, so that D1 ∩ D2 = a0, we can assume that the equations of the lines D1,D2,DI,DJ are of the form  
关于i和j，值得注意的是，它们允许以一定的交叉比定义两条直线的角度。实际上，考虑e中两条不同的实线d1和d2，让di和dj是连接d1 d2到i和j的各向同性线。我们将计算交叉比[d1，d2，di，dj]。为此，我们只需计算d1、d2、di、dj与任何不通过d1 d2的线相交所得到的四个点的交叉比。如有必要，通过改变帧，使d1 d2=a0，我们可以假定线d1、d2、di、dj的方程为

|  |  |  |
| --- | --- | --- |
| y Y | = = | m1x, M1X， |
| y Y | = = | m2x, 小精灵， |
| y y Y | = =  = = |  |

leaving the cases m1 = ∞ and m2 = ∞ as a simple exercise. If we choose z = 0 as the intersecting line, we need to compute the cross-ratio of the points (D1)∞ = (1,m1,0),  
将案例m1=∞和m2=∞留作简单练习。如果我们选择z=0作为相交线，我们需要计算点的交叉比（d1）∞=（1，m1，0），

(D2)∞ = (1,m2,0), I = (1,−i,0), and J = (1,i,0), and we get  
（d2）∞=（1，m2,0），i=（1，−i，0）和j=（1，i，0），我们得到

,  
，

that is,  
也就是说，

.  
.

However, since m1 and m2 are the slopes of the lines D1 and D2, it is well known that if θ is the (oriented) angle between D1 and D2, then  
然而，由于m1和m2是直线d1和d2的斜率，众所周知，如果θ是d1和d2之间的（定向）角，那么

.  
.

Thus, we have  
因此，我们

,  
，

that is,  
也就是说，

One can check that the formula still holds when m1 = ∞ or m2 = ∞, and also when  
我们可以检查当m1=∞或m2=∞时公式是否仍然有效，以及当

D1 = D2. The formula  
d1=d2。公式

[D1,D2,DI,DJ] = ei2θ  
[d1，d2，di，dj]=ei2θ

is known as Laguerre’s formula.  
被称为拉盖尔公式。

If U denotes the group {eiθ | −π ≤ θ ≤ π} of complex numbers of modulus 1, recall that the map Λ: R → U defined such that  
如果u表示模1复数的eiθ−π≤θ≤π组，则回想图∧：r→u定义如下：

Λ(t) = eit  
∧（t）=eit

is a group homomorphism such that Λ−1(1) = 2kπ, where k ∈ Z. The restriction  
是一个群同态，使得∧−1（1）=2kπ，其中k∈z。限制

Λ: ] − π, π[ → (U − {−1})  
∧：−π，π[→（U−−1）

of Λ to ] − π, π[ is a bijection, and its inverse will be denoted by  
其中∧到]−π，π[是双射，其逆时针表示为

logU : (U − {−1}) → ] − π, π[ .  
对数单位：（U−−1）→]−π，π[。

For stating Proposition 25.28 more conveniently, we extend logU to U by letting logU(−1) = π, even though the resulting function is not continuous at −1!. Then we can write  
为了更方便地说明命题25.28，我们通过让logu（−1）=π将logu扩展到u，即使结果函数在−1处不是连续的！然后我们可以写

.  
.

If the orientation of the plane E is reversed, θ becomes π − θ, and since  
如果平面e的方向相反，θ变为π−θ，并且

ei2(π−θ) = e2iπ−i2θ = e−i2θ,  
e i2（π−θ）=e2iπ−i2θ=e−i2θ，

logU(ei2(π−θ)) = −logU(ei2θ), and  
logu（ei2（π−θ））=−logu（ei2θ），和

.  
.

In all cases, we have  
在所有情况下，我们

,  
，

a formula due to Cayley. We summarize the above in the following proposition.  
凯莱的公式。我们将在下面的命题中总结上述内容。

Proposition 25.28. Given any two lines D1,D2 in a real Euclidean plane E,, letting  
提案25.28。给定任意两条线d1，d2在一个真正的欧几里得平面e中，让

DI and DJ be the isotropic lines in joining the intersection point D1 ∩ D2 of D1 and D2 to the circular points I and J, if θ is the angle of the two lines D1, D2, we have  
di和dj是将d1和d2的交点d1 d2连接到圆点i和j的各向同性线，如果θ是两条线d1和d2的夹角，我们得到

[D1,D2,DI,DJ] = ei2θ,  
[d1，d2，di，dj]=ei2θ，

known as Laguerre’s formula, and independently of the orientation of the plane, we have  
被称为拉盖尔公式，独立于平面的方向，我们有

,  
，

known as Cayley’s formula.  
被称为凯莱公式。

In particular, note that θ = π/2 iff [D1,D2,DI,DJ] = −1, that is, if (D1,D2,DI, DJ) forms a harmonic division. Thus, two lines D1 and D2 are orthogonal iff they form a harmonic division with DI and DJ.  
特别注意，θ=π/2 iff[d1，d2，di，dj]=−1，也就是说，如果（d1，d2，di，dj）形成一个谐波除法。因此，两条线d1和d2是正交的iff，它们与di和dj形成一个谐波分区。

The above considerations show that it is not necessary to assume that E, is a real Euclidean plane to define the angle of two lines and orthogonality. Instead, it is enough to assume that two complex conjugate points I,J on the line H at infinity are given. We say that hI,Ji provides a similarity structure on EeC. Note in passing that a circle can be defined as a conic in EeC that contains the circular points I,J. Indeed, the equation of a conic is of the form  
上述考虑表明，不必假定e是一个真正的欧几里得平面来定义两条直线的夹角和正交性。相反，假设在无穷远的H线上有两个复共轭点i，j就足够了。我们说，hi，ji在eec上提供了一个相似的结构。请注意，在包含圆点i，j的EEC中，圆可以定义为圆锥曲线。实际上，圆锥曲线方程的形式是

ax2 + by2 + cxy + dxz + eyz + fz2 = 0.  
ax2+by2+cxy+dxz+eyz+fz2=0。

If this conic contains the circular points I = (1,−i,0) and J = (1,i,0), we get the two equations  
如果这个圆锥曲线包含圆点i=（1，−i，0）和j=（1，i，0），我们得到两个方程

,  
，

from which we get 2ic = 0 and a = b, that is, c = 0 and a = b. The resulting equation  
从中我们得到2ic=0和a=b，即c=0和a=b。所得方程

ax2 + ay2 + dxz + eyz + fz2 = 0  
ax2+ay2+dxz+eyz+fz2=0

is indeed that of a circle.  
确实是一个圆。

Instead of using the function logU : (U − {−1}) → ] − π, π[ as logarithm, one may use the complex logarithm function log: C∗ → B, where C∗ = C − {0} and  
不用函数log u:（u−−1）→]−π，π[作为对数，可以使用复数对数函数log c:c→b，其中c=c−0和

B = {x + iy | x,y ∈ R, −π < y ≤ π}.  
b=x+iy x，y∈r，−π<y≤π。

Indeed, the restriction of the complex exponential function z 7→ ez to B is bijective, and thus, log is well-defined on C∗ (note that log is a homeomorphism from C − {x | x ∈ R, x ≤ 0} onto {x + iy | x,y ∈ R, −π < y < π}, the interior of B). Then Cayley’s formula reads as  
实际上，复指数函数z 7→ez对b的约束是双射的，因此，对数在c上定义得很好（注意，对数是从c−x x∈r，x≤0到x+iy x，y∈r，−π<y<π，b的内部的同态）。凯莱的公式是

,  
，

with a ± in front when the plane is nonoriented. Observe that this formula allows the definition of the angle of two complex lines (possibly a complex number) and the notion of orthogonality of complex lines. In this case, note that the isotropic lines are orthogonal to themselves!  
当平面没有定向时，前面有一个？.注意，这个公式允许定义两条复杂线（可能是复数）的角度和复杂线的正交性概念。在这种情况下，请注意各向同性线与其自身是正交的！

The definition of orthogonality of two lines D1,D2 in terms of (D1,D2, DI,DJ) forming a harmonic division can be used to give elegant proofs of various results. Cayley’s formula can even be used in computer vision to explain modeling and calibrating cameras! (see Faugeras [60]). As an illustration, consider a triangle (a,b,c), and recall that the line a0 passing through a and orthogonal to (b,c) is called the altitude of a, and similarly for b and c. It is well known that the altitudes a0,b0,c0 intersect in a common point called the orthocenter of the triangle (a,b,c). This can be shown in a number of ways using the circular points. Indeed, letting , and denote the points at infinity of the  
用（d1，d2，di，dj）来定义两条直线d1，d2的正交性，形成一个调和除法，可以很好地证明各种结果。凯莱的公式甚至可以用于计算机视觉解释建模和校准相机！（见Faugeras[60]）。作为一个例子，考虑一个三角形（A，B，C），并回想一下，穿过A并与（B，C）正交的线a0被称为A的高度，同样地，对于B和C也是如此。众所周知，高度a0，b0，c0在一个称为三角形正交中心（A，B，C）的公共点相交。这可以通过使用圆点的多种方式来显示。实际上，让并表示

lines hb,ci,ha,bi, ha,ci, a0,b0, and c0, we have  
行HB、CI、HA、BI、HA、CI、A0、B0和C0，我们有

,  
，

and it is easy to show that there is an involution σ of the line at infinity such that  
很容易证明在无穷远处有一条线的对合σ，这样

|  |  |  |
| --- | --- | --- |
| σ(I) 西格玛（I） | = = | J, J |
| σ(J) （j） | = = | I, 我， |
| σ(bc∞) σ(ab∞) σ(ac∞) σ（bc∞）σ（ab∞）σ（ac∞） | = =  = =  = = | a0 , A0  ∞ 无穷大  c0 , C0  ∞ 无穷大  b0 . B0。  ∞ 无穷大 |

Then, it can be shown that the lines a0,b0,c0 are concurrent. For more details and other results, notably on the conics, see Sidler [156], Berger [12], and Samuel [138].  
然后，可以看出，行a0、b0、c0是并发的。有关更多细节和其他结果，尤其是圆锥曲线，请参见Sidler[156]、Berger[12]和Samuel[138]。

The generalization of what we just did to real Euclidean spaces E, of dimension n is simple. Let (a0,...,an+1) be any projective frame for such that (a0,...,an−1) arises from an orthonormal basis (u1,...,un) of →−E and the hyperplane at infinity H corresponds to xn+1 = 0 (where (x1,...,xn+1) are the homogeneous coordinates of a point with respect to (a0,...,an+1)). Consider the points belonging to the intersection of the real quadric Σ of equation  
我们刚才对实欧几里得空间e，即维n所做的推广很简单。设（a0，…，an+1）为任意投影框架，使（a0，…，an-1）由→−e的正交基（u1，…，un）产生，无穷大h处的超平面对应于xn+1=0（其中（x1，…，xn+1）是点相对于（a0，…，an+1）的齐次坐标）。考虑方程实二次∑的交点



with the hyperplane at infinity xn+1 = 0. For such points,  
超平面在无穷大xn+1=0。对于这些问题，

= 0 and xn+1 = 0.  
=0和xn+1=0。

Such points belong to a quadric called the absolute quadric of EeC, and denoted by Ω. Any line containing any point on the absolute quadric is called an isotropic line. Then, given any two coplanar lines D1 and D2 in E, these lines intersect the hyperplane at infinity H in two points (D1)∞ and (D2)∞, and the line ∆ joining (D1)∞ and (D2)∞ intersects the absolute quadric Ω in two conjugate points I∆ and J∆ (also called circular points). It can be shown that the angle θ between D1 and D2 is defined by Laguerre’s formula:  
这些点属于称为EEC绝对二次曲线的二次曲线，并用Ω表示。任何在绝对二次曲面上包含任何点的线称为各向同性线。然后，在e中任意两条共面线d1和d2，这些线在无穷大h处的两点（d1）∞和（d2）∞与超平面相交，而线∆连接（d1）∞和（d2）∞与绝对二次方Ω在两个共轭点i∆和j∆（也称为圆点）相交。可以看出，d1和d2之间的角度θ由拉盖尔公式定义：

[(D1)∞,(D2)∞,I∆,J∆] = [D1,D2,DI∆,DJ∆] = ei2θ,  
[（d1）∞，（d2）∞，i∆，j∆]=[d1，d2，di∆，dj∆]=ei2θ，

where DI∆ and DJ∆ are the lines joining the intersection D1∩D2 of D1 and D2 to the circular points I∆ and J∆.  
式中，di∆和dj∆是将d1和d2的交点d1 d2连接到圆点i∆和j∆的直线。

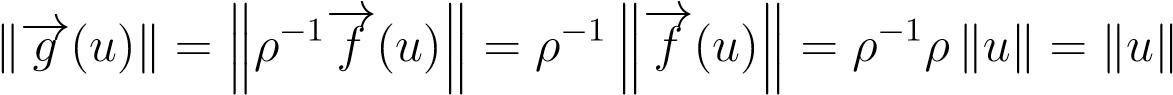
As in the case of a plane, the above considerations show that it is not necessary to assume that E, is a real Euclidean space to define the angle of two lines and orthogonality. Instead, it is enough to assume that a nondegenerate real quadric Ω in the hyperplane at infinity H and without real points is given. In particular, when n = 3, the absolute quadric Ω is a nondegenerate real conic consisting of complex points at infinity. We say that Ω provides a similarity structure on EeC.  
对于平面，上述考虑表明，不必假设e是一个真正的欧几里得空间来定义两条直线的角度和正交性。相反，假设超平面上无穷大H且不带实点的非退化实二次曲面Ω就足够了。特别地，当n=3时，绝对二次方Ω是由无穷远的复点组成的非退化实二次曲线。我们说，Ω在EEC上提供了一个相似的结构。

It is also possible to show that the real projectivities of EeC that leave both the hyperplane H at infinity and the absolute quadric Ω (globally) invariant form a group which is none other than the group of affine similarities; see Lehmann and Bkouche [112] (Chapter 10, page 321), and Berger [11] (Chapter 8, Proposition 8.8.6.4).  
也可以证明，将超平面h保持在无穷远处的欧共体的实射影率和绝对二次欧（全局）不变的欧共体形成一个除了仿射相似性组以外的组；见Lehmann和Bkouche[112]（第10章，第321页）和Berger[11]（第8章，提案8.8.6.4）。

Definition 25.14. Let (→−E,→−E,h−,−i) be a Euclidean affine space of finite dimension. An∈ →− affine similarity of (E, E) is an invertible affine map f GA(E) such that if f is the linear map associated with f, then there is some positive real ρ > 0 satisfying the condition  
定义25.14.设（→−e、→−e、h−、−i）为有限维欧几里德仿射空间。（e，e）的∈→−仿射相似性是可逆仿射映射f ga（e），如果f是与f相关的线性映射，则存在满足条件的正实ρ>0。

for all u ∈ →−E. The number ρ is called the ratio of the affine similarity f.  
对于所有u∈→−e，数ρ称为仿射相似性f的比值。

If f ∈ GA(E) is an affine similarity of ratio ρ, let →−g = ρ−1→−f . Since ρ > 0, we have  
如果f∈ga（e）是比率ρ的仿射相似度，则让→−g=ρ−1→−f。既然ρ>0，我们有



for all u ∈ →−E, and by Proposition 11.12, the map →−g = ρ−1→−f is an isometry; that is, →−g ∈ O(E).  
对于所有u∈→−e，根据命题11.12，图→−g=ρ−1→−f是一个等距测量；即→−g∈o（e）。

Consequently, every affine similarity f of E can be written as the composition of an isometry (a member of O(E)), a central dilatation, and a translation. For example, when n = 2, a similarity is a transformation of the form  
因此，e的每一个仿射相似性f都可以写成等距线（o（e）的一个成员）、中心扩张和翻译的组成部分。例如，当n=2时，相似度是形式的转换。

,  
，

with. We have the following result showing that the affine similarities of the plane can be viewed as special kinds of projectivities of CP2.  
用。结果表明，平面的仿射相似性可以看作是CP2的特殊射影性质。

Proposition 25.29. If a projectivity h of CP2 leaves the set of circular points {I,J} fixed and maps the affine space R2 into itself (where R2 is viewed as the subspace of all points (x,y,1) with x,y ∈ R), then h is an affine similarity.  
提案25.29。如果cp2的射影度h离开一组圆点i，j固定，并将仿射空间r2映射到自身（其中r2被视为所有点（x，y，1）的子空间，其中x，y∈r），则h是仿射相似性。

Proof. The fact that h leaves the set of circular points {I,J} fixed means that either h(I) = I and h(J) = J or h(I) = J and h(J) = I. If we define I0 and J0 by  
证据。H离开圆点集i，j固定的事实意味着h（i）=i和h（j）=j或h（i）=j和h（j）=i。如果我们定义i0和j0

0) and   
0）和

where 1, then the fact that h leaves the set of circular points {I,J} fixed is equivalent to  
式中1，则h离开圆点集i，j固定的事实等于

h(I) = I0 and h(J) = J0.  
h（i）=i0和h（j）=j0。

Assume that h is represented by the invertible matrix  
假设h由可逆矩阵表示

.  
.

Then h(I) = I0 and h(J) = J0 means that there is some nonzero scalars λ,µ ∈ C such  
那么h（i）=i0和h（j）=j0意味着存在一些非零的标量λ，μ∈c这样

and .  
而且。

We obtain the following equations:  
我们得到以下方程：

.  
.

By adding the two equations on the first row we obtain  
通过在第一行添加两个方程，我们得到

λ + µ = 2a,  
λ+μ=2a，

by subtracting the first equation from the second on the second row we obtain  
通过从第二行的第二个方程中减去第一个方程，我们得到

,  
，

so we get  
所以我们得到



By subtracting the first equation from the second on the first row we obtain  
从第一行的第二个方程中减去第一个方程，我们得到

µ − λ = 2ia0,  
礹−λ=2IA0，

and by adding the equations on the second row we obtain  
把第二行的方程相加，我们得到



and since 1, we have = 1, so we get  
从1开始，我们有=1，所以我们得到



By adding and subtracting the equations on the third row we obtain  
通过对第三行的方程进行加减，我们得到

c = c0 = 0.  
c=c0=0。

Sinceassume thatA is invertible,c00 = 1, and we conclude thatc00 = 06 , and since A is determined up to a nonzero scalar we may  
假设a是可逆的，c00=1，我们得出结论，that00=06，由于a被确定为非零标量，我们可以

.  
.

If h maps R2 into itself, then  
如果h将r2映射到自身中，则

must be real for all x,y ∈ R, which implies that a,b,a00,b00 ∈ R.   
必须是所有x，y∈r的实数，这意味着a，b，a00，b00∈r。

The following proposition from Berger [11] (Chapter 8, Proposition 8.8.5.1) gives a convenient characterization of the affine similarities.  
Berger[11]提出的以下命题（第8章，命题8.8.5.1）方便地描述了仿射相似性。

Proposition 25.30. Let (E,→−E,h−,−i) be a Euclidean affine space of finite dimension n ≥  
提案25.30。设（e，→−e，h−，−i）为有限维n≥的欧几里德仿射空间

2. An affine map is an affine similarity iff preserves orthogonality; that is, for any two vectors u,v ∈ E, if hu,vi = 0, then h f (u), f (v)i = 0.  
2。仿射映射是保持正交性的仿射相似性，即对于任意两个向量u，v∈e，如果hu，v i=0，则h f（u），f（v）i=0。

Proof.hu,vi = 0Assume that, then h→−f (uf),∈→−fGA(v)i(E= 0) is an affine map such that for any two vectors. Fix any nonzero u ∈ →−E and consider the linear formu,v ∈ →−E, ifϕu  
证明：hu，v i=0假设，那么h→−f（uf），yger→−fga（v）i（e=0）是一个仿射映射，对于任意两个向量。固定任意非零u∈→−e，并考虑线性形式u，v∈→−e，如果

given by ϕu(v) = h→−f (u),→−f (v)i, v ∈ →−E.  
由u（v）=h→−f（u），→−f（v）i，v∈→−e给出。

Since →−f is invertible, ϕu(u) 6= 0. For any v ∈ →−E such that hu,vi = 0, we have  
因为→−f是可逆的，所以u（u）6=0。对于任何v∈→−e，如hu，vi=0，我们有

ϕu(v) = h→−f (u),→−f (v)i = 0,  
⑨u（v）=h→−f（u），→−f（v）i=0，

thus ϕu is a nonzero linear form vanishing on the hyperplane H orthogonal to u, which is the kernel of the linear form v → h7 u,vi. Therefore, there is some nonzero scalar ρ(u) ∈ R such that ϕu(v) = ρ(u)hu,vi for all v ∈ →−E.  
因此，在与u正交的超平面h上，是一个非零线性形式消失，它是线性形式v→h7 u，vi的核心。因此，有一些非零的标量ρ（u）∈r，这样，所有v∈¨u（v）=ρ（u）hu，vi都是。

Evaluating ϕu at u, we see that ρ(u) > 0. If we can show that ρ(u) is a constant ρ > 0 independent of u, we will have shown that  
在u处评估u，我们发现ρ（u）>0。如果我们能证明ρ（u）是一个独立于u的常数ρ>0，我们将证明

h→−f (u),→−f (v)i = ρhu,vi for all u,v ∈ →−E,  
h→−f（u），→−f（v）i=ρhu，vi表示所有u，v∈→−e，

and we will be done.  
我们就完了。

independent, and let us evaluateSince dim(E) ≥ 2, pick v to be any nonzero vector inh→−f (u + v),→−f (w)i for any→−E such that. We haveu and v are linearly  
独立，并让我们评估，因为dim（e）≥2，选择v为任意非零矢量inh→−f（u+v），对于任意→−e，选择→−f（w）i。我们有u和v是线性的。

h→−f (u + v),→−f (w)i = ϕu+v(w)  
H→−F（U+V），→−F（W）I=\_U+V（W）

= ρ(u + v)hu + v,wi  
=ρ（u+v）hu+v，wi

= ρ(u + v)hu,wi + ρ(u + v)hv,wi  
=ρ（u+v）hu，wi+ρ（u+v）hv，wi

and  
和

h→−f (u + v),→−f (w)i = h→−f (u) + →−f (v),→−f (w)i  
H→−F（U+V），→−F（W）I=H→−F（U）+→−F（V），→−F（W）I

= h→−f (u),→−f (w)i + h→−f (v),→−f (w)i = ρ(u)hu,wi + ρ(v)hv,wi,  
=h→−f（u），→−f（w）i+h→−f（v），→−f（w）i=ρ（u）hu，wi+ρ（v）hv，wi，

so we get  
所以我们得到

h(ρ(u + v) − ρ(u))u + (ρ(u + v) − ρ(v))v,wi = 0 for all w ∈ →−E,  
h（ρ（u+v）−ρ（u））u+（ρ（u+v）−ρ（v））v，wi=0，对于所有w∈→−e，

which implies that  
这意味着

(ρ(u + v) − ρ(u))u + (ρ(u + v) − ρ(v))v = 0.  
（ρ（u+v）-ρ（u））u+（ρ（u+v）-ρ（v））v=0.

Since u and v are linearly independent, we must have  
既然u和v是线性无关的，我们必须

ρ(u + v) = ρ(u) = ρ(v).  
ρ（u+v）=ρ（u）=ρ（v）。

This proves that ρ(u) is a constant ρ independent of u, as claimed.  
这证明了ρ（u）是一个与u无关的常数。

The converse is trivial.   
相反，这是微不足道的。

→−Remark:f ∈ O(E)Letdoes not admit the eigenvalue 1, thenf ∈ GA(E) be an affine similarity of ratiof has a unique fixed point.ρ. If either ρ = 16 or ρ = 1 and  
→−备注：f∈o（e）Let不承认特征值1，那么f∈ga（e）是一个仿射相似度的比率有一个唯一的不动点。ρ。如果ρ=16或ρ=1和

any originIndeed, we havea ∈ E, the point→−f = ρ→−ag +for someu is a fixed point ofρ > 0 and some linear isometryf iff →−g ∈ O(E), so for  
任何一个原点，我们都有一个∈e，点→−f=ρ→−a g+对于someu是一个ρ>0的固定点和一些线性等距iff→−g∈o（e），因此

f(a + u) = a + u  
F（A+U）=A+U

iff f(a) + →−f (u) = a + u  
iff f（a）+→−f（u）=a+u

iff   
敌我识别

iff  
敌我识别

(→−g − ρ−1id)(  
（→−G−ρ−1id）（

The linear map →−g −ρ−1id is singular iff6 ρ−1 →−is an eigenvalue or →−g , and since→− →−g ∈→−O−(E) its eigenvalues have modulus 1, so if ρ = 1 or if ρ = 1 is not an eigenvalue of g , then g ρ−1id is invertible, and then there is a unique u ∈ E such that  
线性映射→−g−ρ−1id是奇异的，如果6ρ−1→−是一个特征值或→−g，并且由于→−→−g∈→−o−（e）其特征值具有模1，因此如果ρ=1或如果ρ=1不是G的特征值，则Gρ−1id是可逆的，然后有一个唯一的u∈e，这样

(→−g − ρ−1id)(  
（→−G−ρ−1id）（

For more details on the use of absolute quadrics to obtain some very sophisticated results, the reader should consult Berger [11, 12], Pedoe [132], Samuel [138], Coxeter [43], Sidler [156], Tisseron [170], Lehmann and Bkouche [112], and, of course, Volume II of Veblen and Young [178], which also explains how some non-Euclidean geometries are obtained by chosing the absolute quadric in an appropriate fashion (after Cayley and Klein).  
关于使用绝对四次曲面获得一些非常复杂的结果的更多细节，读者应咨询Berger[11，12]、Pedoe[132]、Samuel[138]、Coxeter[43]、Sidler[156]、Tisseron[170]、Lehmann和Bkouche[112]，当然，还应咨询Veblen和Young[178]的第二卷，其中还包括平素一些非欧几里得几何是如何通过以适当的方式选择绝对二次曲面获得的（在凯莱和克莱因之后）。

## 25.16 Some Applications of Projective Geometry 25.16射影几何的一些应用

Projective geometry is definitely a jewel of pure mathematics and one of the major mathematical achievements of the nineteenth century. It turns out to be a prerequisite for algebraic geometry, but to our surprise (and pleasure), it also turns out to have applications in engineering. In this short section we summarize some of these applications.  
射影几何无疑是纯数学的瑰宝，是十九世纪数学的主要成就之一。结果证明这是代数几何的先决条件，但令我们惊讶（和高兴）的是，它也被证明在工程中有应用。在这一小段中，我们将总结其中的一些应用程序。

We first discuss applications of projective geometry to camera calibration, a crucial problem in computer vision. Our brief presentation follows quite closely Trucco and Verri [172] (Chapter 2 and Chapter 6). One should also consult Faugeras [60], or Jain, Katsuri, and Schunck [97].  
我们首先讨论了射影几何在摄像机标定中的应用，摄像机标定是计算机视觉中的一个关键问题。我们的简短介绍与Trucco和Verri[172]非常接近（第2章和第6章）。还应咨询Faugeras[60]或Jain、Katsuri和Schunck[97]。

The pinhole (or perspective) model of a camera is a typical example from computer vision that can be explained very simply in terms of projective transformations. A pinhole camera consists of a point O called the center or focus of projection, and a plane π (not containing O) called the image plane. The distance f from the image plane π to the center O is called the focal length. The line through O and perpendicular to π is called the optical axis, and the point o, intersection of the optical axis with the image plane is called the principal point or image center. The way the camera works is that a point P in 3D space is projected onto the image plane (the film) to a point p via the central projection of center O.  
相机的针孔（或透视）模型是计算机视觉的一个典型例子，可以很简单地用投影变换来解释。针孔相机由一个称为投影中心或焦点的点O和一个称为图像平面的平面π（不包含O）组成。从图像平面π到中心O的距离f称为焦距。穿过O并垂直于π的线称为光轴，光轴与像平面的交点称为主点或像中心。相机的工作方式是通过中心O的中心投影将三维空间中的点P投影到图像平面（胶片）上的点P。

It is assumed that an orthonormal frame Fc is attached to the camera, with its origin at O and its z-axis parallel to the optical axis. Such a frame is called the camera reference frame. With respect to the camera reference frame, it is very easy to write the equations relating the coordinates (x,y) (omitting z = f) of the image p (in the image plane π) of a point P of coordinates (X,Y,Z):  
假设一个正交帧fc连接到相机，其原点在o，Z轴平行于光轴。这种帧称为相机参考帧。对于摄像机参考帧，很容易写出关于坐标点p（x，y，z）的图像p（在图像平面π中）的坐标（x，y）（省略z=f）的方程：

.  
.

Typically, points in 3D space are defined by their coordinates not with respect to the camera reference frame, but with respect to another frame Fw, called the world reference frame.  
通常，三维空间中的点是由它们的坐标定义的，而不是相对于相机参考帧，而是相对于另一帧fw，称为世界参考帧。

However, for most computer vision algorithms, it is necessary to know the coordinates of a point in 3D space with respect to the camera reference frame. Thus, it is necessary to know the position and orientation of the camera with respect to the frame Fw. The position and orientation of the camera are given by some affine transformation (R,T) mapping the frame Fw to the frame Fc, where R is a rotation matrix and T is a translation vector. Furthermore, the coordinates of an image point are typically known in terms of pixel coordinates, and it is also necessary to transform the coordinates of an image point with respect to the camera reference frame to pixel coordinates. In summary, it is necessary to know the transformation that maps a point P in world coordinates (w.r.t. Fw) to pixel coordinates.  
然而，对于大多数计算机视觉算法来说，有必要知道三维空间中一点相对于相机参考帧的坐标。因此，有必要知道相机相对于帧FW的位置和方向。摄像机的位置和方向由一些仿射变换（r，t）给出，将帧fw映射到帧fc，其中r是旋转矩阵，t是平移向量。此外，图像点的坐标通常以像素坐标的形式已知，并且还需要将图像点相对于相机参考帧的坐标转换为像素坐标。总之，有必要知道将世界坐标（W.R.T.FW）中的点P映射到像素坐标的转换。

This transformation of world coordinates to pixel coordinates turns out to be a projective transformation that depends on the extrinsic and the intrinsic parameters of the camera. The extrinsic parameters of a camera are the location and orientation of the camera with respect to the world reference frame Fw. It is given by an affine map (in fact, a rigid motion, see Chapter 12, Section 26.2). The intrinsic parameters of a camera are the parameters needed to link the pixel coordinates of an image point to the corresponding coordinates in the camera reference frame. If Pw = (Xw,Yw,Zw) and Pc = (Xc,Yc,Zc) are the coordinates of the 3D point P with respect to the frames Fw and Fc, respectively, we can write  
世界坐标到像素坐标的转换是一种投影变换，它依赖于相机的外在和内在参数。相机的外部参数是相机相对于世界参考帧fw的位置和方向。它由仿射图给出（事实上，刚性运动，见第12章第26.2节）。相机的内部参数是将图像点的像素坐标链接到相机参考帧中相应坐标所需的参数。如果pw=（xw，yw，zw）和pc=（xc，yc，zc）分别是三维点p相对于帧fw和fc的坐标，我们可以写

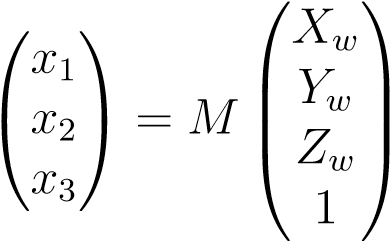
Pc = R(Pw − T).  
pc=r（pw−t）。

|  |  |  |
| --- | --- | --- |
| is given by 由给出 |  |  |
| x X | = = | −(xim − ox)sx, −（XIM−OX）SX， |
| y Y | = = | −(yim − oy)sy, −（Yim−Oy）系统， |

Neglecting distorsions possibly introduced by the optics, the correspondence between the coordinates (x,y) of the image point with respect to Fc and the pixel coordinates (xim,yim)  
忽略光学可能引入的畸变，图像点相对于fc的坐标（x，y）与像素坐标（xim，yim）之间的对应关系。

where (ox,oy) are the pixel coordinates the principal point o and sx,sy are scaling parameters.  
其中（ox，oy）是像素坐标，主点o和sx，sy是缩放参数。

After some simple calculations, the upshot of all this is that the transformation between the homogeneous coordinates (Xw,Yw,Zw,1) of a 3D point and its homogeneous pixel coordinates (x1,x2,x3) is given by  
经过一些简单的计算，得出的结论是，三维点的齐次坐标（xw，yw，zw，1）与其齐次像素坐标（x1，x2，x3）之间的转换由下式得出：



where the matrix M, known as the projection matrix, is a 3 × 4 matrix depending on R, T, ox,oy, f (the focal length), and sx,sy (for the derivation of this equation, see Trucco and Verri [172], Chapter 2).  
其中，矩阵m（称为投影矩阵）是一个3×4的矩阵，取决于r、t、ox、oy、f（焦距）和sx、sy（关于该方程的推导，见Trucco和Verri[172]，第2章）。

The problem of estimating the extrinsic and the instrinsic parameters of a camera is known as the camera calibration problem. It is an important problem in computer vision.  
摄像机的外参数和内参数估计问题称为摄像机标定问题。这是计算机视觉中的一个重要问题。

Now, using the equations  
现在，使用这些方程

|  |  |  |
| --- | --- | --- |
| x X | = = | −(xim − ox)sx, −（XIM−OX）SX， |
| y Y | = = | −(yim − oy)sy, −（Yim−Oy）系统， |

we get  
我们得到

,  
，

relating the coordinates w.r.t. the camera reference frame to the pixel coordinates. This suggests using the parameters fx = f/sx and fy = f/sy instead of the parameters f,sx,sy. In fact, all we need are the parameters fx = f/sx and α = sy/sx, called the aspect ratio. Without loss of generality, it can also be assumed that (ox,oy) are known. Then we have a total of eight parameters.  
将相机参考帧的坐标W.R.T.与像素坐标相关联。这建议使用参数fx=f/sx和fy=f/sy，而不是参数f、sx、sy。实际上，我们需要的只是参数fx=f/sx和α=sy/sx，称为展弦比。在不丧失一般性的情况下，也可以假定（Ox，Oy）是已知的。然后我们一共有八个参数。

One way of solving the calibration problem is to try estimating fx,α, the rotation matrix R, and the translation vector T from N image points (xi,yi), projections of N suitably chosen world points (Xi,Yi,Zi), using the system of equations obtained from the projection matrix. It turns out that if N ≥ 7 and the points are not coplanar, the rank of the system is 7, and the system has a nontrivial solution (up to a scalar) that can be found using SVD methods (see Chapter 20, Trucco and Verri [172], or Jain, Katsuri, and Schunck [97]).  
解决校准问题的一种方法是尝试从N个图像点（XI，YI）估计FX、α、旋转矩阵R和平移向量T，使用从投影矩阵获得的方程系统，N个适当选择的世界点（XI，Yi，ZI）的投影。结果表明，如果n≥7且点不是共面的，则系统的秩为7，并且系统有一个非平凡解（达到一个标量），可以使用SVD方法找到该解（见第20章，Trucco和Verri[172]或Jain、Katsuri和Schunck[97]）。

Another method consists in estimating the whole projection matrix M, which depends on 11 parameters, and then extracting extrinsic and intrinsic parameters. Again, SVD methods are used (see Trucco and Verri [172], and Faugeras [60]).  
另一种方法是估计整个投影矩阵M，它依赖于11个参数，然后提取外部和内部参数。同样，使用SVD方法（见Trucco和Verri[172]和Faugeras[60]）。

Cayley’s formula can also be used to solve the calibration cameras, as explained in Faugeras [60]. Other problems in computer vision can be reduced to problems in projective geometry (see Faugeras [60]).  
如Faugeras[60]所述，Cayley的公式也可用于解决校准摄像头问题。计算机视觉中的其他问题可以简化为射影几何中的问题（见Faugeras[60]）。

In computer graphics, it is also necessary to convert the 3D world coordinates of a point to a two-dimensional representation on a view plane. This is achieved by a so-called viewing system using a projective transformation. For details on viewing systems see Watt [183] or Foley, van Dam, Feiner, and Hughes [64].  
在计算机图形学中，还需要将点的三维世界坐标转换为视图平面上的二维表示。这是通过使用投影变换的所谓观察系统实现的。有关查看系统的详细信息，请参见瓦特[183]或福利、范达姆、费纳和休斯[64]。

Projective spaces are also the right framework to deal with rational curves and rational surfaces. Indeed, in the projective framework it is easy to deal with vanishing denominators and with “infinite” values of the parameter(s).  
射影空间也是处理有理曲线和有理曲面的合适框架。实际上，在射影框架中，很容易处理消失分母和参数的“无限”值。

It is much less obvious that projective geometry has applications to efficient communication, error-correcting codes, and cryptography, as very nicely explained by Beutelspacher and Rosenbaum [22]. We sketch these applications very briefly, referring our readers to [22] for details. We begin with efficient communication. Suppose that eight students would like to exchange information to do their homework economically. The idea is that each student solves part of the exercises and copies the rest from the others (which we do not recommend, of course!). It is assumed that each student solves his part of the homework at home, and that the solutions are communicated by phone. The problem is to minimize the number of phone calls. An obvious but expensive method is for each student to call each of the other seven students. A much better method is to imagine that the eight students are the vertices of a cube, say with coordinates from {0,1}3. There are three types of edges:  
正如Beutelspacher和Rosenbaum[22]很好地解释的那样，射影几何在有效通信、纠错码和密码学方面的应用就不那么明显了。我们非常简单地概述了这些应用程序，详细信息请参阅[22]。我们从有效的沟通开始。假设有八个学生愿意交换信息以经济地完成家庭作业。这个想法是每个学生解决部分练习，并从其他人那里复制其余的（当然，我们不推荐！）.假设每个学生在家里完成自己的家庭作业，并通过电话沟通解决方案。问题是尽量减少通话次数。一个明显但昂贵的方法是让每个学生给另外七个学生打电话。一个更好的方法是假设八个学生是一个立方体的顶点，比如坐标为0,1 3。边缘有三种类型：

1. Those parallel to the z-axis, called type 1; 2. Those parallel to the y-axis, called type 2;  
1。那些平行于z轴的，称为1型；2型。平行于y轴的，称为2型；

3. Those parallel to the x-axis, called type 3.  
三。那些平行于x轴的，称为3型。

The communication can proceed in three rounds as follows: All nodes connected by type 1 edges exchange solutions; all nodes connected by type 2 edges exchange solutions; and finally all nodes connected by type 3 edges exchange solutions.  
通信可以分三轮进行：所有节点通过类型1边缘交换解决方案连接；所有节点通过类型2边缘交换解决方案连接；最后所有节点通过类型3边缘交换解决方案连接。

It is easy to see that everybody has all the answers at the end of the three rounds. Furthermore, each student is involved only in three calls (making a call or receiving it), and the total number of calls is twelve.  
很容易看出，在三轮比赛结束时，每个人都有所有的答案。此外，每个学生只参与三个电话（打一个电话或接一个电话），总的电话数是12个。

In the general case, N nodes would like to exchange information in such a way that eventually every node has all the information. A good way to to this is to construct certain finite projective spaces, as explained in Beutelspacher and Rosenbaum [22]. We pick q to be an integer (for instance, a prime number) such that there is a finite projective space of any dimension over the finite field of order q. Then, we pick d such that  
在一般情况下，n个节点希望以这样的方式交换信息，最终每个节点都拥有所有的信息。一个很好的方法是构造某些有限射影空间，如Beutelspacher和Rosenbaum[22]所述。我们把q选为一个整数（例如质数），这样在q阶的有限域上任何维都有一个有限的射影空间，然后，我们把d选为

qd−1 < N ≤ qd.  
qd−1<n≤qd。

Since q is prime, there is a projective space P(Kd+1) of dimension d over the finite field K of order q, and letting H be the hyperplane at infinity in P(Kd+1), we pick a frame P1,...,Pd in H. It turns out that the affine space A = P(Kd+1) − H has qd points. Then the communication nodes can be identified with points in the affine space A. Assuming for simplicity that N = qd, the algorithm proceeds in d rounds. During round i, each node Q ∈ A sends the information it has received to all nodes in A on the line QPi.  
由于q是素数，在q阶的有限域k上有一个维数d的投影空间p（kd+1），在p（kd+1）中设h为无穷远的超平面，我们选取一帧p1，…，pd in h，结果表明仿射空间a=p（kd+1）−h有qd点。然后利用仿射空间A中的点来识别通信节点，为了简单起见，假设n=qd，算法进行D轮运算。在第一轮中，每个节点q∈a将其接收到的信息发送到在线qpi中的所有节点。

It can be shown that at the end of the d rounds, each node has the total information, and that the total number of transactions is at most  
可以看出，在D轮结束时，每个节点都有总的信息，并且事务的总数最多是

(q − 1)logq(N)N.  
（q−1）logq（n）n.

Other applications of projective spaces to communication systems with switches are described in Chapter 2, Section 8, of Beutelspacher and Rosenbaum [22]. Applications to error-correcting codes are described in Chapter 5 of the same book. Introducing even the most elementary notions of coding theory would take too much space. Let us simply say that the existence of certain types of good codes called linear [n,n−r]-codes with minimum distance d is equivalent to the existence of certain sets of points called (n,d − 1)-sets in the finite projective space P({0,1}r). For the sake of completeness, a set of n points in a projective space is an (n,s)-set if s is the largest integer such that every subset of s points is projectively independent. For example, an (n,3)-set is a set of n points no three of which are collinear, but at least four of them are coplanar.  
射影空间在具有开关的通信系统中的其他应用，如Beutelspacher和Rosenbaum[22]第2章第8节所述。纠错码的应用在同一本书的第5章中进行了描述。即使引入编码理论的最基本概念也会占用太多的空间。让我们简单地说，具有最小距离d的某些类型的称为线性[n，n-r]的好代码的存在等价于有限射影空间p（0,1 r）中称为（n，d-1）-集的某些点集的存在。为了完备性，如果s是最大整数，则射影空间中n个点的集合是（n，s）-集，这样s点的每个子集都是射影独立的。例如，（n，3）-集是一组n点，其中没有三个是共线的，但至少有四个是共面的。

Other applications of projective geometry to cryptography are given in Chapter 6 of Beutelspacher and Rosenbaum [22].  
射影几何在密码学中的其他应用在Beutelspacher和Rosenbaum[22]的第6章中给出。

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Chapter 26  
第二十六章

# The Cartan–Dieudonn´e Theorem 卡坦-狄翁定理

In this chapter the structure of the orthogonal group is studied in more depth. In particular, we prove that every isometry in O(n) is the composition of at most n reflections about hyperplanes (for n ≥ 2, see Theorem 26.1). This important result is a special case of the “Cartan–Dieudonn´e theorem” (Cartan [33], Dieudonn´e [51]). We also prove that every rotation in SO(n) is the composition of at most n flips (for n ≥ 3).  
本章对正交群的结构进行了较深入的研究。特别地，我们证明了O（n）中的每一个等值线都是超平面上至多n个反射的组合（n≥2，见定理26.1）。这个重要的结果是“卡坦-迪乌顿定理”的一个特例（卡坦[33]，迪乌顿[51]）。我们还证明了so（n）中的每一个旋转都是最多n个翻转（n≥3）的组合。

Affine isometries are defined, and their fixed points are investigated. First, we characterize the set of fixed points of an affine map. Then we show that the Cartan–Dieudonn´e theorem can be generalized to affine isometries: Every rigid motion in Is(n) is the composition of at most n affine reflections if it has a fixed point, or else of at most n + 2 affine reflections. We prove that every rigid motion in SE(n) is the composition of at most n affine flips (for n ≥ 3).  
定义了仿射等距线，研究了它们的不动点。首先，我们描述了仿射映射的不动点集。然后证明了卡坦-迪乌顿定理可以推广到仿射等轴测：在is（n）中的每一个刚性运动都是至多n个具有固定点的仿射反射的组合，或者至多n+2个仿射反射的组合。我们证明了SE（n）中的每一个刚性运动都是至多n个仿射翻转（n≥3）的组合。

## 26.1 The Cartan–Dieudonn´e Theorem for Linear Isometries 26.1线性等轴测的卡坦-迪乌顿定理

The fact that the group O(n) of linear isometries is generated by the reflections is a special case of a theorem known as the Cartan–Dieudonn´e theorem. Elie Cartan proved a version of this theorem early in the twentieth century. A proof can be found in his book on spinors [33], which appeared in 1937 (Chapter I, Section 10, pages 10–12). Cartan’s version applies to nondegenerate quadratic forms over R or C. The theorem was generalized to quadratic forms over arbitrary fields by Dieudonn´e [51]. One should also consult Emil Artin’s book [6], which contains an in-depth study of the orthogonal group and another proof of the Cartan–Dieudonn´e theorem.  
线性等距图的O（n）组是由反射产生的，这是一个称为卡坦-迪乌登定理的定理的特例。伊莱·卡坦在二十世纪初证明了这个定理的一个版本。他在1937年出版的关于旋转器的书[33]中找到了证据（第一章，第10节，第10-12页）。Cartan的版本适用于r或c上的非退化二次型。Dieudonn'e[51]将该定理推广到任意场上的二次型。我们还应该参考埃米尔·阿丁的书[6]，其中包括对正交群的深入研究和卡坦-迪乌登定理的另一个证明。

Theorem 26.1. Let E be a Euclidean space of dimension n ≥ 1. Every isometry f ∈ O(E) that is not the identity is the composition of at most n reflections. When n ≥ 2, the identity is the composition of any reflection with itself.  
定理26.1。设e为尺寸n≥1的欧几里得空间。每一个非同一性的等距f∈o（e）至多是n个反射的组成。当n≥2时，同一性是任何反射本身的组成。

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Proof. We proceed by induction on n. When n = 1, every isometry f ∈ O(E) is either the identity or −id, but −id is a reflection about H = {0}. When n ≥ 2, we have id = s ◦ s for every reflection s. Let us now consider the case where n ≥ 2 and f is not the identity. There are two subcases.  
证据。我们对n进行归纳，当n=1时，每一个等距f∈o（e）要么是恒等式，要么是−id，但−id是关于h=0\_的反映。当n≥2时，每个反射s的id=s s。现在让我们考虑n≥2和f不是同一性的情况。有两个子类。

Case 1. The map f admits 1 as an eigenvalue, i.e., there is some nonnull vector w such that f(w) = w. In this case, let H be the hyperplane orthogonal to w, so that E = H ⊕Rw. We claim that f(H) ⊆ H. Indeed, if  
案例1。映射f承认1为特征值，即存在一些非零向量w，使得f（w）=w。在这种情况下，让h是与w正交的超平面，因此e=h rw。我们声称F（H）H。事实上，如果

v · w = 0  
V·W=0

for any v ∈ H, since f is an isometry, we get  
对于任何v∈h，因为f是一个等距，我们得到

f(v) · f(w) = v · w = 0,  
F（V）·F（W）=V·W=0，

and since f(w) = w, we get f(v) · w = f(v) · f(w) = 0,  
因为f（w）=w，我们得到f（v）·w=f（v）·f（w）=0，

and thus f(v) ∈ H. Furthermore, since f is not the identity, f is not the identity of H. Since H has dimension n − 1, by the induction hypothesis applied to H, there are at most k ≤ n − 1 reflections s1,...,sk about some hyperplanes H1,...,Hk in H, such that the restriction of f to H is the composition sk ◦···◦ s1. Each si can be extended to a reflection in E as follows: If H = Hi ⊕ Li (where Li = Hi⊥, the orthogonal complement of Hi in H), L = Rw, and Fi = Hi ⊕ L, since H and L are orthogonal, Fi is indeed a hyperplane, E = Fi ⊕ Li = Hi ⊕ L ⊕ Li, and for every u = h + λw ∈ H ⊕ L = E, since  
因此f（v）∈h。此外，由于f不是恒等式，f不是h的恒等式。由于h的维数为n-1，根据应用于h的诱导假设，在h中的某些超平面h1，…，h k最多有k≤n-1反射s1，…，sk，因此f对h的限制是组成sk·····s1。每个si可以扩展到e中的反射，如下所示：如果h=hi li（其中li=hi，h中hi的正交补码）、l=rw和fi=hi l，因为h和l是正交的，fi确实是一个超平面，e=fi li=hi l li，并且对于每个u=h+λw∈h l=e，因为

si(h) = pHi(h) − pLi(h),  
si（h）=phi（h）−pli（h）、

we can define si on E such that  
我们可以在e上定义si，这样

si(h + λw) = pHi(h) + λw − pLi(h),  
si（h+λw）=phi（h）+λw−pli（h），

and since h ∈ H, w ∈ L, Fi = Hi ⊕ L, and H = Hi ⊕ Li, we have  
既然h∈h，w∈l，fi=hi l，h=hi li，我们有

si(h + λw) = pFi(h + λw) − pLi(h + λw),  
si（h+λw）=pfi（h+λw）−pli（h+λw）、

which defines a reflection about Fi = Hi ⊕ L. Now, since f is the identity on L = Rw, it is immediately verified that f = sk ◦ ··· ◦ s1, with k ≤ n − 1. See Figure 26.1.  
它定义了关于fi=hi l的反射。现在，由于f是l=rw上的标识，因此立即验证f=sk····s1，k≤n−1。见图26.1。

Case 2. The map f does not admit 1 as an eigenvalue, i.e., f(u) =6 u for all u = 06 . Pick any w = 06 in E, and let H be the hyperplane orthogonal to f(w) − w. Since f is an isometry, we have kf(w)k = kwk, and by Lemma 12.2, we know that s(w) = f(w), where s is the reflection about H, and we claim that s ◦ f leaves w invariant. Indeed, since s2 = id, we have  
案例2。映射f不承认1为特征值，即f（u）=6 u（对于所有u=06）。选取e中的任意w=06，设h为与f（w）−w正交的超平面。由于f是一个等距线，我们得到kf（w）k=kwk，根据引理12.2，我们知道s（w）=f（w），其中s是关于h的反射，我们声称s f留下w不变量。事实上，因为s2=id，我们有

s(f(w)) = s(s(w)) = w.  
S（F（W））=S（S（W））=W。

See Figure 26.2.  
见图26.2。

w

L

H

i

L

i

F

i

H

h

p

(

h

)

H

i

L

-

p

(

h

)

i

s (h)

i

*λ*

w

h

+

*λ*

w

*λ*

w

Figure 26.1: An illustration of how to extend the reflection si of Case 1 in Theorem 26.1 to E. The result of this extended reflection is the bold green vector.  
图26.1：关于如何将定理26.1中情况1的反射si扩展到e的说明。扩展反射的结果是粗体绿色矢量。

Since s2 = id, we cannot have s◦f = id, since this would imply that f = s, where s is the identity on H, contradicting the fact that f is not the identity on any vector. Thus, we are back to Case 1. Thus, there are k ≤ n−1 hyperplane reflections such that s◦f = sk ◦···◦s1, from which we get f = s ◦ sk ◦ ··· ◦ s1,  
因为s2=id，我们不能有s f=id，因为这意味着f=s，其中s是h上的恒等式，这与f不是任何向量上的恒等式的事实相矛盾。因此，我们回到案例1。因此，存在k≤n−1超平面反射，因此s f=sk·····s1，从中我们得到f=s sk·····s1，

with at most k + 1 ≤ n reflections.   
最多k+1≤n反射。

Remarks:  
评论：

1. A slightly different proof can be given. Either f is the identity, or there is some nonnull vector u such that f(u) =6 u. In the second case, proceed as in the second part of the proof, to get back to the case where f admits 1 as an eigenvalue.  
   可以给出稍微不同的证明。要么f是恒等式，要么有一些非零向量u，这样f（u）=6u。在第二种情况下，继续进行证明的第二部分，回到f承认1为特征值的情况。
2. Theorem 26.1 still holds if the inner product on E is replaced by a nondegenerate symmetric bilinear form ϕ, but the proof is a lot harder; see Section 28.9.  
   定理26.1仍然适用，如果e上的内积被非退化对称双线性形式\_替换，但证明要困难得多；见第28.9节。
3. The proof of Theorem 26.1 shows more than stated. If 1 is an eigenvalue of f, for any eigenvector w associated with 1 (i.e., f(w) = w, w = 0)6 , then f is the composition of k ≤ n − 1 reflections about hyperplanes Fi such that Fi = Hi ⊕ L, where L is the line Rw and the Hi are subspaces of dimension n − 2 all orthogonal to L (the Hi are hyperplanes in H). This situation is illustrated in Figure 26.3.  
   定理26.1的证明比所说的要多。如果1是f的特征值，对于与1（即f（w）=w，w=0）6相关的任何特征向量w，则f是关于超平面f i的k≤n-1反射的组成，因此fi=hi l，其中l是线rw，hi是尺寸n-2的子空间，均与l正交（hi是hy垂直高度）。这种情况如图26.3所示。

If 1 is not an eigenvalue of f, then f is the composition of k ≤ n reflections about hyperplanes H,F1,...,Fk−1, such that Fi = Hi ⊕ L, where L is a line intersecting H, and the Hi are subspaces of dimension n−2 all orthogonal to L (the Hi are hyperplanes in L⊥). This situation is illustrated in Figure 26.4.  
如果1不是f的特征值，那么f是关于超平面h，f1，…，f k−1的k≤n反射的组成，这样fi=hi l，其中l是与h相交的线，hi是尺寸n−2的子空间，都与l正交（hi是l中的超平面）。这种情况如图26.4所示。

w

f

(

w

)

f

(

w

)

-

w

H

Figure 26.2: The construction of the hyperplane H for Case 2 of Theorem 26.1.  
图26.2：定理26.1的情形2的超平面h的构造。

w

H

H

i

H

j

F

j

L

w

u

s (u)

i

s (u)

s (u)

i

j

Figure 26.3: An isometry f as a composition of reflections, when 1 is an eigenvalue of f.  
图26.3：当1是f的特征值时，作为反射组成的等距线f。

1. It is natural to ask what is the minimal number of hyperplane reflections needed to obtain an isometry f. This has to do with the dimension of the eigenspace Ker(f − id) associated with the eigenvalue 1. We will prove later that every isometry is the composition of k hyperplane reflections, where  
   很自然地，我们会问，获得等距f所需的超平面反射的最小数目是多少。这与特征值1相关的特征空间ker（f-id）的维数有关。稍后我们将证明每个等距线都是k超平面反射的组成，其中

k = n − dim(Ker(f − id)),  
k=n−dim（ker（f−id）），

and that this number is minimal (where n = dim(E)).  
这个数字是最小的（其中n=dim（e））。

When n = 2, a reflection is a reflection about a line, and Theorem 26.1 shows that every isometry in O(2) is either a reflection about a line or a rotation, and that every rotation is the product of two reflections about some lines. In general, since det(s) = −1 for a reflection s, when n ≥ 3 is odd, every rotation is the product of an even number less than or equal  
当n=2时，反射是关于一条直线的反射，定理26.1表明O（2）中的每个等距线要么是关于一条直线的反射，要么是关于一个旋转的反射，并且每个旋转都是关于一些直线的两个反射的乘积。一般来说，由于反射s的Det（s）=-1，当n≥3为奇数时，每个旋转都是小于或等于偶数的乘积。

H

H

j

F

j

L

L

w

f(w)

Figure 26.4: An isometry f as a composition of reflections when 1 is not an eigenvalue of f. Note that the pink plane H is perpendicular to f(w) − w.  
图26.4：当1不是f的特征值时，作为反射组成的等距f。注意，粉红色平面h垂直于f（w）−w。

to n − 1 of reflections, and when n is even, every improper orthogonal transformation is then − 1 of reflections. product of an odd number less than or equal to  
到反射的n−1，当n为偶数时，每一个不正确的正交变换都是反射的−1。奇数小于或等于的积

In particular, for n = 3, every rotation is the product of two reflections about planes. When n is odd, we can say more about improper isometries. Indeed, when n is odd, every improper isometry admits the eigenvalue −1. This is because iff(u)kλ, then=Ekuis a Euclidean space ofk for every u ∈ E, if λ finite dimension and f : E → E is an isometry, because k is any eigenvalue of f and u is an eigenvector associated with  
特别是，对于n=3，每次旋转都是平面上两个反射的产物。当n是奇数时，我们可以说更多关于不适当等距的内容。实际上，当n为奇数时，每一个不适当的等距测量都承认特征值−1。这是因为if f（u）kλ，那么=ekui是k的欧几里得空间，对于每个u∈e，如果λ有限维和f:e→e是一个等距，因为k是f的任何特征值，u是与

kf(u)k = kλuk = |λ|kuk = kuk,  
kf（u）k=kλuk=λkuk=kuk，

which implies |λ| = 1, since u 6= 0. Thus, the real eigenvalues of an isometry are either  
这意味着λ=1，因为u 6=0。因此，等距测量的实际特征值可以是

real root. As a consequence, the characteristic polynomial det(  
真正的根。因此，特征多项式（

+1root, which is either +1 ordet(orf) −is the product of the eigenvalues, the real roots cannot all be +1, and thus1. However, it is well known that polynomials of odd degree always have some−1. Since f is an improper isometry, det(f − λid) off) =f −has some real1, and since−1 is anf,  
+1根，要么是+1，要么是（ORF）−是特征值的乘积，实际根不能都是+1和Thus1。然而，众所周知，奇数度的多项式总是有一些−1。因为f是一个不正确的等距测量，所以det（f−λid）off）=f−有一些real1，因为−1是anf，

eigenvalue ofthere is some nonnull eigenvectorproof, we see that the hyperplanef. Going back to the proof of Theorem 26.1, sincew such that f(w) = −w. Using the second part of thew = −−21wis an eigenvalue ofis in fact orthogonalH,F1,...,Fk−1  
这里的特征值是一些非空的特征向量，我们看到了超平面。回到定理26.1的证明，因为f（w）=−w。使用w=−21wis的第二部分，一个事实上正交的特征值，f1，…，fk−1

H orthogonal to f(w) −  
h与f（w）正交-

to w, and thus f is the product of k ≤ n reflections about hyperplanesH, and the Hi are hyperplanes in such that Fi = Hi ⊕ L, where L is a line orthogonal to  
到w，因此f是关于超平面的k≤n反射的产物，hi是超平面，其中fi=hi l，其中l是与

H = L⊥ orthogonal to L. However, k must be odd, and so k − 1 is even, and thus then is odd, an composition of the reflections about F1,...,Fk−1 is a rotation. Thus, when improper isometry is the composition of a reflection about a hyperplane H with a rotation consisting of reflections about hyperplanes F1,...,Fk−1 containing a line, L, orthogonal to  
H=L与L正交。但是，K必须是奇数，因此K−1是偶数，因此是奇数，关于f1，…，fk−1的反射的组成是旋转。因此，当不适当的等距测量是关于超平面H的反射的组成，其旋转包括关于超平面F1的反射，…，fk−1，其中包含一条直线，l，与

H. In particular, when n = 3, every improper orthogonal transformation is the product of a rotation with a reflection about a plane orthogonal to the axis of rotation.  
特别是，当n=3时，每一个不适当的正交变换都是一个旋转与一个与旋转轴正交的平面上的反射的乘积。

Using Theorem 26.1, we can also give a rather simple proof of the classical fact that in a Euclidean space of odd dimension, every rotation leaves some nonnull vector invariant, and thus a line invariant.  
利用定理26.1，我们还可以给出一个相当简单的证明，证明在奇数维的欧几里得空间中，每一个旋转都会留下一些非零向量不变量，因此是一个线不变量。

If λ is an eigenvalue of f, then the following lemma shows that the orthogonal complement Eλ(f)⊥ of the eigenspace associated with λ is closed under f.  
如果λ是f的特征值，那么下面的引理表明与λ相关的特征空间的正交补码eλ（f）在f下闭合。

Proposition 26.2. Let E be a Euclidean space of finite dimension n, and let f : E → E be an isometry. For any subspace F of E, if f(F) = F, then f(F ⊥) ⊆ F ⊥ and E = F ⊕ F ⊥.  
提案26.2.设e为有限维n的欧几里得空间，设f:e→e为等距线。对于e的任何子空间f，如果f（f）=f，则f（f）f和e=f f。

Proof. We just have to prove that if w ∈ E is orthogonal to every u ∈ F, then f(w) is also orthogonal to every u ∈ F. However, since f(F) = F, for every v ∈ F, there is some u ∈ F such that f(u) = v, and we have  
证据。我们只需要证明，如果w∈e与每一个u∈f正交，那么f（w）也与每一个u∈f正交。然而，由于f（f）=f，对于每一个v∈f，有一些u∈f，这样f（u）=v，我们得到

f(w) · v = f(w) · f(u) = w · u,  
f（w）·v=f（w）·f（u）=w·u，

since f is an isometry. Since we assumed that w ∈ E is orthogonal to every u ∈ F, we have  
因为f是等距线。既然我们假设w∈e与每一个u∈f是正交的，我们有

w · u = 0,  
w·u=0，

and thus f(w) · v = 0,  
因此f（w）·v=0，

and this for every v ∈ F. Thus, f(F ⊥) ⊆ F ⊥. The fact that E = F ⊕ F ⊥ follows from Lemma 11.11.   
对于每一个v∈f，因此，f（f）f。e=f\_f这一事实源自引理11.11。

Lemma 26.2 is the starting point of the proof that every orthogonal matrix can be diagonalized over the field of complex numbers. Indeed, if λ is any eigenvalue of f, then f(Eλ(f)) = Eλ(f), where Eλ(f) is the eigenspace associated with λ, and thus the orthogonal Eλ(f)⊥ is closed under f, and E = Eλ(f) ⊕ Eλ(f)⊥. The problem over R is that there may not be any real eigenvalues. However, when n is odd, the following lemma shows that every rotation admits 1 as an eigenvalue (and similarly, when n is even, every improper orthogonal transformation admits 1 as an eigenvalue).  
引理26.2是证明每个正交矩阵都可以在复数域上对角化的起点。实际上，如果λ是f的特征值，那么f（eλ（f））=eλ（f），其中eλ（f）是与λ相关的特征空间，因此正交eλ（f）在f下闭合，e=eλ（f）eλ（f）。R上的问题是可能没有任何实际的特征值。然而，当n是奇数时，下面的引理表明每个旋转都承认1为特征值（同样，当n是偶数时，每个不适当的正交变换都承认1为特征值）。

Proposition 26.3. Let E be a Euclidean space.  
提案26.3.设e为欧几里得空间。

1. If E has odd dimension n = 2m + 1, then every rotation f admits 1 as an eigenvalue and the eigenspace F of all eigenvectors left invariant under f has an odd dimension 2p + 1. Furthermore, there is an orthonormal basis of E, in which f is represented by a matrix of the form  
   如果e的奇数维数n=2 m+1，那么每个旋转f都承认1为特征值，而f下所有特征向量的特征空间f都是奇数维数2p+1。此外，还有一个e的正交基，其中f由形式的矩阵表示。

,  
，

where R2(m−p) is a rotation matrix that does not have 1 as an eigenvalue.  
其中，r2（m-p）是一个旋转矩阵，没有1作为特征值。

1. If E has even dimension n = 2m, then every improper orthogonal transformation f admits 1 as an eigenvalue and the eigenspace F of all eigenvectors left invariant under f has an odd dimension 2p + 1. Furthermore, there is an orthonormal basis of E, in which f is represented by a matrix of the form  
   如果e的维数为偶数n=2 m，则每一个不适当的正交变换f都承认1为特征值，而f下所有特征向量的特征空间f都是奇数维2p+1。此外，还有一个e的正交基，其中f由形式的矩阵表示。

,  
，

where S2(m−p)−1 is an improper orthogonal matrix that does not have 1 as an eigenvalue.  
其中s2（m−p）−1是不适当的正交矩阵，没有1作为特征值。

Proof. We prove only (1), the proof of (2) being similar. Since f is a rotation and n = 2m+1 is odd, by Theorem 26.1, f is the composition of an even number less than or equal to 2m of reflections. From Lemma 23.15, recall the Grassmann relation  
证据。我们只证明（1），证明（2）相似。由于f是一个旋转，n=2m+1是奇数，根据定理26.1，f是小于或等于2m反射的偶数的组合。从引理23.15，回忆格拉斯曼关系

dim(M) + dim(N) = dim(M + N) + dim(M ∩ N),  
尺寸（m）+尺寸（n）=dim（m+n）+尺寸（m n）

where M and N are subspaces of E. Now, if M and N are hyperplanes, their dimension is n − 1, and thus dim(M ∩ N) ≥ n − 2. Thus, if we intersect k ≤ n hyperplanes, we see that the dimension of their intersection is at least n − k. Since each of the reflections is the identity on the hyperplane defining it, and since there are at most 2m = n − 1 reflections, their composition is the identity on a subspace of dimension at least 1. This proves that 1 is an eigenvalue of f. Let F be the eigenspace associated with 1, and assume that its dimension is q. Let G = F ⊥ be the orthogonal of F. By Lemma 26.2, G is stable under f, and E = F ⊕ G. Using Lemma 11.10, we can find an orthonormal basis of E consisting of an orthonormal basis for G and orthonormal basis for F. In this basis, the matrix of f is of the form  
其中m和n是e的子空间。现在，如果m和n是超平面，那么它们的尺寸是n-1，因此dim（m n）≥n-2。因此，如果我们与k≤n超平面相交，我们可以看到它们相交的尺寸至少为n−k。由于每个反射都是定义它的超平面上的同一性，并且由于反射最多为2 m=n−1，因此它们的组成是Dimen子空间上的同一性。SION至少1.这证明了1是f的特征值。让f是与1相关的特征空间，并假设其维数为q。让g=f是f的正交。由引理26.2，g在f下是稳定的，e=f\_g。利用引理11.10，我们可以找到由正交b组成的e的正交基。对于g和f的正交基，asis。在这个基中，f的矩阵是形式的。

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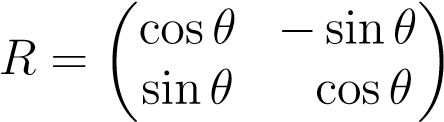
Thus, det(f) = det(R), and R must be a rotation, since f is a rotation and det(f) = 1. Now, if f left some vector u = 06 in G invariant, this vector would be an eigenvector for 1, and we would have u ∈ F, the eigenspace associated with 1, which contradicts E = F ⊕ G. Thus, by the first part of the proof, the dimension of G must be even, since otherwise, the restriction of f to G would admit 1 as an eigenvalue. Consequently, q must be odd, and R does not admit 1 as an eigenvalue. Letting q = 2p + 1, the lemma is established.   
因此，det（f）=det（r），r必须是一个旋转，因为f是一个旋转，det（f）=1。现在，如果f在g不变量中留下一个向量u=06，这个向量将是1的一个特征向量，我们将得到u∈f，这个与1相关的特征空间，它与e=f g相矛盾，因此，根据证明的第一部分，g的维数必须是偶数，否则，f对g的限制是偶数。将1作为特征值。因此，q必须是奇数，r不接受1作为特征值。设q=2p+1，建立引理。

An example showing that Lemma 26.3 fails for n even is the following rotation matrix (when n = 2):  
一个例子表明，引理26.3对于n是失败的，甚至是如下的旋转矩阵（当n=2时）：

.  
.

The above matrix does not have real eigenvalues for θ =6 kπ.  
上面的矩阵没有θ=6 kπ的实特征值。

It is easily shown that for n = 2, with respect to any chosen orthonormal basis (e1, e2), every rotation is represented by a matrix of form  
可以很容易地证明，对于n=2，对于任何选定的正交基（e1，e2），每个旋转都由一个形式的矩阵表示。



where θ ∈ [0,2π[, and that every improper orthogonal transformation is represented by a matrix of the form  
式中θ∈[0,2π[，并且每一个不适当的正交变换都由一个形式的矩阵表示。

.  
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In the first case, we call θ ∈ [0,2π[ the measure of the angle of rotation of R w.r.t. the orthonormal basis (e1, e2). In the second case, we have a reflection about a line, and it is easy to determine what this line is. It is also easy to see that S is the composition of a reflection about the x-axis with a rotation (of matrix R).  
在第一种情况下，我们称之为θ∈[0,2π[r w.r.t旋转角度的测量.正交基（e1，e2）.在第二种情况下，我们对一条线进行了反射，很容易确定这条线是什么。也很容易看出，S是围绕X轴旋转（矩阵R）的反射的组合。

 We refrained from calling θ “the angle of rotation,” because there are some subtleties involved in defining rigorously the notion of angle of two vectors (or two lines). For example, note that with respect to the “opposite basis” (e2, e1), the measure θ must be changed to 2π − θ (or −θ if we consider the quotient set R/2π of the real numbers modulo  
我们避免称θ为“旋转角度”，因为严格定义两个向量（或两条线）的角度概念涉及到一些微妙之处。例如，请注意，对于“相反基”（e2，e1），如果我们考虑实数模的商集r/2π，则测量θ必须更改为2π−θ（或−θ）。

2π).  
2π）。

It is easily shown that the group SO(2) of rotations in the plane is abelian. First, recall that every plane rotation is the product of two reflections (about lines), and that every isometry in O(2) is either a reflection or a rotation. To alleviate the notation, we will omit the composition operator ◦, and write rs instead of r ◦ s. Now, if r is a rotation and s is a reflection, rs being in O(2) must be a reflection (since det(rs) = det(r)det(s) = −1), and thus (rs)2 = id, since a reflection is an involution, which implies that  
很容易证明平面上的转动群是阿贝尔的。首先，回想一下，每个平面的旋转都是两个反射（关于直线）的乘积，而O（2）中的每个等距线都是反射或旋转。为了减少符号，我们将省略组合运算符，并写r s而不是r\_s。现在，如果r是一个旋转，s是一个反射，r在o（2）中必须是一个反射（因为det（rs）=det（r）det（s）=1），因此（rs）2=id，因为反射是一个对合，这意味着那个

srs = r−1.  
SRS=R−1。

Then, given two rotations r1 and r2, writing r1 as r1 = s2s1 for two reflections s1,s2, we have  
然后，给定两个转动R1和R2，将R1写为两个反射的R1=S2S1，s1，s2，我们得到

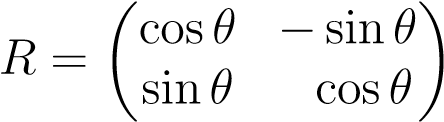
r1r2r1−1 = s2s1r2(s2s1)−1 = s2s1r2s−1 1s−2 1 = s2s1r2s1s2 = s2r2−1s2 = r2,  
r1r2r1−1=s2s1 r2（s2s1）−1=s2s1r2−1 1s−2 1=s2s1r2s1s2=s2r2−1s2=r2，

since srs = r−1 for all reflections s and rotations r, and thus r1r2 = r2r1.  
因为所有反射和旋转的s r s=r−1，因此r1r2=r2r1。

We can also perform the following calculation, using some elementary trigonometry:  
我们还可以使用一些初等三角法执行以下计算：

.  
.

The above also shows that the inverse of a rotation matrix  
上面还显示了旋转矩阵的逆矩阵



is obtained by changing θ to −θ (or 2π − θ). Incidentally, note that in writing a rotation r as the product of two reflections r = s2s1, the first reflection s1 can be chosen arbitrarily, since, and rs1 is a reflection.  
通过将θ更改为−θ（或2π−θ）获得。顺便说一句，请注意，在写一个旋转r作为两个反射r=s2s1的乘积时，第一个反射s1可以任意选择，因为，rs1是一个反射。

For n = 3, the only two choices for p are p = 1, which corresponds to the identity, or p = 0, in which case f is a rotation leaving a line invariant. This line D is called the axis of  
对于n=3，p的唯一两个选择是p=1，它对应于恒等式，或者p=0，在这种情况下，f是一个保留直线不变量的旋转。这条线称为

*D*

*θ*

*/*

2

*u*

*R*

(

*u*

)

Figure 26.5: 3D rotation as the composition of two reflections.  
图26.5：两个反射组成的三维旋转。

rotation. The rotation R behaves like a two-dimensional rotation around the axis of rotation. Thus, the rotation R is the composition of two reflections about planes containing the axis of rotation D and forming an angle θ/2. This is illustrated in Figure 26.5.  
旋转。旋转R的行为类似于围绕旋转轴的二维旋转。因此，旋转r是包含旋转d轴并形成角度θ/2的平面上的两个反射的组成。如图26.5所示。

The measure of the angle of rotation θ can be determined through its cosine via the formula cosθ = u · R(u),  
旋转角θ的测量可以通过其余弦公式cosθ=u·r（u）来确定。

where u is any unit vector orthogonal to the direction of the axis of rotation. However, this does not determine θ ∈ [0,2π[ uniquely, since both θ and 2π − θ are possible candidates. What is missing is an orientation of the plane (through the origin) orthogonal to the axis of rotation.  
其中u是与旋转轴方向垂直的任何单位向量。然而，由于θ和2π−θ都是可能的候选者，因此这不能确定θ∈[0,2π[唯一的]。缺少的是与旋转轴垂直的平面方向（通过原点）。

In the orthonormal basis of the lemma, a rotation is represented by a matrix of the form  
在引理的正交基中，旋转由形式的矩阵表示。

.  
.

Remark: For an arbitrary rotation matrix A, since a11 + a22 + a33 (the trace of A) is the sum of the eigenvalues of A, and since these eigenvalues are cosθ +isinθ, cosθ −isinθ, and 1, for some θ ∈ [0,2π[, we can compute cosθ from  
注：对于任意旋转矩阵a，由于a11+a22+a33（a的迹线）是a的特征值之和，并且由于这些特征值是cosθ+isinθ，cosθ−isinθ，和1，对于某些θ∈[0,2π[，我们可以从

1 + 2cosθ = a11 + a22 + a33.  
1+2cosθ=A11+A22+A33。

It is also possible to determine the axis of rotation (see the problems).  
也可以确定旋转轴（参见问题）。

An improper transformation is either a reflection about a plane or the product of three reflections, or equivalently the product of a reflection about a plane with a rotation, and we noted in the discussion following Theorem 26.1 that the axis of rotation is orthogonal to the plane of the reflection. Thus, an improper transformation is represented by a matrix of the form  
不适当的变换是关于一个平面的反射或三个反射的乘积，或者是关于一个平面的反射与一个旋转的乘积，我们在定理26.1的讨论中注意到，旋转轴与反射面的平面正交。因此，不适当的变换用形式矩阵表示。

.  
.

When n ≥ 3, the group of rotations SO(n) is not only generated by hyperplane reflections, but also by flips (about subspaces of dimension n − 2). We will also see, in Section 26.2, that every proper affine rigid motion can be expressed as the composition of at most n flips, which is perhaps even more surprising! The proof of these results uses the following key lemma.  
当n≥3时，旋转组so（n）不仅由超平面反射产生，也由翻转产生（关于尺寸n-2的子空间）。我们还将在第26.2节中看到，每一个适当的仿射刚性运动都可以表示为最多n个翻转的组合，这可能更令人惊讶！这些结果的证明使用以下关键引理。

Proposition 26.4. Given any Euclidean space E of dimension n ≥ 3, for any two reflections h1 and h2 about some hyperplanes H1 and H2, there exist two flips f1 and f2 such that h2 ◦ h1 = f2 ◦ f1.  
提案26.4.对于尺寸n≥3的任何欧几里得空间e，对于某些超平面h1和h2的任意两个反射h1和h2，存在两个翻转f1和f2，使得h2 h1=f2 f1。

Proof. If h1 = h2, it is obvious that  
证据。如果h1=h2，很明显

h1 ◦ h2 = h1 ◦ h1 = id = f1 ◦ f1  
h1 h2=h1 h1=id=f1 f1

for any flip f1. If h1 =6 h2, then H1 ∩ H2 = F, where dim(F) = n − 2 (by the Grassmann relation). We can pick an orthonormal basis (e1,...,en) of E such that (e1,...,en−2) is an orthonormal basis of F. We can also extend (e1,...,en−2) to an orthonormal basis  
对于任何翻转F1。如果h1=6 h2，则h1 h2=f，其中dim（f）=n−2（根据格拉斯曼关系）。我们可以选择e的正交基（e1，…，en），这样（e1，…，en-2）就是f的正交基。我们也可以将（e1，…，en-2）扩展到正交基。

(e1,...,en−2,u1,v1) of E, where (e1,...,en−2,u1) is an orthonormal basis of H1, in which case  
e的（e1，…，en-2，u1，v1），其中（e1，…，en-2，u1）是h1的正态基，在这种情况下

en−1 = cosθ1 u1 + sinθ1 v1, en = sinθ1 u1 − cosθ1 v1,  
en−1=cosθ1 u1+sinθ1 v1，en=sinθ1 u1−cosθ1 v1，

for some θ1 ∈ [0,2π]. See Figure 26.6  
对于某些θ1∈[0,2π]。见图26.6

Since h1 is the identity on H1 and v1 is orthogonal to H1, it follows that h1(u1) = u1, h1(v1) = −v1, and we get  
因为h1是h1上的单位，v1与h1是正交的，所以h1（u1）=u1，h1（v1）=-v1，我们得到

,  
，

After some simple calculations, we get  
经过一些简单的计算，我们得到

h1(en−1) = cos2θ1 en−1 + sin2θ1 en, h1(en) = sin2θ1 en−1 − cos2θ1 en.  
h1（en-1）=cos2θ1 en-1+sin2θ1 en，h1（en）=sin2θ1 en-1−cos2θ1 en。

e

2

e

2

e

1

e

1

e

1

e

3

e

3

u

1

u

1

v

1

v

1

F

H

H

1

2

Figure 26.6: An illustration of the hyperplanes H1, H2, their intersection F, and the two orthonormal basis utilized in the proof of Proposition 26.4.  
图26.6：证明26.4的超平面h1、h2及其交点f和两个正交基的图解。

As a consequence, the matrix A1 of h1 over the basis (e1,...,en) is of the form  
因此，基（e1，…，en）上的h1矩阵a1的形式为

.  
.

Similarly, the matrix A2 of h2 over the basis (e1,...,en) is of the form  
同样，基（e1，…，en）上的h2矩阵a2的形式为

.  
.

Observe that both A1 and A2 have the eigenvalues −1 and +1 with multiplicity n − 1. The trick is to observe that if we change the last entry in In−2 from +1 to −1 (which is possible since n ≥ 3), we have the following product A2A1:  
观察A1和A2都具有多重性n-1的特征值−1和+1。诀窍是观察到，如果我们将−2中的最后一个条目从+1更改为−1（这是可能的，因为n≥3），我们得到以下产品A2A1：

.  
.

Now, the two matrices above are clearly orthogonal, and they have the eigenvalues −1,−1, and +1 with multiplicity n−2, which implies that the corresponding isometries leave invariant a subspace of dimension n − 2 and act as −id on its orthogonal complement (which has dimension 2). This means that the above two matrices represent two flips f1 and f2 such that h2 ◦ h1 = f2 ◦ f1. See Figure 26.7.   
现在，上面的两个矩阵显然是正交的，它们具有多重性n−2的特征值−1、−1和+1，这意味着相应的等轴测保持不变的维度n−2的子空间，并在其正交补码（具有维度2）上充当−id。这意味着上述两个矩阵表示两个翻转f1和f2，使得h2 h1=f2 f1。见图26.7。

e

2

e

1

e

3

F

H

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2

h

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1

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1

f

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e

(

)

f

1

1

i.

(

)

)

ii.

(

(

iii.

)

Figure 26.7: The conversion of the hyperplane reflection h1 into the flip or 180◦ rotation around the green axis in the e2e3-plane. The green axis corresponds to the restriction of the eigenspace associated with eigenvalue 1.  
图26.7：将超平面反射h1转换为翻转或围绕e2e3平面中绿色轴180°旋转。绿轴对应于与特征值1相关的特征空间的限制。

Using Lemma 26.4 and the Cartan–Dieudonn´e theorem, we obtain the following characterization of rotations when n ≥ 3.  
利用引理26.4和卡坦-迪乌顿定理，我们得到了当n≥3时旋转的以下特征。

Theorem 26.5. Let E be a Euclidean space of dimension n ≥ 3. Every rotation f ∈ SO(E) is the composition of an even number of flips f = f2k ◦···◦ f1, where 2k ≤ n. Furthermore, if u = 06 is invariant under f (i.e., u ∈ Ker(f − id)), we can pick the last flip f2k such that , where F2k is the subspace of dimension n − 2 determining f2k.  
定理26.5。设e为尺寸n≥3的欧几里得空间。每个旋转f∈so（e）是偶数个翻转f=f2k········f1的组合，其中2k≤n。此外，如果u=06在f（即u∈ker（f−id））下不变，我们可以选取最后一个翻转f2k，其中f2k是确定f2k的维度n−2的子空间。

Proof. By Theorem 26.1, the rotation f can be expressed as an even number of hyperplane reflections f = s2k◦s2k−1◦···◦s2◦s1, with 2k ≤ n. By Lemma 26.4, every composition of two reflections s2i ◦ s2i−1 can be replaced by the composition of two flips f2i ◦ f2i−1 (1 ≤ i ≤ k), which yields f = f2k ◦ ··· ◦ f1, where 2k ≤ n.  
证据。根据定理26.1，旋转f可以表示为偶数个超平面反射f=s2 k s2k−1····s2 s1，其中2k≤n。根据引理26.4，两个反射的每一个组成s2 i s2i−1可以被两个翻转的组成f2i f2i−1（1≤i≤k）所代替，其中h产生f=f2k····f1，其中2k≤n。

Assume that f(u) = u, with u = 06 . We have already made the remark that in the case where 1 is an eigenvalue of f, the proof of Theorem 26.1 shows that the reflections si can be chosen so that si(u) = u. In particular, if each reflection si is a reflection about the hyperplane Hi, we have u ∈ H2k−1 ∩ H2k. Letting F = H2k−1 ∩ H2k, pick an orthonormal basis (e1,...,en−3,en−2) of F, where  
假设f（u）=u，其中u=06。我们已经指出，在1是f的特征值的情况下，定理26.1的证明表明可以选择反射si，以便si（u）=u。特别是，如果每个反射si是关于超平面hi的反射，我们有u∈h2k h2k。让f=h2k−1h2k，选取f的正交基（e1，…，en-3，en-2），其中

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26.2. AFFINE ISOMETRIES (RIGID MOTIONS)  
26.2。仿射等距线（刚性运动）

The proof of Lemma 26.4 yields two flips f2k−1 and f2k such that  
引理26.4的证明产生了两个翻转f2k−1和f2k，这样

f2k(en−2) = −en−2 and s2k ◦ s2k−1 = f2k ◦ f2k−1,  
f2k（en-2）=-en-2和s2k s2k-1=f2k f2k-1，

since the (n − 2)th diagonal entry in both matrices is −1, which means that , where F2k is the subspace of dimension n − 2 determining f2k. Since u = kuken−2, we also have.   
因为这两个矩阵中的（n-2）第（n-2）个对角线项都是−1，这意味着，其中f2k是确定f2k的维度n-2的子空间。由于u=kuken−2，我们也有。

Remarks:  
评论：

1. It is easy to prove that if f is a rotation in SO(3) and if D is its axis and θ is its angle of rotation, then f is the composition of two flips about lines D1 and D2 orthogonal to D and making an angle θ/2.  
   可以很容易地证明，如果f是在so（3）中的旋转，如果d是它的轴，θ是它的旋转角，那么f是关于与d正交的d1和d2线的两个翻转的组合，并形成一个角θ/2。
2. It is natural to ask what is the minimal number of flips needed to obtain a rotation f (when n ≥ 3). As for arbitrary isometries, we will prove later that every rotation is the composition of k flips, where  
   自然会问，获得旋转f所需的最小翻转次数是多少（当n≥3时）。对于任意等距图，我们稍后将证明每个旋转都是k翻转的组合，其中

k = n − dim(Ker(f − id)),  
k=n−dim（ker（f−id）），

and that this number is minimal (where n = dim(E)).  
这个数字是最小的（其中n=dim（e））。

We now turn to affine isometries.  
现在我们来看看仿射等距图。

## 26.2 Affine Isometries (Rigid Motions) 26.2仿射等距图（刚性运动）

In the remaining sections we study affine isometries. First, we characterize the set of fixed points of an affine map. Using this characterization, we prove that every affine isometry f can be written uniquely as  
在剩下的部分，我们研究仿射等距线。首先，我们描述了仿射映射的不动点集。利用这个特征，我们证明了每一个仿射同构f都可以唯一地写成

f = t ◦ g, with t ◦ g = g ◦ t,  
f=t\_g，其中t\_g=g\_t，

where g is an isometry having a fixed point, and t is a translation by a vector τ such that →−f (τ) = τ, and with some additional nice properties (see Theorem 26.10). This is a generalization of a classical result of Chasles about (proper) rigid motions in R3 (screw motions). We prove a generalization of the Cartan–Dieudonn´e theorem for the affine isometries: Every isometry in Is(n) can be written as the composition of at most n affine reflections if it has a fixed point, or else as the composition of at most n+2 affine reflections. We also prove that every rigid motion in SE(n) is the composition of at most n affine flips (for n ≥ 3). This is somewhat surprising, in view of the previous theorem.  
其中g是一个具有固定点的等距测量，t是一个矢量τ的平移，使得→−f（τ）=τ，并具有一些附加的优良性质（见定理26.10）。这是关于r3（螺旋运动）中（适当的）刚性运动的裂缝经典结果的推广。我们证明了仿射等距线的卡坦-迪乌顿定理的推广：在is（n）中的每个等距线如果有固定点，可以写成至多n个仿射反射的合成，或者写成至多n+2个仿射反射的合成。我们还证明了SE（n）中的每一个刚性运动都是至多n个仿射翻转（n≥3）的组合。根据前面的定理，这有点令人惊讶。

Definition 26.1. Given any two nontrivial Euclidean affine spaces E and F of the same finite dimension n, a function f : E → F is an affine isometry (or rigid map) if it is an affine map and  
定义26.1.给定任意两个非平凡欧几里得仿射空间e和f的相同有限维n，函数f:e→f是仿射等值线（或刚性映射），如果它是仿射映射，并且

,  
，

for all a,b ∈ E. When E = F, an affine isometry f : E → E is also called a rigid motion.  
对于所有a，b∈e，当e=f时，仿射等距f:e→e也被称为刚性运动。

Thus, an affine isometry is an affine map that preserves the distance. This is a rather strong requirement. In fact, we will show that for any function f : E → F, the assumption that  
因此，仿射等值线是保持距离的仿射图。这是一个相当强烈的要求。事实上，我们将证明对于任何函数f:e→f，假设

,  
，

for all a,b ∈ E, forces f to be an affine map.  
对于所有a，b∈e，强制f是仿射映射。

Remark: Sometimes, an affine isometry is defined as a bijective affine isometry. When E and F are of finite dimension, the definitions are equivalent.  
注：有时，仿射等值线被定义为双射仿射等值线。当e和f为有限维时，定义是等价的。

The following simple lemma is left as an exercise.  
下面的简单引理是作为练习留下的。

Proposition 26.6. Given any two nontrivial Euclidean affine spaces E and F of the same finite dimension , an affine map f : E → F is an affine isometry iff its associated linear map f : E → F is an isometry. An affine isometry is a bijection.  
提案26.6.给定任意两个相同有限维的非平凡欧几里德仿射空间e和f，仿射映射f:e→f是仿射等值线，而其相关线性映射f:e→f是等值线。仿射等值线是双射。

Let us now consider affine isometries→− f : E → E. If →−f is a rotation, we call f a proper (or direct) affine isometry, and if f is an improper linear isometry, we call f an improper (or skew) affine isometry. It is easily shown that the set of affine isometries f : E → E forms a group, and those for which →−f is a rotation is a subgroup. The group of affine isometries, or rigid motions, is a subgroup of the affine group GA(E), denoted by Is(E) (or Is(n) when E = En). In Snapper and Troyer [157] the group of rigid motions is denoted by Mo(E). Since we denote the group of affine bijections as GA(E), perhaps we should denote the group of affine isometries by IA(E) (or EA(E)!). The subgroup of Is(E) consisting of the direct rigid motions is also a subgroup of SA(E), and it is denoted by SE(E) (or SE(n), when E = En). The translations are the affine isometries f for which = id, the identity map on →−E. The following lemma is the counterpart of Lemma 11.12 for isometries between Euclidean vector spaces.  
现在让我们考虑仿射等轴测→−f:e→e。如果→−f是一个旋转，我们称之为适当（或直接）仿射等轴测，如果f是一个不适当的线性等轴测，我们称之为不适当（或歪斜）仿射等轴测。可以很容易地看出，仿射等距图f:e→e构成一个群，其中→−f是一个旋转的群是一个子群。仿射轴测或刚性运动组是仿射组ga（e）的一个子组，表示为is（e）（e=e n时为（n）。在Snapper和Troyer[157]中，刚性运动组用mo（e）表示。既然我们把仿射双射群表示为ga（e），也许我们应该用ia（e）（或ea（e）！）来表示仿射等轴测群。.由直接刚性运动组成的IS（E）子组也是SA（E）子组，当E=e n时，用SE（E）（或SE（N）表示。翻译是仿射等轴测图f，其中=id，在→−e上的标识映射。下面的引理是引理11.12的对应项，用于欧几里得向量空间之间的等轴测图。

Proposition 26.7. Given any two nontrivial Euclidean affine spaces E and F of the same finite dimension n, for every function f : E → F, the following properties are equivalent:  
提案26.7。给定任意两个非平凡欧几里得仿射空间e和f，对于每个函数f:e→f，下列性质是等价的：

(1) f is an affine map and kf−−−−−(a)f(→b)k = k→−abk, for all a,b ∈ E.  
（1）f是仿射图，kf−−−−（a）f（→b）k=k→−abk，对于所有a，b∈e。

, for all a,b ∈ E.  
，对于所有a，b∈e。

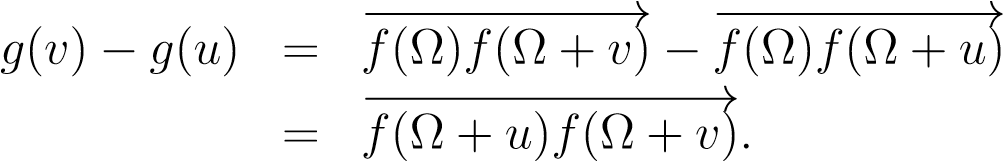
Proof. Obviously, (1) implies (2). In order to prove that (2) implies (1), we proceed as follows. First, we pick some arbitrary point Ω ∈ E. We define the map g: →−E → →−F such that for all u ∈ E. Since  
证据。显然，（1）意味着（2）。为了证明（2）意味着（1），我们继续如下。首先，我们选取一些任意的点Ω∈E，定义了图G：→−E→→−F，这样对于所有的u∈E。



26.3. FIXED POINTS OF AFFINE MAPS  
26.3。仿射映射的不动点

for all u ∈ →−E, f will be affine if we can show thatg is a linear isometry. g is linear, and f will be an affine isometry if we can show that  
对于所有u∈→−e，如果我们能证明g是一个线性等距，f将是仿射的。G是线性的，如果我们能证明，F是仿射等距线。

Observe that  
注意



Then, the hypothesis  
那么，假设

k−−−−−f(a)f(→b)k = k→−abk  
K−−−F（A）F（→B）K=K→−ABK

for all a,b ∈ E, implies that  
对于所有a，b∈e，意味着

.  
.

Thus, g preserves the distance. Also, by definition, we have  
因此，G保持距离。而且，根据定义，我们

g(0) = 0.  
G（0）=0。

Thus, we can apply Lemma 11.12, which shows that g is indeed a linear isometry, and thus f is an affine isometry.   
因此，我们可以应用引理11.12，它表明g确实是一个线性等距，因此f是一个仿射等距。

In order to understand the structure of affine isometries, it is important to investigate the fixed points of an affine map.  
为了了解仿射等距图的结构，研究仿射图的不动点十分重要。

## 26.3 Fixed Points of Affine Maps 26.3仿射图的不动点

Recall that denotes the eigenspace of the linear map →−f associated with the scalar  
表示线性映射的特征空间→−f与标量相关

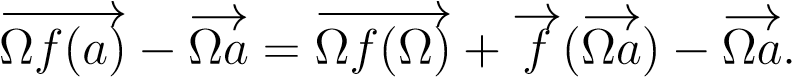
Ker →− →− . Given some origin Ω ∈ E, since∈ →−E such that →−f (u) = u. Clearly, 1, that is, the subspace consisting of all vectors u  
KER––––––。给定一个原点，Ω∈e，因为∈→−e使得→−f（u）=u。很明显，1，也就是说，由所有向量u组成的子空间

f(a) = f(Ω + −Ω→a) = f(Ω) + →−f (−Ω→a),  
F（A）=F（欧姆+欧姆→A）=F（欧姆）+→F（−欧姆→A），

we have), and thus  
我们有），因此

.  
.

From the above, we get  
从上面我们可以看到



Using this, we show the following lemma, which holds for arbitrary affine spaces of finite dimension and for arbitrary affine maps.  
利用这个，我们给出了以下引理，它适用于有限维的任意仿射空间和任意仿射映射。

Proposition 26.8. Let E be any affine space of finite dimension. For every affine map f : E → E, let Fix(f) = {a ∈ E | f(a) = a} be the set of fixed points of f. The following properties hold:  
提案26.8。设e为有限维的任意仿射空间。对于每一个仿射映射f:e→e，让fix（f）=a∈e f（a）=a为f的不动点集。以下属性成立：

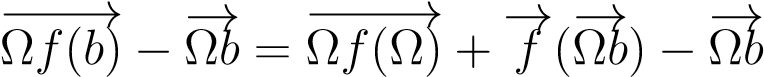
1. If f has some fixed point a, so that Fix(f) 6= ∅, then Fix(f) is an affine subspace of E such that  
   如果f有不动点a，那么fix（f）6=∅，那么fix（f）是e的仿射子空间，这样

Fix(,  
修复，（

where is the eigenspace of the linear map →−f for the eigenvalue 1.  
其中，线性映射的特征空间→−f表示特征值1。

1. The affine map f has a unique fixed point iff = Ker .  
   仿射映射f有一个唯一的固定点iff=ker。

Proof. (1) Since the identity  
证据。（1）自身份



holds for all Ω,b ∈ E, if f(a) = a, then −−−af(→a) = 0, and thus, letting Ω = a, for any b ∈ E we have and so  
对于所有的Ω，b∈e，如果f（a）=a，那么−−af（→a）=0，因此，对于任何b∈e，让Ω=a，我们有，因此

iff  
敌我识别

−−−af(→b) − →−ab = 0  
−−af（→b）−→−ab=0

iff  
敌我识别

→−f (→−ab) →−ab = 0  
→−F（→−AB）→−AB=0

iff  
敌我识别

,  
，

which proves that  
这证明了

Fix(.  
修理（…）

(2) Again, fix some origin Ω. Some a satisfies f(a) = a iff  
（2）再次，固定一些原点Ω。一些a满足f（a）=iff

−−−Ωf(→a) − −Ω→a = 0  
−−ΩF（→A）−−Ω→A=0

iff  
敌我识别

−−−−Ωf(Ω) +→ →−f (−Ω→a) −Ω→a = 0,  
−−−欧F（欧）+→→−F（−欧→A）−欧→A=0，

which can be rewritten as  
可以重写为

We have = Ker id is injective, and since →−E has finite  
我们有=KER ID是内射的，因为→−E是有限的

dimension, f − id is also surjective, and thus, there is indeed some a ∈ E such that  
维度，f−id也是主观的，因此，确实有一些a∈e这样

,  
，

and it is unique, since →−f − id is injective. Conversely, if f has a unique fixed point, say a, from  
它是独一无二的，因为→−F−ID是注射的。相反，如果f有一个唯一的固定点，比如a，from

,  
，

we have (Ω) = Ω, and since a is the unique fixed point of f, we must have a = Ω, which shows that →−f − id is injective.   
我们有（Ω）=Ω，由于a是f的唯一固定点，我们必须有a=Ω，这表明→−f−id是内射的。

Remark: The fact that E has finite dimension is used only to prove (2), and (1) holds in general.  
注：E有有限维的事实仅用于证明（2），（1）一般成立。

If an affine isometry f leaves some point fixed, we can take such a point Ω as the origin, and then f(Ω) = Ω and we can view f as a rotation or an improper orthogonal transformation, depending on the nature of →−f . Note that it is quite possible that Fix(f) = ∅. For example, nontrivial translations have no fixed points. A more interesting example is provided by the composition of a plane reflection about a line composed with a a nontrivial translation parallel to this line.  
如果一个仿射等值线f离开某个固定点，我们可以取这样一个点Ω作为原点，然后f（Ω）=Ω，我们可以将f视为旋转或不适当的正交变换，这取决于→−f的性质。注意，很有可能固定（f）=∅。例如，非平凡的翻译没有固定点。一个更有趣的例子是由一条平行于这条线的非平凡平移组成的线的平面反射组成。

Otherwise, we will see in Theorem 26.10 that every affine isometry is the (commutative) composition of a translation with an affine isometry that always has a fixed point.  
否则，我们将在定理26.10中看到，每个仿射等值线都是一个具有固定点的仿射等值线的翻译的（交换）组合。

## 26.4 Affine Isometries and Fixed Points 26.4仿射等距线和固定点

Let E be an affine space. Given any two affine subspaces F,G, if F and G are orthogonal complements in E, which means that →−F and →−G are orthogonal subspaces of →−E such that →−E = →−F ⊕ →−G, for any point Ω ∈ F, we define q: E → →−G such that  
设e为仿射空间。给定任意两个仿射子空间f，g，如果f和g是e中的正交互补，这意味着→−f和→−g是→−e的正交子空间，因此→−e=→−f→−g，对于任何点Ω∈f，我们定义q:e→→−g以便

.  
.

Note that q(a) is independent of the choice of Ω ∈ F, since we have  
注意，q（a）独立于Ω∈f的选择，因为我们有

,  
，

and for any Ω1 ∈ F, we have  
对于任何Ω1∈f，我们有

,  
，

and since , this shows that  
从那以后，这表明

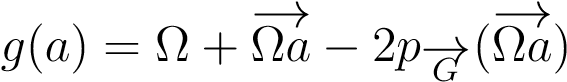
.  
.

Then the map g: E → E such that g(a) = a − 2q(a), or equivalently  
然后图g:e→e，这样g（a）=a−2q（a），或相等

,  
，

does not depend on the choice of Ωin F, we note that g is identified with the symmetry with respect to∈ F. If we identify E to →−E by choosing any origin Ω→−F and parallel to →−G.  
不依赖于f中的Ω的选择，我们注意到g与∈f对称。如果我们通过选择任何原点Ω→−f并与→−g平行来确定e到→−e。

Thus, the map g is an affine isometry, and it is called the affine orthogonal symmetry about F. Since  
因此，图G是一个仿射等值线，它被称为关于f的仿射正交对称。



the (linear) symmetry about the subspacefor all Ω ∈ F and for all a ∈ E, we note that the linear map→−F (the direction of F), and parallel to→−g associated with→−G (gtheis direction of G).  
关于子空间的（线性）对称，对于所有的Ω∈f和所有的a∈e，我们注意到线性映射→−f（f的方向），并平行于→−g与→−g（g的gtheis方向）相关联。

Remark: The map p: E → F such that p(a) = a − q(a), or equivalently  
备注：图P:E→F，使得P（A）=A−Q（A），或等效

,  
，

is also independent of Ω ∈ F, and it is called the affine orthogonal projection onto F.  
也独立于Ω∈F，称为F上的仿射正交投影。

The following amusing lemma shows the extra power afforded by affine orthogonal symmetries: Translations are subsumed! Given two parallel affine subspaces F1 and F2 in E, letting →−F be the common direction of F1 and F2 and →−G = →−F ⊥ be its orthogonal comple-  
下面有趣的引理显示了仿射正交对称所提供的额外能力：翻译被包含在内！给定e中的两个平行仿射子空间f1和f2，让→−f为f1和f2的公共方向，→−g=→−f为其正交配位。-

ment, for anybetween23.16). We define theF1 anda ∈FF1is independent of the choice of, the affine subspacedistance between F1aand+→−GFintersects2 asa →−abkF. It is easily seen that the distanceF1, and that it is the minimum of2 in a single point b (see Lemma k  
在23.16之间）。我们定义了f1和∈ff1独立于f1aa和+之间的仿射子空间距离→−gf与s2 asa→−abkf的选择。很容易看出距离f1，在一个点b中是2的最小值（参见引理k

2 in  
2英寸

k−xy→k for all x ∈ F1 and all y ∈ F2.  
所有x∈f1和所有y∈f2的k−xy→k。

Proposition 26.9.defined by the vector Given any affine space2→−ab, where →−ab is any vector perpendicular to the common directionE, if f : E → FE2, thenand gg: E◦ f→is a translationE are affine→−F orthogonal symmetries about parallel affine subspaces F1 and  
命题26.9.由给定任意仿射空间2→−ab的矢量定义，其中→−ab是垂直于公共方向的任意矢量，如果f:e→fe2，那么和gg:e f→是一个平移，即仿射→−f关于平行仿射子空间f1和

of F1 and F2 such that k→−abk is the distance betweenτ is obtained as the composition of two affineF1 and F2, with a ∈ F1 and b ∈ F2.  
其中k→−abk是τ之间的距离，由两个仿射f1和f2组成，其中a∈f1和b∈f2。

Conversely, every translation by a vector  
相反，每一个矢量的翻译

orthogonal symmetries about parallel affine subspaces F1 and F2 whose common direction is orthogonal to→−abkτ/2=. →−ab, for some a ∈ F1 and some b ∈ F2 such that the distance between F1 and F2 is k  
平行仿射子空间f1和f2的正交对称性，其公共方向与→−abkτ/2=正交。→−a b，对于一些a∈f1和一些b∈f2，使得f1和f2之间的距离为k

Proof. We observed earlier that the linear maps →−f and →−g associated with f and g are the linear reflections about the directions of F1 and F2. However, F1 and F2 have the same  
证据。我们之前观察到，与f和g相关的线性映射→−f和→−g是关于f1和f2方向的线性反射。然而，f1和f2是一样的。

direction, and soevery reflection is an involution, we have→−f = →−g . Since −−g ◦→f = →−−−gg◦→◦f→−f=and sinceid, proving that→−f ◦ →−g g=◦→−ffaboutis a translation. If◦ →−f =F2id, because, and it is  
方向，所以反射是对合的，我们有→−f=→−g。由于−−g→f=→−−−gg→f→−f=和sinceid，证明→−f→−g=→−ffabout是一个翻译。如果→−f=f2id，因为，它是

we pick a ∈ F1, then g ◦ f(a) = g(a), the affine reflection of   
我们选取a∈f1，然后g f（a）=g（a），仿射

distance betweeneasily checked thatF1gand◦ f is the translation by the vectorF2. The second part of the lemma is left as an easy exercise.τ = ag(a) whose norm is twice the  
检查之间的距离，其中f1gand f是矢量2的平移。引理的第二部分留作一个简单的练习。τ=Ag（a），其范数是

We conclude our quick study of affine isometries by proving a result that plays a major role in characterizing the affine isometries. This result may be viewed as a generalization of Chasles’s theorem about the direct rigid motions in E3.  
通过对仿射等距图的研究，证明了仿射等距图在表征仿射等距图中起着重要作用。这个结果可以被看作是关于E3中直接刚性运动的Chales定理的推广。

Theorem 26.10. Let E be a Euclidean affine space of finite dimension n. For every affine  
定理26.10。对于每个仿射，设e为有限维n的欧几里得仿射空间。

isometry f : →−Ef (→τ) =E, there is a unique affine isometryτ (i.e., τ ∈ Ker), such that the setgE: Eof direction→ EFix(and a unique translationg) = {a ∈ E | g(a) = t = tτ, with a} of fixed points of g is a nonempty affine subspace of  
等距f：→−ef（→τ）=e，存在一个唯一的仿射等距τ（即τ∈ker），使得集合ge:e of方向→efix（和一个唯一的平移g）=a∈e g（a）=t=tτ，其中g的固定点a是一个非空的仿射子空间

→−G = Ker ,  
→−G=KER，

and such that  
这样的话

f = t ◦ g and t ◦ g = g ◦ t.  
f=t\_g和t\_g=g\_t。

Furthermore, we have the following additional properties:  
此外，我们还有以下附加属性：

1. f = g and τ = 0 iff f has some fixed point, i.e., iff Fix(f) 6= ∅.  
   f=g，τ=0，iff f有固定点，即iff fix（f）6=∅。
2. If f has no fixed points, i.e., Fix(f) = ∅, then dim Ker .  
   如果F没有固定点，即固定（F）=∅，则调暗KER。

Proof. The proof rests on the following two key facts:  
证据。证据基于以下两个关键事实：

(1) If we can find some x ∈ E such that belongs to Ker , we get the existence of g and τ.  
（1）如果我们能找到一些x∈e，那么我们就得到了g和τ的存在性。

, and the spaces Ker and  
以及空格和

Im are orthogonal. This implies the uniqueness of g and τ.  
我是正交的。这意味着g和τ的唯一性。

First, we prove that for every isometry h: →−E → →−E, Ker(h − id) and Im(h − id) are orthogonal and that  
首先，我们证明对于每一个等距测量，h：→−e→−e，ker（h−id）和im（h−id）是正交的，并且

→−E = Ker(h − id) ⊕ Im(h − id).  
→−E=KER（H−ID）IM（H−ID）。

Recall that dim = dim(Kerϕ) + dim(Imϕ),  
回想一下dim=dim（ker\_）+dim（im\_）

for any linear map ; see Theorem 5.11. To show that we have a direct sum, we prove orthogonality. Let u ∈ Ker(h − id), so that , and compute  
对于任何线性映射，请参见定理5.11。为了证明我们有一个直接和，我们证明了正交性。让u∈ker（h-id），这样，然后计算

u · (h(v) − v) = u · h(v) − u · v = h(u) · h(v) − u · v = 0,  
U·（H（V）−V）=U·H（V）−U·V=H（U）·H（V）−U·V=0，

since h(u) = u and h is an isometry.  
因为h（u）=u和h是等距测量。

Next, assume that there is some x ∈ E such that belongs to the space Ker . If we define g: E → E such that  
接下来，假设有一些x∈e，它属于空间ker。如果我们将g:e→e定义为

g = t(−τ) ◦ f,  
g=t（−τ）f，

we have g(x) = f(x) − τ = x,  
我们有g（x）=f（x）−τ=x，

sinceis equivalent to x = f(x) − τ. As a composition of affine isometries, g is an affine isometry, x is a fixed point of g, and since τ ∈ Ker , we have  
sinceis等于x=f（x）−τ。作为仿射等距图的一个组成部分，G是仿射等距图，X是G的不动点，由于τ∈ker，我们得到

→−f (τ) = τ,  
→−f（τ）=τ，

and since  
从那以后

g(b) = f(b) − τ  
g（b）=f（b）−τ

is an affine subspace offor all b ∈ E, we have →−gE=with direction Ker→−f . Since g has some fixed point= Kerx, by Lemma 26.8, Fix(→− . We also haveg)  
是所有b∈e的仿射子空间，我们有→−ge=和方向ker→−f。由于g有一些固定点=kerx，根据引理26.8，fix（→−。我们也有g）

f(b) = g(b) + τ for all b ∈ E, and thus  
f（b）=g（b）+τ表示所有b∈e，因此

(g ◦ tτ)(b) = g(b + τ) = g(b) + →−g (τ) = g(b) + →−f (τ) = g(b) + τ = f(b),  
（g\_tτ）（b）=g（b+τ）=g（b）+→−g（τ）=g（b）+→−f（τ）=g（b）+τ=f（b）、

and  
和

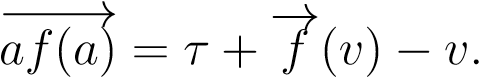
(tτ ◦ g)(b) = g(b) + τ = f(b),  
（tτg）（b）=g（b）+τ=f（b），

which proves that t ◦ g = g ◦ t.  
证明t g=g t。

To prove the existence of x as above, pick any arbitrary point a ∈ E. Since  
为了证明上述x的存在，选取任意点a∈e。

→−E = Ker ,  
→E=KER，

there is a unique vector τ ∈ Ker and some v ∈ →−E such that  
有一个唯一的向量τ∈ker和一些v∈→−e，这样



For any x ∈ E, since we also have  
对于任何x∈e，因为我们也有

,  
，

we get  
我们得到

,  
，

which can be rewritten as  
可以重写为

.  
.

If we let −ax→ = −v, that is, x = a − v, we get  
如果我们让−a x→−v，也就是x=a−v，我们得到



with τ ∈ Ker .  
与τ∈ker。

Finally, we show that τ is unique. Assume two decompositions (g1,τ1) and (g2,τ2). Since  
最后，我们证明τ是唯一的。假设两个分解（g1，τ1）和（g2，τ2）。自从

, we have Ker( id) = Ker . Since g1 has some fixed point b, we get  
，我们有ker（id）=ker。因为g1有固定点b，我们得到

f(b) = g1(b) + τ1 = b + τ1,  
f（b）=g1（b）+τ1=b+τ1，

that is, bf−−−(→b) = τ1, and τ1 ∈ . Similarly, for some fixed point c of g2, we get cf(c) = τ2 and cf(c) ∈ Ker . Then we have  
也就是说，bf−−（→b）=τ1，τ1∈。同样地，对于g2的某个不动点c，我们得到了cf（c）=τ2和cf（c）∈ker。然后我们有了

,  
，

which shows that  
这表明

τ2 − τ1 ∈ Ker ,  
τ2−τ1∈ker，

and thus that τ2 = τ1, since we have shown that  
因此，τ2=τ1，因为我们已经证明

→−E = Ker .  
→−E=KER。

The fact that (a) holds is a consequence of the uniqueness of g and τ, since f and 0 clearly satisfy the required conditions. That (b) holds follows from Lemma 26.8 (2), since the affine map f has a unique fixed point iff = Ker .   
（a）成立的事实是g和τ的唯一性的结果，因为f和0显然满足要求的条件。这（b）符合引理26.8（2），因为仿射映射f有一个唯一的固定点iff=ker。

The determination of x is illustrated in Figure 26.8.  
X的测定如图26.8所示。

*a*

*x*

-

v

f

(

v

)

f(v) - v

*a*

(

*v*

)

−

*v*

*f*

+

*τ*

*f*

(

*x*

)

*f*

(

*a*

)

*τ*

*f*

(

*a*

)+

K

e

r

*f*

−

id

*a*

+

I

m

*f*

−

id

a

f

(

a

)

f

(

a

)

f

(

x

)

Figure 26.8: Affine rigid motion as f = t ◦ g, where g has some fixed point x.  
图26.8：仿射刚性运动，f=t\_g，其中g有一些固定点x。

Remarks:  
评论：

1. Note that Keriff Fix(g) consists of a single element, which is the unique fixed point of f. However, even if f is not a translation, f may not have any fixed points. For example, this happens when E is the affine Euclidean plane and f is the composition of a reflection about a line composed with a nontrivial translation parallel to this line.  
   请注意，keriff fix（g）由单个元素组成，这是f的唯一固定点。但是，即使f不是翻译，f也可能没有任何固定点。例如，当e是仿射欧几里得平面，f是关于一条线的反射的合成，该线由平行于该线的非平凡平移组成。
2. The fact that E has finite dimension is used only to prove (b).  
   事实上，E的有限维仅用于证明（b）。
3. It is easily checked that Fix(g) consists of the set of points x such that is minimal.  
   可以很容易地检查fix（g）是否由点X组成，从而使其最小。

In the affine Euclidean plane it is easy to see that the affine isometries (besides the identity) are classified as follows. An affine isometry f that has a fixed point is a rotation if it is a direct isometry; otherwise, it is an affine reflection about a line. If f has no fixed point, then it is either a nontrivial translation or the composition of an affine reflection about a line with a nontrivial translation parallel to this line.  
在仿射欧几里得平面上，很容易看出仿射等距线（除了同一性）被分类如下。一个具有固定点的仿射等距f是一个旋转，如果它是一个直接等距；否则，它是关于一条直线的仿射反射。如果f没有不动点，那么它要么是一个非平凡平移，要么是关于一条线的仿射反射的组合，其中一条线的非平凡平移与此线平行。

In an affine space of dimension 3 it is easy to see that the affine isometries (besides the identity) are classified as follows. There are three kinds of affine isometries that have a fixed point. A proper affine isometry with a fixed point is a rotation around a line D (its set of fixed points), as illustrated in Figure 26.9.  
在维数3的仿射空间中，很容易看出仿射等距线（除了恒等式）被分类如下。有三种具有固定点的仿射等距线。具有固定点的适当仿射等值线是围绕线D（其固定点集）旋转的，如图26.9所示。

*D*

*a*

*f*

(

*a*

)

Figure 26.9: 3D proper affine rigid motion with line D of fixed points (rotation).  
图26.9：固定点D线（旋转）的三维适当仿射刚性运动。

An improper affine isometry with a fixed point is either an affine reflection about a plane H (the set of fixed points) or the composition of a rotation followed by an affine reflection about a plane H orthogonal to the axis of rotation D, as illustrated in Figures 26.10 and 26.11. In the second case, there is a single fixed point O = D ∩ H.  
具有固定点的不适当仿射等值线是关于平面H（固定点集）的仿射反射，或者是关于与旋转轴D正交的平面H的旋转后的仿射反射的组合，如图26.10和26.11所示。在第二种情况下，有一个固定点o=d h。

There are three types of affine isometries with no fixed point. The first kind is a nontrivial translation. The second kind is the composition of a rotation followed by a nontrivial translation parallel to the axis of rotation D. Such an affine rigid motion is proper, and is called a screw motion. A screw motion is illustrated in Figure 26.12.  
仿射等角图有三种类型，没有固定点。第一种是非平凡的翻译。第二类是旋转的组合，随后是平行于旋转轴d的非平凡平移。这种仿射刚性运动是适当的，称为螺旋运动。图26.12说明了螺钉的运动。

26.5. THE CARTAN–DIEUDONNE THEOREM FOR AFFINE ISOMETRIES´  
26.5。仿射等距线的卡坦-迪乌顿定理

*a*

*f*

(

*a*

)

*H*

Figure 26.10: 3D improper affine rigid motion with a plane H of fixed points (reflection).  
图26.10：固定点平面H（反射）的三维不适当仿射刚性运动。

*D*

*a*

*O*

*H*

f(a)

Figure 26.11: 3D improper affine rigid motion with a unique fixed point.  
图26.11：具有唯一固定点的三维不适当仿射刚性运动。

The third kind is the composition of an affine reflection about a plane followed by a nontrivial translation by a vector parallel to the direction of the plane of the reflection, as illustrated in Figure 26.13.  
第三类是关于平面的仿射反射的合成，随后是平行于反射平面方向的向量的非平凡平移，如图26.13所示。

This last transformation is an improper affine isometry.  
最后一个转换是一个不正确的仿射同构。

## 26.5 The Cartan–Dieudonn´e Theorem for Affine Isometries 26.5仿射等轴测的卡坦-迪乌顿定理

The Cartan–Dieudonn´e theorem also holds for affine isometries, with a small twist due to translations. The reader is referred to Berger [11], Snapper and Troyer [157], or Tisseron [170] for a detailed treatment of the Cartan–Dieudonn´e theorem and its variants.  
卡坦-迪乌登定理也适用于仿射等距线，由于翻译，它有一个小的扭曲。读者可以参考Berger[11]、Snapper和Troyer[157]或Tisseron[170]来详细处理Cartan–Dieudonn'e定理及其变体。

Theorem 26.11. Let E be an affine Euclidean space of dimension n ≥ 1. Every affine isometry f ∈ Is(E) that has a fixed point and is not the identity is the composition of at most n affine reflections. Every affine isometry f ∈ Is(E) that has no fixed point is the  
定理26.11。设e为尺寸n≥1的仿射欧几里德空间。每一个仿射同构f∈都是（e）有一个固定的点，而不是同一性，至多是n个仿射反射的组成。每一个没有固定点的仿射同构f∈是（e）是

*D*

*τ*

*a*

*f*

(

*a*

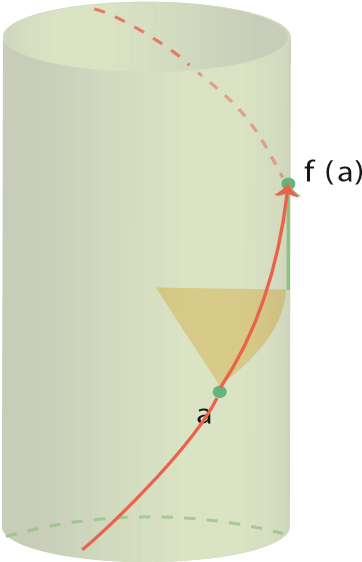
)

*g*

(

*a*

)



(i.)  
（一）

(ii.)  
（二）

Figure 26.12: 3D proper affine rigid motion with no fixed point (screw motion). The second illustration demonstrates that a screw motion produces a helix path along the surface of a cylinder.  
图26.12：没有固定点的三维适当仿射刚性运动（螺旋运动）。第二幅图显示了螺杆运动沿圆柱表面产生螺旋路径。

*g*

(

*a*

)

*H*

*a*

*τ*

*f*

(

*a*

)

Figure 26.13: 3D improper affine rigid motion with no fixed points.  
图26.13：没有固定点的三维不适当仿射刚性运动。

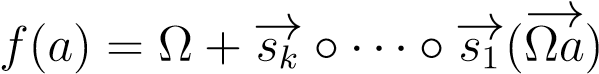
26.5. THE CARTAN–DIEUDONNE THEOREM FOR AFFINE ISOMETRIES´  
26.5。仿射等距线的卡坦-迪乌顿定理

composition of at most n + 2 affine reflections. When n ≥ 2, the identity is the composition of any reflection with itself.  
最多由n+2个仿射反射组成。当n≥2时，同一性是任何反射本身的组成。

Proof. First, we use Theorem 26.10. If f has a fixed point Ω, we choose Ω as an origin and work in the vector space EΩ. Since f behaves as a linear isometry, the result follows from Theorem 26.1. More specifically, we can write hyperplane reflections →−si . We define the affine reflections si such that  
证据。首先，我们使用定理26.10。如果F有一个固定点Ω，我们选择Ω作为原点，并在向量空间EΩ中工作。由于f表现为一个线性等距测量，其结果来自定理26.1。更具体地说，我们可以写超平面反射→−si。我们将仿射反射定义为

si(a) = Ω + →−si (Ω−→a)  
Si（A）=Ω+→Si（Ω-→A）

for all a ∈ E, and we note that f = sk ◦ ··· ◦ s1, since  
对于所有a∈e，我们注意到f=sk····s1，因为

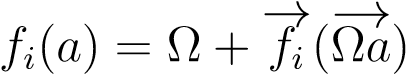


for all a ∈ E. If f has no fixed point, then for some affine isometry g that has a fixed point Ω and some translation t = tτ, with f (τ) = τ. By the argument just given, we can write g = sk ◦ ··· ◦ s1 for some affine reflections (at most n). However, by Lemma 26.9, the translation t = tτ can be achieved by two affine reflections about parallel hyperplanes, and thus f = sk+2 ◦ ··· ◦ s1, for some affine reflections (at most n + 2).   
对于所有a∈e，如果f没有不动点，那么对于某些具有不动点Ω的仿射等距G，某些平移t=tτ，其中f（τ）=τ。根据刚才给出的论点，我们可以写出g=sk·····s1来表示一些仿射反射（最多n）。然而，在引理26.9中，对于某些仿射反射（最多n+2），平移t=tτ可以通过两个关于平行超平面的仿射反射来实现，因此f=sk+2····s1。

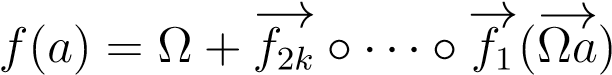
When n ≥ 3, we can also characterize the affine isometries in SE(n) in terms of affine flips. Remarkably, not only we can do without translations, but we can even bound the number of affine flips by n.  
当n≥3时，我们也可以用仿射翻转来描述SE（n）中的仿射等距图。值得注意的是，我们不仅可以不翻译，而且还可以将仿射翻转的数量限制为n。

Theorem 26.12. Let E be a Euclidean affine space of dimension n ≥ 3. Every affine rigid motion f ∈ SE(E) is the composition of an even number of affine flips f = f2k ◦ ··· ◦ f1, where 2k ≤ n.  
定理26.12。设e为维数n≥3的欧氏仿射空间。每一个仿射刚性运动f∈se（e）是偶数个仿射翻转f=f2k···f1的组合，其中2k≤n。

Proof. As in the proof of Theorem 26.11, we distinguish between the two cases where f has some fixed point or not. If f has a fixed point Ω, we apply Theorem 26.5. More specifically, we can write →−f = f−→2k ◦ ··· ◦ →−f1 for some flips. We define the affine flips fi such that  
证据。在定理26.11的证明中，我们区分了F是否有固定点的两种情况。如果f有一个固定点Ω，我们应用定理26.5。更具体地说，我们可以为一些翻转写→−f=f−→2K···→−f1。我们定义仿射翻转fi，以便



for all a ∈ E, and we note that f = f2k ◦ ··· ◦ f1, since  
对于所有a∈e，我们注意到f=f2k···f1，因为



for all a ∈ E.  
对于所有a∈e。

If f does not have a fixed point, as in the proof of Theorem 26.11, we get  
如果f没有固定点，如定理26.11的证明，我们得到

f = tτ ◦ f2k ◦ ··· ◦ f1,  
f=tτf2k···f1，

for some affine flips fi. We need to get rid of the translation. However,, and by the second part of Theorem 26.5, we can assume that , where F2k is the direction of the affine subspace defining the affine flip f2k. Finally, appealing to Lemma 26.9, since  
对于一些仿射翻转fi。我们得把翻译掉。然而，根据定理26.5的第二部分，我们可以假设，其中f2k是定义仿射翻转f2k的仿射子空间的方向。最后，吸引引理26.9，因为

, the translation tτ can be expressed as the compositionof two affine flips andabout the two parallel subspaces Ω + and Ω +, whose distance is kτk/2. However, since and f2k are both the identity on Ω + F2k, we must have , and thus  
，平移tτ可以表示为两个仿射翻转和两个平行子空间Ω+和Ω+的组合，其距离为kτk/2。但是，既然和f2k都是Ω+f2k上的标识，我们必须有，因此

f = tτ ◦ f2k ◦ f2k−1 ◦ ··· ◦ f1  
f=tτf2k f2k−1···f1

= f20k ◦ f20k−1 ◦ f2k ◦ f2k−1 ◦ ··· ◦ f1 = f20k ◦ f2k−1 ◦ ··· ◦ f1,  
=f20k f20k−1 f2k f2k−1···f1=f20k f2k−1··f1，

since and = id, since f2k is an affine symmetry.   
因为和=id，因为f2k是仿射对称。

Remark: It is easy to prove that if f is a screw motion in SE(3), D its axis, θ is its angle of rotation, and τ the translation along the direction of D, then f is the composition of two affine flips about lines D1 and D2 orthogonal to D, at a distance kτk/2 and making an angle θ/2.  
注：可以很容易地证明，如果f是在se（3）中的螺旋运动，d是它的轴，θ是它的旋转角，τ是沿着d的方向的平移，那么f是关于d1和d2直线的两个仿射翻转的组合，在kτk/2的距离上，作一个θ/2角。

Chapter 27  
第二十七章

# Isometries of Hermitian Spaces 埃尔米特空间等距图

## 27.1 The Cartan–Dieudonn´e Theorem, Hermitian Case 27.1卡坦-迪乌登定理，赫米提亚案例

The Cartan-Dieudonn´e theorem can be generalized (Theorem 27.2), but this requires allowing new types of hyperplane reflections that we call Hermitian reflections. After doing so, every isometry in U(n) can always be written as a composition of at most n Hermitian reflections (for n ≥ 2). Better yet, every rotation in SU(n) can be expressed as the composition of at most 2n − 2 (standard) hyperplane reflections! This implies that every unitary transformation in U(n) is the composition of at most 2n−1 isometries, with at most one Hermitian reflection, the other isometries being (standard) hyperplane reflections. The crucial Proposition 12.2 is false as is, and needs to be amended. The QR-decomposition of arbitrary complex matrices in terms of Householder matrices can also be generalized, using a trick.  
卡坦-迪乌顿定理可以推广（定理27.2），但这需要允许新类型的超平面反射，我们称之为厄米特反射。这样做之后，u（n）中的每个等距线都可以写成至多n个厄米特反射（n≥2）的组合。更好的是，su（n）中的每个旋转可以表示为最多2n-2（标准）超平面反射的组成！这意味着u（n）中的每一个单位变换都是至多2n-1等距线的组成，其中至多一个厄米特反射，其他等距线是（标准）超平面反射。关键的12.2号提案是错误的，需要修改。利用一个技巧，也可以推广任意复杂矩阵在户主矩阵方面的二维分解。

In order to generalize the Cartan–Dieudonn´e theorem and the QR-decomposition in terms of Householder transformations, we need to introduce new kinds of hyperplane reflections. This is not really surprising, since in the Hermitian case, there are improper isometries whose determinant can be any unit complex number. Hyperplane reflections are generalized as follows.  
为了从户主变换的角度推广卡坦-迪乌顿定理和二维分解，需要引入新的超平面反射。这并不令人惊讶，因为在厄米提亚的例子中，有不适当的等距线，其行列式可以是任何单位复数。超平面反射一般如下。

Definition 27.1. Let E be a Hermitian space of finite dimension. For any hyperplane H, for any nonnull vector w orthogonal to H, so that E = H ⊕G, where G = Cw, a Hermitian reflection about H of angle θ is a linear map of the form ρH,θ : E → E, defined such that  
定义27.1.设e为有限维的厄米空间。对于任何超平面h，对于正交于h的任何非零矢量w，因此e=h\_g，其中g=cw，关于θ角h的厄米反射是形式为ρh，θ：e→e的线性映射，其定义如下：

ρH,θ(u) = pH(u) + eiθpG(u),  
ρh，θ（u）=ph（u）+eiθpg（u），

for any unit complex number eiθ = 1 (6 i.e. θ =6 k2π). For any nonzero vector w ∈ E, we denote by ρw,θ the Hermitian reflection given by ρH,θ, where H is the hyperplane orthogonal to w.  
对于任何单位复数e iθ=1（6即θ=6 k2π）。对于任何非零矢量w∈e，我们用ρw，θ表示由ρh，θ给出的厄米反射，其中h是与w正交的超平面。

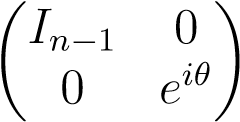
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八百八十五

Since u = pH(u) + pG(u), the Hermitian reflection ρw,θ is also expressed as  
由于u=ph（u）+pg（u），厄米反射ρw，θ也表示为

or as   
或作为

Note that the case of a standard hyperplane reflection is obtained when eiθ = −1, i.e., θ = π.  
注意，标准超平面反射的情况是在e iθ=−1，即θ=π时得到的。

We leave as an easy exercise to check that ρw,θ is indeed an isometry, and that the inverse of ρw,θ is ρw,−θ. If we pick an orthonormal basis (e1,...,en) such that (e1,...,en−1) is an orthonormal basis of H, the matrix of ρw,θ is  
作为一个简单的练习，我们将检查ρw，θ确实是一个等距测量，并且ρw，θ的倒数是ρw，−θ。如果我们选取一个正交基（e1，…，en），使得（e1，…，en-1）是h的正交基，那么ρw，θ的矩阵是



We now come to the main surprise. Given any two distinct vectors u and v such that kusing two Hermitian reflections!uk = kvk, there isn’t always a hyperplane reflection mapping u to v, but this can be done  
我们现在主要的惊喜是。给任何两个截然不同的向量u和v，使kusing两个厄米提反射！Uk=kvk，不总是有一个超平面反射映射u到v，但这可以做到。

Proposition 27.1. Let E be any nontrivial Hermitian space.  
提案27.1.让我们成为任何非平凡的隐士空间。

1. thatthe (usual) reflectionFor any two vectorss(u) = eiθv. u,vs about the hyperplane orthogonal to the vector∈ E such that u =6 v and kuk = kvk, if u · vv=−eiθe−|uiθu· vis such|, then  
   任意两个向量（u）=eiθv.u，vs关于与向量正交的超平面的（通常）反射∈e，使得u=6v，kuk=kvk，如果u·vv=−eiθe−uiθu·vis，那么
2. For any nonnull vector v ∈ E, for any unit complex number eiθ 6= 1, there is a Hermitian reflection ρv,θ such that  
   对于任何非零向量v∈e，对于任何单位复数eiθ6=1，存在厄米反射ρv，θ，这样

ρv,θ(v) = eiθv.  
ρv，θ（v）=eiθv。

As a consequence, for u and v as in (1), we have ρv,−θ ◦ s(u) = v.  
因此，对于（1）中的u和v，我们有ρv，−θs（u）=v。

Proof. (1) Consider the (usual) reflection about the hyperplane orthogonal to w = v−e−iθu. We have  
证据。（1）考虑与w=v−e−iθu正交的超平面的（通常）反射。

.  
.

We need to compute  
我们需要计算

−2u · (v − e−iθu) and (v − e−iθu) · (v − e−iθu).  
−2u·（v−e−iθu）和（v−e−iθu）·（v−e−iθu）。

Since u · v = eiθ|u · v|, we have  
既然u·v=eiθu·v，我们有

e−iθu · v = |u · v| and eiθv · u = |u · v|.  
e−iθu·v=u·v和eiθv·u=u·v。

Using the above and the fact that kuk = kvk, we get  
利用上述和kuk=kvk的事实，我们得到

−2u · (v − e−iθu) = 2eiθiθ(kkuukk22−−2|uu ·· v,v|),  
−2u·（v−e−iθu）=2eiθiθ（kkukk22−2 uu··v，v），

= 2e  
= 2e

and  
和

(v e−iθu) (v − e−iθu) = kvk2 + kuk2 − e−iθu · v − eiθv · u,  
（v e−iθu）（v−e−iθu）=kvk2+kuk2−e−iθu·v−eiθv·u，

2  
二

= 2(kuk − |u · v|),  
=2（kuk−u·v），

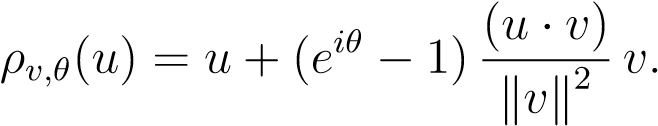
and thus,  
因此，

.  
.

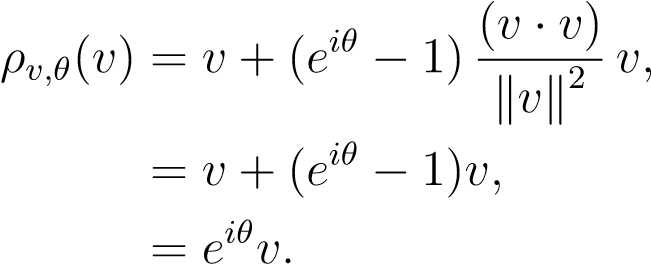
But then,  
但是后来，

and s(u) = eiθv, as claimed.  
和s（u）=eiθv，如权利要求所述。

(2) This part is easier. Consider the Hermitian reflection  
（2）这部分比较容易。想想赫敏的倒影



We have  
我们有



Thus, ρv,θ(v) = eiθv. Since ρv,θ is linear, changing the argument v to eiθv, we get  
因此，ρv，θ（v）=eiθv。由于ρv，θ是线性的，将参数v改为eiθv，我们得到

ρv,−θ(eiθv) = v,  
ρv，−θ（eiθv）=v，

and thus, ρv,−θ ◦ s(u) = v.   
因此，ρv，−θs（u）=v。

Remarks:  
评论：

1. If we use the vector v + e−iθu instead of v − e−iθu, we get s(u) = −eiθv.  
   如果我们使用向量v+e−iθu而不是v−e−iθu，我们得到s（u）=−eiθv。
2. Certain authors, such as Kincaid and Cheney [100] and Ciarlet [41], use the vector instead of our vector v + e−iθu. The effect of this choice is that they also get  
   某些作者，如Kincaid和Cheney[100]和Ciarlet[41]，使用矢量而不是我们的矢量v+e-iθu。这种选择的效果是，他们还可以

s(u) = −e v.  
S（U）=-E V.

1. If v = kuke1, where e1 is a basis vector, u · e1 = a1, where a1 is just the coefficient of u over the basis vector e1. Then, since u · e1 = eiθ|a1|, the choice of the plus sign in the vector kuke1 + e−iθu has the effect that the coefficient of this vector over e1 is kuk+|a1|, and no cancellations takes place, which is preferable for numerical stability (we need to divide by the square norm of this vector).  
   如果v=kuke1，其中e1是基向量，u·e1=a1，其中a1只是基向量e1上u的系数。那么，由于u·e1=e iθa1，在矢量kuke1+e iθu中选择加号，会影响到该矢量在e1上的系数是kuk+a1，并且不会发生取消，这对于数值稳定性更可取（我们需要除以该矢量的平方范数）。

The last part of Proposition 27.1 shows that the Cartan–Dieudonn´e is salvaged, since we can send u to v by a sequence of two Hermitian reflections when u =6 v and kuk = kvk, and since the inverse of a Hermitian reflection is a Hermitian reflection. Actually, because we are over the complex field, a linear map always have (complex) eigenvalues, and we can get a slightly improved result.  
第27.1号提案的最后一部分表明，卡坦-迪乌登的E被挽救了，因为当u=6 v和kuk=kvk时，我们可以通过两个厄米特反射序列将u发送到v，并且由于厄米特反射的倒数是厄米特反射。实际上，由于我们在复场上，线性映射总是有（复）特征值，我们可以得到一个稍微改进的结果。

Theorem 27.2. Let E be a Hermitian space of dimension n ≥ 1. Every isometry f ∈ U(E) is the composition f = ρn ◦ ρn−1 ◦ ··· ◦ ρ1 of n isometries ρj, where each ρj is either the identity or a Hermitian reflection (possibly a standard hyperplane reflection). When n ≥ 2, the identity is the composition of any hyperplane reflection with itself.  
定理27.2。设e为尺寸n≥1的厄米空间。每一个等距f∈u（e）是n个等距ρj的组成f=ρnρn−1····ρ1，其中每个ρj是同一性或厄米特反射（可能是标准超平面反射）。当n≥2时，同一性是任何超平面反射本身的组成。

Proof. We prove by induction on n that there is an orthonormal basis of eigenvectors (u1,...,un) of f such that f(uj) = eiθjuj,  
证据。我们通过n的归纳证明了f的特征向量（u1，…，un）有一个正交基，这样f（uj）=eiθjuj，

where eiθj is an eigenvalue associated with uj, for all j, 1 ≤ j ≤ n.  
式中，eiθj是与uj相关的特征值，对于所有j，1≤j≤n。

When n = 1, every isometry f ∈ U(E) is either the identity or a Hermitian reflection ρθ, since for any nonnull vector u, we have f(u) = eiθu for some θ. We let u1 be any nonnull unit vector.  
当n=1时，每一个等距f∈u（e）要么是恒等式，要么是厄米反射ρθ，因为对于任何非零向量u，对于某些θ，我们有f（u）=eiθu。我们让u1是任何非空的单位向量。

Let us now consider the case where n ≥ 2. Since C is algebraically closed, the characteristic polynomial det(f − λid) of f has n complex roots which must be the form1 eiθ, since they have absolute value 1. Pick any such eigenvalue1 eiθ , and pick any eigenvector u1 = 06 of f for eiθ of unit length. If F = Cu1 is the subspace spanned by u1, we have f(F) = F, since f(u1) = eiθ1u1. Since f(F) = F and f is an isometry, it is easy to see that f(F ⊥) ⊆ F ⊥, and by Proposition 13.13, we have E = F ⊕ F ⊥. Furthermore, it is obvious that the restriction of f to F ⊥ is unitary. Since dim(F ⊥) = n − 1, we can apply the induction hypothesis to F ⊥, and we get an orthonormal basis of eigenvectors (u2,...,un) for F ⊥ such that  
现在让我们考虑n≥2的情况。因为c是代数闭的，所以f的特征多项式det（f-λid）有n个复数根，因为它们的绝对值为1，所以它们必须是形式1 eiθ。选取任意一个这样的特征值1 eiθ，对于单位长度的eiθ，选取任意一个f的特征向量u1=06。如果f=cu1是u1所跨越的子空间，我们得到f（f）=f，因为f（u1）=eiθ1u1。因为f（f）=f和f是一个等距，很容易看出f（f）f，根据命题13.13，我们得到e=f f。此外，F对F的限制是单一的。由于dim（f）=n-1，我们可以将诱导假设应用于f，我们得到f的特征向量（u2，…，un）的正态基，这样

f(uj) = eiθjuj,  
f（uj）=eiθjuj，

where eiθj is an eigenvalue associated with uj, for all j, 2 ≤ j ≤ n Since E = F ⊕ F ⊥ and F = Cu1, the claim is proved. But then, if ρj is the Hermitian reflection about the hyperplane Hj orthogonal to uj and of angle θj, it is obvious that  
式中，eiθj是与uj相关的特征值，对于所有j，2≤j≤n，因为e=f\_f和f=cu1，证明了该权利要求。但是，如果ρj是与uj正交的超平面hj和θj角的厄米反射，那么很明显

f = ρθn ◦ ··· ◦ ρθ1.  
f=ρθn···ρθ1.

When n ≥ 2, we have id = s ◦ s for every reflection s.   
当n≥2时，每个反射s的id=s\_s。

Remarks:  
评论：

1. Any isometry f ∈ U(n) can be express as f = ρθ◦g, where g ∈ SU(n) is a rotation, and ρθ is a Hermitian reflection. Indeed, by the above theorem, with respect to the basis (u1,...,un), det(f) = ei(θ1+···+θn), and letting θ = θ1 +···+θn and ρθ be the Hermitian reflection about the hyperplane orthogonal to u1 and of angle θ, since ρθ ◦ ρ−θ = id, we have  
   任何等距f∈u（n）都可以表示为f=ρθ\_g，其中g∈su（n）是一个旋转，而ρθ是一个厄米反射。实际上，根据上述定理，关于基（u1，…，un），Det（f）=ei（θ1+·····+θn），并让θ=θ1+······+θn和ρθ是关于与u1正交的超平面和角θ的厄米反射，因为ρθρ−θ=id，我们有

f = (ρθ ◦ ρ−θ) ◦ f = ρθ ◦ (ρ−θ ◦ f).  
f=（ρθρ−θ）f=ρθ（ρ−θf）。

LettingbetweengS=1 ×ρ−SUθ ◦f(, it is obvious that det(n) and U(n), where S1g) = 1is the unit circle (which corresponds to the. As a consequence, there is a bijection group of complex numbers eiθ of unit length). In fact, it is a homeomorphism.  
LettingBetweengs=1×ρ−suθf（，很明显，det（n）和u（n），其中s1g）=1是单位圆（对应于。因此，有一个复数的双射群，单位长度为eiθ）。事实上，它是同态的。

1. We abandoned the style of proof used in theorem 26.1, because in the Hermitian case, eigenvalues and eigenvectors always exist, and the proof is simpler that way (in the real case, an isometry may not have any real eigenvalues!). The sacrifice is that the theorem yields no information on the number of (standard) hyperplane reflections. We shall rectify this situation shortly.  
   我们放弃了定理26.1中使用的证明方式，因为在厄米提亚情况下，特征值和特征向量总是存在的，而且这种证明方式更简单（在实际情况下，等距测量可能没有任何实际特征值！）.牺牲是，该定理不产生有关（标准）超平面反射数的信息。我们将很快纠正这种情况。

We will now reveal the beautiful trick (found in Mneimn´e and Testard [124]) that allows us to prove that every rotation in SU(n) is the composition of at most 2n − 2 (standard) hyperplane reflections. For what follows, it is more convenient to denote a standard reflection about the hyperplane H as hu (it is trivial that these do not depend on the choice of u in H⊥). Then, given any two distinct orthogonal vectors u,v such that kuk = kvk, consider the composition ρv,−θ ◦ρu,θ. The trick is that this composition can be expressed as two standard hyperplane reflections! This wonderful fact is proved in the next Proposition.  
我们现在将展示一个漂亮的技巧（在mneimn'e和testard[124]中发现），它允许我们证明su（n）中的每个旋转都是至多2n-2（标准）超平面反射的组成。对于以下内容，将超平面h的标准反射表示为h u更为方便（这些不依赖于h中u的选择，这是微不足道的）。然后，给任意两个不同的正交向量u，v，使kuk=kvk，考虑组成ρv，−θρu，θ。诀窍是这种合成可以表示为两个标准超平面反射！这个奇妙的事实在下一个命题中得到了证明。

Proposition 27.3. Let E be a nontrivial Hermitian space. For any two distinct orthogonal vectors u,v such that kuk = kvk, we have  
提案27.3.让我们成为一个非平凡的隐士空间。对于任意两个不同的正交向量u，v，如kuk=kvk，我们有

ρv,−θ ◦ ρu,θ = hv−u ◦ hv−e−iθu = hu+v ◦ hu+eiθv.  
ρv，−θρu，θ=hv−u hv−e−iθu=hu+v hu+eiθv.

Proof. Since u and v are orthogonal, each one is in the hyperplane orthogonal to the other, and thus,  
证据。因为u和v是正交的，所以每个都在与另一个正交的超平面中，因此，

ρu,θ(u) = eiθu, ρu,θ(v) = v,  
ρu，θ（u）=eiθu，ρu，θ（v）=v，

ρv,−θ(u) = u, ρv,−θ(v) = e−iθv, hv−u(u) = v, hv−u(v) = u,  
ρv，−θ（u）=u，ρv，−θ（v）=e−iθv，hv−u（u）=v，hv−u（v）=u，

hv−e−iθu(u) = eiθv,  
hv−e−iθu（u）=eiθv，

hv−e−iθu(v) = e−iθu.  
hv−e−iθu（v）=e−iθu。

Consequently, using linearity,  
因此，使用线性，

ρv,−θ ◦ ρu,θ(u) = eiθu, ρv,−θ ◦ ρu,θ(v) = e−iθv,  
ρv，−θρu，θ（u）=e iθu，ρv，−θρu，θ（v）=e-iθv，

hv−u ◦ hv−e−iθu(u) = eiθu, hv−u ◦ hv−e−iθu(v) = e−iθv, and since both ρv,−θ ◦ρu,θ and hv−u ◦hv−e−iθu are the identity on the orthogonal complement of {u,v}, they are equal. Since we also have  
hv−u hv−e−iθu（u）=eiθu，hv−u hv−e−iθu（v）=e−iθv，并且由于ρv、−θρu、θ和hv−u hv−e−iθu是u、v的正交补码上的恒等式，因此它们是相等的。因为我们也有

hu+v(u) = −v, hu+v(v) = −u,  
hu+v（u）=-v，hu+v（v）=-u，

hu+eiθv(u) = −eiθv,  
hu+eiθv（u）=−eiθv，

hu+eiθv(v) = −e−iθu,  
hu+e iθv（v）=−e−iθu，

it is immediately verified that  
立即证实

hv−u ◦ hv−e−iθu = hu+v ◦ hu+eiθv.  
hv−u hv−e−iθu=hu+v hu+eiθv.

We will use Proposition 27.3 as follows.  
我们将使用27.3号提案，如下所示。

Proposition 27.4. Let E be a nontrivial Hermitian space, and let (u1,...,un) be some orthonormal basis for E. For any θ1,...,θn such that θ1 + ··· + θn = 0, if f ∈ U(n) is the isometry defined such that  
提案27.4.设e为非平凡厄米空间，设（u1，…，u n）为e的一些正交基，对于任何θ1，…，θn，使θ1+·····+θn=0，如果f∈u（n）是等距线，定义如下：

f(uj) = eiθjuj,  
f（uj）=eiθjuj，

for all j, 1 ≤ j ≤ n, then f is a rotation (f ∈ SU(n)), and  
对于所有j，1≤j≤n，那么f是一个旋转（f∈su（n）），并且

f = ρun,θn ◦ ··· ◦ ρu1,θ1  
f=ρun，θn····ρu1，θ1

= ρun,−(θ1+···+θn−1) ◦ ρun−1,θ1+···+θn−1 ◦ ··· ◦ ρu2,−θ1 ◦ ρu1,θ1  
=ρun，−（θ1+···+θn−1）ρun−1，θ1+····+θn−1··································

= hun−un−1 ◦ hun−e−i(θ1+···+θn−1)un−1 ◦ ··· ◦ hu2−u1 ◦ hu2−e−iθ1u1 = hun−1+un ◦ hun−1+ei(θ1+···+θn−1)un ◦ ··· ◦ hu1+u2 ◦ hu1+eiθ1u2.  
=hun−un−1 hun−e−i（θ1+·····+θn−1）un−1·································································

Proof. It is obvious from the definitions that  
证据。从定义上看，很明显

f = ρun,θn ◦ ··· ◦ ρu1,θ1,  
f=ρun，θn····ρu1，θ1，

and since the determinant of f is  
既然f的行列式是

D(f) = eiθ1 ···eiθn = ei(θ1+···+θn)  
d（f）=eiθ1···eiθn=ei（θ1+···+θn）

and θ1 + ··· + θn = 0, we have D(f) = e0 = 1, and f is a rotation. Letting  
θ1+·····+θn=0，我们得到d（f）=e0=1，f是一个旋转。出租

fk = ρuk,−(θ1+···+θk−1) ◦ ρuk−1,θ1+···+θk−1 ◦ ··· ◦ ρu3,−(θ1+θ2) ◦ ρu2,θ1+θ2 ◦ ρu2,−θ1 ◦ ρu1,θ1,  
fk=ρuk，−（θ1+···+θk−1）ρuk−1，θ1+····+θk−1····················································

|  |  |
| --- | --- |
| we prove by induction on k, 2 ≤ k ≤ n, that 通过对k，2≤k≤n的诱导，我们证明 |  |
|  iθjuj e iθjuj e   γ  fk(uj) = e−i(θ1+···+θk−1)uk fk（uj）=e−i（θ1+····+θk−1）英国  uj UJ | if 1 ≤ j ≤ k − 1, if j = k, and if k + 1 ≤ j ≤ n. 如果1≤j≤k−1，如果j=k，如果k+1≤j≤n。 |

The base case was treated in Proposition 27.3. Now, the proof of Proposition 27.3 also showed that  
基本情况在提案27.3中进行了处理。现在，27.3号提案的证明也表明

ρuk+1,−(θ1+···+θk) ◦ ρuk,θ1+···+θk(uk) = ei(θ1+···+θk)uk, ρuk+1,−(θ1+···+θk) ◦ ρuk,θ1+···+θk(uk+1) = e−i(θ1+···+θk)uk+1,  
ρuk+1，−（θ1+····+θk）ρuk，θ1+····+θk（uk）=e i（θ1+·····+θk）uk，ρuk+1，−（θ1+·····+θk）ρuk，θ1+·······+θk（uk+1）=e−i（θ1+····+θk）uk+1，

and thus, using the induction hypothesis for k (2 ≤ k ≤ n − 1), we have  
因此，利用k（2≤k≤n-1）的诱导假设，我们得出

fk+1(uj) = ρuk+1,−(θ1+···+θk) ◦ ρuk,θ1+···+θk ◦ fk(uj) = eiθjuj, 1 ≤ j ≤ k − 1, fk+1(uk) = ρuk+1,−(θ1+···+θk) ◦ ρuk,θ1+···+θk ◦ fk(uk) = ei(θ1+···+θk)e−i(θ1+···+θk−1)uk = eiθkuk, fk+1(uk+1) = ρuk+1,−(θ1+···+θk) ◦ ρuk,θ1+···+θk ◦ fk(uk+1) = e−i(θ1+···+θk)uk+1, fk+1(uj) = ρuk+1,−(θ1+···+θk) ◦ ρuk,θ1+···+θk ◦ fk(uj) = uj, k + 1 ≤ j ≤ n, which proves the induction step.  
fk+1(uj) = ρuk+1,−(θ1+···+θk) ◦ ρuk,θ1+···+θk ◦ fk(uj) = eiθjuj, 1 ≤ j ≤ k − 1, fk+1(uk) = ρuk+1,−(θ1+···+θk) ◦ ρuk,θ1+···+θk ◦ fk(uk) = ei(θ1+···+θk)e−i(θ1+···+θk−1)uk = eiθkuk, fk+1(uk+1) = ρuk+1,−(θ1+···+θk) ◦ ρuk,θ1+···+θk fk（uk+1）=e−i（θ1+····+θk）uk+1，fk+1（uj）=ρuk+1，−（θ1+······+θk）ρuk，θ1+······+θk fk（uj）=uj，k+1≤j≤n，证明了诱导步骤。

As a summary, we proved that  
总之，我们证明了

(eiθjuj if 1 ≤ j ≤ n − 1,  
（eiθjuj，如果1≤j≤n-1，

fn(uj) = e−i(θ1+···+θn−1)un when j = n,  
当j=n时，fn（uj）=e−i（θ1+····+θn−1）un，

but since θ1 +···+θn = 0, we have θn = −(θ1 +···+θn−1), and the last expression is in fact  
但由于θ1+····+θn=0，我们得到了θn=−（θ1+····+θn-1），最后一个表达式实际上是

fn(un) = eiθnun. Therefore, we proved that  
fn（un）=eiθnun。因此，我们证明

f = ρun,θn ◦ ··· ◦ ρu1,θ1 = ρun,−(θ1+···+θn−1) ◦ ρun−1,θ1+···+θn−1 ◦ ··· ◦ ρu2,−θ1 ◦ ρu1,θ1,  
f=ρun，θn··········································································

and using Proposition 27.3, we also have  
利用27.3号提案，我们也有

f = ρun,−(θ1+···+θn−1) ◦ ρun−1,θ1+···+θn−1 ◦ ··· ◦ ρu2,−θ1 ◦ ρu1,θ1  
F=ρun，−（θ1+···+θn−1）ρun−1，θ1+·····+θn−1·································

= hun−un−1 ◦ hun−e−i(θ1+···+θn−1)un−1 ◦ ··· ◦ hu2−u1 ◦ hu2−e−iθ1u1  
=hun−un−1 hun−e−i（θ1+······+θn−1）un−1····hu2 hu2−e−iθ1U1

= hun−1+un ◦ hun−1+ei(θ1+···+θn−1)un ◦ ··· ◦ hu1+u2 ◦ hu1+eiθ1u2,  
=hun−1+un hun−1+ei（θ1+····+θn−1）un····hu1+u2 hu1+eiθ1u2，

which completes the proof.   
这就完成了证明。

We finally get our improved version of the Cartan–Dieudonn´e theorem.  
我们最后得到了改进版的卡坦-迪乌顿定理。

Theorem 27.5. Let E be a Hermitian space of dimension n ≥ 1. Every rotation f ∈ SU(E) different from the identity is the composition of at most 2n−2 standard hyperplane reflections. Every isometry f ∈ U(E) different from the identity is the composition of at most 2n − 1 isometries, all standard hyperplane reflections, except for possibly one Hermitian reflection.  
定理27.5。设e为尺寸n≥1的厄米空间。每一个不同于同一性的旋转f∈su（e）是2n-2标准超平面反射的组成。每一个与同一性不同的等距f∈u（e），最多是2n-1等距的组成，所有标准超平面反射，除了可能的一个厄米特反射。

When n ≥ 2, the identity is the composition of any reflection with itself.  
当n≥2时，同一性是任何反射本身的组成。

Proof. By Theorem 27.2, f ∈ SU(n) can be written as a composition  
证据。根据定理27.2，f∈su（n）可以写成一个组合

ρun,θn ◦ ··· ◦ ρu1,θ1,  
ρun，θn····ρu1，θ1，

where (u1,...,un) is an orthonormal basis of eigenvectors. Since f is a rotation, det(f) = 1, and this implies that θ1 + ··· + θn = 0. By Proposition 27.4,  
其中（u1，…，un）是特征向量的正交基。因为f是一个旋转，Det（f）=1，这意味着θ1+····+θn=0。根据27.4号提案，

f = hun−un−1 ◦ hun−e−i(θ1+···+θn−1)un−1 ◦ ··· ◦ hu2−u1 ◦ hu2−e−iθ1u1,  
F=hun−un−1 hun−e−i（θ1+······+θn−1）un−1·····························

a composition of 2n − 2 hyperplane reflections. In general, if f ∈ U(n), by the remark after Theorem 27.2, f can be written as f = ρθ ◦ g, where g ∈ SU(n) is a rotation, and ρθ is a Hermitian reflection. We conclude by applying what we just proved to g.   
2n-2超平面反射的组成。一般来说，如果f∈u（n），通过定理27.2后的注释，f可以写成f=ρθg，其中g∈su（n）是一个旋转，而ρθ是一个厄米反射。最后，我们将刚才证明的应用于G。

As a corollary of Theorem 27.5, the following interesting result can be shown (this is not hard, do it!). First, recall that a linear map f : E → E is self-adjoint (or Hermitian) iff f = f∗. Then, the subgroup of U(n) generated by the Hermitian isometries is equal to the group  
作为定理27.5的推论，可以得出以下有趣的结果（这并不难，做吧！）首先，回想一下线性映射f:e→e是自伴（或厄米提安）iff=f。然后，由厄米提亚等距线生成的u（n）子群等于该群

SU(n)± = {f ∈ U(n) | det(f) = ±1}.  
su（n）±=f∈u（n）det（f）=±1。

Equivalently, SU(n)± is equal to the subgroup of U(n) generated by the hyperplane reflections.  
同样地，su（n）±等于由超平面反射产生的u（n）的子群。

This problem had been left open by Dieudonn´e in [50]. Evidently, it was settled since the publication of the third edition of the book [50].  
Dieudonn'e在[50]年公开了这个问题。显然，这本书自第三版出版以来就已得到解决。

Inspection of the proof of Proposition 26.4 reveals that this Proposition also holds for Hermitian spaces. Thus, when n ≥ 3, the composition of any two hyperplane reflections is equal to the composition of two flips. As a consequence, a version of Theorem 26.5 holds for rotations in a Hermitian space of dimension at least 3.  
对26.4号命题的证明的检验表明，这个命题也适用于赫敏空间。因此，当n≥3时，任意两个超平面反射的合成等于两个翻转的合成。因此，定理26.5的一个版本适用于至少3维的厄米空间中的旋转。

Theorem 27.6. Let E be a Hermitan space of dimension n ≥ 3. Every rotation f ∈ SU(E) is the composition of an even number of flips f = f2k ◦···◦f1, where k ≤ n−1. Furthermore, if u = 06 is invariant under f (i.e. u ∈ Ker(f − id)), we can pick the last flip f2k such that , where F2k is the subspace of dimension n − 2 determining f2k.  
定理27.6。设e为尺寸n≥3的埃尔米坦空间。每个旋转f∈su（e）是偶数个翻转f=f2k·····f1的组合，其中k≤n−1。此外，如果u=06在f（即u∈ker（f−id））下是不变的，我们可以选取最后一个翻转f2k，其中f2k是确定f2k的n−2维的子空间。

Proof. It is identical to that of Theorem 26.5, except that it uses Theorem 27.5 instead of Theorem 26.1. The second part of the Proposition also holds, because if u = 06 is an eigenvector of f for 1, then u is one of the vectors in the orthonormal basis of eigenvectors used in 27.2. The details are left as an exercise.   
证据。它与定理26.5相同，只是它使用了定理27.5而不是定理26.1。命题的第二部分也成立，因为如果u=06是f的1的特征向量，那么u是27.2中使用的特征向量的正交基中的向量之一。细节留作练习。

We now show that the QR-decomposition in terms of (complex) Householder matrices holds for complex matrices. We need the version of Proposition 27.1 and a trick at the end of the argument, but the proof is basically unchanged.  
我们现在证明，用（复杂）户主矩阵进行的QR分解适用于复杂矩阵。我们需要27.1号提案的版本和在辩论结束时的技巧，但是证据基本上是不变的。

Proposition 27.7. Let E be a nontrivial Hermitian space of dimension n. Given any orthonormal basis (e1,...,en), for any n-tuple of vectors (v1,...,vn), there is a sequence of n − 1 isometries h1,...,hn−1, such that hi is a hyperplane reflection or the identity, and if (r1,...,rn) are the vectors given by  
提案27.7.设e为维数n的非平凡厄米空间。给定任意正交基（e1，…，en），对于向量（v1，…，vn）的任意n元组，有一个n-1等距线h1，…，hn-1的序列，使得hi是超平面反射或恒等式，如果（r1，…，rn）是

rj = hn−1 ◦ ··· ◦ h2 ◦ h1(vj) 1 ≤ j ≤ n,  
rj=hn−1·····h2 h1（vj）1≤j≤n，

then every rj is a linear combination of the vectors (e1,...,ej), (1 ≤ j ≤ n). Equivalently, the matrix R whose columns are the components of the rj over the basis (e1,...,en) is an upper triangular matrix. Furthermore, if we allow one more isometry hn of the form  
然后，每个RJ是向量（e1，…，ej），（1≤j≤n）的线性组合。等价地，其列是基上RJ的分量的矩阵r（e1，…，en）是上三角矩阵。此外，如果我们允许形式的另一个等值线hn

hn = ρen,ϕn ◦ ··· ◦ ρe1,ϕ1  
hn=ρen，n···ρe1，\_

after h1,...,hn−1, we can ensure that the diagonal entries of R are nonnegative.  
在h1，…，hn−1之后，我们可以确保r的对角线项是非负的。

Proof. The proof is very similar to the proof of Proposition 12.3, but it needs to be modified a little bit since Proposition 27.1 is weaker than Proposition 12.2. We explain how to modify the induction step, leaving the base case and the rest of the proof as an exercise.  
证据。证明与12.3号提案的证明非常相似，但由于27.1号提案弱于12.2号提案，因此需要对其稍作修改。我们将解释如何修改归纳步骤，将基本情况和其他证明作为练习。

As in the proof of Proposition 12.3, the vectors (e1,...,ek) form a basis for the subspace denoted as Uk0, the vectors (ek+1,...,en) form a basis for the subspace denoted as Uk00, the subspaces and are orthogonal, and . Let  
在命题12.3的证明中，向量（e1，…，ek）构成表示为UK0的子空间的基础，向量（ek+1，…，en）构成表示为UK00的子空间的基础，子空间和是正交的，以及。让

uk+1 = hk ◦ ··· ◦ h2 ◦ h1(vk+1).  
UK+1=hk····h2 h1（vk+1）。

We can write  
我们可以写信

,  
，

where u0k+1 ∈ Uk0 and. Let  
其中U0K+1∈UK0和。让

, and .  
，和。

If, we let hk+1 = id. Otherwise, by Proposition 27.1, there is a  
如果，我们让hk+1=id。否则，根据27.1号提案，有一个

unique hyperplane reflection hk+1 such that  
独特的超平面反射hk+1

hk+1(u00k+1) = eiθk+1rk+1,k+1 ek+1,  
hk+1（u00k+1）=eiθk+1rk+1，k+1 ek+1，

where hk+1 is the reflection about the hyperplane Hk+1 orthogonal to the vector  
其中，hk+1是垂直于向量的超平面hk+1的反射。

.  
.

At the end of the induction, we have a triangular matrix R, but the diagonal entries eiθjrj,j of R may be complex. Letting  
在归纳的最后，我们得到了一个三角形的矩阵r，但是r的对角线项eiθj r j，j可能是复杂的。出租

hn+1 = ρen,−θn ◦ ··· ◦ ρe1,−θ1,  
hn+1=ρen，−θn····ρe1，−θ1，

we observe that the diagonal entries of the matrix of vectors  
我们观察到向量矩阵的对角项



is triangular with nonnegative entries.   
是带有非负项的三角形。

Remark: For numerical stability, it is preferable to use wk+1 = rk+1,k+1 ek+1 + e−iθk+1u00k+1 instead of wk+1 = rk+1,k+1 ek+1 − e−jiθk+1u00k+1. The effect of that choice is that the diagonalj entries in R will be of the form −eiθ rj,j = ei(θ +π)rj,j. Of course, we can make these entries nonegative by applying  
备注：对于数值稳定性，最好使用wk+1=rk+1，k+1ek+1+e−iθk+1U00k+1，而不是wk+1=rk+1，k+1ek+1−e−jiθk+1U00k+1。这种选择的效果是，r中的对角线j项的形式为−eiθrj，j=ei（θ+π）rj，j。当然，我们可以通过应用

hn+1 = ρen,π−θn ◦ ··· ◦ ρe1,π−θ1  
hn+1=ρen，π−θn···ρe1，π−θ1

after hn.  
在hn之后。

As in the Euclidean case, Proposition 27.7 immediately implies the QR-decomposition for arbitrary complex n×n-matrices, where Q is now unitary (see Kincaid and Cheney [100], Golub and Van Loan [80], Trefethen and Bau [171], or Ciarlet [41]).  
在欧几里得案例中，命题27.7立即暗示了任意复杂n×n矩阵的qr分解，其中q现在是一元的（见Kincaid和Cheney[100]、Golub和van Loan[80]、Trefethen和Bau[171]或Ciarlet[41]）。

Proposition 27.8. For every complex n × n-matrix A, there is a sequence H1,...,Hn−1 of matrices, where each Hi is either a Householder matrix or the identity, and an upper triangular matrix R, such that  
提案27.8.对于每个复杂的n×n矩阵a，有一个矩阵的序列h1，…，hn−1，其中每个hi都是一个户主矩阵或恒等式，以及一个上三角矩阵r，这样

R = Hn−1 ···H2H1A.  
R=hn−1···h2h1a。

As a corollary, there is a pair of matrices Q,R, where Q is unitary and R is upper triangular, such that A = QR (a QR-decomposition of A). Furthermore, R can be chosen so that its diagonal entries are nonnegative. This can be achieved by a diagonal matrix D with entries such that |dii| = 1 for i = 1,...,n, and we have A = QeRe with  
作为推论，有一对矩阵q，r，其中q是一元的，r是上三角的，因此a=qr（a的qr分解）。此外，可以选择r，使其对角线项为非负。这可以通过一个对角矩阵d来实现，其中的条目为d i i=1代表i=1，…，n，我们有a=qere

Qe = H1 ···Hn−1D, Re = D∗R,  
qe=h1···hn−1d，re=d r，

where Re is upper triangular and has nonnegative diagonal entries  
其中，re是上三角形，具有非负对角线条目

Proof. It is essentially identical to the proof of Proposition 12.4, and we leave the details as an exercise. For the last statement, observe that hn ◦ ··· ◦ h1 is also an isometry.   
证据。这与12.4号提案的证明基本相同，我们将细节留作练习。对于最后一个陈述，观察hn·····h1也是一个等距测量。

As in the Euclidean case, the QR-decomposition has applications to least squares problems. It is also possible to convert any complex matrix to bidiagonal form.  
在欧几里得的例子中，QR分解应用于最小二乘问题。也可以将任何复杂的矩阵转换为双对角形式。

## 27.2 Affine Isometries (Rigid Motions) 27.2仿射等距图（刚性运动）

In this section, we study very briefly the affine isometries of a Hermitian space. Most results holding for Euclidean affine spaces generalize without any problems to Hermitian spaces.  
在这一部分中，我们非常简单地研究了厄米空间的仿射等距线。欧几里得仿射空间的大多数结果都推广到埃尔米特空间，没有任何问题。

The characterization of the set of fixed points of an affine map is unchanged. Similarly, every affine isometry f (of a Hermitian space) can be written uniquely as  
仿射映射不动点集的特征不变。类似地，每一个仿射等距f（厄米空间的）都可以唯一地写为

f = t ◦ g, with t ◦ g = g ◦ t,  
f=t\_g，其中t\_g=g\_t，

where g is an isometry having a fixed point, and t is a translation by a vector τ such that →−f (τ) = τ, and with some additional nice properties (see Proposition 27.13). A generalization  
其中，g是具有固定点的等距测量，t是矢量τ的平移，使得→−f（τ）=τ，并具有一些额外的优良性质（见命题27.13）。泛化

27.2. AFFINE ISOMETRIES (RIGID MOTIONS)  
27.2。仿射等距线（刚性运动）

of the Cartan–Dieudonn´e theorem can easily be shown: every affine isometry in Is(n,C) can be written as the composition of at most 2n − 1 isometries if it has a fixed point, or else as the composition of at most 2n+1 isometries, where all these isometries are affine hyperplane reflections except for possibly one affine Hermitian reflection. We also prove that every rigid motion in SE(n,C) is the composition of at most 2n − 2 flips (for n ≥ 3).  
关于卡坦-迪乌顿定理，可以很容易地证明：IS（n，c）中的每一个仿射等值线，如果有一个固定点，可以写成至多2n-1等距线的组合，或者写成至多2n+1等距线的组合，其中所有这些等距线都是仿射超平面反射，不包括可能有一个仿射厄米特反射。我们还证明了SE（n，c）中的每一个刚性运动都是至多2n−2翻转（n≥3）的组成。

Definition 27.2. Given any two nontrivial Hermitian affine spaces E and F of the same finite dimension n, a function f : E → F is an affine isometry (or rigid map) iff it is an affine map and  
定义27.2.给定任意两个具有相同有限维n的非平凡厄米仿射空间e和f，函数f:e→f是仿射等值线（或刚性映射），如果它是仿射映射，并且

,  
，

for all a,b ∈ E. When E = F, an affine isometry f : E → E is also called a rigid motion.  
对于所有a，b∈e，当e=f时，仿射等距f:e→e也被称为刚性运动。

Thus, an affine isometry is an affine map that preserves the distance. This is a rather strong requirement, but unlike the Euclidean case, not strong enough to force f to be an affine map.  
因此，仿射等值线是保持距离的仿射图。这是一个相当强烈的要求，但与欧几里得的情况不同，它的强度不足以迫使f成为仿射映射。

The following simple Proposition is left as an exercise.  
下面的简单命题留作练习。

Proposition 27.9. Given any two nontrivial Hermitian affine spaces E and F of the same finite dimension , an affine map f : E → F is an affine isometry iff its associated linear map f : E → F is an isometry. An affine isometry is a bijection.  
提案27.9.给定任意两个具有相同有限维的非平凡厄米仿射空间e和f，仿射映射f:e→f是仿射等值线，而其相关线性映射f:e→f是等值线。仿射等值线是双射。

As in the Euclidean case, given an affine isometry→− f : E → E, if →−f is a rotation, we call f a proper (or direct) affine isometry, and if f is a an improper linear isometry, we call f a an improper (or skew) affine isometry. It is easily shown that the set of affine isometries f : E → E forms a group, and those for which →−f is a rotation is a subgroup. The group of affine isometries, or rigid motions, is a subgroup of the affine group GA(E,C) denoted as Is(E,C) (or Is(n,C) when E = Cn). The subgroup of Is(E,C) consisting of the direct rigid motions is also a subgroup of SA(E,C), and it is denoted as SE(E,C) (or SE(n,C), when E = Cn). The translations are the affine isometries f for which →−f = id, the identity map on →−E. The following Proposition is the counterpart of Proposition 13.14 for isometries between Hermitian vector spaces.  
在欧几里得例子中，给定一个仿射等值线→−f:e→e，如果→−f是一个旋转，我们称之为适当（或直接）仿射等值线，如果f是一个不适当的线性等值线，我们称之为不适当（或歪斜）仿射等值线。可以很容易地看出，仿射等距图f:e→e构成一个群，其中→−f是一个旋转的群是一个子群。仿射等轴测群或刚性运动群是仿射群ga（e，c）的一个子群，表示为is（e，c）（或e=c n时为（n，c）。由直接刚性运动组成的IS（E，C）子群也是SA（E，C）子群，当E=CN时，它表示为SE（E，C）（或SE（N，C）。翻译是仿射等距f，其中→−f=id，在→−e上的标识映射。下面的命题是13.14号命题的对应命题，用于Hermitian向量空间之间的等距。

Proposition 27.10. Given any two nontrivial Hermitian affine spaces E and F of the same finite dimension n, for every function f : E → F, the following properties are equivalent:  
提案27.10。给定任意两个具有相同有限维n的非平凡厄米仿射空间e和f，对于每一个函数f:e→f，下列性质是等价的：

(1) f is an affine map and , for all a,b ∈ E.  
（1）f是仿射映射，对于所有a，b∈e。

, and there is some Ω ∈ E such that  
，还有一些Ω∈e，这样

,  
，

for all a,b ∈ E.  
对于所有a，b∈e。

Proof. Obviously, (1) implies (2). The proof that that (2) implies (1) is similar to the proof of Proposition 26.7, but uses Proposition 13.14 instead of Proposition 11.12. The details are left as an exercise.   
证据。显然，（1）意味着（2）。证明（2）暗示（1）类似于26.7号提案的证明，但使用13.14号提案而不是11.12号提案。细节留作练习。

Inspection of the proof shows immediately that Proposition 26.8 holds for Hermitian spaces. For the sake of completeness, we restate the Proposition in the complex case.  
对证据的检验立即表明26.8号提案适用于赫米特空间。为了完整起见，我们在复杂的情况下重述了这个命题。

Proposition 27.11. Let E be any complex affine space of finite dimension For every affine map f : E → E, let Fix(f) = {a ∈ E | f(a) = a} be the set of fixed points of f. The following properties hold:  
提案27.11.设e为每个仿射映射f:e→e的任意有限维仿射空间，设fix（f）=a∈e f（a）=a为f的不动点集，其性质如下：

1. If f has some fixed point a, so that Fix(f) =6 ∅, then Fix(f) is an affine subspace of  
   如果f有固定点a，那么fix（f）=6∅，那么fix（f）是

E such that  
这样的话

Fix(f) = a + E(1,→−f ) = a + Ker(→−f − id),  
固定（F）=A+E（1，→−F）=A+KER（→−F−ID）

where E(1,→−f ) is the eigenspace of the linear map →−f for the eigenvalue 1.  
其中e（1，→−f）是线性映射的特征空间→−f是特征值1的特征空间。

1. The affine map f has a unique fixed point iff E(1,→−f ) = Ker(→−f − id) = {0}.  
   仿射映射f有一个唯一的固定点iff e（1，→−f）=ker（→−f−id）=0。

Affine orthogonal symmetries are defined just as in the Euclidean case, and Proposition  
仿射正交对称的定义与欧几里得的情形一样，以及命题

26.9 also applies to complex affine spaces.  
26.9也适用于复杂仿射空间。

Proposition 27.12. Given any affine complex space E, if f : E → E and g: E → E are affine orthogonal symmetries about parallel affine subspaces→−→− F1 and F2, then g ◦ f is a translation defined by the vector 2ab, where ab is any vector perpendicular to the common direction →−F of F1 and F2 such that is the distance between F1 and F2, with a ∈ F1 and b ∈ F2. Conversely, every translation by a vector τ is obtained as the composition of two affine orthogonal symmetries about parallel affine subspaces F1 and F2 whose common direction is orthogonal to→− τ = →−ab, for some a ∈ F1 and some b ∈ F2 such that the distance betwen F1 and.  
提案27.12。给定任意仿射复空间e，如果f:e→e和g:e→e是关于平行仿射子空间→→→−f1和f2的仿射正交对称性，则g f是向量2ab定义的平移，其中ab是垂直于f1和f2的公共方向→−f的任意向量，因此t是f1和f2之间的距离，a∈f1和b∈f2。相反，对于一些a∈f1和一些b∈f2，向量τ的每一个平移都是作为平行仿射子空间f1和f2的两个仿射正交对称的组成，其公共方向与→−τ→−ab正交，从而使得f1和f1之间的距离。

It is easy to check that the proof of Proposition 26.10 also holds in the Hermitian case.  
很容易证实26.10号命题的证明也适用于赫米特案。

Proposition 27.13. Let E be a Hermitian affine space of finite dimension n. For every affine isometry f : E →→−E, there is a unique affine isometry∈ →− − g: E → E and a unique{ ∈ translation t = tτ, with f (τ) = τ (i.e., τ Ker( f id)), such that the set Fix(g) = a  
提案27.13。设e为有限维n的厄米特仿射空间，对于每一个仿射同构f:e→→−e，都有一个唯一的仿射同构∈→−−g:e→e和一个唯一的∈翻译t=tτ，其中f（τ）=τ（即τker（f id）），使得集fix（g）=a

E, | g(a) = a} of fixed points of g is a nonempty affine subspace of E of direction  
E，G（A）=A G的不动点是方向E的非空仿射子空间

→−G = Ker(→−f − id) = E(1,→−f ),  
→−G=KER（→−F−ID）=E（1，→−F），

and such that  
这样的话

f = t ◦ g and t ◦ g = g ◦ t.  
f=t\_g和t\_g=g\_t。

Furthermore, we have the following additional properties:  
此外，我们还有以下附加属性：

27.2. AFFINE ISOMETRIES (RIGID MOTIONS)  
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1. f = g and τ = 0 iff f has some fixed point, i.e., iff Fix(f) =6 ∅.  
   f=g，τ=0，iff f有固定点，即iff fix（f）=6∅。
2. If f has no fixed points, i.e., Fix(f) = ∅, then dim(Ker(→−f − id)) ≥ 1.  
   如果f没有固定点，即fix（f）=∅，则dim（ker（→−f−id））≥1。

The remarks made in the Euclidean case also apply to the Hermitian case. In particular, the fact that E has finite dimension is only used to prove (b).  
欧几里得案例中的评论也适用于赫米特案例。特别是，e有有限维这一事实仅用于证明（b）。

A version of the Cartan–Dieudonn´e also holds for affine isometries, but it may not be possible to get rid of Hermitian reflections entirely.  
卡坦-迪乌登的一个版本也适用于仿射等距线，但可能无法完全消除赫米特反射。

Theorem 27.14. Let E be an affine Hermitian space of dimension n ≥ 1. Every affine isometry in Is(n,C) can be written as the composition of at most 2n − 1 affine isometries if it has a fixed point, or else as the composition of at most 2n + 1 affine isometries, where all these isometries are affine hyperplane reflections except for possibly one affine Hermitian reflection. When n ≥ 2, the identity is the composition of any reflection with itself.  
定理27.14。设e为尺寸n≥1的仿射厄米空间。IS（n，c）中的每一个仿射等值线可以写成至多2n-1个仿射等距线的组成，如果它有一个固定点，或者写成至多2n+1个仿射等距线的组成，其中所有这些等距线都是仿射超平面反射，除了可能有一个仿射厄米提亚参考线。选择。当n≥2时，同一性是任何反射本身的组成。

Proof. The proof is very similar to the proof of Theorem 26.11, except that it uses Theorem 27.5 instead of Theorem 26.1. The details are left as an exercise.   
证据。证明与定理26.11的证明非常相似，只是它使用了定理27.5而不是定理26.1。细节留作练习。

When n ≥ 3, as in the Euclidean case, we can characterize the affine isometries in SE(n,C) in terms of flips, and we can even bound the number of flips by 2n − 2.  
当n≥3时，如欧几里得案例中，我们可以用翻转来描述SE（n，c）中的仿射等距图，甚至可以用2n-2绑定翻转的数量。

Theorem 27.15. Let E be a Hermitian affine space of dimension n ≥ 3. Every rigid motion f ∈ SE(E,C) is the composition of an even number of affine flips f = f2k ◦ ··· ◦ f1, where k ≤ n − 1.  
定理27.15。设e为尺寸n≥3的厄米特仿射空间。每一个刚性运动f∈se（e，c）是偶数个仿射翻转f=f2k····f1的组成，其中k≤n−1。

Proof. It is very similar to the proof of theorem 26.12, but it uses Proposition 27.6 instead of Proposition 26.5. The details are left as an exercise.   
证据。它与定理26.12的证明非常相似，但它使用了命题27.6而不是命题26.5。细节留作练习。

A more detailed study of the rigid motions of Hermitian spaces of dimension 2 and 3 would seem worthwhile, but we are not aware of any reference on this subject.  
对2维和3维厄米特空间的刚性运动进行更详细的研究似乎是值得的，但我们不知道这方面的任何参考。

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Chapter 28  
第二十八章

# The Geometry of Bilinear Forms; Witt’s Theorem; The Cartan–Dieudonn´e Theorem 双线性形式的几何；维特定理；卡坦-迪乌登定理

## 28.1 Bilinear Forms 28.1双线性形式

In this chapter, we study the structure of a K-vector space E endowed with a nondegenerate  
在本章中，我们研究了具有非退化性的k向量空间e的结构。

inner product. Unlike the case of an inner product, there may be nonzero vectorsbilinear formthat ϕ(u,u) = 0ϕ: Eso the map×E → Ku(for any field7→ ϕ(u,u) can no longer be interpreted as a notion of squareK), which can be viewed as a kind of generalizedu ∈ E such  
内部产品。与内积的情况不同，可能存在非零向量双线性形式，即，图×e→ku（对于任何场7→（u，u）不再被解释为平方的概念），这可以被看作是一种推广∈e，例如

length (also, ϕ(u,u) may not be real and positive!). However, the notion of orthogonality survives: we say that u,v ∈ E are orthogonal iff ϕ(u,v) = 0. Under some additional conditions on ϕ, it is then possible to split E into orthogonal subspaces having some special properties. It turns out that the special cases where ϕ is symmetric (or Hermitian) or skewsymmetric (or skew-Hermitian) can be handled uniformly using a deep theorem due to Witt (the Witt decomposition theorem (1936)).  
长度（也可以是，\_（u，u）可能不是真的和正的！）然而，正交性的概念仍然存在：我们说u，v∈e是正交的iff\_（u，v）=0。在一些附加条件下，可以将e分解成具有一些特殊性质的正交子空间。结果表明，当\_为对称（或厄米提亚）或偏对称（或斜厄米提亚）时，由于witt（witt分解定理（1936）），可以使用深层定理统一处理。

We begin with the very general situation of a bilinear form ϕ: E×F → K, where K is an arbitrary field, possibly of characteristric 2. Actually, even though at first glance this may appear to be an uncessary abstraction, it turns out that this situation arises in attempting to prove properties of a bilinear map ϕ: E ×E → K, because it may be necessary to restrict ϕ to different subspaces U and V of E. This general approach was pioneered by Chevalley [37], E. Artin [6], and Bourbaki [24]. The third source was a major source of inspiration, and many proofs are taken from it. Other useful references include Snapper and Troyer [157], Berger [12], Jacobson [96], Grove [83], Taylor [169], and Berndt [14].  
我们从双线性形式的一般情况开始，其中k是一个任意场，可能具有特征2。事实上，尽管乍一看这似乎是一个不确定的抽象，但事实证明，这种情况是在试图证明双线性映射的性质时出现的，因为可能有必要将π限制到e的不同子空间u和v。这种一般方法是由Chevalley[37]、E.Artin[6]和Bourbaki[24]开创的。第三个来源是一个主要的灵感来源，许多证据都是从中获得的。其他有用的参考文献包括Snapper和Troyer[157]、Berger[12]、Jacobson[96]、Grove[83]、Taylor[169]和Berndt[14]。

is aDefinition 28.1.bilinear form iff the following conditions hold: For allGiven two vector spaces E and F over a fieldu,u ,uK∈, a mapE, all v,vϕ: E,v×∈FF→, forK  
是定义28.1.双线性形式iff的以下条件成立：对于所有给定的两个向量空间e和f，在一个域u，u，uk∈，a mape，all v，v\_：e，v×∈ff→，fork上

1 2 1 2  
1 2 1 2

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all λ,µ ∈ K, we have  
所有的λ，μ∈k，我们有

ϕ(u1 + u2,v) = ϕ(u1,v) + ϕ(u2,v)  
⑨（u1+u2，v）=⑨（u1，v）+\_（u2，v）

ϕ(u,v1 + v2) = ϕ(u,v1) + ϕ(u,v2) ϕ(λu,v) = λϕ(u,v) ϕ(u,µv) = µϕ(u,v).  
⑨（u，v1+v2）＝（u，v1）＋（u，v2）（λu，v）＝（u，v）（u，μv）＝\_（u，v）。

A bilinear form as in Definition 28.1 is sometimes called a pairing. The first two conditions imply that ϕ(0,v) = ϕ(u,0) = 0 for all u ∈ E and all v ∈ F.  
定义28.1中的双线性形式有时称为配对。前两个条件意味着，所有u∈e和所有v∈f的\_（0，v）=（u，0）=0。

If E = F, observe that  
如果e=f，观察

ϕ(λu + µv,λu + µv) = λϕ(u,λu + µv) + µϕ(v,λu + µv)  
\_（λu+μv，λu+μv）=λ（u，λu+μv）+\_（v，λu+μv）

= λ2ϕ(u,u) + λµϕ(u,v) + λµϕ(v,u) + µ2ϕ(v,v).  
=λ2\_（u，u）+λ\_（u，v）+λ\_（v，u）+\_2（v，v）。

If we let λ = µ = 1, we get  
如果我们让λ=μ=1，我们得到

ϕ(u + v,u + v) = ϕ(u,u) + ϕ(u,v) + ϕ(v,u) + ϕ(v,v).  
⑨（u+v，u+v）=⑨（u，u）+（u，v）+（v，u）+（v，v）。

If ϕ is symmetric, which means that  
如果φ是对称的，这意味着

ϕ(u,v) = ϕ(v,u) for all u,v ∈ E,  
所有u，v∈e，的（u，v）=（v，u）

then  
然后

2ϕ(u,v) = ϕ(u + v,u + v) − ϕ(u,u) − ϕ(v,v). (∗)  
2\_（u，v）=\_（u+v，u+v）-（u，u）-（v，v）。（）

The function Φ defined such that  
函数Φ定义如下：

Φ(u) = ϕ(u,u) u ∈ E,  
Φ（u）＝（u，u）u∈e，

is called the quadratic form associated with ϕ. If the field K is not of characteristic 2, then ϕ is completely determined by its quadratic form Φ. The symmetric bilinear form ϕ is called the polar form of Φ. This suggests the following definition.  
被称为二次型。如果场k不属于特征2，那么，根据其二次型Φ完全确定。对称双线性形式，称为Φ的极性形式。这表明了以下定义。

Definition 28.2. A function Φ: E → K is a quadratic form on E if the following conditions hold:  
定义28.2.如果满足以下条件，函数Φ：e→k是e上的二次型：

1. We have Φ(λu) = λ2Φ(u), for all u ∈ E and all λ ∈ E.  
   我们有Φ（λu）=λ2Φ（u），对于所有的u∈e和所有的λ∈e。
2. The map ϕ0 given by ϕ0(u,v) = Φ(u+v)−Φ(u)−Φ(v) is bilinear. Obviously, the map ϕ0 is symmetric.  
   由\_（u，v）=Φ（u+v）−Φ（u）−Φ（v）给出的图\_为双线性。显然，图\_是对称的。

Since Φ(x + x) = Φ(2x) = 4Φ(x), we have  
因为Φ（x+x）=Φ（2x）=4Φ（x），我们有

ϕ0(u,u) = 2Φ(u) u ∈ E.  
⑨0（u，u）=2Φ（u）u∈e.

If the field K is not of characteristic 2, then is the unique symmetric bilinear form such that that ϕ(u,u) = Φ(u) for all u ∈ E. The bilinear form is called the polar form of Φ. In this case, there is a bijection between the set of bilinear forms on E and the set of quadratic forms on E.  
如果场k不属于特征2，则为唯一对称双线性形式，使得所有u∈e的（u，u）=Φ（u）。双线性形式称为Φ的极性形式。在这种情况下，E上的双线性形式集和E上的二次形式集之间存在双射。

If K is a field of characteristic 2, then ϕ0 is alternating, which means that  
如果k是特征2的场，那么\_是交替的，这意味着

ϕ0(u,u) = 0 for all u ∈ E.  
⑨0（u，u）=0表示所有u∈e。

Thus if K is a field of characteristic 2, then Φ cannot be recovered from the symmetric bilinear form ϕ0.  
因此，如果k是特征2的场，那么Φ就不能从对称双线性形式\_中恢复。

If (e1,...,en) is a basis of E, it is easy to show that  
如果（e1，…，en）是e的基础，很容易证明

.  
.

This shows that the quadratic form Φ is completely determined by the scalars Φ(ei) and ϕ0(ei,ej) (i =6 j). Furthermore, given any bilinear form ψ: E × E → K (not necessarily symmetric) we can define a quadratic form Φ by setting Φ(x) = ψ(x,x), and we immediately check that the symmetric bilinear form ϕ0 associated with Φ is given by ϕ0(u,v) = ψ(u,v)+ ψ(v,u). Using the above facts, it is not hard to prove that given any quadratic form Φ, there is some (nonsymmetric) bilinear form ψ such that Φ(u) = ψ(u,u) for all u ∈ E (see Bourbaki [24], Section §3.4, Proposition 2). Thus, quadratic forms are more general than symmetric bilinear forms (except in characteristic = 2).6  
由此可见，二次型Φ完全由量角器Φ（ei）和磴0（ei，ej）（i=6 j）决定。此外，对于任意双线性形式ψ：e×e→k（不一定是对称的），我们可以通过设置Φ（x）=ψ（x，x）来定义二次型Φ，并立即检查与Φ相关的对称双线性形式\_0是否由痻0（u，v）=ψ（u，v）+ψ（v，u）给出。利用上述事实，不难证明给定任何二次型Φ，存在一些（非对称）双线性型ψ，使得所有u∈e的Φ（u）=ψ（u，u）（见Bourbaki[24]第3.4节，命题2）。因此，二次型比对称双线性型更为普遍（特征值为2的情况除外）。

Definition 28.3. Given any bilinear form ϕ: E ×E → K where K is a field of any characteristic, we say that ϕ is alternating if  
定义28.3.给定双线性形式，其中k是任何特征的场，我们认为，如果

ϕ(u,u) = 0 for all u ∈ E,  
⑨（u，u）=0表示所有u∈e，

and skew-symmetric if  
和斜对称if

ϕ(v,u) = −ϕ(u,v) for all u,v ∈ E.  
⑨（v，u）=−⑨（u，v）表示所有u，v∈e。

If K is a field of any characteristic, the identity  
如果k是一个有任何特征的场，那么

ϕ(u + v,u + v) = ϕ(u,u) + ϕ(u,v) + ϕ(v,u) + ϕ(v,v)  
⑨（U+V，U+V）=⑨（U，U）+（U，V）+（V，U）+（V，V）

shows that if ϕ is alternating, then  
表明如果\_是交替的，那么

ϕ(v,u) = −ϕ(u,v) for all u,v ∈ E,  
所有u，v∈e，的（v，u）=−（u，v）

that is, ϕ is skew-symmetric. Conversely, if the field K is not of characteristic 2, then a skew-symmetric bilinear map is alternating, since ϕ(u,u) = −ϕ(u,u) implies ϕ(u,u) = 0.  
也就是说，η是斜对称的。相反，如果场k不属于特征2，则斜对称双线性映射是交替的，因为\_（u，u）=-（u，u）表示（u，u）=0。

An important consequence of bilinearity is that a pairing yields a linear map from E into F ∗ and a linear map from F into E∗ (where E∗ = HomK(E,K), the dual of E, is the set of linear maps from E to K, called linear forms).  
双线性的一个重要结果是，配对产生一个从E到F的线性映射，和一个从F到E的线性映射（其中e=homk（e，k），e的对偶，是一组从E到K的线性映射，称为线性形式）。

Definition 28.4. Given a bilinear map ϕ: E × F → K, for every u ∈ E, let lϕ(u) be the linear form in F ∗ given by  
定义28.4.给定双线性映射，对于每一个u e，设l\_（u）为f中的线性形式，由

lϕ(u)(y) = ϕ(u,y) for all y ∈ F,  
l\_（u）（y）=（u，y）表示所有y∈f，

and for every v ∈ F, let rϕ(v) be the linear form in E∗ given by  
对于每一个v∈f，设r（v）为e中的线性形式，由

rϕ(v)(x) = ϕ(x,v) for all x ∈ E.  
对于所有x∈e，r\_（v）（x）=（x，v）。

Because ϕ is bilinear, the maps lϕ : E → F ∗ and rϕ : F → E∗ are linear.  
因为\_是双线性的，所以图l\_：e→f和r\_：f→e\_是线性的。

Definition 28.5. A bilinear map ϕ: E×F → K is said to be nondegenerate iff the following conditions hold:  
定义28.5.双线性图\_：e×f→k称为非简并iff，条件如下：

1. For every u ∈ E, if ϕ(u,v) = 0 for all v ∈ F, then u = 0, and  
   对于每一个u∈e，如果所有v∈f的（u，v）=0，则u=0，并且
2. For every v ∈ F, if ϕ(u,v) = 0 for all u ∈ E, then v = 0.  
   对于每一个v∈f，如果所有u∈e的（u，v）=0，则v=0。

The following proposition shows the importance of lϕ and rϕ.  
下面的命题说明了“l”和“r”的重要性。

Proposition 28.1. Given a bilinear map ϕ: E × F → K, the following properties hold:  
提案28.1.给定双线性映射，其中：e×f→k，以下属性保持不变：

1. The map lϕ is injective iff Property (1) of Definition 28.5 holds.  
   图l\_是定义28.5中的注射iff属性（1）。
2. The map rϕ is injective iff Property (2) of Definition 28.5 holds.  
   图r\_是定义28.5的注射iff属性（2）。
3. The bilinear form ϕ is nondegenerate and iff lϕ and rϕ are injective.  
   双线性形式\_是非退化的，如果l\_和r\_是注射剂。
4. If the bilinear form ϕ is nondegenerate and if E and F have finite dimensions, then dim(E) = dim(F), and lϕ : E → F ∗ and rϕ : F → E∗ are linear isomorphisms.  
   如果双线性形式\_是非退化的，并且如果e和f有有限的尺寸，那么dim（e）=dim（f），l\_：e→f和r\_：f→e是线性同构。

Proof. (a) Assume that (1) of Definition 28.5 holds. If lϕ(u) = 0, then lϕ(u) is the linear form whose value is 0 for all y; that is,  
证据。（a）假设（1）定义28.5成立。如果l\_（u）=0，则l（u）是线性形式，其值为0，表示所有y；即，

lϕ(u)(y) = ϕ(u,y) = 0 for all y ∈ F,  
对于所有y∈f，l\_（u）（y）=（u，y）=0，

and by (1) of Definition 28.5, we must have u = 0. Therefore, lϕ is injective. Conversely, if lϕ is injective, and if  
根据定义28.5的（1），我们必须得到u=0。因此，L\_为注射型。相反，如果l\_是注射的，并且如果

lϕ(u)(y) = ϕ(u,y) = 0 for all y ∈ F,  
对于所有y∈f，l\_（u）（y）=（u，y）=0，

then lϕ(u) is the zero form, and by injectivity of lϕ, we get u = 0; that is, (1) of Definition 28.5 holds.  
那么l\_（u）是零形式，通过l\_的注入率，我们得到u=0；也就是说，定义28.5的（1）成立。

1. The proof is obtained by swapping the arguments of ϕ.  
   证据是通过交换\_的论点获得的。
2. This follows from (a) and (b).  
   这源于（a）和（b）。
3. If E and F are finite dimensional, then dim(E) = dim(E∗) and dim(F) = dim(F ∗). Since ϕ is nondegenerate, lϕ : E → F ∗ and rϕ : F → E∗ are injective, so dim(E) ≤ dim(F ∗) = dim(F) and dim(F) ≤ dim(E∗) = dim(E), which implies that  
   如果e和f是有限维，那么dim（e）=dim（e）和dim（f）=dim（f）。由于\_是非退化的，l\_：e→f和r\_：f→e是注射剂，因此dim（e）≤dim（f）=dim（f）和dim（f）≤dim（e）=dim（e），这意味着

dim(E) = dim(F),  
dim（e）=dim（f）

and thus, lϕ : E → F ∗ and rϕ : F → E∗ are bijective.   
因此，L\_：E→F和R\_：F→E\_是双射的。

As a corollary of Proposition 28.1, we have the following characterization of a nondegenerate bilinear map. The proof is left as an exercise.  
作为命题28.1的一个推论，我们有一个非退化双线性映射的以下特征。证据留作练习。

Proposition 28.2. Given a bilinear map ϕ: E × F → K, if E and F have the same finite dimension, then the following properties are equivalent:  
提案28.2.给定双线性映射，如果e和f具有相同的有限维，则下列属性等效：

1. The map lϕ is injective.  
   图l\_是注射剂。
2. The map lϕ is surjective.  
   地图“L”是推测性的。
3. The map rϕ is injective.  
   地图R\_是注射剂。
4. The map rϕ is surjective.  
   地图“R”是推测性的。
5. The bilinear form ϕ is nondegenerate.  
   双线性形式为非退化形式。

Observe that in terms of the canonical pairing between E∗ and E given by  
观察e和e之间的规范配对

hf,ui = f(u), f ∈ E∗,u ∈ E,  
hf，ui=f（u），f e，u e，

(and the canonical pairing between F ∗ and F), we have  
（和f和f之间的规范配对），我们有

ϕ(u,v) = hlϕ(u),vi = hrϕ(v),ui u ∈ E,v ∈ F.  
⑨（u，v）=hl（u），vi=hr（v），ui u∈e，v∈f.

Proposition 28.3. Given a bilinear map ϕ: E × F → K, if ϕ is nondegenerate and E and F are finite-dimensional, then dim(E) = dim(F) = n, and for every basis (e1,...,en) of E, there is a basis (f1,...,fn) of F such that ϕ(ei,fj) = δij, for all i,j = 1,...,n.  
提案28.3.给定双线性映射，如果\_为非退化映射，且e和f为有限维，则dim（e）=dim（f）=n，对于e的每个基（e1，…，en），f都有一个基（f1，…，fn），因此，对于所有i，j=1，…，n，（ei，fj）=δij。

Proof. Since ϕ is nondegenerate, by Proposition 28.1 we have dim(E) = dim(F) = n, and by Proposition 28.2, the linear map rϕ is bijective. Then, if () is the dual basis (in E∗) of the basis (e1,...,en), the vectors (f1,...,fn) given by fi = rϕ−1(e∗i ) form a basis of F, and we have  
证据。由于\_是非退化的，根据命题28.1，我们有dim（e）=dim（f）=n，根据命题28.2，线性映射r\_是双射的。那么，如果（）是基（e1，…，en）的对偶基（e），由f i=r\_−1（e i）给出的向量（f1，…，fn）构成f的基，我们有

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as claimed.   
如要求。

If E = F and ϕ is symmetric, then we have the following interesting result.  
如果e=f和\_是对称的，那么我们得到了以下有趣的结果。

Theorem 28.4. Given any bilinear form ϕ: E×E → K with dim(E) = n, if ϕ is symmetric (possibly degenerate) and K does not have characteristic 2, then there is a basis (e1,...,en) of E such that ϕ(ei,ej) = 0, for all i =6 j.  
定理28.4.给定任何双线性形式，其中，dim（e）=n，如果\_对称（可能退化），且k不具有特征2，则e有一个基（e1，…，en），使得所有i=6 j，\_（ei，ej）=0。

Proof. We proceed by induction on n ≥ 0, following a proof due to Chevalley. The base case n = 0 is trivial. For the induction step, assume that n ≥ 1 and that the induction hypothesis holds for all vector spaces of dimension n−1. If ϕ(u,v) = 0 for all u,v ∈ E, then the statement holds trivially. Otherwise, since K does not have characteristic 2, equation  
证据。我们通过n≥0的诱导进行，根据Chevalley的证明。基本情况n=0无关紧要。对于归纳步骤，假设n≥1，并且归纳假设适用于维度n-1的所有向量空间。如果所有的u，v∈e的\_（u，v）=0，那么该语句就无关紧要了。否则，因为k没有特征2，方程

2ϕ(u,v) = ϕ(u + v,u + v) − ϕ(u,u) − ϕ(v,v) (∗)  
2\_（u，v）=（u+v，u+v）−（u，u）−（v，v）（）

show that there is some nonzero vector e1 ∈ E such that ϕ(e1,e1) = 06 since otherwise ϕ would vanish for all u,v ∈ E. We claim that the set  
证明有一些非零向量e1∈e，这样，因为如果没有，那么，对于所有u，v∈e，θ（e1，e1）=06将消失。我们声称

H = {v ∈ E | ϕ(e1,v) = 0}  
h=v∈e \_（e1，v）=0

has dimension n − 1, and that e1 ∈/ H.  
具有尺寸n-1，并且e1∈/h。

This is because  
这是因为

H = Ker(lϕ(e1)),  
H=Ker（l\_（e1）），

where lϕ(e1) is the linear form in E∗ determined by e1. Since ϕ(e1,e1) = 06 , we have e1 ∈/ H, the linear form lϕ(e1) is not the zero form, and thus its kernel is a hyperplane H (a subspace of dimension n − 1). Since dim(H) = n − 1 and e1 ∈/ H, we have the direct sum  
式中，L\_（e1）是e中由e1确定的线性形式。既然ω（e1，e1）=06，我们得到了e1∈/h，线性形式lω（e1）不是零形式，因此它的核是一个超平面h（维度n-1的子空间）。由于dim（h）=n−1和e1∈/h，我们得到了直接和

E = H ⊕ Ke1.  
e=h ke1。

By the induction hypothesis applied to H, we get a basis (e2,...,en) of vectors in H such that ϕ(ei,ej) = 0, for all i =6 j with 2 ≤ i,j ≤ n. Since ϕ(e1,v) = 0 for all v ∈ H and since ϕ is symmetric, we also have ϕ(v,e1) = 0 for all v ∈ H, so we obtain a basis (e1,...,en) of E such that ϕ(ei,ej) = 0, for all i =6 j.   
通过应用于h的归纳假设，我们得到h中向量的一个基（e2，…，e n），使得所有i=6 j，其中2≤i，j≤n，因此，对于所有v∈h，由于（e1，v）=0，并且由于\_是对称的，因此，对于所有v∈h，我们也有\_（v，e1）=0，因此我们得到e的一个基（e1，…，en），以便⑨（ei，ej）=0，对于所有i=6 J。

If E and F are finite-dimensional vector spaces and if (e1,...,em) is a basis of E and (f1,...,fn) is a basis of F then the bilinearity of ϕ yields  
如果e和f是有限维向量空间，如果（e1，…，em）是e的基础，并且（f1，…，fn）是f的基础，那么\_的双线性度产生

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This shows that ϕ is completely determined by the n × m matrix M = (mij) with mij = ϕ(ej,fi), and in matrix form, we have  
这表明，\_完全由n×m矩阵m=（mij）确定，其中mij=\_（ej，fi），在矩阵形式中，我们有

ϕ(x,y) = x>M>y = y>Mx,  
⑨（x，y）=x>m>y=y>mx，

where x and y are the column vectors associated with (x1,...,xm) ∈ Km and (y1,...,yn) ∈ Kn. As in Section 11.1, we are committing the slight abuse of notation of letting x denote both the vector and the column vector associated with (x1,...,xn) (and similarly for y).  
其中x和y是与（x1，…，xm）∈km和（y1，…，yn）∈kn相关的列向量。如第11.1节所述，我们犯了一个小错误，即x表示与（x1，…，xn）相关的向量和列向量（与y类似）。

Definition 28.6. If (e1,...,em) is a basis of E and (f1,...,fn) is a basis of F, for any bilinear form ϕ: E × F → K, the n × m matrix M = (mij) given by mij = ϕ(ej,fi) for i = 1,...,n and j = 1,...,m is called the matrix of ϕ with respect to the bases (e1,...,em) and (f1,...,fn).  
定义28.6.如果（e1，…，e m）是e的基，而（f1，…，f n）是f的基，对于任何双线性形式，如果（f1，…，fn）是f的基，那么对于i=1，…，n和j=1，…，m的n×m矩阵m=（mij）由mij=\_（ej，fi）给出，则m被称为关于基（e1，…，em）和（f1，…，fn）的\_矩阵。

The following fact is easily proved.  
以下事实很容易证明。

Proposition 28.5. If m = dim(E) = dim(F) = n, then ϕ is nondegenerate iff the matrix M is invertible iff det(M) = 06 .  
提案28.5。如果m=dim（e）=dim（f）=n，那么，如果矩阵m是可逆的，那么，如果矩阵m是非退化的，那么，iff det（m）=06。

As we will see later, most bilinear forms that we will encounter are equivalent to one whose matrix is of the following form:  
正如我们稍后将看到的，我们将遇到的大多数双线性形式等价于其矩阵为以下形式的形式：

1. In, −In.  
   在−在。
2. If p + q = n, with p,q ≥ 1,  
   如果p+q=n，其中p，q≥1，
3. If n = 2m,   
   如果n=2米，
4. If n = 2m,  
   如果n=2米，

.  
.

If we make changes of bases given by matrices P and Q, so that x = Px0 and y = Qy0, then the new matrix expressing ϕ is P >MQ. In particular, if E = F and the same basis is used, then the new matrix is P >MP. This shows that if ϕ is nondegenerate, then the determinant of ϕ is determined up to a square element.  
如果我们改变矩阵p和q给出的碱基，使x=px0和y=qy0，那么新的矩阵式\_是p>mq。特别是，如果e=f并且使用相同的基，那么新的矩阵是p>mp。这表明，如果\_是非退化的，那么\_的行列式被确定为一个平方元素。

Observe that if ϕ is a symmetric bilinear form (E = F) and if K does not have characteristic 2, then by Theorem 28.4, there is a basis of E with respect to which the matrix M representing ϕ is a diagonal matrix. If K = R or K = C, this allows us to classify completely the symmetric bilinear forms. Recall that Φ(u) = ϕ(u,u) for all u ∈ E.  
注意，如果\_是对称双线性形式（e=f），并且k没有特征2，那么根据定理28.4，有一个e的基础，代表\_的矩阵m是对角矩阵。如果k=r或k=c，这允许我们完全分类对称双线性形式。回想一下，所有u∈e的Φ（u）=（u，u）。

Proposition 28.6. Given any bilinear form ϕ: E × E → K with dim(E) = n, if ϕ is symmetric and K does not have characteristic 2, then there is a basis (e1,...,en) of E such that  
提案28.6.给定任何双线性形式，直径（e）=n，如果直径（e）=n，且k不具有特征2，则e有一个基（e1，…，en），以便

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for some λi ∈ K − {0} and with r ≤ n. Furthermore, if K = C, then there is a basis  
对于某些λi∈k−0且r≤n.此外，如果k=c，则有一个基

(e1,...,en) of E such that  
（e1，…，en）的

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，

and if K = R, then there is a basis (e1,...,en) of E such that  
如果k=r，那么e有一个基（e1，…，en），这样

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with 0 ≤ p,q and p + q ≤ n.  
0≤P，Q和P+Q≤N。

Proof. The first statement is a direct consequence of Theorem 28.4. If K = C, then every λi has a square root µi, and if replace ei by ei/µi, we obtained the desired form.  
证据。第一个陈述是定理28.4的直接结果。如果k=c，那么每个λi都有一个平方根μi，如果用ei/μi替换ei，我们就得到了所需的形式。

If K = R, then there are two cases:  
如果k=r，则有两种情况：

1. If λi > 0, let µi be a positive square root of λi and replace ei by ei/µi.  
   如果λi>0，则将μi设为λi的正平方根，并将ei替换为ei/μi。
2. If λi < 0, et µi be a positive square root of −λi and replace ei by ei/µi.  
   如果λi<0，则etμi为−λi的正平方根，并将ei替换为ei/μi。

In the nondegenerate case, the matrices corresponding to the complex and the real case are, In,−In, and Ip,q. Observe that the second statement of Proposition 28.6 holds in any field in which every element has a square root. In the case K = R, we can show that(p,q) only depends on ϕ.  
在非退化情况下，对应于复数和实数的矩阵是，in、−in和ip，q。注意，命题28.6的第二个陈述存在于每个元素都有平方根的任何域中。在k=r的情况下，我们可以证明（p，q）仅取决于\_。

Definition 28.7. Let ϕ: E×E → R be any symmetric real bilinear form. For any subspace U of E, we say that ϕ is positive definite on U iff ϕ(u,u) > 0 for all nonzero u ∈ U, and we say that ϕ is negative definite on U iff ϕ(u,u) < 0 for all nonzero u ∈ U. Then, let  
定义28.7.设a:e×e→r为任意对称实双线性形式。对于e的任何子空间u，我们说，对于所有非零u∈u，在u iff（u，u）>0上，\_是正定的，并且我们说，在u iff（u，u）<0，对于所有非零u∈u，\_是负定的。那么，让

r = max{dim(U) | U ⊆ E, ϕ is positive definite on U}  
r=max dim（u）u e，ω在u上为正定

and let s = max{dim(U) | U ⊆ E, ϕ is negative definite on U}  
设s=max dim（u）u e，\_在u上为负定

Proposition 28.7. (Sylvester’s inertia law) Given any symmetric bilinear form ϕ: E×E → R with dim(E) = n, for any basis (e1,...,en) of E such that  
提案28.7.（西尔维斯特惯量定律）给定任意对称双线性形式，ω：e×e→r，dim（e）=n，对于e的任何基（e1，…，en），这样

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with 0 ≤ p,q and p + q ≤ n, the integers p,q depend only on ϕ; in fact, p = r and q = s, with r and s as defined above.  
当0≤p，q和p+q≤n时，整数p，q仅取决于\_；事实上，p=r和q=s，其中r和s如上文所定义。

Proof. If we let U be the subspace spanned by (e1,...,ep), then ϕ is positive definite on U, so r ≥ p. Similarly, if we let V be the subspace spanned by (ep+1,...,ep+q), then ϕ is negative definite on V , so s ≥ q.  
证据。如果我们将u设为（e1，…，ep）所跨越的子空间，那么，在u上，ω是正定的，所以r≥p。同样，如果我们将v设为（ep+1，…，ep+q）所跨越的子空间，那么，在v上，ω是负定的，所以s≥q。

Next, if W1 is any subspace of maximum dimension such that ϕ is positive definite on  
下一步，如果w1是最大尺寸的任何子空间，那么，在

W1, and if we let V 0 be the subspace spanned by (ep+1,...,en), then ϕ(u,u) ≤ 0 on V 0, so W1 ∩ V 0 = (0), which implies that dim(W1) + dim(V 0) ≤ n, and thus, r + n − p ≤ n; that is, r ≤ p. Similarly, if W2 is any subspace of maximum dimension such that ϕ is negative definite on W2, and if we let U0 be the subspace spanned by (e1,...,ep,ep+q+1,...,en), then ϕ(u,u) ≥ 0 on U0, so W2 ∩ U0 = (0), which implies that s + n − q ≤ n; that is, s ≤ q. Therefore, p = r and q = s, as claimed   
w1，如果我们让v 0为（ep+1，…，en）所跨越的子空间，那么v 0上的瓒（u，u）≤0，那么w1 v 0=（0），这意味着dim（w1）+dim（v 0）≤n，因此，r+n−p≤n；也就是说，r≤p。同样，如果w2是最大尺寸的任何子空间，使得瓒在w2上为负定，如果我们让u0是（e1，…，ep，ep+q+1，…，en）所跨越的子空间，那么，在u0上，那么，ω（u，u）≥0，那么，w2 u0=（0），这意味着s+n−q≤n；也就是说，s≤q。因此，p=r，q=s，正如所声称的那样。

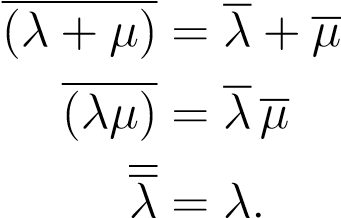
These last two results can be generalized to ordered fields. For example, see Snapper and Troyer [157], Artin [6], and Bourbaki [24].  
最后两个结果可以推广到有序域。例如，参见Snapper和Troyer[157]、Artin[6]和Bourbaki[24]。

28.2. SESQUILINEAR FORMS  
28.2。倍线性形式

## 28.2 Sesquilinear Forms 28.2倍线性形式

In order to accomodate Hermitian forms, we assume that some involutive automorphism,  
为了适应赫米特形式，我们假设一些对合自同构，

λ 7→ λ, of the field K is given. This automorphism of K satisfies the following properties:  
给出了K场的λ7→λ。k的自同构满足以下性质：



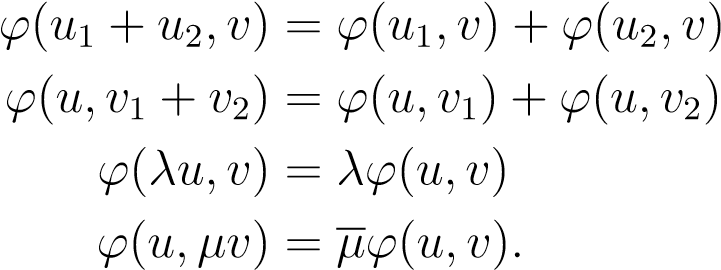
Since any field automorphism maps the multiplicative unit 1 to itself, we have 1 = 1.  
因为任何场的自同构都将乘法单位1映射到自身，所以我们有1=1。

If the automorphism λ 7→ λ is the identity, then we are in the standard situation of a bilinear form. When K = C (the complex numbers), then we usually pick the automorphism of C to be conjugation; namely, the map  
如果自同构λ7→λ是恒等式，则我们处于双线性形式的标准情形。当k=c（复数）时，我们通常选择c的自同构作为共轭，即映射

a + ib 7→ a − ib.  
A+Ib 7→A−Ib。

Definition 28.8. Given two vector spaces E and F over a field K with an involutive au-  
定义28.8.给定具有对合Au的K场上的两个向量空间e和f-

tomorphism λ 7→ λ, a mapu,u1ϕ,u:E∈×EF, all→v,vK1is a (right),v2 ∈ F, for allsesquilinear formλ,µ ∈ K, we haveiff the following conditions hold: For all  
tomorphismλ7→λ，a mapu，u1\_，u:e∈×e f，all→v，vk1为a（右），v2∈f，对于所有二次线性形式λ，\_∈k，我们有以下条件保持：对于所有



Again, ϕ(0,v) = ϕ(u,0) = 0. If E = F, then we have  
再说一遍，\_（0，v）=（u，0）=0。如果e=f，那么我们有

.  
.

If we let λ = µ = 1 and then λ = 1,µ = −1, we get  
如果我们让λ=μ=1，然后λ=1，μ=−1，我们得到

ϕ(u + v,u + v) = ϕ(u,u) + ϕ(u,v) + ϕ(v,u) + ϕ(v,v) ϕ(u − v,u − v) = ϕ(u,u) − ϕ(u,v) − ϕ(v,u) + ϕ(v,v),  
⑨（u+v，u+v）=（u，u）+（u，v）+（v，u）+（v，v）（u−v，u−v）=（u，u）−（u，v）−（v，u）+（v，v）

so by subtraction, we get  
所以通过减法，我们得到

2(ϕ(u,v) + ϕ(v,u)) = ϕ(u + v,u + v) − ϕ(u − v,u − v) for u,v ∈ E.  
2（\_（u，v）+\_（v，u））=\_（u+v，u+v）−（u−v，u−v）表示u，v∈e。

If we replace v by λv (with λ 6= 0), we get  
如果用λv代替v（用λ6=0），我们得到

and by combining the above two equations, we get  
把这两个方程结合起来，我们得到

. (∗)  
.（）

If the automorphism λ 7→ λ is not the identity, then there is some λ ∈ K such thatϕ is completelyλ−λ 6= 0, and if K is not of characteristic 2, then we see that the sesquilinear form determined by its restriction to the diagonal (that is, the set of valuesIn the special case where K = C, we can pick λ = i, and we get {ϕ(u,u) | u ∈ E}).  
如果自同构式λ7→λ不是恒等式，则存在一些λ∈k，使得\_完全是λ−λ6=0，如果k不属于特征2，则我们看到由其对对角线的限制所确定的倍线性形式（即，在特殊情况下，k=c的值集，我们可以选择λ=i，得到（u，u）u∈e）。

4ϕ(u,v) = ϕ(u + v,u + v) − ϕ(u − v,u − v) + iϕ(u + λv,u + λv) − iϕ(u − λv,u − λv).  
4\_（u，v）=（u+v，u+v）-（u-v，u-v）+i\_（u+λv，u+λv）-i（u-λv，u-λv）。

Remark: If the automorphism λ 7→ϕλis symmetric.is the identity, then in general ϕ is not determined by its value on the diagonal, unless  
注：如果自同构式λ7→λ是对称的。是恒等式，那么一般来说，\_不是由对角线上的值决定的，除非

In the sesquilinear setting, it turns out that the following two cases are of interest:  
在倍线性设置中，发现以下两种情况是有意义的：

1. We have  
   我们有

ϕ(v,u) = ϕ(u,v), for all u,v ∈ E,  
⑨（v，u）=⑨（u，v），对于所有u，v∈e，

in which case we say that ϕ is Hermitian. In the special case where K = C and the involutive automorphism is conjugation, we see that ϕ(u,u) ∈ R, for u ∈ E.  
在这种情况下，我们称之为\_为赫米特。在k=c的特殊情况下，对合自同构是共轭的，我们可以看到，对于u∈e，ω（u，u）∈r。

1. We have  
   我们有

ϕ(v,u) = −ϕ(u,v), for all u,v ∈ E,  
⑨（v，u）=−⑨（u，v），对于所有u，v∈e，

in which case we say that ϕ is skew-Hermitian.  
在这种情况下，我们称之为倾斜赫米特。

We observed that in characteristic different from 2, a sesquilinear form is determined by its restriction to the diagonal. For Hermitian and skew-Hermitian forms, we have the following kind of converse.  
我们观察到，在不同于2的特征中，倍线性形式是由它对对角线的限制决定的。对于厄米提亚和歪厄米提亚形式，我们有以下几种相反的形式。

Proposition 28.8. If ϕ is a nonzero Hermitian or skew-Hermitian form and if ϕ(u,u) = 0  
提案28.8.如果η是非零厄米式或斜厄米式，并且如果η（u，u）=0

for all u ∈ E, then K is of characteristic 2 and the automorphism λ 7→ λ is the identity.  
对于所有u∈e，则k为特征2，自同构性λ7→λ为同一性。

Proof. We give the proof in the Hermitian case, the skew-Hermitian case being left as an exercise. Assume that ϕ is alternating. From the identity  
证据。我们在赫米特的案例中给出了证明，歪斜的赫米特案例作为练习被留下。假设\_是交替的。从身份上

ϕ(u + v,u + v) = ϕ(u,u) + ϕ(u,v) + ϕ(u,v) + ϕ(v,v),  
⑨（U+V，U+V）=⑨（U，U）+（U，V）+（U，V）+（V，V）、

we get  
我们得到

ϕ(u,v) = −ϕ(u,v) for all u,v ∈ E.  
⑨（u，v）=-全部u，v∈e。

SinceFor anyϕ is not the zero form, there exist some nonzero vectorsλ ∈ K, we have u,v ∈ E such that ϕ(u,v) = 1.  
既然任何瓒不是零形式，存在一些非零向量λ∈k，我们有u，v∈e，因此瓒（u，v）=1。

λϕ(u,v) = ϕ(λu,v) = −ϕ(λu,v) = −λϕ(u,v),  
哿（u，v）=哿（λu，v）=哿（λu，v）=哿（u，v）

28.2. SESQUILINEAR FORMS  
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and since ϕ(u,v) = 1, we get  
既然\_（u，v）=1，我们得到

λ = −λ for all λ ∈ K.  
λ=所有λ的−λ∈k。

For λ = 1, we get 1 = −1, which means that K has characterictic 2. But then  
对于λ=1，我们得到1=−1，这意味着k具有特征2。但是那时

λ = −λ = λ for all λ ∈ K,  
λ=−λ=所有λ的λ∈k，

so the automorphism λ 7→ λ is the identity.   
所以自同构式λ7→λ是同一性。

The definition of the linear maps lϕ and rϕ requires a small twist due to the automorphism  
线性映射的定义l\_和r\_由于自同构需要一个小的扭曲。

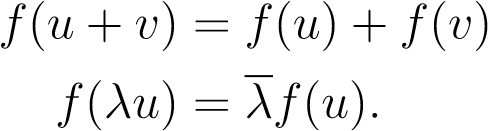
λ 7→ λ.  
λ7→λ。

Definition 28.9. Given a vector space E over a field K with an involutive automorphism  
定义28.9.给出了具有对合自同构的K域上的向量空间E

λ 7→ λ, we define the K-vector space E as E with its abelian group structure, but with scalar multiplication given by  
λ7→λ，我们用它的阿贝尔群结构将k向量空间e定义为e，但用下式给出的标量乘法

(λ,u) 7→ λu.  
（λ，u）7→λu。

Given two K-vector spaces E and F, a semilinear map f : E → F is a function, such that for all u,v ∈ E, for all λ ∈ K, we have  
给定两个k向量空间e和f，一个半线性映射f:e→f是一个函数，这样对于所有u，v∈e，对于所有λ∈k，我们得到



Because λ = λ, observe that a function f : E → F is semilinear iff it is a linear map  
因为λ=λ，观察函数f:e→f是半线性的，如果它是线性映射

f : E → F. The K-vector spaces E and E are isomorphic, since any basis (ei)i∈I of E is also a basis of E.  
f:e→f。k向量空间e和e是同构的，因为e的任何基（e i）i∈i也是e的基。

The maps lϕ and rϕ are defined as follows:  
图L\_和R\_定义如下：

For every u ∈ E, let lϕ(u) be the linear form in F ∗ defined so that  
对于每一个u e，设l\_（u）为f中定义的线性形式，以便

lϕ(u)(y) = ϕ(u,y) for all y ∈ F,  
l\_（u）（y）=（u，y）表示所有y∈f，

and for every v ∈ F, let rϕ(v) be the linear form in E∗ defined so that  
对于每一个v∈f，设r（v）为e中的线性形式，因此

rϕ(v)(x) = ϕ(x,v) for all x ∈ E.  
对于所有x∈e，r\_（v）（x）=（x，v）。

The reader should check that because we used ϕ(u,y) in the definition of lϕ(u)(y), the function lϕ(u) is indeed a linear form in F ∗. It is also easy to check that lϕ is a linear  
读者应检查，因为我们在l\_（u）（y）的定义中使用了\_（u，y），函数l\_（u）实际上是f\_中的线性形式。也可以很容易地检查l\_是线性的

map lϕ : E → F ∗, and that rϕ is a linear map rϕ : F → E∗ (equivalently, lϕ : E → F ∗ and rϕ : F → E∗ are semilinear).  
图l\_：e→f\_，而r\_是线性图r\_：f→e（相当于，l\_：e→f\_和r\_：f→e是半线性的）。

The notion of a nondegenerate sesquilinear form is identical to the notion for bilinear forms. For the convenience of the reader, we repeat the definition.  
非退化倍线性形式的概念与双线性形式的概念相同。为了方便读者阅读，我们重复定义。

Definition 28.10. A sesquilinear map ϕ: E × F → K is said to be nondegenerate iff the following conditions hold:  
定义28.10.当满足以下条件时，称倍线性映射为非退化的iff:e×f→k：

1. For every u ∈ E, if ϕ(u,v) = 0 for all v ∈ F, then u = 0, and  
   对于每一个u∈e，如果所有v∈f的（u，v）=0，则u=0，并且
2. For every v ∈ F, if ϕ(u,v) = 0 for all u ∈ E, then v = 0.  
   对于每一个v∈f，如果所有u∈e的（u，v）=0，则v=0。

Proposition 28.1 translates into the following proposition. The proof is left as an exercise. Proposition 28.9. Given a sesquilinear map ϕ: E ×F → K, the following properties hold:  
28.1号提案转化为以下提案。证据留作练习。提案28.9.给定一个倍线性映射，其中：e×f→k，以下属性保持不变：

1. The map lϕ is injective iff Property (1) of Definition 28.10 holds.  
   图l\_是定义28.10的注射iff属性（1）。
2. The map rϕ is injective iff Property (2) of Definition 28.10 holds.  
   图R\_是定义28.10的注射iff属性（2）。
3. The sesquilinear form ϕ is nondegenerate and iff lϕ and rϕ are injective.  
   倍线性形式\_是非退化的，如果l\_和r\_是注射剂。
4. If the sesquillinear form ϕ is nondegenerate and if E and F have finite dimensions,  
   如果倍半线性形式\_是非退化的，并且如果e和f有有限的尺寸，

then dim(E) = dim(F), and lϕ : E → F ∗ and rϕ : F → E∗ are linear isomorphisms.  
那么dim（e）=dim（f），l\_：e→f和r\_：f→e是线性同构。

Propositions 28.2 and 28.3 also generalize to sesquilinear forms. We also have the following version of Theorem 28.4, whose proof is left as an exercise.  
命题28.2和28.3也归纳为倍线性形式。我们还有下面的定理28.4版本，它的证明留作练习。

Theorem 28.10. Given any sesquilinear form ϕ: E × E → K with dim(E) = n, if ϕ is Hermitian and K does not have characteristic 2, then there is a basis (e1,...,en) of E such that ϕ(ei,ej) = 0, for all i =6 j.  
定理28.10。给定任意倍线性形式，其中，dim（e）=n，如果\_为Hermitian且k不具有特征2，则e有一个基（e1，…，en），因此，所有i=6 j，\_（ei，ej）=0。

As in Section 28.1, if E and F are finite-dimensional vector spaces and if (e1,...,em) is a basis of E and (f1,...,fn) is a basis of F then the sesquilinearity of ϕ yields  
如第28.1节所述，如果e和f是有限维向量空间，并且如果（e1，…，em）是e的基础，并且（f1，…，fn）是f的基础，那么，θ的倍线性产生

.  
.

This shows that ϕ is completely determined by the n × m matrix M = (mij) with mij = ϕ(ej,fi), and in matrix form, we have  
这表明，\_完全由n×m矩阵m=（mij）确定，其中mij=\_（ej，fi），在矩阵形式中，我们有

ϕ(x,y) = x>M>y = y∗Mx,  
⑨（x，y）=x>m>y=y mx，

where x and y are the column vectors associated with (x1,...,xm) ∈ Km and (y1,...,yn) ∈ Kn, and y∗ = y>. As earlier, we are committing the slight abuse of notation of letting x denote both the vector and the column vector associated with (x1,...,xn)  
其中x和y是与（x1，…，xm）km和（y1，…，yn）kn和y=y>相关的列向量。如前所述，我们犯了一个小小的错误，即x表示与（x1，…，xn）相关联的向量和列向量。

(and similarly for y).  
（与Y类似）。

Definition 28.11. If (e1,...,em) is a basis of E and (f1,...,fn) is a basis of F, for any sesquilinear form ϕ: E × F → K, the n × m matrix M = (mij) given by mij = ϕ(ej,fi) for i = 1,...,n and j = 1,...,m is called the matrix of ϕ with respect to the bases (e1,...,em) and (f1,...,fn).  
定义28.11.如果（e1，…，e m）是e的基，（f1，…，f n）是f的基，对于任何倍线性形式，\_：e×f→k，对于i=1，…，n和j=1，…，m的n×m矩阵m=（mij）由mij=\_（ej，fi）给出，m被称为关于基（e1，…，em）和（f1，…，fn）的\_矩阵。

Proposition 28.5 also holds for sesquilinear forms and their matrix representations.  
命题28.5也适用于倍线性形式及其矩阵表示。

Observe that if ϕ is a Hermitian form (E = F) and if K does not have characteristic 2, then by Theorem 28.10, there is a basis of E with respect to which the matrix M representing ϕ is a diagonal matrix. If K = C, then these entries are real, and this allows us to classify completely the Hermitian forms.  
注意，如果\_是厄米形式（e=f），并且k没有特征2，那么根据定理28.10，有一个e的基础，表示\_的矩阵m是对角矩阵。如果k=c，那么这些条目是真实的，这允许我们完全分类赫米特形式。

Proposition 28.11. Given any Hermitian form ϕ: E × E → C with dim(E) = n, there is a basis (e1,...,en) of E such that  
提案28.11.给定任何厄米特式\_：e×e→c，dim（e）=n，e有一个基（e1，…，en），这样

,  
，

with 0 ≤ p,q and p + q ≤ n.  
0≤P，Q和P+Q≤N。

The proof of Proposition 28.11 is the same as the real case of Proposition 28.6. Sylvester’s inertia law (Proposition 28.7) also holds for Hermitian forms: p and q only depend on ϕ.  
第28.11号提案的证明与第28.6号提案的真实情况相同。西尔维斯特惯性定律（命题28.7）也适用于赫米特形式：p和q仅取决于\_。

## 28.3 Orthogonality 28.3正交性

In this section we assume that we are dealing with a sesquilinear form ϕ: E × F → K. We  
在本节中，我们假设我们处理的是一个双方程形式，即：e×f→k。

allow the automorphism λ →7 λ to be the identity, in which case ϕ is a bilinear form. This way, we can deal with properties shared by bilinear forms and sesquilinear forms in a uniform fashion. Orthogonality is such a property.  
允许自同构λ→7λ为恒等式，在这种情况下，Ⅷ为双线性形式。这样，我们就可以以统一的方式处理双线性形式和倍线性形式共享的属性。正交性就是这样一种性质。

Definition 28.12. Given a sesquilinear form ϕ: E×F → K, we say that two vectors u ∈ E and v ∈ F are orthogonal (or conjugate) if ϕ(u,v) = 0. Two subsets E0 ⊆ E and F 0 ⊆ F are orthogonal if ϕ(u,v) = 0 for all u ∈ E0 and all v ∈ F 0. Given a subspace U of E, the right orthogonal space of U, denoted U⊥, is the subspace of F given by  
定义28.12.给出了一个二次线性形式，即：e×f→k，我们认为两个向量u∈e和v∈f是正交的（或共轭的），前提是（u，v）=0。两个子集e0 e和f 0 f是正交的，如果所有u∈e0和所有v∈f 0的ω（u，v）=0。给定e的子空间u，u的右正交空间，表示u，是f的子空间，由

U⊥ = {v ∈ F | ϕ(u,v) = 0 for all u ∈ U},  
u=v∈f（u，v）=0表示所有u∈u，

and given a subspace V of F, the left orthogonal space of V , denoted V ⊥, is the subspace of E given by  
给定f的子空间v，v的左正交空间，表示为v，是e的子空间，由

V ⊥ = {u ∈ E | ϕ(u,v) = 0 for all v ∈ V }.  
V=U∈E（U，V）=0表示所有V∈V。

When E and F are distinct, there is little chance of confusing the right orthogonal subspace U⊥ of a subspace U of E and the left orthogonal subspace V ⊥ of a subspace V of F. However, if E = F, then ϕ(u,v) = 0 does not necessarily imply that ϕ(v,u) = 0, that is, orthogonality is not necessarily symmetric. Thus, if both U and V are subsets of E, there is a notational ambiguity if U = V . In this case, we may write U⊥r for the right orthogonal and U⊥l for the left orthogonal.  
当e和f不同时，几乎不可能混淆e的子空间u的右正交子空间u和f的子空间v的左正交子空间v。然而，如果e=f，则\_（u，v）=0并不一定意味着（v，u）=0，也就是说，正交性不必要。对称。因此，如果u和v都是e的子集，那么如果u=v，就有一个符号模糊性。在这种情况下，我们可以为右正交写u\_r，为左正交写u\_l。

The above discussion brings up the following point: When is orthogonality symmetric?  
以上讨论提出了以下几点：正交对称是什么时候？

If ϕ is bilinear, it is shown in E. Artin [6] (and in Jacobson [96]) that orthogonality is symmetric iff either ϕ is symmetric or ϕ is alternating (ϕ(u,u) = 0 for all u ∈ E).  
如果ω是双线性的，则在e.artin[6]和jacobson[96]中表明，正交性是对称的iff，其中，ω是对称的，或者，ω是交替的（对于所有u∈e，u=0）。

If ϕ is sesquilinear, the answer is more complicated. In addition to the previous two cases, there is a third possibility:  
如果π是倍线性的，答案就更复杂了。除前两种情况外，还有第三种可能性：

) for all u,v ∈ E,  
）对于所有u，v∈e，

where is some nonzero element in K. We say that-Hermitian. Observe that  
K中的非零元素在哪里，我们称之为厄米提安。注意

,  
，

so if ϕ is not alternating, then ϕ(u,u) = 06 for some u, and we must have = 1. The most common cases are  
因此，如果\_不是交替的，那么\_（u，u）=06对于一些u，我们必须有=1。最常见的情况是

1. = 1, in which case ϕ is Hermitian, and  
   =1，在这种情况下，η为Hermitian，并且
2. 1, in which case ϕ is skew-Hermitian.  
   1，在这种情况下，η是歪厄米提安。

If ϕ is alternating and K is not of characteristic 2, then equation (∗) from Section 28.2  
如果φ是交替的，k不属于特征2，则第28.2节中的方程式（）

implies that the automorphism λ 7→ λ must be the identity if ϕ is nonzero. If so, ϕ is skew-symmetric, so  
意味着自同构λ7→λ必须是单位，如果\_为非零。如果是的话，\_是斜对称的，所以

In summary, if ϕ is either symmetric, alternating, or -Hermitian, then orthogonality is symmetric, and it makes sense to talk about the orthogonal subspace U⊥ of U. Observe that if-Hermitian, then  
综上所述，如果ω是对称的、交替的或-hermitian，那么正交性是对称的，讨论u的正交子空间u是有意义的。观察如果hermitian，那么

.  
.

This is because  
这是因为

,  
，

so.  
所以。

If E and F are finite-dimensional with bases (e1,...,em) and (f1,...,fn), and if ϕ is represented by the n × m matrix M, then-Hermitian iff  
如果e和f是有限维的，有基（e1，…，e m）和（f1，…，f n），并且如果ρ由n×m矩阵m表示，那么Hermitian iff

,  
，

where M∗ = (M)> (as usual). This captures the following kinds of familiar matrices:  
式中m=（m）>（通常）。这捕获了以下类型的熟悉矩阵：

1. Symmetric matrices (  
   对称矩阵（
2. Skew-symmetric matrices (  
   斜对称矩阵（
3. Hermitian matrices (  
   厄米特矩阵（
4. Skew-Hermitian matrices (  
   偏斜厄米特矩阵（

Going back to a sesquilinear form ϕ: E × F → K, for any subspace U of E, it is easy to check that  
回到倍线性形式，对于e的任何子空间u，很容易检查

* 1. ⊆ (U⊥)⊥,  
     （u），

and that for any subspace V of F, we have  
对于f的任何子空间v，我们有

* 1. ⊆ (V ⊥)⊥.  
     （V）。

For simplicity of notation, we write U⊥⊥ instead of (U⊥)⊥ (and V ⊥⊥ instead of (V ⊥)⊥).  
为了简化表示法，我们编写u而不是（u）（和v而不是（v））。

Given any two subspaces U1 and U2 of E, if U1 ⊆ U2, then . Indeed, if then ϕ(u2,v) = 0 for all u2 ∈ U2, and since U1 ⊆ U2 this implies that ϕ(u1,v) = 0 for all u1 ∈ U1, which shows that . Similarly for any two subspaces V1,V2 of F, if V1 ⊆ V2, then. As a consequence,  
给定e的任意两个子空间u1和u2，如果u1 u2，那么。实际上，如果所有的u2∈u2，那么，既然u1 u2，那么意味着所有的u1∈u1，v=0，这表明。同样，对于任意两个子空间v1，v2 of f，如果v1 v2，那么。因此，

U⊥ = U⊥⊥⊥, V ⊥ = V ⊥⊥⊥.  
U=U，V=V。

First, we have U⊥ ⊆ U⊥⊥⊥. Second, from U ⊆ U⊥⊥, we get U⊥⊥⊥ ⊆ U⊥, so U⊥ = U⊥⊥⊥.  
首先，我们有u\_u\_\_。其次，从u u，我们得到u u，所以u=u。

The other equation is proved is a similar way.  
另一个方程也被证明是类似的。

Observe that ϕ is nondegenerate iff E⊥ = {0} and F ⊥ = {0}. Furthermore, since  
观察到，如果e=0和f=0不退化。此外，因为

ϕ(u + x,v) = ϕ(u,v) ϕ(u,v + y) = ϕ(u,v)  
⑨（u+x，v）=⑨（u，v）⑨（u，v+y）=⑨（u，v）

for any x ∈ F ⊥ and any y ∈ E⊥, we see that we obtain by passing to the quotient a sesquilinear form  
对于任何x f和任何y e，我们看到我们通过传递给商得到一个倍线性形式。

[ϕ]: (E/F ⊥) × (F/E⊥) → K  
[⑨]：（E/F）×（F/E）→K

which is nondegenerate.  
这是不退化的。

Proposition 28.12. For any sesquilinear form ϕ: E × F → K, the space E/F ⊥ is finitedimensional iff the space F/E⊥ is finite-dimensional; if so, dim(E/F ⊥) = dim(F/E⊥).  
提案28.12。对于任何倍线性形式：e×f→k，空间e/f是有限维的，如果空间f/e是有限维的，那么dim（e/f）=dim（f/e）。

Proof. Since the sesquilinear form [ϕ]: (E/F ⊥) × (F/E⊥) → K is nondegenerate, the maps  
证据。由于sesquilinear形式[]：（e/f）×（f/e）→k是非退化的，因此地图

l[ϕ] : (E/F ⊥) → (F/E⊥)∗ and r[ϕ] : (F/E⊥) → (E/F ⊥)∗ are injective. If dim(E/F ⊥) = m, then dim(E/F ⊥) = dim((E/F ⊥)∗), so by injectivity of r[ϕ], we have dim(F/E⊥) =  
L[]：（E/F）→（F/E）和R[]：（F/E）→（E/F）为注射型。如果dim（e/f）=m，那么dim（e/f）=dim（（e/f）），那么通过r[\_]的注入率，我们得到dim（f/e）=

dim((F/E⊥)) ≤ m. A similar reasoning using the injectivity of l[ϕ] applies if dim(F/E⊥) = n,  
dim（（f/e）≤m。如果dim（f/e）=n，使用l[]的注入率的类似推理适用。

and we get dim(E/F ⊥) = dim((E/F ⊥)) ≤ n. Therefore, dim(E/F ⊥) = m is finite iff dim(F/E⊥) = n is finite, in which case m = n by Proposition 28.1(d).   
我们得到dim（e/f）=dim（（e/f）≤n。因此，dim（e/f）=m是有限的，iff dim（f/e）=n是有限的，在这种情况下，根据命题28.1（d），m=n。

If U is a subspace of a space E, recall that the codimension of U is the dimension of E/U, which is also equal to the dimension of any subspace V such that E is a direct sum of  
如果u是空间e的子空间，回想一下u的余维是e/u的维数，也等于任何子空间v的维数，这样e是

U and V (E = U ⊕ V ).  
u和v（e=u v）。

Proposition 28.12 implies the following useful fact.  
建议28.12暗示了以下有用的事实。

Proposition 28.13. Let ϕ: E×F → K be any nondegenerate sesquilinear form. A subspace U of E has finite dimension iff U⊥ has finite codimension in F. If dim(U) is finite, then codim(U⊥) = dim(U), and U⊥⊥ = U.  
提案28.13。设\_：e×f→k为任何非退化倍线性形式。e的子空间u有有限维，如果u在f中有有限的余维。如果dim（u）是有限的，那么codim（u）=dim（u），u=u。

Proof. Since ϕ is nondegenerate E⊥ = {0} and F ⊥ = {0}, so Proposition 28.12 applied to the restriction of ϕ to U × F implies that a subspace U of E has finite dimension iff U⊥ has finite codimension in F, and that if dim(U) is finite, then codim(U⊥) = dim(U). Since U⊥ and U⊥⊥ are orthogonal, and since codim(U⊥) is finite, dim(U⊥⊥) is finite and we have dim( ) = codim(U⊥⊥⊥) = codim(U⊥) = dim(U). Since U ⊆ U⊥⊥, we must have  
证据。既然\_是非退化的e=0和f=0，那么适用于\_到u×f的限制的命题28.12意味着e的子空间u具有有限的维数iff u在f中具有有限的余维数，并且如果dim（u）是有限的，那么codim（u）=dim（u）。由于u和u是正交的，并且由于codim（u）是有限的，所以dim（u）是有限的，我们有dim（）=codim（u）=codim（u）=dim（u）。既然你你，我们必须

U = U .   
u=u。

Proposition 28.14. Let ϕ: E ×F → K be any sesquilinear form. Given any two subspaces U and V of E, we have  
提案28.14.设\_：e×f→k为任意倍线性形式。对于任意两个子空间u和v of e，我们有

(U + V )⊥ = U⊥ ∩ V ⊥.  
（U+V）=U V。

Furthermore, if ϕ is nondegenerate and if U and V are finite-dimensional, then  
此外，如果\_是非退化的，并且u和v是有限尺寸，那么

(U ∩ V )⊥ = U⊥ + V ⊥.  
（u v）=u+v。

Proof. If w ∈ (U + V )⊥, then ϕ(u + v,w) = 0 for all u ∈ U and all v ∈ V . In particular, with v = 0, we have ϕ(u,w) = 0 for all u ∈ U, and with u = 0, we have ϕ(v,w) = 0 for all v ∈ V , so w ∈ U⊥ ∩ V ⊥. Conversely, if w ∈ U⊥ ∩ V ⊥, then ϕ(u,w) = 0 for all u ∈ U and ϕ(v,w) = 0 for all v ∈ V . By bilinearity, ϕ(u + v,w) = ϕ(u,w) + ϕ(v,w) = 0, which shows that w ∈ (U + V )⊥. Therefore, the first identity holds.  
证据。如果w∈（u+v），那么对于所有u∈u和所有v∈v，w=0。特别是，当v=0时，所有u∈u都有（u，w）=0，当u=0时，所有v∈v都有（v，w）=0，因此w∈u v。相反，如果w∈u v，则所有u∈u和（v，w）=0，所有v∈v。根据双线性度，ω（u+v，w）＝（u，w）＋（v，w）=0，表示w∈（u+v）。因此，第一个身份是成立的。

Now, assume that ϕ is nondegenerate and that U and V are finite-dimensional, and let W = U⊥ + V ⊥. Using the equation that we just established and the fact that U and V are finite-dimensional, by Proposition 28.13, we get  
现在，假设\_是非退化的，u和v是有限维，并假设w=u+v。利用我们刚刚建立的方程和u和v是有限维的事实，通过28.13号命题，我们得到

W ⊥ = U⊥⊥ ∩ V ⊥⊥ = U ∩ V.  
W=U=U V。

We can apply Proposition 28.12 to the restriction of ϕ to U × W (since U⊥ ⊆ W and  
我们可以将第28.12号提案应用于对\_至u×w的限制（因为u w和

W ⊥ ⊆ U), and we get  
W U），我们得到

dim(U/W ⊥) = dim(U/(U ∩ V )) = dim(W/U⊥).  
dim（u/w）=dim（u/（u v））=dim（w/u）。

If T is a supplement of U⊥ in W so that W = U⊥ ⊕T and if S is a supplement of W in E so that E = W ⊕ S, then codim(W) = dim(S), dim(T) = dim(W/U⊥), and we have the direct sum  
如果t是w中u的一个补充，使w=u t，如果s是e中w的补充，使e=w s，那么codim（w）=dim（s），dim（t）=dim（w/u），我们得到了直接和

E = U⊥ ⊕ T ⊕ S  
E=U T S

which implies that  
这意味着

dim(T) = codim(U⊥) − dim(S) = codim(U⊥) − codim(W)  
dim（t）=codim（u）−dim（s）=codim（u）−codim（w）

so dim(U/(U ∩ V )) = dim(W/U⊥) = codim(U⊥) − codim(W),  
所以dim（u/（u v））=dim（w/u）=codim（u）−codim（w），

and since codim(U⊥) = dim(U), we deduce that  
由于codim（u）=dim（u），我们推断

dim(U ∩ V ) = codim(W).  
dim（u v）=codim（w）。

However, by Proposition 28.13, we have dim(U ∩ V ) = codim((U ∩ V )⊥), so codim(W) = codim((U ∩ V )⊥), and since W ⊆ W ⊥⊥ = (U ∩ V )⊥, we must have W = (U ∩ V )⊥, as claimed.   
然而，根据第28.13号提案，我们有dim（u v）=codim（（u v）），因此codim（w）=codim（（u v）），既然w w w=（u v），我们必须有w=（u v），如所声称的。

In view of Proposition 28.12, we can make the following definition.  
根据28.12号提案，我们可以作出以下定义。

Definition 28.13. Let ϕ: E × F → K be any sesquilinear form. If E/F ⊥ and F/E⊥ are finite-dimensional, then their common dimension is called the rank of the form ϕ. If E/F ⊥ and F/E⊥ have infinite dimension, we say that ϕ has infinite rank.  
定义28.13.设\_：e×f→k为任意倍线性形式。如果e/f和f/e是有限维，那么它们的共同维数称为形式的秩。如果e/f和f/e有无穷大的维数，我们称之为\_有无穷大的秩。

Not surprisingly, the rank of ϕ is related to the ranks of lϕ and rϕ.  
不奇怪，\_的等级与l\_和r\_的等级有关。

Proposition 28.15. Let ϕ: E × F → K be any sesquilinear form. If ϕ has finite rank r, then lϕ and rϕ have the same rank, which is equal to r.  
提案28.15。设\_：e×f→k为任意倍线性形式。如果\_具有有限等级R，则l\_和r\_具有相同等级，等于r。

Proof. Because for every u ∈ E,  
证据。因为对于每一个u∈e，

lϕ(u)(y) = ϕ(u,y) for all y ∈ F,  
l\_（u）（y）=（u，y）表示所有y∈f，

and for every v ∈ F, rϕ(v)(x) = ϕ(x,v) for all x ∈ E,  
对于所有x∈e，对于每个v∈f，r\_（v）（x）=（x，v）

it is clear that the kernel of lϕ : E → F ∗ is equal to F ⊥ and that, the kernel of rϕ : F → E∗ is equal to E⊥. Therefore, rank(lϕ) = dim(Imlϕ) = dim(E/F ⊥) = r, and similarly rank(rϕ) = dim(F/E⊥) = r.   
很明显，l\_：e→f的核等于f\_，r\_：f→e\_的核等于e\_。因此，等级（L\_）=dim（iml）=dim（e/f）=r，类似等级（r）=dim（f/e）=r。

Remark: If the sesquilinear form ϕ is represented by the matrix n × m matrix M with respect to the bases (e1,...,em) in E and (f1,...,fn) in F, it can be shown that the matrix representing lϕ with respect to the bases (e1,...,em) and (, and that the matrix representing rϕ with respect to the bases (f1,...,fn) and (. It follows that the rank of ϕ is equal to the rank of M.  
注：如果倍线性形式\_由矩阵n×m表示，矩阵m关于e中的基（e1，…，em）和f中的（f1，…，fn），则可以证明矩阵代表l\_关于基（e1，…，em）和（，矩阵代表r\_关于baSES（F1，…，FN）和（.由此可知，\_的等级等于m的等级。

## 28.4 Adjoint of a Linear Map 28.4线性图的伴随

Let E1 and E2 be two K-vector spaces, and let ϕ1 : E1×E1 → K be a sesquilinear form on E1 and ϕ2 : E2 ×E2 → K be a sesquilinear form on E2. It is also possible to deal with the more general situation where we have four vector spaces E1,F1,E2,F2 and two sesquilinear forms ϕ1 : E1 ×F1 → K and ϕ2 : E2 ×F2 → K, but we will leave this generalization as an exercise.  
设e1和e2为两个k向量空间，设\_:e1×e1→k为e1上的倍线性形式，\_:e2×e2→k为e2上的倍线性形式。也可以处理更一般的情况，即我们有四个向量空间e1、f1、e2、f2和两个倍线性形式\_:e1×f1→k和\_:e2×f2→k，但我们将把这个推广留作练习。

We also assume that lϕ1 and rϕ1 are bijective, which implies that that ϕ1 is nondegenerate. This is automatic if the space E1 is finite dimensional and ϕ1 is nondegenerate.  
我们还假设l\_和r\_是双射的，这意味着\_是非退化的。如果空间e1为有限维，而\_为非退化空间，则此操作为自动操作。

Given any linear map f : E1 → E2, for any fixed u ∈ E2, we can consider the linear form in given by x 7→ ϕ2(f(x),u), x ∈ E1.  
给定任意一个线性映射f:e1→e2，对于任意一个固定的u∈e2，我们可以考虑x 7→\_（f（x），u），x∈e1给出的线性形式。

Since is bijective, there is a unique y ∈ E1 (because the vector spaces E1 and  
由于是双目标的，所以有一个唯一的y∈e1（因为向量空间e1和

E1 only differ by scalar multiplication), so that  
e1只因标量乘法而不同），因此

ϕ2(f(x),u) = ϕ1(x,y), for all x ∈ E1.  
\_（f（x），u）=\_（x，y），对于所有x∈e1。

If we denote this unique y ∈ E1 by f∗l(u), then we have  
如果我们用f l（u）表示这个唯一的y∈e1，那么我们有

ϕ2(f(x),u) = ϕ1(x,f∗l(u)), for all x ∈ E1, and all u ∈ E2.  
\_（f（x），u）=\_（x，f l（u）），对于所有x∈e1和所有u∈e2。

Thus, we get a function f∗l : E2 → E1. We claim that this function is a linear map. For any v1,v2 ∈ E2, we have  
因此，我们得到一个函数f l:e2→e1。我们声称这个函数是一个线性映射。对于任何v1，v2∈e2，我们有

ϕ2(f(x),v1 + v2) = ϕ2(f(x),v1) + ϕ2(f(x),v2)  
\_（f（x），v1+v2）=\_（f（x），v1）+\_（f（x），v2）

= ϕ1(x,f∗l(v1)) + ϕ1(x,f∗l(v2))  
=\_1（x，f\_l（v1））+\_1（x，f\_l（v2））

= ϕ1(x,f∗l(v1) + f∗l(v2)) = ϕ1(x,f∗l(v1 + v2)),  
=\_1（x，f l（v1）+f l（v2））=\_1（x，f l（v1+v2）），

for all x ∈ E1. Since rϕ1 is injective, we conclude that  
对于所有x∈e1。既然R\_1是注射剂，我们得出结论：

f∗l(v1 + v2) = f∗l(v1) + f∗l(v2).  
f l（v1+v2）=f l（v1）+f l（v2）。

For any λ ∈ K, we have  
对于任何λ∈k，我们有

,  
，

for all x ∈ E1. Since rϕ1 is injective, we conclude that  
对于所有x∈e1。既然R\_1是注射剂，我们得出结论：

f∗l(λv) = λf∗l(v).  
f l（λv）=λf l（v）。

28.4. ADJOINT OF A LINEAR MAP  
28.4。线性映射的伴随

Therefore, f∗l is linear. We call it the left adjoint of f.  
因此，f l是线性的。我们称之为f的左伴随。

Now, for any fixed u ∈ E2, we can consider the linear form in given by  
现在，对于任何固定的u∈e2，我们可以考虑下式中的线性形式。

x 7→ ϕ2(u,f(x)) x ∈ E1.  
x 7→\_（u，f（x））x∈e1.

Since is bijective, there is a unique y ∈ E1 so that  
因为是双目标的，所以有一个唯一的y∈e1，所以

ϕ2(u,f(x)) = ϕ1(y,x), for all x ∈ E1.  
\_（u，f（x））=\_（y，x），对于所有x∈e1。

If we denote this unique y ∈ E1 by f∗r(u), then we have  
如果我们用f r（u）表示这个唯一的y∈e1，那么我们有

ϕ2(u,f(x)) = ϕ1(f∗r(u),x), for all x ∈ E1, and all u ∈ E2.  
\_（u，f（x））=\_（f r（u），x），对于所有x∈e1和所有u∈e2。

Thus, we get a function f∗r : E2 → E1. As in the previous situation, it easy to check that f∗r is linear. We call it the right adjoint of f. In summary, we make the following definition.  
因此，我们得到一个函数f r:e2→e1。和前面的情况一样，很容易检查f r是线性的。我们称之为f的右伴随。总之，我们做出如下定义。

Definition 28.14. Let E1 and E2 be two K-vector spaces, and let ϕ1 : E1 × E1 → K and ϕ2 : E2 × E2 → K be two sesquilinear forms. Assume that lϕ1 and rϕ1 are bijective, so that ϕ1 is nondegnerate. For every linear map f : E1 → E2, there exist unique linear maps f∗l : E2 → E1 and f∗r : E2 → E1, such that  
定义28.14.设e1和e2为两个k向量空间，设\_:e1×e1→k和\_:e2×e2→k为两个倍线性形式。假设l\_1和r\_1是双射的，因此，\_1是不偏袒的。对于每个线性映射f:e1→e2，都存在唯一的线性映射f l:e2→e1和f r:e2→e1，这样

ϕ2(f(x),u) = ϕ1(x,f∗l(u)), for all x ∈ E1, and all u ∈ E2 ϕ2(u,f(x)) = ϕ1(f∗r(u),x), for all x ∈ E1, and all u ∈ E2.  
\_（f（x），u）=\_（x，f l（u）），对于所有x∈e1，并且所有u∈e2（u，f（x））=\_（f r（u），x），对于所有x∈e1，和所有u∈e2。

The map f∗l is called the left adjoint of f, and the map f∗r is called the right adjoint of f.  
图f\_l称为f的左伴随，图f\_r称为f的右伴随。

If E1 and E2 are finite-dimensional with bases (e1,...,em) and (f1,...,fn), then we can work out the matrices A∗l and A∗r corresponding to the left adjoint f∗l and the right adjoint f∗r of f. Assumine that f is represented by the n × m matrix A, ϕ1 is represented by the m × m matrix M1, and ϕ2 is represented by the n × n matrix M2. Since  
如果e1和e2是基（e1，…，em）和（f1，…，f n）的有限维，那么我们就可以算出对应于f的左伴f l和右伴f r的矩阵a l和a r。假设f由n×m矩阵a表示，那么，ω1由m×m矩阵m1表示，\_用n×n矩阵m2表示。自从

ϕ1(x,f∗l(u)) = (A∗lu)∗M1x = u∗(A∗l)∗M1x  
⑨1（x，f l（u））=（a lu）m1x=u（a l）m1x

ϕ2(f(x),u) = u∗M2Ax  
\_（f（x），u）=u m2ax

we find that (A∗l)∗M1 = M2A, that is (A∗l)∗ = M2AM1−1, and similarly  
我们发现（a l）m1=m2a，即（a l）=m2am1−1，并且类似地

ϕ1(f∗r(u),x) = x∗M1A∗ru  
\_1（f\_r（u），x）=x m1a\_ru

ϕ2(u,f(x)) = (Ax)∗M2u = x∗A∗M2u,  
\_（u，f（x））=（ax）m2u=x a m2u，

we have M1A∗r = A∗M2, that is A∗r = (M1)−1A∗M2. Thus, we obtain  
我们有m1 a r=a m2，即a r=（m1）−1a m2。因此，我们得到

A∗l = (M1∗)−1A∗M2∗  
a l=（m1）−1a m2

A∗r = (M1)−1A∗M2.  
a r=（m1）−1a m2。

If ϕ1 and ϕ2 are symmetric bilinear forms, then f∗l = f∗r. This also holds if ϕ is -Hermitian. Indeed, since ϕ2(u,f(x)) = ϕ1(f∗r(u),x),  
如果\_1和\_2是对称双线性形式，则f l=f r。如果\_是-厄米提安，这也适用。事实上，由于\_（u，f（x））=\_（f r（u），x），

we get  
我们得到

,  
，

and since λ 7→ λ is an involution, we get  
既然λ7→λ是对合的，我们得到

ϕ2(f(x),u) = ϕ1(x,f∗r(u)).  
\_（f（x），u）=\_（x，f r（u））。

Since we also have ϕ2(f(x),u) = ϕ1(x,f∗l(u)),  
因为我们还有\_（f（x），u）=\_（x，f l（u）），

we obtain ϕ1(x,f∗r(u)) = ϕ1(x,f∗l(u)) for all x ∈ E1, and all u ∈ E2,  
对于所有x∈e1和所有u∈e2，我们得到\_（x，f r（u））=\_（x，f l（u）），

and since ϕ1 is nondegenerate, we conclude that f∗l = f∗r. Whenever f∗l = f∗r, we use the simpler notation f∗.  
既然\_1是非退化的，我们得出的结论是f l=f r。每当f l=f r时，我们使用更简单的符号f。

If f : E1 → E2 and g: E1 → E2 are two linear maps, we have the following properties:  
如果f:e1→e2和g:e1→e2是两个线性映射，则我们具有以下特性：

1. + g)∗l = f∗l + g∗l id∗l = id  
   +g）l=f l+g l id l=id

(λf)∗l = λf∗l,  
（λf）l=λf l，

and similarly for right adjoints. If E3 is another space, ϕ3 is a sesquilinear form on E3, and if lϕ2 and rϕ2 are bijective, then for any linear maps f : E1 → E2 and g: E2 → E3, we have  
同样地，对于正确的邻接。如果e3是另一个空间，那么，在e3上，\_是一个双线性形式，如果l\_和r\_是双射的，那么对于任何线性映射f:e1→e2和g:e2→e3，我们有

1. ◦ f)∗l = f∗l ◦ g∗l,  
   \_f）l=f l g l，

and similarly for right adjoints. Furthermore, if E1 = E2 = E andl ϕ: rE × E → K is  
同样地，对于正确的邻接。此外，如果e1=e2=e和l\_：re×e→k为

-Hermitian, for any linear map f : E → E (recall that in this case f∗ = f∗ = f∗), we have  
-Hermitian，对于任何线性映射f:e→e（回想一下，在本例中f=f=f），我们有



## 28.5 Isometries Associated with Sesquilinear Forms 28.5与倍线性形式相关的等轴测图

The notion of adjoint is a good tool to investigate the notion of isometry between spaces equipped with sesquilinear forms. First, we define metric maps and isometries.  
伴随概念是研究具有倍线性形式的空间之间等距概念的一个很好的工具。首先，我们定义度量图和等轴测图。

Definition 28.15. If (E1,ϕ1) and (E2,ϕ2) are two pairs of spaces and sesquilinear maps ϕ1 : E1 × E1 → K and ϕ2 : E2 × E2 → K, a metric map from (E1,ϕ1) to (E2,ϕ2) is a linear map f : E1 → E2 such that  
定义28.15.如果（e1，\_）和（e2，\_）是两对空间和双线性映射，那么（e1，\_）到（e2，\_）的公制映射是线性映射f:e1→e2，因此

ϕ1(u,v) = ϕ2(f(u),f(v)) for all u,v ∈ E1.  
所有u，v∈e1的\_（u，v）=\_（f（u），f（v））。

We say that ϕ1 and ϕ2 are equivalent iff there is a metric map f : E1 → E2 which is bijective.  
我们说，如果有一个公制图f:e1→e2是双射的，那么\_和\_是等效的。

Such a metric map is called an isometry.  
这样一个度量图被称为等距图。

28.5. ISOMETRIES ASSOCIATED WITH SESQUILINEAR FORMS  
28.5。与倍线性形式相关的等轴测图

The problem of classifying sesquilinear forms up to equivalence is an important but very difficult problem. Solving this problem depends intimately on properties of the field K, and a complete answer is only known in a few cases. The problem is easily solved for K = R, K = C. It is also solved for finite fields and for K = Q (the rationals), but the solution is surprisingly involved!  
将倍线性形式分类到等价形式是一个重要而困难的问题。解决这个问题与K场的性质密切相关，只有在少数情况下才知道完整的答案。对于k=r，k=c，这个问题很容易解决。对于有限域和k=q（有理数），这个问题也可以解决，但令人惊讶的是，这个问题涉及到了这个问题！

It is hard to say anything interesting if ϕ1 is degenerate and if the linear map f does not have adjoints. The next few propositions make use of natural conditions on ϕ1 that yield a useful criterion for being a metric map.  
很难说有什么有趣的东西，如果ω1退化，如果线性映射f没有邻接。接下来的几个命题利用了第5.1条的自然条件，得出了作为度量图的有用标准。

Proposition 28.16. With the same assumptions as in Definition 28.14 (which imply that ϕ1 is nondegenerate), if f : E1 → E2 is a bijective linear map, then we have  
提案28.16。如果F:e1→e2是一个双射线性映射，那么我们有

ϕ1(x,y) = ϕ2(f(x),f(y)) for all x,y ∈ E1 iff f−1 = f∗l = f∗r.  
所有x，y∈e1 iff−1=f l=f r时，\_（x，y）=\_（f（x），f（y））。

Proof. We have ϕ1(x,y) = ϕ2(f(x),f(y))  
证据。我们有\_（x，y）=\_（f（x），f（y））

iff ϕ1(x,y) = ϕ2(f(x),f(y)) = ϕ1(x,f∗l(f(y))  
iff\_1（x，y）=\_（f（x），f（y））=\_（x，f\_l（f（y））

iff  
敌我识别

ϕ1(x,(id − f∗l ◦ f)(y)) = 0 for all x ∈ E1 and all y ∈ E2.  
所有x e1和所有y e2中，1（x，（id−f l f）（y））=0。

Since ϕ1 is nondegenerate, we must have  
既然\_是非退化的，我们必须

f∗l ◦ f = id,  
F L F=身份证，

which implies that f−1 = f∗l. Similarly,  
这意味着f−1=f l。类似地，

ϕ1(x,y) = ϕ2(f(x),f(y))  
\_（x，y）=\_（f（x），f（y））

iff ϕ1(x,y) = ϕ2(f(x),f(y)) = ϕ1(f∗r(f(x)),y)  
iff\_1（x，y）=\_（f（x），f（y））=\_（f r（f（x）），y）

iff  
敌我识别

ϕ1((id − f∗r ◦ f)(x),y) = 0 for all x ∈ E1 and all y ∈ E2.  
⑨1（（id−f r f）（x），y）=0，表示所有x∈e1和所有y∈e2。

Since ϕ1 is nondegenerate, we must have  
既然\_是非退化的，我们必须

f∗r ◦ f = id,  
F R F=ID，

which implies that f−1 = f∗r. Therefore, f−1 = f∗l = f∗r. For the converse, do the computations in reverse.   
这意味着f−1=f r。因此，f−1=f l=f r。相反，进行反向计算。

As a corollary, we get the following important proposition.  
作为推论，我们得到以下重要命题。

Proposition 28.17. If ϕ: E ×E → K is a sesquilinear map, and if lϕ and rϕ are bijective, for every bijective linear map f : E → E, then we have  
提案28.17。如果ω：e×e→k是一个双线性映射，并且如果l瓒和r瓒是双线性映射，那么对于每个双线性映射f:e→e，我们有

ϕ(f(x),f(y)) = ϕ(x,y) for all x,y ∈ E iff f−1 = f∗l = f∗r.  
所有x，y∈e iff f−1=f l=f r时，（f（x），f（y））=（x，y）。

We also have the following facts.  
我们还有以下事实。

Proposition 28.18. (1) If ϕ: E × E → K is a sesquilinear map and if lϕ is injective, then for every linear map f : E → E, if  
提案28.18。（1）如果ω：e×e→k是一个双线性映射，如果lω是内射的，那么对于每个线性映射f:e→e，如果

ϕ(f(x),f(y)) = ϕ(x,y) for all x,y ∈ E, (∗)  
所有x，y∈e，（）

then f is injective.  
那么F是注射剂。

(2) If E is finite-dimensional and if ϕ is nondegenerate, then the linear maps f : E → E satisfying (∗) form a group. The inverse of f is given by f−1 = f∗.  
（2）如果e是有限维，如果\_是非退化的，那么线性映射f:e→e满足（）形成一个群。f的倒数由f−1=f给出。

Proof. (1) If f(x) = 0, then  
证据。（1）如果f（x）=0，则

ϕ(x,y) = ϕ(f(x),f(y)) = ϕ(0,f(y)) = 0 for all y ∈ E.  
所有y∈e时，ω（x，y）＝（f（x），f（y））＝（0，f（y））=0。

Since lϕ is injective, we must have x = 0, and thus f is injective.  
既然l\_是注射剂，我们必须有x=0，因此f是注射剂。

(2) If E is finite-dimensional, since a linear map satisfying (∗) is injective, it is a bijection.  
（2）如果e是有限维，因为满足（）的线性映射是内射的，它是双射的。

By Proposition 28.17, we have f−1 = f∗. We also have ϕ(f(x),f(y)) = ϕ((f∗ ◦ f)(x),y) = ϕ(x,y) = ϕ((f ◦ f∗)(x),y) = ϕ(f∗(x),f∗(y)),  
根据命题28.17，我们得到f−1=f。我们还有（f（x），f（y））=（f（f）（x），y）=（x，y）=（f f）（x），y）=（f（x），f（y）），

which shows that f∗ satisfies (∗). If ϕ(f(x),f(y)) = ϕ(x,y) for all x,y ∈ E and ϕ(g(x),g(y))  
这表明f满足（）。如果所有x，y∈e和（g（x），g（y））=\_（x，y），则

= ϕ(x,y) for all x,y ∈ E, then we have  
=（x，y）对于所有x，y∈e，那么我们有

ϕ((g ◦ f)(x),(g ◦ f)(y)) = ϕ(f(x),f(y)) = ϕ(x,y) for all x,y ∈ E.  
所有x，y∈e的（（g f）（x），（g f）（y））=\_（f（x），f（y））=（x，y）。

Obviously, the identity map idE satisfies (∗). Therefore, the set of linear maps satisfying (∗) is a group.  
显然，标识映射ide满足（）。因此，满足（）的线性映射集是一组。

The above considerations motivate the following definition.  
上述考虑激发了以下定义。

Definition 28.16. Let ϕ: E × E → K be a sesquilinear map, and assume that E is finitedimensional and that ϕ is nondegenerate. A linear map f : E → E is an isometry of E (with respect to ϕ) iff ϕ(f(x),f(y)) = ϕ(x,y) for all x,y ∈ E.  
定义28.16.设\_：e×e→k为倍线性映射，假设e为有限维，\_为非退化。线性图f:e→e是e（关于\_）iff\_（f（x），f（y））=\_（x，y）的等距图，适用于所有x，y∈e。

The set of all isometries of E is a group denoted by Isom(ϕ).  
e的所有等轴测集是一组由isom（a）表示的组。

28.5. ISOMETRIES ASSOCIATED WITH SESQUILINEAR FORMS  
28.5。与倍线性形式相关的等轴测图

If ϕ is symmetric, then the group Isom(ϕ) is denoted O(ϕ) and called the orthogonal group of ϕ. If ϕ is alternating, then the group Isom(ϕ) is denoted Sp(ϕ) and called the symplectic group of -Hermitian, then the group Isom(ϕ) is denoted U) and called the -unitary group of ϕ. When = 1, we drop and just say unitary group.  
如果直径对称，则组isom（直径）表示为o（直径），称为直径的正交组。如果\_是交替的，则组isom（\_）表示为sp（\_），称为-Hermitian的辛群，则组isom（\_）表示为u）并称为\_的-单一群。当=1时，我们放弃，只说一元群。

If (e1,...,en) is a basis of E, ϕ is the represented by thel r n × n matrix M, and f is represented by the n × n matrix A, since A−1 = A∗ = A∗ = M−1A∗M, then we find that f ∈ Isom(ϕ) iff  
如果（e1，…，e n）是e的一个基，那么\_是由thel r n×n矩阵m表示的，而f是由n×n矩阵a表示的，因为a−1=a=a=m−1a m，那么我们发现f∈isom（）iff

A∗MA = M,  
a ma=m，

and A−1 is given by A−1 = M−1A∗M.  
a−1由a−1=m−1a m给出。

More specifically, we define the following groups, using the matrices Ip,q,Jm,m and Am,m defined at the end of Section 28.1.  
更具体地说，我们使用第28.1节末尾定义的矩阵ip、q、jm、m和am、m来定义以下组。

1. K = R. We have  
   K=R，我们有

O(n) = {A ∈ Mn(R) | A>A = In}  
o（n）=a∈mn（r）a>a=in

O(p,q) = {A ∈ Mp+q(R) | A>Ip,qA = Ip,q}  
o（p，q）=a∈mp+q（r）a>ip，qa=ip，q

Sp(2n,R) = {A ∈ M2n(R) | A>Jn,nA = Jn,n}  
sp（2n，r）=a∈m2n（r）a>jn，na=jn，n

SO(n) = {A ∈ Mn(R) | A>A = In, det(A) = 1}  
所以（n）=a∈mn（r）a>a=in，det（a）=1

SO(p,q) = {A ∈ Mp+q(R) | A>Ip,qA = Ip,q, det(A) = 1}.  
所以（p，q）=a∈mp+q（r）a>ip，qa=ip，q，det（a）=1。

The group O(n) is the orthogonal group, Sp(2n,R) is the real symplectic group, and SO(n) is the special orthogonal group. We can define the group  
群O（n）是正交群，sp（2n，r）是实辛群，所以（n）是特殊正交群。我们可以定义这个组

{A ∈ M2n(R) | A>An,nA = An,n},  
a∈m2n（r）a>a n，na=an，n，

but it is isomorphic to O(n,n).  
但它与O（N，N）同构。

1. K = C. We have  
   K=C，我们有

U(n) = {A ∈ Mn(C) | A∗A = In}  
u（n）=a∈mn（c）a a=in

U(p,q) = {A ∈ Mp+q(C) | A∗Ip,qA = Ip,q}  
u（p，q）=a∈mp+q（c）a ip，qa=ip，q

Sp(2n,C) = {A ∈ M2n(C) | A>Jn,nA = Jn,n}  
sp（2n，c）=a∈m2n（c）a>jn，na=jn，n

SU(n) = {A ∈ Mn(C) | A∗A = In, det(A) = 1}  
su（n）=a∈mn（c）a a=in，det（a）=1

SU(p,q) = {A ∈ Mp+q(C) | A∗Ip,qA = Ip,q, det(A) = 1}.  
su（p，q）=a∈mp+q（c）a ip，qa=ip，q，det（a）=1。

The group U(n) is the unitary group, Sp(2n,C) is the complex symplectic group, and SU(n) is the special unitary group.  
群U（n）是一元群，sp（2n，c）是复辛群，su（n）是特殊的一元群。

It can be shown that if A ∈ Sp(2n,R) or if A ∈ Sp(2n,C), then det(A) = 1.  
可以看出，如果a∈sp（2n，r）或a∈sp（2n，c），那么det（a）=1。

## 28.6 Totally Isotropic Subspaces 28.6完全各向同性子空间

In this section, we deal with -Hermitian forms, ϕ: E × E → K. In general, E may have subspaces U such that U ∩ U⊥ = (0)6 , or worse, such that U ⊆ U⊥ (that is, ϕ is zero on U). We will see that such subspaces play a crucial in the decomposition of E into orthogonal subspaces.  
在本节中，我们讨论了-厄米特形式，:e×e→k。一般来说，e可能有子空间u，使得u（0）6，或更糟，这样u u（即，\_在u上为零）。我们将看到这些子空间在将e分解为正交子空间中起着关键作用。

Definition 28.17. Given an -Hermitian forms ϕ: E × E → K, a nonzero vector u ∈ E is said to be isotropic if ϕ(u,u) = 0. It is convenient to consider 0 to be isotropic. Given any subspace U of E, the subspace rad(U) = U ∩ U⊥ is called the radical of U. We say that  
定义28.17.给定一个-厄米式，如果\_（u，u）=0，则非零矢量u∈e称为各向同性。将0考虑为各向同性是很方便的。对于e的任何子空间u，子空间rad（u）=u u称为u的根式。

1. U is degenerate if rad(U) = (0) (6 equivalently if there is some nonzero vector u ∈ U such that x ∈ U⊥). Otherwise, we say that U is nondegenerate.  
   如果rad（u）=（0）（6等价，如果有一些非零向量u∈u，那么u是退化的，这样x∈u）。否则，我们就说u是非退化的。
2. U is totally isotropic if U ⊆ U⊥ (equivalently if the restriction of ϕ to U is zero).  
   如果u u，u是完全各向同性的（如果\_到u的限制为零，则等于）。

By definition, the trivial subspace U = (0) (= {0}) is nondegenerate. Observe that a subspace U is nondegenerate iff the restriction of ϕ to U is nondegenerate. A degenerate subspace is sometimes called an isotropic subspace. Other authors say that a subspace U is isotropic if it contains some (nonzero) isotropic vector. A subspace which has no nonzero isotropic vector is often called anisotropic. The space of all isotropic vectors is a cone often called the light cone (a terminology coming from the theory of relativity). This is not to be confused with the cone of silence (from Get Smart)! It should also be noted that some authors (such as Serre) use the term isotropic instead of totally isotropic. The apparent lack of standard terminology is almost as bad as in graph theory!  
根据定义，平凡子空间u=（0）（=0）是非退化的。观察到子空间u是非简并的，而如果对u的限制是非简并的。退化子空间有时称为各向同性子空间。其他作者说，如果子空间U包含一些（非零）各向同性向量，则它是各向同性的。没有非零各向同性向量的子空间通常称为各向异性。所有各向同性向量的空间都是一个圆锥，通常称为光锥（一个术语来自相对论）。不要把这和沉默的锥体混淆（从聪明做起）！还应注意的是，有些作者（如SERRE）使用了“各向同性”一词，而不是“完全各向同性”。明显缺乏标准术语几乎和图论一样糟糕！

It is clear that any direct sum of pairwise orthogonal totally isotropic subspaces is totally isotropic. Thus, every totally isotropic subspace is contained in some maximal totally isotropic subspace. Here is another fact that we will use all the time: if V is a totally isotropic subspace and if U is a subspace of V , then U is totally isotropic.  
很明显，任何成对正交完全各向同性子空间的直接和都是完全各向同性的。因此，每个完全各向同性子空间都包含在一些最大的完全各向同性子空间中。还有一个事实，我们将一直使用：如果v是一个完全各向同性的子空间，如果u是v的子空间，那么u是完全各向同性的。

This is because by definition V is isotropic if V ⊆ V ⊥, and since U ⊆ V we get V ⊥ ⊆ U⊥, so U ⊆ V ⊆ V ⊥ ⊆ U⊥, which shows that U is totally isotropic.  
这是因为根据定义，如果v v，v是各向同性的，既然u，我们得到v u，那么u v u，这表明u是完全各向同性的。

First, let us show that in order to sudy an -Hermitian form on a space E, it suffices to restrict our attention to nondegenerate forms.  
首先，让我们证明，为了在空间E上苏迪安-赫米特形式，它足以限制我们对非退化形式的关注。

Proposition 28.19. Given an -Hermitian form ϕ: E × E → K on E, we have:  
提案28.19。给定一个-厄米提亚式，在e上：e×e→k，我们有：

1. If U and V are any two orthogonal subspaces of E, then  
   如果u和v是e的任意两个正交子空间，那么

rad(U + V ) = rad(U) + rad(V ).  
rad（u+v）=rad（u）+rad（v）。

1. rad(rad(E)) = rad(E).  
   rad（rad（e））=rad（e）。
2. If U is any subspace supplementary to rad(E), so that  
   如果u是rad（e）的补充子空间，那么

E = rad(E) ⊕ U,  
e=rad（e）u，

then U is nondegenerate, and rad(E) and U are orthogonal.  
那么u是非退化的，rad（e）和u是正交的。

Proof. (a) If U and V are orthogonal, then U ⊆ V ⊥ and V ⊆ U⊥. We get  
证据。（a）如果u和v是正交的，则u v和v u。我们得到

rad(U + V ) = (U + V ) ∩ (U + V )⊥  
rad（u+v）=（u+v）（u+v）

= (U + V ) ∩ U⊥ ∩ V ⊥  
=（U+V）U V

= U ∩ U⊥ ∩ V ⊥ + V ∩ U⊥ ∩ V ⊥  
=U U V+V U V

= U ∩ U⊥ + V ∩ V ⊥  
=U U+V V

= rad(U) + rad(V ).  
=rad（u）+rad（v）。

1. By definition, rad(E) = E⊥, and obviously E = E⊥⊥, so we get  
   根据定义，rad（e）=e，显然e=e，所以我们得到

rad(rad(E)) = E⊥ ∩ E⊥⊥ = E⊥ ∩ E = E⊥ = rad(E).  
rad（rad（e））=e=e=rad（e）。

1. If E = rad(E)⊕U, by definition of rad(E), the subspaces rad(E) and U are orthogonal. From (a) and (b), we get rad(E) = rad(E) + rad(U).  
   如果e=rad（e）u，根据rad（e）的定义，子空间rad（e）和u是正交的。从（a）和（b），我们得到rad（e）=rad（e）+rad（u）。

Since rad(U) = U ∩ U⊥ ⊆ U and since rad(E) ⊕ U is a direct sum, we have a direct sum  
因为rad（u）=u u u和rad（e）u是一个直接和，所以我们有一个直接和

rad(E) = rad(E) ⊕ rad(U),  
rad（e）=rad（e）rad（u）、

which implies that rad(U) = (0); that is, U is nondegenerate.   
这意味着rad（u）=0；也就是说，u是非退化的。

Proposition 28.19(c) shows that the restriction of ϕ to any supplement U of rad(E) is nondegenerate and ϕ is zero on rad(U), so we may restrict our attention to nondegenerate forms.  
命题28.19（c）表明，对rad（e）的任何补充u的限制为非退化形式，而对rad（u）的限制为零，因此我们可以将注意力限制在非退化形式。

The following is also a key result.  
以下也是一个关键结果。

Proposition 28.20. Given an -Hermitian form ϕ: E × E → K on E, if U is a finitedimensional nondegenerate subspace of E, then E = U ⊕ U⊥.  
提案28.20。在e上给出了一个-厄米式，如果u是e的有限维非退化子空间，那么e=u u。

Proof. By hypothesis, the restriction ϕU of ϕ to U is nondegenerate, so the semilinear map rϕU : U → U∗ is injective. Since U is finite-dimensional, rϕU is actually bijective, so for every v ∈ E, if we consider the linear form in U∗ given by u 7→ ϕ(u,v) (u ∈ U), there is a unique v0 ∈ U such that ϕ(u,v0) = ϕ(u,v) for all u ∈ U;  
证据。根据假设，对u的限制是非退化的，所以半线性映射r\_u:u→u\_是内射的。由于u是有限维的，r\_u实际上是双射的，所以对于每一个v∈e，如果我们考虑u中由u 7给出的线性形式（u，v）（u∈u），就有一个唯一的v0∈u，使得所有u∈u的\_（u，v0）=（u，v）；

that is, ϕ(u,v − v0) = 0 for all u ∈ U, so v − v0 ∈ U⊥. It follows that v = v0 + v − v0, with v0 ∈ U and v0 −v ∈ U⊥, and since U is nondegenerate U ∩U⊥ = (0), and E = U ⊕U⊥.   
也就是说，对于所有u∈u，那么v−v0∈u来说，（u，v−v0）=0。由此可知，v=v0+v−v0，其中v0∈u和v0−v∈u，由于u是非退化的u u=（0），并且e=u u。

As a corollary of Proposition 28.20, we get the following result.  
作为28.20号命题的推论，我们得到如下结果。

Proposition 28.21. Given an -Hermitian form ϕ: E×E → K on E, if ϕ is nondegenerate and if U is a finite-dimensional subspace of E, then rad(U) = rad(U⊥), and the following conditions are equivalent:  
提案28.21。给定-Hermitian形式，在e上：e×e→k，如果\_是非退化的，并且如果u是e的有限维子空间，那么rad（u）=rad（u），并且下列条件是等效的：

1. U is nondegenerate.  
   u是非退化的。
2. U⊥ is nondegenerate.  
   U是非退化的。
3. E = U ⊕ U⊥.  
   E=U U。

Proof. By definition, rad(U⊥) = U⊥ ∩ U⊥⊥, and since ϕ is nondegenerate and U is finitedimensional, U⊥⊥ = U, so rad(U⊥) = U⊥ ∩ U⊥⊥ = U ∩ U⊥ = rad(U).  
证据。根据定义，rad（u）=u u，由于\_是非退化的，u是有限维，u=u，所以rad（u）=u=u=rad（u）。

By Proposition 28.20, (i) implies (iii). If E = U ⊕ U⊥, then rad(U) = U ∩ U⊥ = (0), so U is nondegenerate and (iii) implies (i). Since rad(U⊥) = rad(U), (iii) also implies (ii).  
根据28.20号提案，（i）暗示（iii）。如果e=u u，那么rad（u）=u u=（0），那么u是非退化的，并且（i i i）意味着（i）。因为rad（u）=rad（u），（iii）也意味着（ii）。

Now, if U⊥ is nondegenerate, we have U⊥ ∩ U⊥⊥ = (0), and since U ⊆ U⊥⊥, we get  
现在，如果u是非退化的，我们就得到u=（0），而且既然u u，我们得到

U ∩ U⊥ ⊆ U⊥⊥ ∩ U⊥ = (0),  
U U U=0，

which shows that U is nondegenerate, proving the implication (ii) =⇒ (i).   
这表明U是非简并的，证明其含义（i i）=⇒（i）。

If E is finite-dimensional, we have the following results.  
如果e是有限维的，我们得到以下结果。

Proposition 28.22. Given an -Hermitian form ϕ: E × E → K on a finite-dimensional space E, if ϕ is nondegenerate, then for every subspace U of E we have  
提案28.22。给定有限维空间e上的-Hermitian形式：e×e→k，如果该形式为非退化形式，则对于e的每个子空间u，我们都有

1. dim(U) + dim(U⊥) = dim(E).  
   dim（u）+dim（u）=dim（e）。
2. U⊥⊥ = U.  
   U=U.

Proof. (i) Since ϕ is nondegenerate and E is finite-dimensional, the semilinear map lϕ : E →  
证据。（i）由于\_是非退化的，e是有限维，半线性图l\_：e→

E∗ is bijective. By transposition, the inclusion i: U → E yields a surjection r: E∗ → U∗  
E是双主题的。通过转位，包涵体i:u→e产生一个推测r:e→u

(with r(f) = f ◦ i for every f ∈ E∗; the map f ◦ i is the restriction of the linear form f to  
（其中r（f）=f i对于每一个f∈e映射f i是线性形式f对

U). It follows that the semilinear map r ◦ lϕ : E → U∗ given by  
U）。由此可知，半线性图r\_l\_：e→u由下式给出

(r ◦ lϕ)(x)(u) = ϕ(x,u) x ∈ E,u ∈ U  
（r\_l\_）（x）（u）＝（x，u）x∈e，u∈u

is surjective, and its kernel is U⊥. Thus, we have  
是主观性的，它的核心是u。因此，我们

dim(U∗) + dim(U⊥) = dim(E),  
尺寸（U）+尺寸（U）=尺寸（E）

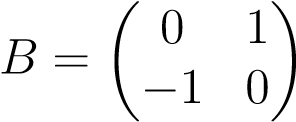
and since dim(U) = dim(U∗) because U is finite-dimensional, we get  
因为dim（u）=dim（u），因为u是有限维，我们得到

dim(U) + dim(U⊥) = dim(U∗) + dim(U⊥) = dim(E).  
dim（u）+dim（u）=dim（u）+dim（u）=dim（e）。

(ii) Applying the above formula to U⊥, we deduce that dim(U) = dim(U⊥⊥). Since U ⊆ U⊥⊥, we must have U⊥⊥ = U.   
（ii）将上述公式应用于u，我们推导出dim（u）=dim（u）。因为u u，我们必须有u=u。

Remark: We already proved in Proposition 28.13 that if U is finite-dimensional, then codim(U⊥) = dim(U) and U⊥⊥ = U, but it doesn’t hurt to give another proof. Observe that (i) implies that dim(U) + dim(rad(U)) ≤ dim(E).  
注：我们已经在28.13号命题中证明了，如果u是有限维的，那么codim（u）=dim（u）和u=u，但给出另一个证明并不伤人。观察（i）表示dim（u）+dim（rad（u））≤dim（e）。

We can now proceed with the Witt decomposition, but before that, we quickly take care of the structure theorem for alternating bilinear forms (the case where ϕ(u,u) = 0 for all u ∈ E). For an alternating bilinear form, the space E is totally isotropic. For example in dimension 2, the matrix  
我们现在可以继续进行维特分解，但在这之前，我们很快地处理了交替双线性形式的结构定理（其中，所有的u∈e，都是ω（u，u）=0的情况）。对于交替双线性形式，空间E是完全各向同性的。例如，在维度2中，矩阵



defines the alternating form given by  
定义由给定的交替形式

ϕ((x1,y1),(x2,y2)) = x1y2 − x2y1.  
⑨（（x1，y1），（x2，y2））=x1y2−x2y1。

This case is surprisingly general.  
这件事非常普遍。

Proposition 28.23. Let ϕ: E × E → K be an alternating bilinear form on E. If u,v ∈ E are two (nonzero) vectors such that ϕ(u,v) = λ = 06 , then u and v are linearly independent. If we let u1 = λ−1u and v1 = v, then ϕ(u1,v1) = 1, and the restriction of ϕ to the plane spanned by u1 and v1 is represented by the matrix  
提案28.23。设ω：e×e→k为e上的一个交替双线性形式，如果u，v∈e是两个（非零）向量，使得ω（u，v）=λ=06，则u和v是线性无关的。如果我们让u1=λ−1u和v1=v，那么（u1，v1）=1，并且\_对u1和v1所跨越平面的限制由矩阵表示。

.  
.

Proof. If u and v were linearly dependent, as u,v = 06 , we could write v = µu for some µ = 06 , but then, since ϕ is alternating, we would have  
证据。如果u和v是线性相关的，如u，v=06，我们可以为一些礹=06写v=礹u，但是，由于礹是交替的，我们将

λ = ϕ(u,v) = ϕ(u,µu) = µϕ(u,u) = 0,  
λ=\_（u，v）=（u，\_u）=（u，u）=0，

contradicting the fact that λ = 06 . The rest is obvious.   
与λ=06这一事实相矛盾。剩下的很明显。

Proposition 28.23 yields a plane spanned by two vectors u1,v1 such that ϕ(u1,u1) = ϕ(v1,v1) = 0 and ϕ(u1,v1) = 1. Such a plane is called a hyperbolic plane. If E is finitedimensional, we obtain the following theorem.  
命题28.23给出了一个由两个向量u1，v1所跨越的平面，使得（u1，u1）=（v1，v1）=0和（u1，v1）=1。这样的平面称为双曲面。如果e是有限维的，我们得到以下定理。

Theorem 28.24. Let ϕ: E × E → K be an alternating bilinear form on a space E of finite dimension n. Then, there is a direct sum decomposition of E into pairwise orthogonal subspaces  
定理28.24。设\_：e×e→k为有限维n空间e上的交替双线性形式，然后将e直接和分解为成对正交子空间。

E = W1 ⊕ ··· ⊕ Wr ⊕ rad(E),  
e=w1···wr rad（e）

where each Wi is a hyperbolic plane and rad(E) = E⊥. Therefore, there is a basis of E of the form  
其中，每个wi是一个双曲面，rad（e）=e。因此，形式的e有一个基础

(u1,v1,...,ur,vr,w1,...,wn−2r),  
（u1，v1，…，ur，vr，w1，…，wn−2r）

with respect to which the matrix representing ϕ is a block diagonal matrix M of the form  
关于该矩阵，表示\_的矩阵是形式的块对角矩阵m。

,  
，

with  
具有

Proof. If ϕ = 0, then E = E⊥ and we are done. Otherwise, there are two nonzero vectors u,v ∈ E such that ϕ(u,v) = 06 , so by Proposition 28.23, we obtain a hyperbolic plane W2 spanned by two vectors u1,v1 such that ϕ(u1,v1) = 1. The subspace W1 is nondegenerate (for example, det(J) = −1), so by Proposition 28.21, we get a direct sum  
证据。如果直径=0，那么e=e我们就完成了。否则，有两个非零向量u，v∈e，因此，根据命题28.23，我们得到一个由两个向量u1，v1所跨越的双曲面w2，这样，a（u1，v1）=1。子空间w1是非退化的（例如，det（j）=-1），因此根据命题28.21，我们得到一个直接和

.  
.

By Proposition 28.14, we also have  
根据28.14号提案，我们也有

.  
.

By the induction hypothesis applied to , we obtain our theorem.   
通过应用归纳假设，我们得到了我们的定理。

The following corollary follows immediately.  
下面的推论紧接着。

Proposition 28.25. Let ϕ: E × E → K be an alternating bilinear form on a space E of finite dimension n.  
提案28.25。设a:e×e→k为有限维n空间e上的交替双线性形式。

1. The rank of ϕ is even.  
   \_的等级是偶数。
2. If ϕ is nondegenerate, then dim(E) = n is even.  
   如果直径不退化，则dim（e）=n为偶数。
3. Two alternating bilinear forms ϕ1 : E1 ×E1 → K and ϕ2 : E2 ×E2 → K are equivalent iff dim(E1) = dim(E2) and ϕ1 and ϕ2 have the same rank.  
   两个交替双线性形式\_:e1×e1→k和\_:e2×e2→k是等效的iff dim（e1）=dim（e2）和\_和\_具有相同的等级。

The only part that requires a proof is part (3), which is left as an easy exercise.  
唯一需要证明的部分是第（3）部分，这是一个简单的练习。

If ϕ is nondegenerate, then n = 2r, and a basis of E as in Theorem 28.24 is called a symplectic basis. The space E is called a hyperbolic space (or symplectic space). Observe that if we reorder the vectors in the basis  
如果ω是非简并的，那么n=2r，定理28.24中e的基称为辛基。空间e称为双曲空间（或辛空间）。注意，如果我们重新排列基向量

(u1,v1,...,ur,vr,w1,...,wn−2r)  
（u1，v1，…，ur，vr，w1，…，wn-2r）

to obtain the basis  
获取基础

(u1,...,ur,v1,...vr,w1,...,wn−2r),  
（u1，…，ur，v1，…，vr，w1，…，wn−2r）

then the matrix representing ϕ becomes  
那么代表\_的矩阵变成

.  
.

This particularly simple matrix is often preferable, especially when dealing with the matrices (symplectic matrices) representing the isometries of ϕ (in which case n = 2r).  
这种特别简单的矩阵通常是可取的，尤其是在处理矩阵（辛矩阵）时，它代表了直径的等距（在这种情况下，n=2r）。

As a warm up for Proposition 28.29 of the next section, we prove an analog of Proposition  
作为下一节28.29号提案的热身，我们证明了一个类似的提案

28.23 in the case of a symmetric bilinear form.  
28.23对于对称双线性形式。

Proposition 28.26. Let ϕ: E×E → K be a nondegenerate symmetric bilinear form with K a field of characteristic different from 2. For any nonzero isotropic vector u, there is another nonzero isotropic vector v such that ϕ(u,v) = 2, and u and v are linearly independent. In the basis (u,v/2), the restriction of ϕ to the plane spanned by u and v/2 is of the form  
提案28.26。设\_：e×e→k为非退化对称双线性形式，k为特征场，与2不同。对于任何非零各向同性向量u，还有另一个非零各向同性向量v，使得（u，v）=2，u和v是线性无关的。在基础（u，v/2）中，对u和v/2所跨越平面的直径限制为：

.  
.

Proof. Since ϕ is nondegenerate, there is some nonzero vector z such that (rescaling z if necessary) ϕ(u,z) = 1. If v = 2z − ϕ(z,z)u,  
证据。由于\_是非退化的，因此有一些非零矢量z（必要时重新缩放z）\_（u，z）=1。如果v=2z（z，z）u，

then since ϕ(u,u) = 0 and ϕ(u,z) = 1, note that  
然后，由于\_（u，u）=0和（u，z）=1，注意

ϕ(u,v) = ϕ(u,2z − ϕ(z,z)u) = 2ϕ(u,z) − ϕ(z,z)ϕ(u,u) = 2,  
⑨（u，v）=⑨（u，2z（z，z）u）=2（u，z）−（z，z）（u，u）=2，

and  
和

ϕ(v,v) = ϕ(2z − ϕ(z,z)u,2z − ϕ(z,z)u) = 4ϕ(z,z) − 4ϕ(z,z)ϕ(u,z) + ϕ(z,z)2ϕ(u,u) = 4ϕ(z,z) − 4ϕ(z,z) = 0.  
⑨（v，v）＝（2z（z，z）U，2z（z，z）U）=4（z，z）−4（z，z）（u，z）+（z，z）2（u，u）=4（z，z）−4（z，z）=0.

If u and z were linearly dependent, as u,z = 06 , we could write z = µu for some µ = 06 , but then, we would have ϕ(u,z) = ϕ(u,µu) = µϕ(u,u) = 0,  
如果u和z是线性相关的，当u，z=06时，我们可以为一些祆=06写z=祆u，但随后，我们将得到祆（u，z）=祆（u，祆u）=0，

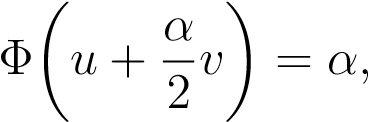
contradicting the fact that ϕ(u,z) = 06 . Then u and v = 2z − ϕ(z,z)u are also linearly independent, since otherwise z could be expressed as a multiple of u. The rest is obvious.   
与之相矛盾的是，\_（u，z）=06。那么u和v=2z（z，z）u也是线性独立的，因为否则z可以表示为u的倍数。其余的很明显。

Proposition 28.26 yields a plane spanned by two vectors u1,v1 such that ϕ(u1,u1) = ϕ(v1,v1) = 0 and ϕ(u1,v1) = 1. Such a plane is called an Artinian plane. Proposition 28.26 also shows that nonzero isotropic vectors come in pair.  
命题28.26给出了一个由两个向量u1，v1所跨越的平面，从而使得\_（u1，u1）=\_（v1，v1）=0和\_（u1，v1）=1。这样的一个平面称为Artian平面。命题28.26还表明非零各向同性向量是成对的。

Proposition 28.26 has the following corollary which has applications in number theory; see Serre [152], Chapter IV.  
命题28.26有以下推论，在数论中有应用；见Serre[152]，第四章。

Proposition 28.27. If Φ is any nondegenerate quadratic form (over a field of characteristic = 26 ) such that there is some nonzero vector x ∈ E with Φ(x) = 0, then for every α ∈ K, there is some y ∈ E such that Φ(y) = α.  
提案28.27。如果Φ是任何非退化二次型（在特征值为26的场上），那么有一些非零向量x∈e，其中Φ（x）=0，那么对于每个α∈k，有一些y∈e，这样Φ（y）=α。

Proof. Since by hypothesis there is some nonzero vector u ∈ E with Φ(u) = 0, by Proposition 28.26 there is another isotropic vector v such that u and v are linearly independent and such that (after rescaling) ϕ(u,v) = 1. Then for any α ∈ K, check that  
证据。由于假设存在一些非零向量u∈e，其中Φ（u）=0，根据命题28.26，存在另一个各向同性向量v，这样u和v是线性无关的，并且（重新缩放后）\_（u，v）=1。那么对于任何α∈k，检查



as desired.   
根据需要。

Remark: Some authors refer to the above plane as a hyperbolic plane. Berger (and others) point out that this terminology is undesirable because the notion of hyperbolic plane already exists in differential geometry and refers to a very different object.  
注：有些作者将上述平面称为双曲面。伯杰（和其他人）指出，这个术语是不可取的，因为双曲平面的概念已经存在于微分几何中，并且指的是一个非常不同的物体。

We leave it as an exercice to figure out that the group of isometries of the Artinian plane, the set of all 2 × 2 matrices A such that  
我们把它作为一个练习，来搞清楚亚第面的等轴测群，所有2×2矩阵的集合A，这样

,  
，

consists of all matrices of the form  
由形式的所有矩阵组成

or .  
或者。

In particular, if K = R, then this group denoted O(1,1) has four connected components.  
特别是，如果k=r，那么表示o（1，1）的这个组有四个相连的组件。

We now turn to the Witt decomposition.  
现在我们来谈谈维特分解。

## 28.7 Witt Decomposition 28.7维特分解

From now on, ϕ: E × E → K is an -Hermitian form. The following assumption will be needed:  
从现在开始，⑨：e×e→k是一个-赫敏形式。需要以下假设：

Property (T). For every u ∈ E, there is some α ∈ K such that.  
属性（t）。对于每一个u∈e，都有一些α∈k这样。

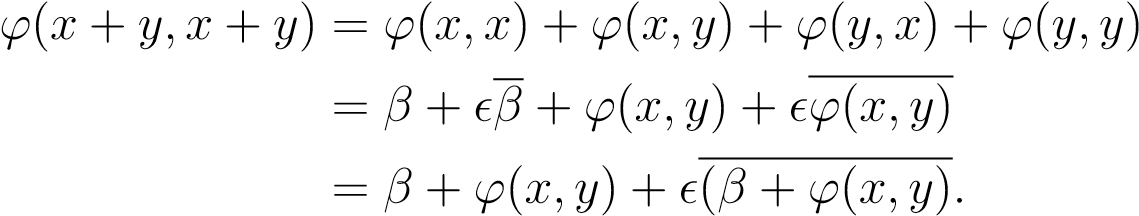
Property (T) is always satisfied if ϕ is alternating, or if K is of characteristic = 26 and 1, with   
如果\_为交替的，或如果k的特征值为26和1，则始终满足性能（t），且

The following (bizarre) technical lemma will be needed.  
需要以下（奇异的）技术引理。

Lemma 28.28. Let ϕ be an -Hermitian form on E and assume that ϕ satisfies property (T). For any totally isotropic subspace U = (0)6 of E, for every x ∈ E not orthogonal to U, and for every α ∈ K, there is some y ∈ U so that  
引理28.28。设a为e上的-厄米式形式，并假设a满足性质（t）。对于任意完全各向同性的子空间u=（0）6 of e，对于每一个不与u正交的x∈e，并且对于每一个α∈k，有一些y∈u，因此



Proof. By property (T), we have for some β ∈ K. For any y ∈ U, since ϕ is -Hermitian, ), and since U is totally isotropic ϕ(y,y) = 0, so we have  
证据。根据性质（t），我们有一些β∈k。对于任何y∈u，因为\_是-厄米特式的，），并且因为u是完全各向同性的（y，y）=0，所以我们有

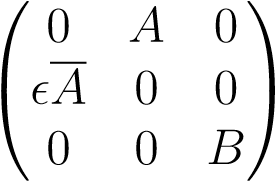


Since x is not orthogonal to U, the function y 7→ ϕ(x,y) + β is not the constant function. Consequently, this function takes the value α for some y ∈ U, which proves the lemma.   
由于x与u不正交，函数y 7→（x，y）+β不是常数函数。因此，这个函数取某个y∈u的α值，证明了引理。

Definition 28.18. Let ϕ be an -Hermitian form on E. A weak Witt decomposition of E is a triple (U,U0,W), such that (i) E = U ⊕ U0 ⊕ W (a direct sum).  
定义28.18.设\_为e上的-厄米形式。e的弱维特分解是三重（u，u0，w），这样（i）e=u u0 w（直接和）。

1. U and U0 are totally isotropic.  
   u和u0是完全各向同性的。
2. W is nondegenerate and orthogonal to U ⊕ U0.  
   w是非简并与u\_u0正交的。

We say that a weak Witt decomposition (U,U0,W) is nontrivial if U = (0)6 and U0 = (0).6 Furthermore, if E is finite-dimensional, then dim(U) = dim(U0) and in a suitable basis, the matrix representing ϕ is of the form  
我们说，弱维特分解（u，u0，w）是非平凡的，如果u=（0）6和u0=（0）。6此外，如果e是有限维的，那么dim（u）=dim（u0），在适当的基础上，表示\_的矩阵是形式。



We say that ϕ is a neutral form if it is nondegenerate, E is finite-dimensional, and if W = (0). In this case, the matrix B is missing.  
我们说，如果是非退化形式，那么\_是中性形式，e是有限维，如果w=（0）。在这种情况下，矩阵B丢失。

A Witt decomposition for which W has no nonzero isotropic vectors (W is anisotropic) is called a Witt decomposition.  
对于w没有非零各向同性向量（w是各向异性的）的witt分解称为witt分解。

Observe that if Φ is nondegenerate, then we have the trivial weak Witt decomposition obtained by letting U = U0 = (0) and W = E. Thus a weak Witt decomposition is informative only if E is not anisotropic (there is some nonzero isotropic vector, i.e. some u = 06 such that Φ(u) = 0), in which case the most informative nontrivial weak Witt decompositions are those for which W is anisotropic and U and U0 are as big as possible.  
观察到，如果Φ是非退化的，那么我们得到了由u=u0=（0）和w=e得到的平凡的弱维特分解。因此，弱维特分解只有当e不是各向异性的（有一些非零的各向同性向量，即一些u=06，使得Φ（u）=0）时才是信息性的，在这种情况下最有用的非平凡弱维特分解是那些w是各向异性的，u和u0是尽可能大的。

Sometimes, we use the notation U1 ⊕⊥ U2 to indicate that in a direct sum U1 ⊕ U2, the subspaces U1 and U2 are orthogonal. Then, in Definition 28.18, we can write that E = (U ⊕ U0) ⊕⊥ W.  
有时，我们用符号u1 u2来表示在直接和u1 u2中，子空间u1和u2是正交的。然后，在定义28.18中，我们可以写出e=（u u0）w。

The first step in showing the existence of a Witt decomposition is this.  
证明维特分解存在的第一步就是这个。

Proposition 28.29. Let ϕ be an -Hermitian form on E, assume that ϕ is nondegenerate and satisfies property (T), and let U be any totally isotropic subspace of E of finite dimension dim(U) = r ≥ 1.  
提案28.29。设\_为e上的-厄米形式，假设\_是非退化的且满足性质（t），设u为有限维dim（u）=r≥1的e的任何完全各向同性子空间。

1. If U0 is any totally isotropic subspace of dimension r and if U0 ∩U⊥ = (0), then U ⊕U0 is nondegenerate, and for any basis (u1,...,ur) of U, there is a basis such that , for all i,j = 1,...,r.  
   如果u0是维r的任何完全各向同性子空间，如果u0 u=（0），那么u u0是非退化的，对于u的任何基（u1，…，ur），都有这样一个基，对于所有i，j=1，…，r。
2. If W is any totally isotropic subspace of dimension at most r and if W ∩ U⊥ = (0), then there exists a totally isotropic subspace U0 with dim(U0) = r such that W ⊆ U0 and U0 ∩ U⊥ = (0).  
   如果w至多是维的任何完全各向同性子空间r，如果w u=（0），则存在一个具有dim（u0）=r的完全各向同性子空间u0，使得w u0和u0 u=（0）。

Proof. (1) Let ϕ0 be the restriction of ϕ to U × U0. Since U0 ∩ U⊥ = (0), for any v ∈ U0, if ϕ(u,v) = 0 for all u ∈ U, then v = 0. Thus, ϕ0 is nondegenerate (we only have to check on the left since-Hermitian). Then, the assertion about bases follows from the version of Proposition 28.3 for sesquilinear forms. Since U is totally isotropic, U ⊆ U⊥, and since U0 ∩ U⊥ = (0), we must have U0 ∩ U = (0), which show that we have a direct sum U ⊕ U0. It remains to prove that U + U0 is nondegenerate. Observe that  
证据。（1）设瘳0为瘳对U×U0的限制。由于u0 u=（0），对于任何v u0，如果所有u u（u，v）=0，则v=0。因此，ω0是非退化的（我们只需检查左侧，因为赫米特）。然后，关于基的断言来自于28.3命题对于倍线性形式的版本。因为u是完全各向同性的，u u，并且由于u0 u=（0），我们必须有u0 u=（0），这表明我们有一个直接和u u0。它仍然需要证明U+U0是非简并的。注意

H = (U + U0) ∩ (U + U0)⊥ = (U + U0) ∩ U⊥ ∩ U0⊥.  
H=（U+U0）（U+U0）=（U+U0）U U0。

Since U is totally isotropic, U ⊆ U⊥, and since U0 ∩ U⊥ = (0), we have  
既然u是完全各向同性的，u u，既然u0 u=（0），我们有

(U + U0) ∩ U⊥ = (U ∩ U⊥) + (U0 ∩ U⊥) = U + (0) = U,  
（U+U0）U=（U U）＋（U0 U）=U+（0）=U，

thus H = U ∩ U0⊥. Since ϕ0 is nondegenerate, U ∩ U0⊥ = (0), so H = (0) and U + U0 is nondegenerate.  
因此，h=u u0。既然\_是非退化的，u u0=（0），那么h=（0）和u+u0是非退化的。

(2) We proceed by descending induction on s = dim(W). The base case s = r is trivial. For the induction step, it suffices to prove that if s < r, then there is a totally isotropic subspace W 0 containing W such that dim(W 0) = s + 1 and W 0 ∩ U⊥ = (0).  
（2）我们在s=dim（w）上通过下降诱导进行。基本情况s=r是微不足道的。对于诱导步骤，它足以证明，如果s<r，那么有一个完全各向同性的子空间w 0包含w，从而dim（w 0）=s+1和w 0 u=（0）。

Since s = dim(W) < dim(U), the restriction of ϕ to U × W is degenerate. Since W ∩ U⊥ = (0), we must have U ∩ W ⊥ = (0)6 . We claim that  
由于s=dim（w）<dim（u），因此，对\_至u×w的限制退化。既然w u=（0），我们必须有u w=（0）6。我们声称

W ⊥ 6⊆ W + U⊥.  
W 6 W+U。

If we had W ⊥ ⊆ W + U⊥,  
如果我们有W W+U，

then because U and W are finite-dimensional and ϕ is nondegenerate, by Proposition 28.13,  
那么，因为u和w是有限维的，并且，根据第28.13号命题，ω是非退化的，

U⊥⊥ = U and W ⊥⊥ = W, so by taking orthogonals, W ⊥ ⊆ W + U⊥ would yield  
u=u和w=w，所以通过取正字法，w w+u将产生

(W + U⊥)⊥ ⊆ W ⊥⊥,  
（W+U），

that is,  
也就是说，

W ⊥ ∩ U ⊆ W,  
W U W，

thus W ⊥ ∩ U ⊆ W ∩ U, and since U is totally isotropic, U ⊆ U⊥, which yields  
因此，w u w u，由于u是完全各向同性的，u u，它产生

W ⊥ ∩ U ⊆ W ∩ U ⊆ W ∩ U⊥ = (0),  
W U W U W=（0）、

contradicting the fact that U ∩ W ⊥ 6= (0).  
与U W 6=（0）这一事实相矛盾。

sinceany vectorTherefore, there is somez ∈ Uz⊥∈, thenW ⊥ U Uu⊥ ∈so thatW ⊥ such thatu+z ∈ Wu /⊥∈andW +uU+⊥Wz /. Since⊥∈∩WU+U= (0)6 U⊆⊥ U(if⊥is totally isotropic, we can add tou+z ∈ W +U⊥u,  
因此，在任何向量机中，都有一些z∈uz uz u u uu uw uu uu uu uw uu wz/。既然w u+u=（0）6u u（如果是完全各向同性的，我们可以加上tou+z w+u u，

∩u ∈ ⊆W + U⊥, a contradiction). Since  
u∈w+u，一个矛盾）。自从

Wandthat0 =u /ϕW(∈uW++z,uKx+ U+⊥is a totally isotropic subspace of dimensionz= () = 0W ⊥. See Figure 28.1. If we write∩ U)⊥, we can invoke Lemma 28.28 to find ax =su+ 1+ z. Furthermore, we claim, thenzx /∈∈WW⊥ ∩+UU⊥such, so  
wndthat0=u/\_w（∈uw++z，ukx+u+）是一个完全各向同性的维数z=（）=0w子空间。见图28.1。如果我们写u），我们可以调用lemma 28.28来找到ax=su+1+z，而且我们声称，那么zx/∈ww+uu这样，所以

that W 0 ∩ U⊥ = 0.  
即w 0 u=0。

U

t

U

0

W

W

z

u

E  
e

Figure 28.1: A schematic illustration of W and x = u + z  
图28.1:w和x=u+z的示意图

Otherwise, we would have y = w + λx ∈ U⊥, for some w ∈ W and some λ ∈ K, and then we would have λx = −w + y ∈yW∈ +U⊥Uand⊥. Ifwλ∈= 06 W, then, we havex ∈ yW∈+WU∩⊥, a contradiction.U⊥ = (0), which  
否则，我们会得到y=w+λx∈u，对于一些w∈w和一些λ∈k，然后我们会得到λx=−w+y∈yw∈+u u and。如果wλ∈=06w，那么我们有x∈yw∈+w u，一个矛盾，u=（0），其中

Therefore, λ = 0, y = w, and since means that y = 0. Therefore, W 0 is the required subspace and this completes the proof.   
因此，λ=0，y=w，因为y=0。因此，w 0是所需的子空间，这就完成了证明。

Here are some consequences of Proposition 28.29. If we set W = (0) in Proposition 28.29(2), then we get the following theorem showing that if E is not anisotropic (there is some nonzero isotropic vector) then weak nontrivial Witt decompositions exist.  
这是28.29号提案的一些结果。如果我们在28.29（2）中设置w=（0），那么我们得到如下定理：如果e不是各向异性的（有一些非零各向同性向量），那么存在弱非平凡的witt分解。

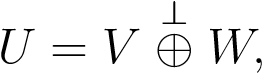
Theorem 28.30. Let ϕ be an -Hermitian form on E which is nondegenerate and satisfies  
定理28.30。设\_为e上非简并满足的-厄米形式。

nondegenerate. As a consequence, ifproperty (T). For any totally isotropic subspaceexists a totally isotropic subspace U0 Eof dimensionis not anisotropic, thenUrofsuch thatE of finite dimensionU ∩ U0 = (0) andr ≥U1⊕, thereU0 is is a weak nontrivial Witt decomposition for E. Furthermore, by Proposition 28.29(1), the block A in the matrix of ϕ is the identity matrix.  
不退化。因此，ifproperty（t）。对于任何完全各向同性的子空间存在一个完全各向同性的子空间，u0 e of维数不是各向异性的，因此，如果有限维数u u0=（0）andr≥u1的e是一个弱的非平凡的witt分解，而且，根据命题28.29（1），矩阵中的a块⑨为单位矩阵。

Proposition 28.31. Any two -Hermitian neutral forms satisfying property (T) defined on spaces of the same dimension are equivalent.  
提案28.31。在同一维空间上定义的满足性质（t）的任何两个埃尔米特中性形式都是等价的。

The following proposition shows that every subspace U of E can be embedded into a nondegenerate subspace. It is needed to prove a version of the Witt extension theorem (Theorem 28.48).  
下面的命题表明，e的每个子空间u都可以嵌入到一个非退化子空间中。需要证明维特推广定理（定理28.48）的一个版本。

Proposition 28.32. Let ϕ be an -Hermitian form on E which is nondegenerate and satisfies property (T). For any subspace U of E of finite dimension, if we write  
提案28.32。设a为e上的一个-厄米形式，它是非退化的且满足属性（t）。对于有限维E的任何子空间u，如果我们写



for some orthogonal complement W of V = rad(U), and if we let r = dim(rad(U)), then there exists a totally isotropic subspace V 0 of dimension r such that V ∩ V 0 = (0), and  
对于一些v=rad（u）的正交补码w，如果我们让r=dim（rad（u）），那么存在一个完全各向同性的子空间v 0，即v v 0=（0），并且

is nondegenerate. Furthermore, any isometry f from U into  
是不退化的。而且，从u到

another space (E0,ϕ0) where ϕ0 is an -Hermitian form satisfying the same assumptions as ϕ can be extended to an isometry on (V ⊕ V 0) ⊕⊥ W.  
另一个空间（e0，\_），其中，\_是符合与\_相同假设的-厄米式形式，可延伸至（v v 0）w上的等距测量。

Proof. Since W is nondegenerate, W ⊥ is also nondegenerate, and V ⊆ W ⊥. Therefore, we can apply Theorem 28.30 to the restriction of ϕ to W ⊥ and to V to obtain the required V 0. We know that V ⊕ V 0 is nondegenerate and orthogonal to W, which is also nondegenerate, so (V ⊕ V 0) ⊕⊥ W = V 0 ⊕ U is nondegenerate.  
证据。由于w是非退化的，w也是非退化的，v w。因此，我们可以将定理28.30应用到\_到w和v的限制，以获得所需的v 0。我们知道v v 0是非简并的，与w正交，w也是非简并的，所以（v v 0）w=v 0 u是非简并的。

We leave the second statement about extending f as an exercise (use the fact that f(U) =  
我们留下第二个关于扩展f的陈述作为练习（使用f（u）这个事实=

), where V1 = f(V ) is totally isotropic of dimension r, to find another totally isotropic susbpace of dimension r such that = (0) and is orthogonal to f(W)).   
，式中，v1=f（v）是尺寸r的完全各向同性，求出另一个尺寸r的完全各向同性的近似值，使之等于（0），并与f（w）正交。

The subspace (V ⊕ V 0) ⊕⊥ W = V 0 ⊕ U is often called a nondegenerate completion of U. The subspace V ⊕ V 0 is called an Artinian space. Proposition 28.29 show that V ⊕ V 0 has a basis (u1,v1,...,ur,vr) consisting of vectors ui ∈ V and vj ∈ V 0 such that ϕ(ui,uj) = δij. The subspace spanned by (ui,vi) is an Artinian plane, so V ⊕ V 0 is the orthogonal direct sum of r Artinian planes. Such a space is often denoted by Ar2r.  
子空间（v v 0）w=v 0 u通常被称为u的非退化完成。子空间v v 0被称为Artian空间。命题28.29表明，v v 0有一个基（u1，v1，…，ur，vr），由向量ui∈v和vj∈v 0组成，从而使得（ui，uj）=δij。（ui，vi）所跨越的子空间是一个Artian平面，因此v v 0是r Artian平面的正交直接和。这种空间通常用ar2r表示。

In order to obtain the stronger version of the Witt decomposition when ϕ has some nonzero isotropic vector and W is anisotropic we now sharpen Proposition 28.29  
为了获得更强大的维特分解形式，当\_有一些非零各向同性向量且w是各向异性时，我们现在尖锐化了命题28.29。

Theorem 28.33. Let ϕ be an -Hermitian form on E which is nondegenerate and satisfies property (T). Let U1 and U2 be two totally isotropic maximal subspaces of E, with U1 or U2 of finite dimension ≥ 1. Write U = U1 ∩ U2, let S1 be a supplement of U in U1 and S2 be a supplement of U in U2 (so that U1 = U ⊕ S1, U2 = U ⊕ S2), and let S = S1 + S2. Then, there exist two subspaces W and D of E such that:  
定理28.33。设a为e上的一个-厄米形式，它是非退化的且满足属性（t）。设u1和u2为e的两个完全各向同性最大子空间，其中有限维的u1或u2≥1。写u=u1 u2，设s1为u1中u的补充，s2为u2中u的补充（使u1=u s1，u2=u s2），设s=s1+s2。然后，存在两个子空间w和d of e，这样：

1. The subspaces S, U + W, and D are nondegenerate and pairwise orthogonal.  
   子空间s、u+w和d是非退化的，并且是成对正交的。
2. We have a direct sum.  
   我们有一个直接的和。
3. The subspace D contains no nonzero isotropic vector (D is anisotropic).  
   子空间d不包含非零各向同性向量（d是各向异性的）。
4. The subspace W is totally isotropic.  
   子空间w是完全各向同性的。

Furthermore, U1 and U2 are both finite dimensional, and we have dim(U1) = dim(U2), dim(W) = dim(U), dim(S1) = dim(S2), and codim(D) = 2dim(F1).  
此外，u1和u2都是有限维的，我们有dim（u1）=dim（u2），dim（w）=dim（u），dim（s1）=dim（s2），codim（d）=2dim（f1）。

Proof. First observe that if X is a totally isotropic maximal subspace of E, then any isotropic vector x ∈ E orthogonal to X must belong to X, since otherwise, X + Kx would be a totally isotropic subspace strictly containing X, contradicting the maximality of X. As a consequence, if xi is any isotropic vector such that xi ∈ Ui⊥ (for i = 1,2), then xi ∈ Ui.  
证据。首先观察到，如果X是E的全迷向最大子空间，那么任何与X正交的各向同性向量X E必须属于X，否则，X+KX将是严格包含X的完全各向同性子空间，与X的最大性相矛盾，因此，如果XI是各向同性的，向量，如Xi uIi（i＝1，2），然后是Xi uUI。

We claim that  
我们声称

= (0) and .  
=（0）和。

Assume that y ∈ S1 is orthogonal to S2. Since U1 = U ⊕ S1 and U1 is totally isotropic, y is orthogonal to U1, and thus orthogonal to U, so that y is orthogonal to U2 = U ⊕ S2. Since S1 ⊆ U1 and U1 is totally isotropic, y is an isotropic vector orthogonal to U2, which by a previous remark implies that y ∈ U2. Then, since S1 ⊆ U1 and U ⊕ S1 is a direct sum, we have  
假设y∈s1与s2正交。由于u1=u\_s1和u1完全各向同性，y与u1正交，因此与u正交，因此y与u2=u\_s2正交。由于s1 u1和u1是完全各向同性的，y是一个正交于u2的各向同性向量，前面的一句话暗示y∈u2。那么，既然s1 u1和u s1是一个直接和，我们有

y ∈ S1 ∩ U2 = S1 ∩ U1 ∩ U2 = S1 ∩ U = (0).  
y∈s1 u2=s1 u1 u2=s1 u=（0）。

Therefore = (0). A similar proof show that is finite-dimensional  
因此=（0）。一个类似的证明表明这是有限维的

(the case where U2 is finite-dimensional is similar), then S1 is finite-dimensional, so by Proposition 28.13,has finite codimension. Since = (0), and since any supplement of has finite dimension, we must have  
（u2是有限维的情况相似），那么s1是有限维，因此根据命题28.13，具有有限的余维。既然=（0），而且由于的任何补充都有有限维，我们必须

dim(S2) ≤ codim(.  
尺寸（s2）≤codim（.

By a similar argument, dim(S1) ≤ dim(S2), so we have  
通过一个类似的论点，dim（s1）≤dim（s2），所以我们有

dim(S1) = dim(S2).  
dim（s1）=dim（s2）。

By Proposition 28.29(1), we conclude that S = S1 + S2 is nondegenerate.  
通过28.29（1）号提案，我们得出S=s1+s2是非退化的。

By Proposition 28.21, the subspace N = S⊥ = (S1 + S2)⊥ is nondegenerate. Since U1 = U ⊕ S1, U2 = U ⊕ S2, and U1,U2 are totally isotropic, U is orthogonal to S1 and to S2, so U ⊆ N. Since U is totally isotropic, by Proposition 28.30 applied to N, there is a totally isotropic subspace W of N such that dim(W) = dim(U), U ∩ W = (0), and U + W is nondegenerate. Consequently, (d) is satisfied by W.  
根据命题28.21，子空间n=s=（s1+s2）是非退化的。由于u1=u\_s1，u2=u\_s2，和u1，u2是完全各向同性的，u与s1和s2正交，所以u\_n。由于u是完全各向同性的，根据适用于n的28.30号命题，有一个完全各向同性的子空间w为n，因此dim（w）=dim（u），u\_w=（0），u+w是非退化的。因此，（d）被w满足。

To satisfy (a) and (b), we pick D to be the orthogonal of U ⊕ W in N. Then, N =  
为了满足（a）和（b），我们选取d作为u\_w在n中的正交，然后，n=

.  
.

As to (c), since D is orthogonal U ⊕ W, D is orthogonal to U, and since D ⊆ N and N is orthogonal to S1 +S2, D is orthogonal to S1, so D is orthogonal to U1 = U ⊕S1. If y ∈ D is any isotropic vector, since , by a previous remark, y ∈ U1, so y ∈ D ∩ U1. But, D ⊆ N with N ∩ (S1 + S2) = (0), and D ∩ (U + W) = (0), so D ∩ (U + S1) = D ∩ U1 = (0), which yields y = 0. The statements about dimensions are easily obtained.   
至于（c），因为d是正交的u\_w，d是正交的u，而且d\_n和n是正交的s1+s2，d是正交的s1，所以d是正交的u1=u\_s1。如果y∈d是任何各向同性向量，因为，根据前面的注释，y∈u1，所以y∈d u1。但是，d n，其中n（s1+s2）=（0），d（u+w）=（0），因此d（u+s1）=d u1=（0），得出y=0。关于尺寸的说明很容易获得。

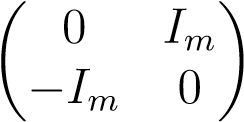
Finally, Theorem 28.33 yields the strong form of the Witt decomposition in which W is anistropic. Given any matrix A ∈ Mn(K), we say that A is definite if x>Ax = 06 for all x ∈ Kn.  
最后，定理28.33给出了witt分解的强形式，其中w是各向异性的。对于任意矩阵a∈mn（k），我们认为a是确定的，如果x>ax=06对于所有x∈kn。

Theorem 28.34. Let ϕ be an -Hermitian form on E which is nondegenerate and satisfies property (T).  
定理28.34。设a为e上的一个-厄米形式，它是非退化的且满足属性（t）。

1. Any two totally isotropic maximal spaces of finite dimension have the same dimension.  
   任意两个完全各向同性的有限维极大空间都有相同的维数。
2. For any totally isotropic maximal subspace U of finite dimension r ≥ 1, there is another totally isotropic maximal subspace U0 of dimension r such that U ∩ U0 = (0), and U ⊕ U0 is nondegenerate. Furthermore, if D = (U ⊕ U0)⊥, then (U,U0,D) is a  
   对于有限维r≥1的完全各向同性最大子空间u，存在另一个完全各向同性最大子空间u0，使得u u0=（0），u u0是非退化的。此外，如果d=（u u0），（u，u0，d）是a

Witt decomposition of E; that is, there are no nonzero isotropic vectors in D (D is anisotropic).  
e的维特分解；也就是说，d中没有非零各向同性向量（d是各向异性的）。

1. If E has finite dimension n ≥ 1 and there is some nonzero isotropic vector for ϕ (E is not anisotropic), then E has a nontrivial Witt decomposition (U,U0,D) as in (2). There is a basis of E such that  
   如果e的有限维数n≥1，且有一些非零各向同性向量用于\_（e不是各向异性的），则e具有非平凡的witt分解（u，u0，d），如（2）中所示。有这样一个基础
   1. if ϕ is alternating ( − and λ = λ for all λ ∈ K), then n = 2m and ϕ is represented by a matrix of the form  
      如果ω是交替的（−和λ=λ代表所有的λ∈k），则n=2 m和ω由形式的矩阵表示。



* 1. if ϕ is symmetric ( and λ = λ for all λ ∈ K), then ϕ is represented by a matrix of the form  
     如果ω是对称的（且所有λ∈k的λ=λ），则用形式的矩阵表示。

,  
，

where either n = 2r and P does not occur, or n > 2r and P is a definite symmetric matrix.  
其中n=2r和p没有出现，或者n>2r和p是一个确定的对称矩阵。

-Hermitian (the involutive automorphism λ 7→ λ is not the identity), then ϕ is represented by a matrix of the form  
-厄米提安（对合自同构λ7→λ不是恒等式），则用形式的矩阵表示。

,  
，

where either n = 2r and P does not occur, or n > 2r and P is a definite matrix such that .  
其中n=2r和p没有出现，或者n>2r和p是一个确定的矩阵，这样。

Proof. Part (1) follows from Theorem 28.33. By Proposition 28.30, we obtain a totally isotropic subspace U0 of dimension r such that U ∩ U0 = (0). By applying Theorem 28.33 to U1 = U and U2 = U0, we get U = W = (0), which proves (2). Part (3) is an immediate consequence of (2).   
证据。第（1）部分来自定理28.33。通过命题28.30，我们得到了维R的完全各向同性子空间u0，这样u u0=（0）。将定理28.33应用于u1=u和u2=u0，得到u=w=（0），证明（2）。第（3）部分是第（2）部分的直接后果。

As a consequence of Theorem 28.34, we make the following definition.  
根据定理28.34，我们做出如下定义。

Definition 28.19. Let E be a vector space of finite dimension n, and let ϕ be an -Hermitian form on E which is nondegenerate and satisfies property (T). The index (or Witt index) ν of ϕ, is the common dimension of all totally isotropic maximal subspaces of E. We have 2ν ≤ n.  
定义28.19.设e为有限维数n的向量空间，且当e为非退化且满足性质（t）的-Hermitian形式时，则当e为非退化且满足性质（t）的-Hermitian形式。ω的指数（或维特指数）为e的所有全向各向同性最大子空间的公共维数。我们有2ν≤n。

Neutral forms only exist if n is even, in which case, ν = n/2. Forms of index ν = 0 have no nonzero isotropic vectors. When K = R, this is satisfied by positive definite or negative definite symmetric forms. When K = C, this is satisfied by positive definite or negative definite Hermitian forms. The vector space of a neutral Hermitian form ( = +1) is an Artinian space, and the vector space of a neutral alternating form is a hyperbolic space.  
中性形式只有当n为偶数时才存在，在这种情况下，ν=n/2。指数的形式v=0没有非零的各向同性向量。当k=r时，用正定或负定对称形式表示。当k=c时，这可由正定或负定厄米式来满足。中性厄米形式（+1）的矢量空间是一个Artian空间，中性交替形式的矢量空间是一个双曲空间。

If the field K is algebraically closed, we can describe all nondegenerate quadratic forms.  
如果K域是代数闭的，我们可以描述所有的非退化二次型。

Proposition 28.35. If K is algebraically closed and E has dimension n, then for every nondegenerate quadratic form Φ, there is a basis (e1,...,en) such that Φ is given by  
提案28.35。如果k是代数闭的，e有维数n，那么对于每一个非退化二次型Φ，都有一个基（e1，…，en），使得Φ由

if n = 2m if n = 2m + 1.  
如果n=2米，如果n=2米+1。

Proof. We work with the polar form ϕ of Φ. Let U1 and U2 be some totally isotropic subspaces such that U1 ∩ U2 = (0) given by Theorem 28.34, and let q be their common dimension. Then, W = U = (0). Since we can pick bases (e1,...eq) in U1 and (eq+1,...,e2q) in U2 such that ϕ(ei,ei+q) = 0, for i,j = 1,...,q, it suffices to proves that dim(D) ≤ 1. If x,y ∈ D with x = 06 , from the identity  
证据。我们使用的极性形式是Φ。设u1和u2为一些完全各向同性的子空间，如定理28.34给出的u1 u2=（0），并设q为它们的公共维数。那么，w=u=（0）。由于我们可以选取U1中的碱基（e1，…eq）和U2中的碱基（eq+1，…，e2q），因此，对于i，j=1，…，q，它足以证明dim（d）≤1。如果x，y∈d，x=06，从恒等式

Φ(y − λx) = Φ(y) − λϕ(x,y) + λ2Φ(x)  
Φ（y−λx）=Φ（y）−λ（x，y）+λ2Φ（x）

and the fact that Φ(x) = 06 since x ∈ D and x = 06 , we see that the equation Φ(y − λy) = 0 has at least one solution. Since Φ(z) = 06 for every nonzero z ∈ D, we get y = λx, and thus dim(D) ≤ 1, as claimed.   
从x∈d和x=06开始，Φ（x）=06，我们发现方程Φ（y-λy）=0至少有一个解。由于Φ（z）=06，对于每一个非零z∈d，我们得到y=λx，因此dim（d）≤1，如权利要求所述。

Proposition 28.35 shows that for every nondegenerate quadratic form Φ over an algebraically closed field, if dim(E) = 2m or dim(E) = 2m + 1 with m ≥ 1, then Φ has some nonzero isotropic vector.  
命题28.35表明，对于代数闭场上的每一个非退化二次型Φ，如果dim（e）=2 m或dim（e）=2 m+1且m≥1，则Φ具有一些非零各向同性向量。

## 28.8 Symplectic Groups 28.8辛群

In this section, we are dealing with a nondegenerate alternating form ϕ on a vector space E of dimension n. As we saw earlier, n must be even, say n = 2m. By Theorem 28.24, there is a direct sum decomposition of E into pairwise orthogonal subspaces  
在这一节中，我们要处理的是一个非退化的交替形式，在维n的向量空间e上。正如我们前面所看到的，n必须是偶数，即n=2米。根据定理28.24，e的直接和分解成成对的正交子空间。

E = W1 ⊕ ···⊥ ⊕⊥ Wm,  
E=w1····wm，

where each Wi is a hyperbolic plane. Each Wi has a basis (ui,vi), with ϕ(ui,ui) = ϕ(vi,vi) = 0 and ϕ(ui,vi) = 1, for i = 1,...,m. In the basis  
其中每个wi都是一个双曲线平面。每个wi都有一个基（ui，vi），其中，当i=1，…，m时，其中，qian（ui，ui）=0和qian（ui，vi）=1。

(u1,...,um,v1,...,vm),  
（u1，…，um，v1，…，vm）

ϕ is represented by the matrix  
用矩阵表示

.  
.

The symplectic group Sp(2m,K) is the group of isometries of ϕ. The maps in Sp(2m,K) are called symplectic maps. With respect to the above basis, Sp(2m,K) is the group of 2m × 2m matrices A such that  
辛群sp（2m，k）为\_等轴测群。SP（2M，K）中的映射称为辛映射。就上述基础而言，sp（2m，k）是由2m×2m矩阵a组成的组，因此

A>Jm,mA = Jm,m.  
a>jm，ma=jm，m。

Matrices satisfying the above identity are called symplectic matrices. In this section, we show that Sp(2m,K) is a subgroup of SL(2m,K) (that is, det(A) = +1 for all A ∈ Sp(2m,K)), and we show that Sp(2m,K) is generated by special linear maps called symplectic transvections.  
满足上述恒等式的矩阵称为辛矩阵。在本节中，我们证明了sp（2 m，k）是sl（2 m，k）的一个子群（即，所有a∈sp（2 m，k）的det（a）=+1），并且我们证明sp（2 m，k）是由称为辛变换的特殊线性映射生成的。

First, we leave it as an easy exercise to show that Sp(2,K) = SL(2,K). The reader should also prove that Sp(2m,K) has a subgroup isomorphic to GL(m,K).  
首先，我们将它作为一个简单的练习来展示sp（2，k）=sl（2，k）。读者还应证明sp（2m，k）与gl（m，k）具有同构子群。

Next we characterize the symplectic maps f that leave fixed every vector in some given hyperplane H, that is,  
接下来，我们描述辛映射f，在给定的超平面h中保持不变的每个向量，也就是说，

f(v) = v for all v ∈ H.  
f（v）=v，表示所有v∈h。

Since ϕ is nondegenerate, by Proposition 28.22, the orthogonal H⊥ of H is a line (that is, dim(H⊥) = 1). For every u ∈ E and every v ∈ H, since f is an isometry and f(v) = v for all v ∈ H, we have  
因为根据命题28.22，\_是非退化的，所以h的正交h是一条线（即，dim（h）=1）。对于每一个u∈e和每一个v∈h，因为f是一个等值线，而f（v）=v对于所有v∈h，我们有

ϕ(f(u) − u,v) = ϕ(f(u),v) − ϕ(u,v)  
Ⅷ（f（u）−u，v）=Ⅷ（f（u），v）−Ⅷ（u，v）

= ϕ(f(u),v) − ϕ(f(u),f(v))  
=\_（f（u），v）-\_（f（u），f（v））

= ϕ(f(u),v − f(v))) = ϕ(f(u),0) = 0,  
=\_（f（u），v−f（v）））=（f（u），0）=0，

which shows that f(u) − u ∈ H⊥ for all u ∈ E. Therefore, f − id is a linear map from E into the line H⊥ whose kernel contains H, which means that there is some nonzero vector w ∈ H⊥ and some linear form ψ such that  
这表明f（u）−u∈h对于所有u∈e，因此f−id是从e到h线的线性映射，h线的内核包含h，这意味着存在一些非零向量w∈h和一些线性形式ψ，因此

f(u) = u + ψ(u)w, u ∈ E.  
f（u）=u+ψ（u）w，u∈e。

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Since f is an isometry, we must have ϕ(f(u),f(v)) = ϕ(u,v) for all u,v ∈ E, which means that  
因为f是一个等距测量，所以我们必须对所有u，v∈e都取\_（f（u），f（v））=\_（u，v），这意味着

ϕ(u,v) = ϕ(f(u),f(v))  
⑨（u，v）=⑨（f（u），f（v））

= ϕ(u + ψ(u)w,v + ψ(v)w)  
=ψ（u+ψ（u）w，v+ψ（v）w）

= ϕ(u,v) + ψ(u)ϕ(w,v) + ψ(v)ϕ(u,w) + ψ(u)ψ(v)ϕ(w,w)  
=\_（u，v）+\_（u）\_（w，v）+\_（v）\_（u，w）+\_（u）\_（v）\_（w，w）

= ϕ(u,v) + ψ(u)ϕ(w,v) − ψ(v)ϕ(w,u),  
=\_（u，v）+ψ（u）\_（w，v）−ψ（v）（w，u）、

which yields ψ(u)ϕ(w,v) = ψ(v)ϕ(w,u) for all u,v ∈ E.  
其中，所有u，v∈e产生ψ（u）\_（w，v）=ψ（v）（w，u）。

Since ϕ is nondegenerate, we can pick some v0 such that ϕ(w,v0) = 06 , and we get ψ(u)ϕ(w,v0) = ψ(v0)ϕ(w,u) for all u ∈ E; that is,  
既然\_是非退化的，我们可以选择一些v0，这样，\_（w，v0）=06，我们得到所有u∈e的ψ（u）\_（w，v0）=ψ（v0）（w，u）；也就是说，

ψ(u) = λϕ(w,u) for all u ∈ E,  
ψ（u）＝\_（w，u）表示所有u∈e，

for some λ ∈ K. Therefore, f is of the form  
对于某些λ∈k，因此，f是形式

f(u) = u + λϕ(w,u)w, for all u ∈ E.  
f（u）=u+λ（w，u）w，表示所有u∈e。

It is also clear that every f of the above form is a symplectic map. If λ = 0, then f = id. Otherwise, if λ = 06 , then f(u) = u iff ϕ(w,u) = 0 iff u ∈ (Kw)⊥ = H, where H is a hyperplane. Thus, f fixes every vector in the hyperplane H. Note that since ϕ is alternating, ϕ(w,w) = 0, which means that w ∈ H.  
很明显，上述形式的每一个f都是一个辛映射。如果λ=0，则f=id。否则，如果λ=06，则f（u）=u iff（w，u）=0 iff u∈（kw）=h，其中h是超平面。因此，f固定了超平面h中的每一个向量。注意，由于\_是交替的，因此\_（w，w）=0，这意味着w∈h。

In summary, we have characterized all the symplectic maps that leave every vector in some hyperplane fixed, and we make the following definition.  
总之，我们已经描述了所有使某个超平面中的每个向量保持不变的辛映射，并且我们做出了以下定义。

Definition 28.20. Given a nondegenerate alternating form ϕ on a space E, a symplectic transvection (of direction w) is a linear map f of the form  
定义28.20。给定空间E上的非简并交替形式，辛矢量（方向W）是形式的线性映射f。

f(u) = u + λϕ(w,u)w, for all u ∈ E,  
f（u）=u+λ（w，u）w，对于所有u∈e，

for some nonzero w ∈ E and some λ ∈ K. If λ = 06 , the subspace of vectors left fixed by f is the hyperplane H = (Kw)⊥. The map f is also denoted τw,λ.  
对于一些非零w∈e和一些λ∈k，如果λ=06，由f固定的向量的子空间是超平面h=（kw）。图f也表示为τw，λ。

Observe that  
注意

τw,λ ◦ τw,µ = τw,λ+µ  
τw，λτw，μ=τw，λ+μ

and τw,λ = id iff λ = 0. The above shows that det(τw,λ) = 1, since when λ = 06 , we have τw,λ = (τw,λ/2)2.  
τw，λ=id iffλ=0。上述结果表明，Det（τw，λ）=1，因为当λ=06时，我们得到τw，λ=（τw，λ/2）2。

Our next goal is to show that if u and v are any two nonzero vectors in E, then there is a simple symplectic map f such that f(u) = v.  
我们的下一个目标是证明，如果u和v是e中的任意两个非零向量，那么有一个简单的辛映射f，这样f（u）=v。

Proposition 28.36. Given any two nonzero vectors u,v ∈ E, there is a symplectic map f such that f(u) = v, and f is either a symplectic transvection, or the composition of two symplectic transvections.  
提案28.36。对于任意两个非零矢量u，v∈e，存在一个辛映射f，使得f（u）=v，f是一个辛变换，或者是两个辛变换的组合。

Proof. There are two cases.  
证据。有两种情况。

Case 1. ϕ(u,v) = 0.6  
案例1。⑨（u，v）=0.6

In this case, u =6 v, since ϕ(u,u) = 0. Let us look for a symplectic transvection of the form τv−u,λ. We want  
在这种情况下，U=6 V，因为\_（U，U）=0。让我们来寻找形式τv−u，λ的辛变换。我们想要

v = u + λϕ(v − u,u)(v − u) = u + λϕ(v,u)(v − u),  
V=U+λ\_（V−U，U）（V−U）=U+λ（V，U）（V−U）、

which yields  
会产生

(λϕ(v,u) − 1)(v − u) = 0.  
（λη（v，u）−1）（v−u）=0.

Since ϕ(u,v) = 06 and ϕ(v,u) = −ϕ(u,v), we can pick λ = ϕ(v,u)−1 and τv−u,λ maps u to v. Case 2. ϕ(u,v) = 0.  
既然\_（u，v）=06和（v，u）=-（u，v），我们可以选择λ=\_（v，u）−1和τv−u，λ将u映射到v。情况2。⑨（u，v）=0.

If u = v, use τu,0 = id. Now, assume u =6 v. We claim that it is possible to pick some w ∈ E such that ϕ(u,w) = 06 and ϕ(v,w) = 06 . Indeed, if (Ku)⊥ = (Kv)⊥, then pick any nonzero vector w not in the hyperplane (Ku)⊥. Othwerwise, (Ku)⊥ and (Kv)⊥ are two distinct hyperplanes, so neither is contained in the other (they have the same dimension), so pick any nonzero vector w1 such that w1 ∈ (Ku)⊥ and w1 ∈/ (Kv)⊥, and pick any nonzero vector w2 such that w2 ∈ (Kv)⊥ and w2 ∈/ (Ku)⊥. If we let w = w1 + w2, then ϕ(u,w) = ϕ(u,w2) = 06 , and ϕ(v,w) = ϕ(v,w1) = 06 . From case 1, we have some symplectic transvection τw−u,λ1 such that τw−u,λ1(u) = w, and some symplectic transvection τv−w,λ2 such that τv−w,λ2(w) = v, so the composition τv−w,λ2 ◦ τw−u,λ1 maps u to v.   
如果u=v，则使用τu，0=id。现在，假设u=6v。我们声称可以选择一些w∈e，这样，（u，w）=06和（v，w）=06。实际上，如果（ku）=（kv），那么选取不在超平面（ku）中的任何非零矢量w。另外，（ku）和（kv）是两个不同的超平面，因此两者都不包含在另一个超平面中（它们具有相同的维数），因此选取任意非零矢量w1，使w1∈（ku）和w1∈/（kv），并选取任意非零矢量w2，使w2∈（kv）和w2∈/（ku）。如果我们让w=w1+w2，那么\_（u，w）=（u，w2）=06，和（v，w）=（v，w1）=06。从情况1来看，我们有一些辛矢量τw−u，λ1，这样τw−u，λ1（u）=w，和一些辛矢量τv−w，λ2，这样τv−w，λ2（w）=v，所以组成τv−w，λ2τw−u，λ1将u映射到v。

Next, we would like to extend Proposition 28.36 to two hyperbolic planes W1 and W2.  
接下来，我们将28.36号命题扩展到两个双曲面w1和w2。

Proposition 28.37. Given any two hyperbolic planes W1 and W2 given by bases (u1,v1) and (u2,v2) (with ϕ(ui,ui) = ϕ(vi,vi) = 0 and ϕ(ui,vi) = 1, for i = 1,2), there is a symplectic map f such that f(u1) = u2, f(v1) = v2, and f is the composition of at most four symplectic transvections.  
提案28.37。给定任意两个由基（u1，v1）和（u2，v2）给出的双曲线平面w1和w2（其中，\_（ui，ui）=（vi，vi）=0和（ui，vi）=1，对于i=1,2，有一个辛映射f，这样f（u1）=u2，f（v1）=v2，并且f至多是四个辛变换的组合。

Proof. From Proposition 28.36, we can map u1 to u2, using a map f which is the composition of at most two symplectic transvections. Say v3 = f(v1). We claim that there is a map g such that g(u2) = u2 and g(v3) = v2, and g is the composition of at most two symplectic transvections. If so, g ◦ f maps the pair (u1,v1) to the pair (u2,v2), and g ◦ f consists of at most four symplectic transvections. Thus, we need to prove the following claim:  
证据。从28.36号提案，我们可以用最多由两个辛变换组成的映射f来映射u1到u2。假设v3=f（v1）。我们认为存在一个图G，这样G（u2）=u2和G（v3）=v2，并且G是最多两个辛矢量的组合。如果是这样，g f将对（u1，v1）映射到对（u2，v2），并且g f最多包含四个辛变换。因此，我们需要证明以下声明：

Claim. If (u,v) and (u,v0) are hyperbolic bases determining two hyperbolic planes, then there is a symplectic map g such that g(u) = u, g(v) = v0, and g is the composition of at most two symplectic transvections. There are two case.  
索赔。如果（u，v）和（u，v0）是决定两个双曲线平面的双曲线基，那么有一个辛映射g，这样g（u）=u，g（v）=v0，g是至多两个辛变换的组合。有两种情况。

Case 1. ϕ(v,v0) = 0.6  
案例1。⑨（v，v0）=0.6

In this case, there is a symplectic transvection τv0−v,λ such that τv0−v,λ(v) = v0. We also have ϕ(u,v0 − v) = ϕ(u,v0) − ϕ(u,v) = 1 − 1 = 0.  
在这种情况下，有一个辛矢量τv0−v，λ，这样τv0−v，λ（v）=v0。我们也有\_（u，v0−v）=（u，v0）−（u，v）=1−1=0。

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Therefore, τv0−v,λ(u) = u, and g = τv0−v,λ does the job.  
因此，τv0−v，λ（u）=u，g=τv0−v，λ起作用。

Case 2. ϕ(v,v0) = 0.  
案例2。⑨（v，v0）=0.

First, check that (u,u + v) is a also hyperbolic basis. Furthermore,  
首先，检查（u，u+v）是否也是双曲线的基础。此外，

ϕ(v,u + v) = ϕ(v,u) + ϕ(v,v) = ϕ(v,u) = −1 = 06 .  
⑨（V，U+V）=⑨（V，U）＋⑨（V，V）＝（V，U）=-1=06.

Thus, there is a symplectic transvection τv,λ1 such that τu,λ1(v) = u + v and τu,λ1(u) = u.  
因此，有一个辛矢量τv，λ1，使得τu，λ1（v）=u+v和τu，λ1（u）=u。

We also have  
我们也有

ϕ(u + v,v0) = ϕ(u,v0) + ϕ(v,v0) = ϕ(u,v0) = 1 = 06 ,  
⑨（u+v，v0）=⑨（u，v0）+（v，v0）=（u，v0）=1=06，

so there is a symplectic transvection τv0−u−v,λ2 such that τv0−u−v,λ2(u + v) = v0. Since  
所以有一个辛矢量τv0−u−v，λ2，这样τv0−u−v，λ2（u+v）=v0。自从

ϕ(u,v0 − u − v) = ϕ(u,v0) − ϕ(u,u) − ϕ(u,v) = 1 − 0 − 1 = 0,  
⑨（u，v0−u−v）=（u，v0）−（u，u）−（u，v）=1−0−1=0，

we have τv0−u−v,λ2(u) = u. Then, the composition g = τv0−u−v,λ2 ◦ τu,λ1 is such that g(u) = u and g(v) = v0.   
我们有τv0−u−v，λ2（u）=u。那么，组成g=τv0−u−v，λ2τu，λ1是这样的g（u）=u和g（v）=v0。

We will use Proposition 28.37 in an inductive argument to prove that the symplectic transvections generate the symplectic group. First, make the following observation: If U is a nondegenerate subspace of E, so that  
我们将在归纳论点中使用命题28.37来证明辛变换产生辛群。首先，做如下观察：如果u是e的非退化子空间，那么

,  
，

and if τ is a transvection of H⊥, then we can form the linear map idU ⊕⊥ τ whose restriction to U is the identity and whose restriction to U⊥ is τ, and idU ⊕⊥ τ is a transvection of E.  
如果τ是H的一个变换，那么我们可以形成一个线性映射iduτ，它对u的限制是同一性，对u的限制是τ，iduτ是e的一个变换。

Theorem 28.38. The symplectic group Sp(2m,K) is generated by the symplectic transvections. For every transvection f ∈ Sp(2m,K), we have det(f) = 1.  
定理28.38。辛群sp（2 m，k）由辛向矢量产生。对于每一个f∈sp（2m，k），我们得到了Det（f）=1。

Proof. Let G be the subgroup of Sp(2m,K) generated by the transvections. We need to prove that G = Sp(2m,K). Let (u1,v1,...,um,vm) be a symplectic basis of E, and let f ∈ Sp(2m,K) be any symplectic map. Then, f maps (u1,v1,...,um,vm) to another symplectic basis (). If we prove that there is some g ∈ G such that g(ui) = u0i and , then f = g and G = Sp(2m,K).  
证据。设g为由变换生成的sp（2m，k）的子群。我们需要证明g=sp（2m，k）。设（u1，v1，…，um，vm）为e的辛基，设f∈sp（2m，k）为任意辛映射。然后，f将（u1，v1，…，um，vm）映射到另一个辛基（）。如果我们证明存在一些g∈g，使得g（ui）=u0i，那么f=g，g=sp（2m，k）。

We use induction on i to prove that there is some gi ∈ G so that gi maps (u1,v1,...,ui,vi) to (  
我们利用I上的归纳法来证明存在一些g i∈g，因此gi映射（u1，v1，…，ui，vi）到（

The base case i = 1 follows from Proposition 28.37.  
基本情况i=1从命题28.37开始。

For the induction step, assume that we have some gi ∈ G mapping (u1,v1,...,ui,vi) to (), and let () be the image of (ui+1,vi+1,...,um,vm) by gi. If U is the subspace spanned by (), then each hyperbolic plane given by (u0i+k,vi0+k) and each hyperbolic plane given by () belongs to U⊥. Using the remark before the theorem and Proposition 28.37, we can find a transvection τ mapping onto and leaving every vector in U fixed. Then, τ ◦ gi maps  
对于归纳步骤，假设我们有一些gi∈g映射（u1，v1，…，ui，vi）到（），并让（）是（ui+1，vi+1，…，um，vm）的gi图像。如果u是用（）表示的子空间，则由（u0i+k，vi0+k）表示的每个双曲面和由（）表示的每个双曲面都属于u。利用定理和命题28.37之前的注释，我们可以找到一个关于u中每一个向量的转移τ映射。然后，τGI图

), establishing the induction step.  
）建立诱导步骤。

For the second statement, since we already proved that every transvection has a determinant equal to +1, this also holds for any composition of transvections in G, and since G = Sp(2m,K), we are done.   
对于第二个陈述，由于我们已经证明了每一个transvection都有一个等于+1的行列式，这也适用于g中transvections的任何组成，并且由于g=sp（2m，k），我们就这样做了。

It can also be shown that the center of Sp(2m,K) is reduced to the subgroup {id,−id}. The projective symplectic group PSp(2m,K) is the quotient group PSp(2m,K)/{id,−id}. All symplectic projective groups are simple, except PSp(2,F2),PSp(2,F3), and PSp(4,F2), see Grove [83].  
也可以证明，sp（2m，k）的中心被缩小到亚组id、−id。投影辛群psp（2 m，k）是商群psp（2 m，k）/id、−id。除psp（2，f2）、psp（2，f3）和psp（4，f2）外，所有辛射影群都很简单，见grove[83]。

The orders of the symplectic groups over finite fields can be determined. For details, see Artin [6], Jacobson [96] and Grove [83].  
有限域上辛群的阶可以确定。有关详细信息，请参见Artin[6]、Jacobson[96]和Grove[83]。

An interesting property of symplectic spaces is that the determinant of a skew-symmetric matrix B is the square of some polynomial Pf(B) called the Pfaffian; see Jacobson [96] and Artin [6]. We leave considerations of the Pfaffian to the exercises.  
辛空间的一个有趣的性质是，斜对称矩阵b的行列式是称为pfafian的多项式pf（b）的平方；见Jacobson[96]和Artin[6]。我们在练习中考虑了pfafian。

We now take a look at the orthogonal groups.  
现在我们来看看正交群。

## 28.9 Orthogonal Groups and the Cartan–Dieudonn´e Theorem 28.9正交群与卡坦-迪乌顿定理

In this section we are dealing with a nondegenerate symmetric bilinear from ϕ over a finitedimensional vector space E of dimension n over a field of characateristic not equal to 2. Recall that the orthogonal group O(ϕ) is the group of isometries of ϕ; that is, the group of linear maps f : E → E such that  
在这一节中，我们讨论的是一个非退化对称双线性，它是在一个不等于2的特征场上，由π在一个尺寸为n的有限维向量空间e上得到的。记住，正交组o（\_）是\_的等轴测图组；也就是说，线性映射组f:e→e这样

ϕ(f(u),f(v)) = ϕ(u,v) for all u,v ∈ E.  
所有u，v∈e的（f（u），f（v））=（u，v）。

The elements of O(ϕ) are also called orthogonal transformations. If M is the matrix of ϕ in any basis, then a matrix A represents an orthogonal transformation iff  
O（\_）的元素也被称为正交变换。如果m是任何基的矩阵，那么矩阵a表示正交变换iff

A>MA = M.  
a>ma=m。

Since ϕ is nondegenerate, M is invertible, so we see that det(A) = ±1. The subgroup  
因为\_是非退化的，m是可逆的，所以我们看到Det（a）=±1。小组

SO(ϕ) = {f ∈ O(ϕ) | det(f) = 1}  
所以（\_）=f∈o（\_）det（f）=1

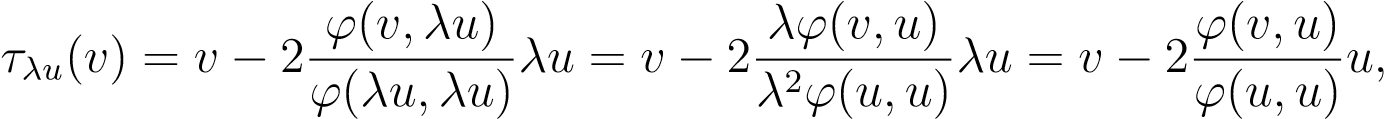
is called the special orthogonal group (of ϕ), and its members are called rotations (or proper orthogonal transformations). Isometries f ∈ O(ϕ) such that det(f) = −1 are called improper orthogonal transformations, or sometimes reversions.  
被称为特殊正交群（的），其成员被称为旋转（或适当的正交变换）。等距f∈o（），使得det（f）=-1被称为不适当的正交变换，或有时逆转。

If H is any nondegenerate hyperplane in E, then D = H⊥ is a nondegenerate line and we have  
如果h是e中的任何非简并超平面，那么d=h是一条非简并线，我们有

.  
.

For any nonzero vector u ∈ D = H⊥ Consider the map τu given by  
对于任何非零向量u∈d=h考虑由

for all v ∈ E. If we replace u by λu with λ = 06 , we have  
对于所有v∈e，如果我们用λu替换u，用λ=06，我们得到



which shows that τu depends only on the line D, and thus only the hyperplane H. Therefore, denote by τH the linear map τu determined as above by any nonzero vector u ∈ H⊥. Note that if v ∈ H, then τH(v) = v,  
这表明τu仅依赖于线d，因此仅依赖于超平面h。因此，用τh表示由任何非零向量u∈h确定的线性图τu。注意，如果v∈h，那么τh（v）=v，

and if v ∈ D, then τH(v) = −v.  
如果v∈d，那么τh（v）=−v。

A simple computation shows that  
简单的计算表明

ϕ(τH(u),τH(v)) = ϕ(u,v) for all u,v ∈ E,  
⑨（τh（u），τh（v））=所有u，v∈e的Ⅷ（u，v）

so τH ∈ O(ϕ), and by picking a basis consisting of u and vectors in H, that det(τH) = −1. It is also clear that τH2 = id.  
因此，τh∈o（），通过选择由u和h中的向量组成的基，该det（τh）=-1。很明显，τh2=id。

Definition 28.21. If H is any nondegenerate hyperplane in E, for any nonzero vector u ∈ H⊥, the linear map τH given by  
定义28.21。如果h是e中的任何非退化超平面，对于任何非零向量u∈h，线性映射τh由

for all v ∈ E  
对于所有v∈e

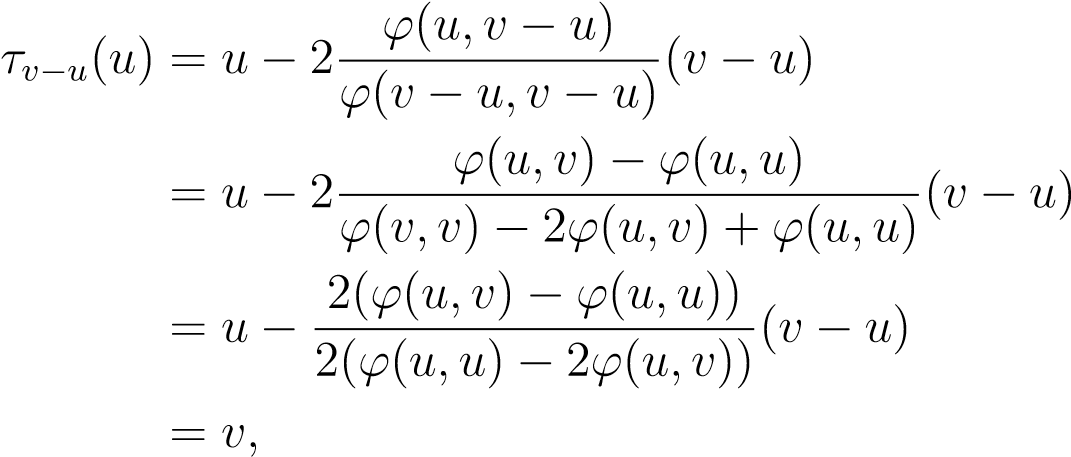
is an involutive isometry of E called the reflection through (or about) the hyperplane H.  
是E的对合等距线，称为通过（或关于）超平面H的反射。

Remarks:  
评论：

1. It can be shown that if f ∈ O(ϕ) leaves every vector in some hyperplane H fixed, then either f = id or f = τH; see Taylor [169] (Chapter 11). Thus, there is no analog to symplectic transvections in the orthogonal group.  
   可以证明，如果f∈o（）使某个超平面h中的每一个向量保持不变，则f=id或f=τh；见Taylor[169]（第11章）。因此，在正交组中没有类似于辛矢量的变换。
2. If K = R and ϕ is the usual Euclidean inner product, the matrices corresponding to hyperplane reflections are called Householder matrices.  
   如果k=r和\_是通常的欧几里得内积，则对应于超平面反射的矩阵称为户主矩阵。

Our goal is to prove that O(ϕ) is generated by the hyperplane reflections. The following proposition is needed.  
我们的目标是证明O（a）是由超平面反射产生的。需要以下建议。

Proposition 28.39. Let ϕ be a nondegenerate symmetric bilinear form on a vector space E. For any two nonzero vectors u,v ∈ E, if ϕ(u,u) = ϕ(v,v) and v − u is nonisotropic, then the hyperplane reflection τH = τv−u maps u to v, with H = (K(v − u))⊥. Proof. Since v − u is not isotropic, ϕ(v − u,v − u) = 06 , and we have  
提案28.39。设a为向量空间e上的非退化对称双线性形式。对于任意两个非零向量u，v∈e，如果a（u，u）=a（v，v）和v−u是非各向同性的，则超平面反射τh=τv−u映射u到v，h=（k（v−u））。证据。由于v−u不是各向同性的，因此，ω（v−u，v−u）=06，我们有



which proves the proposition.   
这证明了这个命题。

We can now obtain a cheap version of the Cartan–Dieudonn´e theorem.  
我们现在可以得到卡坦-迪乌顿定理的廉价版本。

Theorem 28.40. (Cartan–Dieudonn´e, weak form) Let ϕ be a nondegenerate symmetric bilinear form on a K-vector space E of dimension n (char(K) = 26 ). Then, every isometry f ∈ O(ϕ) with f =6 id is the composition of at most 2n − 1 hyperplane reflections.  
定理28.40。（Cartan–Dieudonn'e，弱形式）假设a是尺寸n（char（k）=26）的k向量空间e上的非退化对称双线性形式。然后，每个f=6 id的等距f∈o（）是至多2n-1超平面反射的组成。

Proof. We proceed by induction on n. For n = 0, this is trivial (since O(ϕ) = {id}).  
证据。我们在n上进行归纳，对于n=0，这是微不足道的（因为o（）=id）。

Next, assume that n ≥ 1. Since ϕ is nondegenerate, we know that there is some nonisotropic vector u ∈ E. There are three cases.  
接下来，假设n≥1。由于\_是非退化的，我们知道有一些非各向同性向量u∈e，有三种情况。

Case 1. f(u) = u.  
案例1。F（U）=U。

Since ϕ is nondegenrate and u is nonisotropic, the hyperplane H = (Ku)⊥ is nondegenerate, E = H ⊕⊥ Ku, and since f(u) = u, we must have f(H) = H. The restriction f0 of of f to H is an isometry of H. By the induction hypothesis, we can write  
既然π是非退化的，u是非各向同性的，超平面h=（ku）是非退化的，e=h ku，既然f（u）=u，我们必须有f（h）=h。f到h的限制f0是h的等距测量。通过归纳假设，我们可以写下

,  
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where τi is some hyperplane reflection about a hyperplane, with k ≤ 2n − 3. We can extend each τi0 to a reflection τi about the hyperplane Li ⊕ Ku so that τi(u) = u, and clearly,  
其中，τi是关于超平面的超平面反射，k≤2n-3。我们可以将每个τi0扩展到超平面li\_ku的反射τi，这样τi（u）=u，并且很明显，

f = τk ◦ ··· ◦ τ1.  
F=τk···τ1.

Case 2. f(u) = −u.  
案例2。F（U）=-U。

If τ is the hyperplane reflection about the hyperplane H = (Ku)⊥, then g = τ ◦ f is an isometry of E such that g(u) = u, and we are back to Case (1). Since τ2 = 1 We obtain  
如果τ是关于超平面h=（ku）的超平面反射，那么g=τf是e的等距测量，这样g（u）=u，我们回到情况（1）。因为τ2=1，我们得到

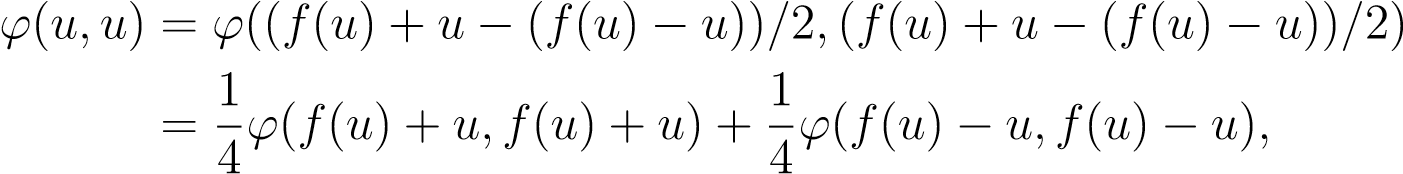
f = τ ◦ τk ◦ ··· ◦ τ1  
f=ττk···τ1

where τ and the τi are hyperplane reflections, with k ≥ 2n − 3, and we get a total of 2n − 2 hyperplane reflections.  
其中，τ和τi是超平面反射，k≥2n−3，我们总共得到2n−2超平面反射。

Case 3. f(u) =6 u and f(u) =6 −u.  
案例3。F（U）=6 U，F（U）=6−U。

Note that f(u) − u and f(u) + u are orthogonal, since  
注意f（u）−u和f（u）+u是正交的，因为

ϕ(f(u) − u,f(u) + u) = ϕ(f(u),f(u)) + ϕ(f(u),u) − ϕ(u,f(u)) − ϕ(u,u) = ϕ(u,u) − ϕ(u,u) = 0. We also have  
Ⅷ（f（u）−u，f（u）+u）=Ⅷ（f（u），f（u））+Ⅷ（f（u），u）−Ⅷ（u，f（u））−Ⅷ（u，u）=Ⅷ（u，u）−Ⅷ（u，u）=0.我们也有



so f(u) + u and f(u) − u cannot be both isotropic, since u is not isotropic. If f(u) − u is not isotropic, then the reflection τf(u)−u is such that  
所以f（u）+u和f（u）−u不能同时是各向同性的，因为u不是各向同性的。如果f（u）−u不是各向同性的，那么反射τf（u）−u是这样的：

τf(u)−u(u) = f(u),  
τf（u）−u（u）=f（u），

and since τf2(u)−u = id, if g = τf(u)−u ◦ f, then g(u) = u, and we are back to case (1). We obtain f = τf(u)−u ◦ τk ◦ ··· ◦ τ1  
既然τf2（u）−u=id，如果g=τf（u）−u f，那么g（u）=u，我们回到情况（1）。我们得到f=τf（u）−uτk···τ1

where τf(u)−u and the τi are hyperplane reflections, with k ≥ 2n − 3, and we get a total of 2n − 2 hyperplane reflections.  
其中，τf（u）−u和τi是超平面反射，k≥2n−3，我们得到了2n−2个超平面反射。

If f(u) + u is not isotropic, then the reflection τf(u)+u is such that  
如果f（u）+u不是各向同性的，那么反射τf（u）+u是这样的：

τf(u)+u(u) = −f(u),  
τf（u）+u（u）=-f（u），

and since τf2(u)+u = id, if g = τf(u)+u ◦ f, then g(u) = −u, and we are back to case (2). We obtain f = τf(u)−u ◦ τ ◦ τk ◦ ··· ◦ τ1  
既然τf 2（u）+u=id，如果g=τf（u）+u\_f，那么g（u）=−u，我们回到情况（2）。我们得到f=τf（u）−uττk···τ1

where τ,τf(u)−u and the τi are hyperplane reflections, with k ≥ 2n−3, and we get a total of 2n − 1 hyperplane reflections. This proves the induction step.   
其中，τ、τf（u）−u和τi是超平面反射，k≥2n−3，我们得到2n−1的超平面反射。这证明了归纳步骤。

The bound 2 1 is not optimal. The strong version of the Cartan–Dieudonn´e theorem says that at most reflections are needed, but the proof is harder. Here is a neat proof due to E. Artin (see [6], Chapter III, Section 4).  
绑定2 1不是最佳的。卡坦-迪乌登定理的强大版本说，大多数情况下需要反思，但证明更难。这是一个整洁的证据，由于E.Artin（见[6]，第三章，第4节）。

Case 1 remains unchanged. Case 2 is slightly different: f(u) − u = 06 is not isotropic. Since ϕ(f(u) + u,f(u) − u) = 0, as in the first subcase of Case (3), g = τf(u)−u ◦ f is such that g(u) = u and we are back to Case 1. This only costs one more reflection.  
案例1保持不变。情况2略有不同：f（u）−u=06不是各向同性的。由于\_（f（u）+u，f（u）−u）=0，如第（3）种情况的第一个亚基，g=τf（u）−u f是这样的，g（u）=u，我们回到了第1种情况。这只需要再考虑一次。

The new (bad) case is:  
新的（坏的）情况是：

Case 3’. f(u) − u is nonzero and isotropic for all nonisotropic u ∈ E. In this case, what saves us is that E must be an Artinian space of dimension n = 2m and that f must be a rotation (f ∈ SO(ϕ)).  
案例3”。f（u）−u是所有非各向同性u∈e的非零和各向同性。在这种情况下，我们省去的是，e必须是尺寸n=2米的Artian空间，f必须是一个旋转（f∈so（））。

If we acccept this fact proved in Proposition 28.43 then pick any hyperplane reflection τ. Then, since f is a rotation, g = τ ◦ f is not a rotation because det(g) = det(τ)det(f) = (−1)(+1) = −1, so g(u) − u is either 0 or not isotropic for some nonisotropic u ∈ E (otherwise, g would be a rotation), we are back to either Case 1 or Case 2, and using the induction hypothesis, we get τ ◦ f = τk ◦ ...,τ1,  
如果我们接受28.43号提案中证明的这个事实，那么选取任何超平面反射τ。那么，既然f是一个旋转，g=τf不是一个旋转，因为det（g）=det（τ）det（f）=（-1）（+1）=−1，所以g（u）−u是0或不是各向同性的，对于一些非各向同性的u∈e（否则，g将是一个旋转），我们返回到情况1或情况2，并且使用归纳假设，我们得到τf=τk…，τ1，

where each *τi* is a hyperplane reflection, and *k* ≤ 2*m*. Since *τ* ◦ *f* is not a rotation, actually *k* ≤ 2*m*−1, and then *f* = *τ* ◦*τk* ◦*...,τ*1, the composition of at most *k* +1 ≤ 2*m* hyperplane reflections.

Therefore, except for the fact that in Case 3’, *E* must be an Artinian space of dimension *n* = 2*m* and that *f* must be a rotation, which has not been proven yet, we proved the following theorem.

**Theorem 28.41.** *(Cartan–Dieudonn´e, strong form) Let ϕ be a nondegenerate symmetric bilinear form on a K-vector space E of dimension n (*char(*K*) = 26 *). Then, every isometry f* ∈ **O**(*ϕ*) *with f* =6 id *is the composition of at most n hyperplane reflections.*

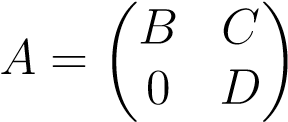
To fill in the gap, we need two propositions.

**Proposition 28.42.** *Let* (*E,ϕ*) *be an Artinian space of dimension* 2*m, and let U be a totally isotropic subspace of dimension m. For any isometry f* ∈ **O**(*ϕ*)*, if f*(*U*) = *U, then* det(*f*) = 1 *(f is a rotation).*

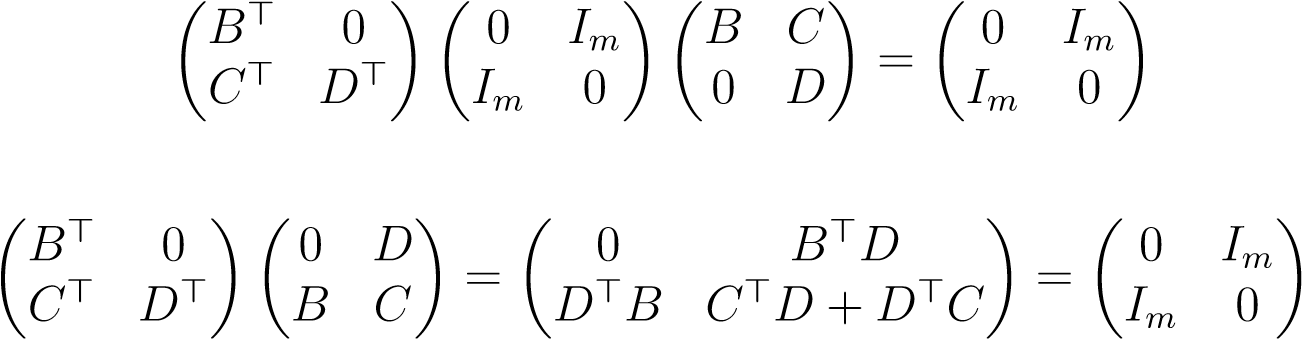
*Proof.* We know that we can find a basis (*u*1*,...,um,v*1*,...,vm*) of *E* such (*u*1*,...,um*) is a basis of *U* and *ϕ* is represented by the matrix

 *.*

Since *f*(*U*) = *U*, the matrix representing *f* is of the form

 *.*

The condition *A*>*Am,mA* = *Am,m* translates as

that is,

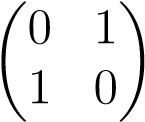
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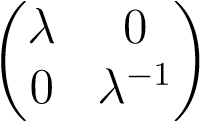
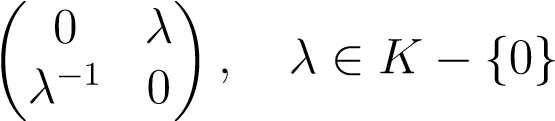
which implies that *B*>*D* = *I*, and so det(*A*) = det(*B*)det(*D*) = det(*B*>)det(*D*) = det(*B*>*D*) = det(*I*) = 1*,*

as claimed

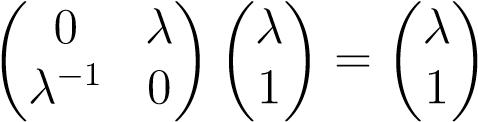
**Proposition 28.43.** *Let ϕ be a nondegenerate symmetric bilinear form on a space E of dimension n, and let f be any isometry f* ∈ **O**(*ϕ*) *such that f*(*u*)−*u is nonzero and isotropic for every nonisotropic vector u* ∈ *E. Then, E is an Artinian space of dimension n* = 2*m, and f is a rotation (f* ∈ **SO**(*ϕ*)*).*

*Proof.* We follow E. Artin’s proof (see [6], Chapter III, Section 4). First, consider the case *n* = 2. Since we are assuming that *E* has some nonzero isotropic vector, by Proposition 28.26, *E* is an Artinian plane and there is a basis in which *ϕ* is represented by the matrix

 *,*

we have *ϕ*((*x*1*,x*2)*,*(*x*1*,x*2)) = 2*x*1*x*2, and the matrices representing isometries are of the form  or *.*

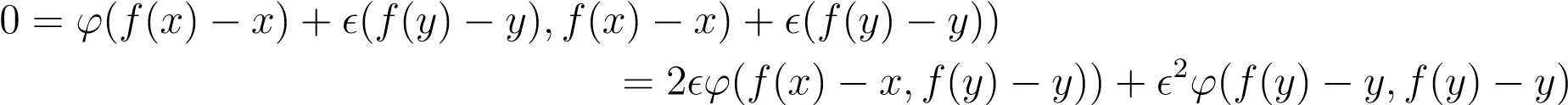
In the second case,

 *,*

but *u* = (*λ,*1) is a nonisotropic vector such that *f*(*u*)−*u* = 0. Therefore, we must be in the first case, and det(*f*) = +1.

Let us now assume that *n* ≥ 3. We are going to prove that *f*(*y*) − *y* is isotropic for all nonzero isotropic vectors *y*. Let *y* be any nonzero isotropic vector. Since *n* ≥ 3, the orthogonal space (*Ky*)⊥ has dimension at least 2, and we know that rad(*Ky*) = rad((*Ky*)⊥), a space of dimension at most 1, which implies that (*Ky*)⊥ contains some nonisotropic vector, say *x*. We have = 0, for 1. Then, by hypothesis, the vectors *f*(*x*) − *x,f*(*x* + *y*) − (*x* + *y*) = *f*(*x*) − *x* + (*f*(*y*) − *y*), and *f*(*x*−*y*)−(*x*−*y*) = *f*(*x*)−*x*−(*f*(*y*)−*y*) are isotropic. The last two vectors can be written

1, so we have

*.*

If we write the two equations corresponding to 1, and then add them up, we get

*ϕ*(*f*(*y*) − *y,f*(*y*) − *y*) = 0*.*