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Generalising Kripke Semantics for Quantified Modal Logics*

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We turn now to what is arguably one of the least well behaved modal languages ever proposed: first-order modal logic.
[?]

Contents

1 Introduction

Modal logic has outgrown its philosophical origins. What used to be the logic of possibility and necessity has become topic-neutral, with applications ranging from the validation of computer programs to the study of mathematical proofs.

Along the way, modal *predicate* logic has lost its role as the centre of investigation, to the point that it is hardly mentioned in many textbooks. Indeed, propositional modal logic itself has emerged as a fragment of first-order predicate logic, with the domain of "worlds" playing the role of "individuals". As emphasized in [?], the distinctive character of modal logic is not its subject matter, but its *perspective*. Statements of modal logic describe relational structures from the inside perspective of a particular node. Modal predicate logic emerges as a somewhat cumbersome hybrid, combining an internal perspective on one class of objects (the domain of the modal operators) with an external perspective on a possibly different class of objects (the domain of quantification).

* These notes were mostly written in 2009–2012, and polished in 2022.

Nevertheless, this hybrid perspective is useful and natural for many applications. When reasoning about time, for example, it is natural to take a perspective that is internal to the structure of times (so that what is true at one point may be false at another), but external to the structure of sticks and stones and people existing at the various times.

Standard Kripke semantics assumes that each "world" w is associated with a domain of "individuals" D_w that somehow exist relative to this world. Formulas like $\exists x \Diamond Fx$ that mix the two kind of quantification have a straightforward interpretation: $\exists x \Diamond Fx$ is true at w iff the individual domain of w contains an object that satisfies Fx at some world accessible from w.

David Lewis [?] once proposed an alternative to Kripke semantics that promises to overcome these limitations. Lewis's key idea was that modal operators simultaneously shift the modal point of evaluation and the reference of singular terms. Informally, $\exists x \Diamond Fx$ is true at w iff the domain of w contains an individual for which there is a *counterpart* at some accessible point w' that satisfies Fx.

Lewis also swapped the traditional, hybrid perspective of Kripke semantics for a thoroughly internal perspective, where statements are evaluated not relative to worlds, but relative to individuals at worlds (see esp. [?: 230-235]). As a consequences, the 'necessity of existence', $\Box \exists y (x = y)$, comes out valid, while basic distribution principles such as $\Box(A \wedge B) \supset \Box A$ become invalid (as noted e.g. in [?] and [?]).

Lewis's internalist semantics has been further developed by Silvio Ghilardi, Giancarlo Meloni, and Giovanna Corsi, who have shown that it has many useful and interesting properties (see [?], [?], [?] [?], [?: 591–616]). However, it goes against the traditional conception of modal predicate logic.

In this essay, I will investigate a semantics that combines the hybrid perspective of classical Kripke semantics with the idea that individuals are tracked across worlds by a counterpart relation. Since I will not assume that the domain of individuals at different worlds are distinct, Kripke semantics emerges as the special case where counterparthood is identity. As we will see, allowing for counterpart relations other than identity results in a fairly simple and intuitive framework that overcomes several shortcomings of standard Kripke semantics.[1]

1 My proposal is inspired by [?], which in turn is inspired by [?]. ([?] summarizes the main results of [?] in English.) Some of the key results announced in [?] and [?] are incorrect; these problems will be repaired. I will also offer a model theory for negative free logics without outer domains, and for languages with individual constants and object-language substitution operators. To incorporate individual constants, Kracht and Kutz [?] switch from counterpart semantics to what Schurz [?] calls *worldline semantics*, where quantifiers range over functions from worlds to individuals; see also [?].

2 Counterpart models

We want to reason about some "worlds", each of which is associated with some "individuals" that are assumed to exist at the relevant world, so that $\Diamond\exists xFx$ is true at a world iff there is some accessible world at which there is an individual satisfying Fx.

A familiar choice point in Kripke semantics is whether we want to allow different individuals to exist at different worlds. This question won't be important in counterpart semantics. But we face an analogous question: whether every individual at some world should have a counterpart at every other world (or at every accessible world).

If we allows for individuals without counterparts at accessible worlds, the next question is what can be said about things that don't exist. The alternatives are well-known from free logic. One option is that if x doesn't exist at w, then every atomic predication Fx is false at w. This is known as a *negative* semantics. Alternatively, one may hold that non-existence is no bar to satisfying predicates, so that Fx may be true at some worlds where x doesn't exist and false at others. The extension of F at a world must therefore be specified not only for things that exist at that world, but also for things that don't exist. This is known as a *positive* semantics. Both approaches have their applications, so I will explore them in tandem.[2]

In positive models, terms are never genuinely empty. Worlds are associated with an *inner domain* of individuals existing at that world, and an *outer domain* of individuals which, although they don't exist, may still fall in the extension of atomic predicates. Every individual at any world will have at least one counterpart at every accessible world, if only in the outer domain.

In negative models, we want to do without the somewhat ghostly outer domains. When we shift the point of evaluation to a world where the value of x has no counterpart, the term becomes empty, and we stipulate that Fx is always false.

A further question in counterpart-theoretic accounts is whether we want to allow for what Allen Hazen calls "internal relations" (see [**?**: 328–330], [**?**: 232f.]). Suppose Dee and Dum are siblings, and consider a possible world that embeds two copies of the actual world, a "left" copy and a "right" copy. We may want to say that this world contains two counterparts of Dee and two of Dum, and that Dee and Dum are necessarily siblings, even though not all counterparts of Dee and Dum at the Left-Right world are siblings of one another.

To model this sort of situations, we need to allow for different ways of locating the individuals from one world at another world. Formally, we will have multiple counterpart relations. One relation will link Dee and Dum to their counterparts in the left copy,

2 There are also *non-valent* options on which atomic predications with empty terms are neither true nor false. The account I will develop is easy to adapt to this approach; see [**?**].

another to their counterparts in the right copy. $\Box Gab$ will be true iff, *relative to every counterpart relation*, all counterparts of a are G-related to all counterparts of b.[3]

Let's define the two kinds of models. As usual, a model combines an abstract frame or structure with an interpretation of our language on that structure. The relevant structures are defined as follows.

Definition 2.1 (Counterpart structure)
A *counterpart structure* is a quintuple $\mathcal{S} = \langle W, R, U, D, K \rangle$, consisting of

1. a non-empty set W (of "points" or "worlds"),
2. a binary ("accessibility") relation R on W,
3. a ("outer domain") function U that assigns to each $w \in W$ a set U_w,
4. a ("inner domain") function D that assigns to each $w \in W$ a set $D_w \subseteq U_w$, and
5. a ("counterpart-inducing") function K that assigns to each pair of points $\langle w, w' \rangle \in R$ a non-empty set $K_{w,w'}$ of ("counterpart") relations $C \subseteq U_w \times U_{w'}$.

$\mathcal{S}$ is *positive* if (i) all outer domains U_w are non-empty and (ii) all counterpart relations are total, in the sense that if $C \in K_{w,w'}$, then for each $d \in U_w$ there is a $d' \in U_{w'}$ with dCd'.

$\mathcal{S}$ is *negative* (or *single-domain*) if $D = U$.

At first, the "counterpart-inducing function" with its associated many counterpart relations may look unfamiliar. Think of this as constructed from a Lewisian counterpart relation in two steps. First, we drop Lewis's requirement of disjoint domains, so that an individual can occur in the domain of many or all worlds. It is then not enough to just specify which individuals are counterparts of other individuals. For example, d at w might have d' as its only counterpart at w', and it might have d'' as its only counterpart at w'', even though d' also exists at w''. Now is d' a counterpart of d? In effect, counterparthood turns into a four-place relation between one individual at one world and another (or the same) individual at another (or the same) world. It proves convenient to represent this by associating each pair of worlds with a "local" counterpart relation between the individuals in the associated domains. In the second step, these local counterpart relation give way to sets of relations in order to allow for internal relations.

3 [?] introduces models with multiple counterpart relations, but stipulates that each relation is actually an injective function, in order to validate the necessity of identity and to get a traditional logic for 'actually'. Multiple counterpart relations are also used in [?] and [?]. It turns out that the introduction of multiple counterpart relations makes little difference to the base logic. In particular, the logic of all positive or negative counterpart models is exactly the same either way. However, multiple counterpart relations will help in the construction of canonical models for stronger logics in section 7, where we will run into a form of Hazen's "problem of internal relations".

The following terminology might help to make all this look more familiar. I will say that in a given model, d' *at* w' *is a counterpart of* d *at* w iff there is a $C \in K_{w,w'}$ such that dCd'. Similarly, a pair of individuals $\langle d'_1, d'_2 \rangle$ *at* w' *is a counterpart of* $\langle d_1, d_2 \rangle$ *at* w iff there is a $C \in K_{w,w'}$ such that $d_1Cd'_1$ and $d_2Cd'_2$. And so on for larger sequences. (Note that an identity pair $\langle d, d \rangle$ at w has $\langle d'_1, d'_2 \rangle$ at w' as counterpart iff there is a $C \in K_{w,w'}$ such that d is C-related to both d'_1 and d'_2.) As we will see, the interpretation of modal formulas can be spelled out directly in terms of this "counterpart relation" between sequences rather than the function $K_{w,w'}$ on which it is officially based.

That counterparthood should be extended to sequences is suggested in [**?**] and [**?**], in response to the problem of internal relations. In the present framework, counterparthood between sequences is a derivative notion. This has the advantage that it immediately rules out some otherwise problematic possibilities. For example, it can never happen that a pair $\langle d_1, d_2 \rangle$ at w has $\langle d'_1, d'_2 \rangle$ at w' as counterpart although by itself, d_1 at w does not have d'_1 at w' as counterpart. Similarly, it can never happen that $\langle d_1, d_2 \rangle$ at w has $\langle d'_1, d'_2 \rangle$ at w' as counterpart while $\langle d_2, d_1 \rangle$ at w does not have $\langle d'_2, d'_1 \rangle$ at w' as counterpart. We also don't have to worry about "gappy" sequences that arise when some things fail to have counterparts. And we automatically get a sensible answer to the question which sequences, in general, matter for the evaluation of a modal formula $\Box A$ at a world: should we consider only individuals denoted by terms in $\Box A$? In what order should the individuals be listed: in order of appearance in $\Box A$? Should we include repetitions if a term occurs more than once in A? And so on.

Next, we define interpretations of the language of quantified modal logic on counterpart structures. To this end, we first have to say what that language is.

> Definition 2.2 (The standard language of QML)
> We assume that there is a denumerable set *Var* of variables, a non-empty set *Pred* of predicates, each associated with an arity. Formulas of the language $\mathcal{L}$ are generated by the rule
>
> $$Px_1 \ldots x_n \mid x{=}y \mid \neg A \mid (A \supset B) \mid \forall x A \mid \Box A,$$
>
> where P is a predicate with arity n, and $x, y, x_1, \ldots, x_n$ are variables.

Notational conventions: I will use 'x', 'y', 'z', 'v' (sometimes with indices or dashes) for members of *Var*, and 'F', 'G', 'P' for members of *Pred* with arity 1, 2 and n, respectively. Formulas involving '$\wedge$', '$\vee$', '$\leftrightarrow$', '$\exists$' and '$\Diamond$' are defined by the usual metalinguistic abbreviations. The order of precedence among connectives is $\neg, \wedge, \vee, \supset$; association is to the right. For any variable x, 'Ex' abbreviates '$\exists y(y{=}x)$', where y is the alphabetically first variable other than x. '$A_1 \wedge \ldots \wedge A_n$' stands for '$A_1$' if $n = 1$, or for '$(A_1 \wedge \ldots \wedge A_{n-1}) \wedge A_n$' if $n > 1$, or for an arbitrary tautology $\top$ (say, '$x{=}x \supset x{=}x$') if $n = 0$. For any expression or set of expressions A, *Var*(A) is the set of variables in (members of) A, and *Varf*(A) is the set of variables with free occurrences in (members of) A.

Individual constants are not explicitly mentioned in definition 2.2. We can simulate individual constants by free variables.

> DEFINITION 2.3 (PREDICATE INTERPRETATION)
> Let $\mathcal{S} = \langle W, R, U, D, K \rangle$ be a counterpart structure. A *predicate interpretation I for $\mathcal{L}$ on $\mathcal{S}$* is a function I that assigns to each world $w \in W$ a function I_w such that
>
> (i) for every non-logical predicate P of $\mathcal{L}$ with arity n, $I_w(P) \subseteq U_w^n$, and
> (ii) $I_w(=) = \{\langle d, d \rangle : d \in U_w\}$.

(For zero-ary predicates P, clause (i) says that $I_w(P) \subseteq U_w^0$. For any U_w, there is exactly one "zero-tuple" in U_w^0, which we may identify with the empty set. So U_w^0 has exactly two subsets, the empty set $\emptyset = 0$ and the unit set of the empty set $\{\emptyset\} = 1$. It is convenient to think of these as truth-values.)

> DEFINITION 2.4 (COUNTERPART MODEL)
> A *counterpart model* $\mathcal{M}$ consists of a counterpart structure $\mathcal{S}$ together with a predicate interpretation I for $\mathcal{L}$ on $\mathcal{S}$.
>
> We call $\mathcal{M}$ *positive* or *negative* in accordance with whether $\mathcal{S}$ is positive or negative.

> DEFINITION 2.5 (ASSIGNMENT)
> A *(variable) assignment on* a set S is a possibly partial function g from *Var* into S.

> DEFINITION 2.6 (VARIANT)
> A variable assignment g' is an *x-variant of an assignment g on* a set S if $g(x) \in S$ and $g'(y) = g(y)$ for all variables y other than x.

When modal operators shift the point of evaluation to another world, variables denote counterparts of the things they originally denoted. Let's introduce an operation that shifts the value of variables to the counterparts of their original value.

Definition 2.7 (Image)
Let $\mathcal{S} = \langle W, R, U, D, K \rangle$ be a counterpart structure, w, w' two worlds in W, and g, g' assignments on $U_w, U_{w'}$ respectively. We say that g' *at* w*'* is an *image of* g *at* w (for short, $w, g \triangleright w', g'$) iff there is a $C \in K_{w,w'}$ such that for every variable x, if $g(x)$ is C-related to some element of $U_{w'}$ then $g(x)Cg'(x)$, otherwise $g'(x)$ is undefined.

In positive models, $g(x)$ is always C-related to some element of $U_{w'}$, for any $C \in K_{w,w'}$.

Since modal operators shift the point of evaluation from one world to another along the accessibility relation, it never matters what counterparts an individual at one world has at another world unless that other world is accessible. This is why I officially stipulated in definition 2.1 that counterparthood is only defined between accessible worlds.

As a consequence, $w, g \triangleright w', g'$ entails wRw': if w' is not accessible from w then there is no $C \in K_{w,w'}$. In the semantics of the box (coming up below), we therefore don't need to mention the accessibility relation R.

Definition 2.8 (Satisfaction)
Let $\mathcal{M} = \langle W, R, U, D, K, I \rangle$ be a counterpart model, w a member of W, and g a variable assignment on U_w. Then we define, for any predicate P, variables $x_1, \ldots, x_n$, and $\mathcal{L}$-formulas A, B,

$$
\begin{array}{ll}
\mathcal{M}, w, g \models Px_1 \ldots x_n & \text{iff } \langle g(x_1), \ldots, g(x_n) \rangle \in I_w(P). \\
\mathcal{M}, w, g \models \neg A & \text{iff } \mathcal{M}, w, g \not\models A. \\
\mathcal{M}, w, g \models A \supset B & \text{iff } \mathcal{M}, w, g \not\models A \text{ or } \mathcal{M}, w, g \models B. \\
\mathcal{M}, w, g \models \forall x A & \text{iff } \mathcal{M}, w, g' \models A \text{ for all } x\text{-variants } g' \text{ of } g \text{ on } D_w. \\
\mathcal{M}, w, g \models \Box A & \text{iff } \mathcal{M}, w', g' \Vdash A \text{ for all } w', g' \text{ such that } w, g \triangleright w', g'.
\end{array}
$$

Validity is truth at all points of evaluation under all interpretations. We allow partial assignment functions for negative logics.

Definition 2.9 (Validity)
A set of $\mathcal{L}$-formulas Γ is *positively valid* in a set Σ of counterpart structures if $\mathcal{S}, I, w, g \models A$ for all $A \in \Gamma$, all $\mathcal{S} = \langle W, R, U, D, K \rangle \in \Sigma$, all interpretations I on $\mathcal{S}$, all worlds $w \in W$ and all total assignments g on U_w.

Γ is *negatively valid* in Σ if $\mathcal{S}, I, w, g \models A$ for all $A \in \Gamma$, all $\mathcal{S} = \langle W, R, U, D, K \rangle \in \Sigma$, all interpretations I on $\mathcal{S}$, all worlds $w \in W$ and all partial assignments g on U_w.

In definition **??**, the "image" relation $\triangleright$ between world-assignment pairs w, g plays the role of the accessibility relation in standard Kripke semantics. This suggests that we could take the $\triangleright$ relation as primitive, as some authors do. I have decided against this, for two reasons.

First, it would blur the distinction between the abstract structures in which $\mathcal{L}$ is interpreted and the interpretation itself. Variable assignments belong to the interpretation of $\mathcal{L}$. They don't represent an independent aspect of a scenario in which $\mathcal{L}$-formulas are interpreted.

One could get around this by replacing the variable assignments g in the semantics by infinite sequences of individuals $\langle d_1, d_2, \ldots \rangle$, with the understanding that the first individual d_1 is the value of the alphabetically first variable, and so on. A second problem would remain: we would have to put the same kinds of constraints on the $\triangleright$ relation that we want to put on counterpart relations between sequences, and these constraints are easier to understand if the $\triangleright$ relation is derived from R and K.

Lemma 2.10 (Locality lemma)
Let A be an $\mathcal{L}$-formula, w a world in a counterpart model $\mathcal{M} = \langle W, R, U, D, K, I \rangle$, and g, g' assignments on U_w such that $g(x) = g'(x)$ for every variable x that is free in A. Then

$$\mathcal{M}, w, g \Vdash A \text{ iff } \mathcal{M}, w, g' \Vdash A.$$

Proof by induction on A.

1. For atomic formulas, the claim is guaranteed directly by definition 2.8.
2. A is $\neg B$. $\mathcal{M}, w, g \Vdash \neg B$ iff $\mathcal{M}, w, g \nVdash B$ by definition 2.8, iff $\mathcal{M}, w, g' \nVdash B$ by induction hypothesis, iff $\mathcal{M}, w, g' \Vdash \neg B$ by definition 2.8.
3. A is $B \supset C$. $\mathcal{M}, w, g \Vdash B \supset C$ iff $\mathcal{M}, w, g \nVdash B$ or $\mathcal{M}, w, g \Vdash C$ by definition 2.8, iff $\mathcal{M}, w, g' \nVdash B$ or $\mathcal{M}, w, g' \Vdash C$ by induction hypothesis, iff $\mathcal{M}, w, g' \Vdash B \supset C$ by definition 2.8.

4. A is $\forall xB$. By definition 2.8, $\mathcal{M}, w, g \Vdash \forall xB$ iff $\mathcal{M}, w, g^{[x\mapsto d]} \Vdash B$ for all $d \in D_w$, where $g^{[x\mapsto d]}$ is the x-variant of g that maps x to d. For each $d \in D_w$, $g^{[x\mapsto d]}$ and $g'^{[x\mapsto d]}$ assign the same value to every variable in B. So by induction hypothesis, $\mathcal{M}, w, g^{[x\mapsto d]} \Vdash B$ for all $d \in D_w$ iff $\mathcal{M}, w, g'^{[x\mapsto d]} \Vdash B$ for all $d \in D_w$, iff $\mathcal{M}, w, g' \Vdash \forall xB$ by definition 2.8.
5. A is $\Box B$. By definition 2.8, $\mathcal{M}, w, g \Vdash \Box B$ iff $\mathcal{M}, w', g^* \Vdash B$ for all w' with wRw' and g^* for which there is a $C \in K_{w,w'}$ such that for every variable x, either $g(x)Cg^*(x)$ or $g(x)$ has no C-counterpart at w' and $g^*(x)$ is undefined. Since $g(x) = g'(x)$ for all variables x in B, each w'-image of g at w agrees with some w'-image of g' on all variables in B and vice versa. So by induction hypothesis, $\mathcal{M}, w', g^* \Vdash B$ for all $\mathcal{M}, w', g^*$ such that $w, g \triangleright w', g^*$ iff $\mathcal{M}, w', g'^* \Vdash B$ for all w', g'^* such that $w, g' \triangleright w', g'^*$, iff $\mathcal{M}, w, g' \Vdash \Box B$ by definition 2.8. ■

Negative models can be "simulated" by positive models, in the following sense. Starting with any negative model $\mathcal{M}^-$, we can create a corresponding positive model $\mathcal{M}^+$ by adding a "null individual" o to the outer domain U_w of every world w that doesn't satisfy any predicates.

Definition 2.11 (Positive transpose)
The *positive transpose* $\mathcal{M}^+$ of a counterpart model $\mathcal{M} = \langle W, R, U, D, K, I \rangle$ is the model $\langle W, R, U^+, D, K^+, I \rangle$ with U^+, K^+ constructed as follows.

Let o be an arbitrary individual (say, the smallest ordinal) not in $\bigcup_w D_w$. For all $w \in W$, $U^+_w = U_w \cup \{o\}$.

For all $\langle w, w' \rangle \in R$, $K^+_{w,w'}$ is the set of relations $C^+ \subseteq U^+_w \times U^+_{w'}$ such that for some $C \in K_{w,w'}$, $C^+ = C \cup \{\langle d, o \rangle : d \in U^+_w \text{ and there is no } d' \in U_{w'} \text{ with } dCd' \}$.

The *positive transpose* g^+ of an assignment function g for $\mathcal{M}$ is the function that "completes" g by setting

$$g^+(x) = \begin{cases} g(x) & \text{if } g(x) \text{ is defined} \\ o & \text{otherwise} \end{cases}$$

Lemma 2.12 (Truth-preservation under transposes)
If $\mathcal{M}^+ = \langle W, R, U^+, D, K^+, I \rangle$ is the positive transpose of a counterpart model $\mathcal{M} = \langle W, R, U, D, K, I \rangle$, then for any world w in $\mathcal{W}$, any assignment g on U_w with positive transpose g^+, and any formula A of $\mathcal{L}$,

$$\mathcal{M}, w, g \Vdash A \text{ iff } \mathcal{M}^+, w, g^+ \Vdash A.$$

Proof by induction on A.

1. A is $Px_1 \dots x_n$. By definition 2.8, $\mathcal{M}, w, g \Vdash Px_1 \dots x_n$ iff $\langle g(x_1), \dots, g(x_n) \rangle \in I_w(P)$. Since $\mathcal{M}$ and $\mathcal{M}^+$ have the same interpretation function I, we have to show that $\langle g(x_1), \dots, g(x_n) \rangle \in I_w(P)$ iff $\langle g^+(x_1), \dots, g^+(x_n) \rangle \in I_w(P)$. If $g(x_i)$ is defined for all $x_1, \dots, x_n$ then this follows from the fact that $g^+(x_i) = g(x_i)$. If some $g(x_i)$ is undefined then $\langle g(x_1), \dots, g(x_n) \rangle$ is undefined and not in $I_w(P)$. Moreover, we then have $g^+(x_i) = o$ and since $I_w(P)$ doesn't contains any tuples involving o, $\langle g^+(x_1), \dots, g^+(x_n) \rangle \notin g^+(P)$.
2. A is $\neg B$. $\mathcal{M}, w, g \Vdash \neg B$ iff $\mathcal{M}, w, g \nVdash B$ by definition 2.8, iff $\mathcal{M}^+, w, g^+ \nVdash B$ by induction hypothesis, iff $\mathcal{M}^+, w, g^+ \Vdash \neg B$ by definition 2.8.
3. A is $B \supset C$. $\mathcal{M}, w, g \Vdash B \supset C$ iff $\mathcal{M}, w, g \nVdash B$ or $\mathcal{M}, w, g \Vdash C$ by definition 2.8, iff $\mathcal{M}^+, w, g^+ \nVdash B$ or $\mathcal{M}^+, w, g^+ \Vdash C$ by induction hypothesis, iff $\mathcal{M}^+, w, g^+ \Vdash B \supset C$ by definition 2.8.
4. A is $\forall x B$. By definition 2.8, $\mathcal{M}, w, g \Vdash \forall x B$ iff $\mathcal{M}, w, g^{[x \mapsto d]} \Vdash B$ for all $d \in D_w$, where $g^{[x \mapsto d]}$ is the x-variant of g that maps x to d. For each $d \in D_w$, ${g^+}^{[x \mapsto d]}$ is the positive transpose of $g^{[x \mapsto d]}$. So by induction hypothesis, $\mathcal{M}, w, g^{[x \mapsto d]} \Vdash B$ for all $d \in D_w$ iff $\mathcal{M}, w, {g^+}^{[x \mapsto d]} \Vdash B$ for all $d \in D_w$, iff $\mathcal{M}^+, w, g^+ \Vdash \forall x B$ by definition 2.8.
5. A is $\Box B$. Assume $\mathcal{M}, w, g \Vdash \Box B$. By definition 2.8, this means that $\mathcal{M}, w', g' \Vdash B$ for all w', g' with $w, g \triangleright w', g'$. We need to show that $\mathcal{M}^+, w', {g^+}' \Vdash B$ for all $w', {g^+}'$ with $w, g^+ \triangleright w', {g^+}'$. So let $w', {g^+}'$ be such that $w, g \triangleright w', {g^+}'$. Since g^+ is total and $\mathcal{M}^+$ a positive structure, this means that for every variable x there is a $C^+ \in K^+_{w,w'}$ with $g^+(x) C^+ {g^+}'(x)$. Let g' be the assignment on $U_{w'}$ that coincides with ${g^+}'$ except that $g'(x)$ is undefined for every variable x for which ${g^+}'(x) = o$. Let $C = \{\langle d, d' \rangle \in C^+ : d \neq o \text{ and } d' \neq o\}$. Now let x be any variable. Assume that there are d, d' with $g(x) = d$ and dCd'. Then $g^+(x) = d$ and dC^+d' and thus ${g^+}'(x) \neq o$, for $\langle d, o \rangle \in C^+$ only if there is no d' with $\langle d, d' \rangle \in C$. So $g'(x) = {g^+}'(x)$, and $g(x) C g'(x)$. On the other hand, assume there are no d, d' with $g(x) = d$ and dCd', either because $g(x)$ is undefined or because $g(x) = d$ and the only d' with $\langle d, d' \rangle \in C^+$ is o. Either way, then ${g^+}'(x) = o$, and so $g'(x)$ is undefined. So for all variables x, if there are d, d' with $g(x) = d$ and dCd' then $g(x) C g'(x)$, otherwise $g'(x)$ is undefined. Since $C \in K_{w,w'}$ by construction of K^+ (definition 2.11), this means that $w, g \triangleright w', g'$. But ${g^+}'$ is the positive transpose of g'. So we've shown that whenever $w, g^+ \triangleright w', {g^+}'$, then there is a g' such that ${g^+}'$ is the positive transpose of g' and $w, g \triangleright w', g'$. We know that $\mathcal{M}, w', g' \Vdash B$. So by induction hypothesis, $w', {g^+}' \Vdash B$. That is, for each $w', {g^+}'$ with $w, g \triangleright w', g'$, $w', {g^+}' \Vdash B$. By definition 2.8, this means that $w, g^+ \Vdash \Box B$.
In the other direction, assume $\mathcal{M}^+, w, g^+ \Vdash \Box B$. That is, $\mathcal{M}^+, w', {g^+}' \Vdash B$ for each $w', {g^+}'$ with $w, g^+ \triangleright w', {g^+}'$. We have to show that $\mathcal{M}, w', g' \Vdash B$ for all w', g' with $w, g \triangleright w', g'$. So let w', g' be such that $w, g \triangleright w', g'$. Then there is a $C \in K_{w,w'}$ such that for every variable x, either $g(x) C g'(x)$ or $g(x)$ has no C-counterpart at w' and $g'(x)$ is undefined. Let ${g'}^+$ be the positive transform of g'. Let $C^+ = C \cup \{\langle d, o \rangle : d \in U^+_w$ and there is no $d' \in U_{w'}$ with dCd' $\}$. By definition 2.11, $C^+ \in K^+_{w,w'}$. For any variable x, if $g(x) C g'(x)$ then both $g(x)$ and $g'(x)$ are defined and thus $g^+(x) = g(x)$ and ${g'}^+(x) = g'(x)$ by definition 2.11; moreover, then $g^+(x) C^+ {g'}^+(x)$ since $C \subseteq C^+$. On the other hand, if $g(x)$ has no C-counterpart at w', so that $g'(x)$ is undefined, then by construction of C^+ and g^+, $g^+(x)$ (which equals $g(x)$ if $g(x)$ is defined, else o) has o as C^+-counterpart at w'; and ${g'}^+(x) = o$; so again $g^+(x) C^+ {g'}^+(x)$. So for every variable

x, there is a $C^+ \in K^+_{w,w'}$ with $g^+(x)C^+g'^+(x)$, and so $w, g^+ \triangleright w', g'^+$. Now we know that $\mathcal{M}^+, w', g^{+'} \Vdash B$ for all $w', g^{+'}$ with $w, g \triangleright w', g^{+'}$. Hence $\mathcal{M}^+, w', g'^+ \Vdash B$. By induction hypothesis, $\mathcal{M}, w', g' \Vdash B$. So we've shown that whenever $w, g \triangleright w', g'$, then $\mathcal{M}, w', g' \Vdash B$. By definition 2.8, this means that $\mathcal{M}, w, g \Vdash \Box B$. ■

(This shows that if a sentence is invalid on a negative structure then it is invalid on its positive transpose. It does not follow that if a sentence is valid on a negative structure then it is valid on its positive transpose: validity on the positive transpose would mean truth at all points under all interpretations of the predicates, including interpretations that make o satisfy some predicates.)

3 Substitution

An odd feature of counterpart semantics, as defined in the previous section, is that it invalidates some classical principles of substitution, such as the "necessity of identity", as formalised by

$$x = y \supset (\Box x = x \supset \Box x = y). \tag{NI}$$

Informally, if x and y denote the same individual, then $\Box x = x$ says that at all accessible worlds, all counterparts of this individual are self-identical, but $\Box x = y$ says that all counterpart of the individual are identical *to one another*.

People sometimes complain that (NI) should be valid because it is an instance of "Leibniz's Law", the schema

$$x = y \supset (A \supset [y/x]A).$$

But if $[y/x]A$ is the formula A with some or all free occurrences of x replaced by y, then this "Law" is invalid even in classical first-order logic, as the following counterexample illustrates:

$$x = y \supset (\exists y(x \neq y) \supset \exists y(y \neq y)). \tag{1}$$

In the substitution of y for x in $\exists y(x \neq y)$, the variable y gets "captured" by the quantifier $\exists y$. A similar kind of capturing happens in (NI), on its counterpart-theoretic interpretation. In counterpart semantics, modal operators effectively function as unselective binders of all variables in their scope, as Lewis [**?**] observed.

There are three common ways to prevent unwanted capturing of substituted terms in classical first-order logic.

The first restricts substitution principles like Leibniz' Law to individual constants. In counterpart semantics, this wouldn't help. We don't officially have individual constants, but if we had, then their referent would plausibly be shifted by modal operators, just like the referent of individual variables. In other words, modal operators are unselective binders of all *singular terms* in their scope, not just of variables.

A second way to prevent unwanted instances of substitution principle in classical first-order logic is to redefine the substitution operation $[y/x]$ so that $[y/x]A$ renames all bound occurrences of y in A before x is replaced by y. We will see in section 3 that this approach also doesn't work in counterpart semantics – as one might already guess given the fact that the box binds *every* variable.

A third option allows restricts substitution principles like Leibniz' Law to cases where the substituted variable doesn't get captured:

$$x = y \supset (A \supset [y/x]A), \text{ provided } y \text{ is free for } x \text{ in } A. \tag{LL}$$

In classical first-order logic, a variable y is *free (to be substituted) for x in A* iff no free occurrence of x in A lies in the scope of a quantifier that binds y. We are going to adopt this third response, but with a different definition of 'free for'.

Given that the box binds all free variables in its scope, y always ends up bound in $[y/x]A$. But modal contexts are not totally opaque. If x and y pick out the same individual d then $\Box Fx$ and $\Box Fy$ are both true iff (roughly) all counterparts of d at all worlds are F. Problems only arise if a modal operator contains both x and y in its scope, and d has multiple counterparts at some world. Informally, $\Box Gxy$ says that (at every accessible world), every d-counterpart is G-related to every d-counterpart. By contrast, $\Box Gxx$ and $\Box Gyy$ only say that every d-counterpart is G-related to itself. As a result, $\Box Gxy \to \Box Gyy$ and $\Box Gxy \to \Box Gxx$ (and $\Diamond Gxx \to \Diamond Gxy$) are true, but their converse is false.

It is hard to directly state conditions under which replacing *some or all* free occurrences of co-referring variables x by y is guaranteed to preserve truth. We can simplify the task by only considering "complete" substitutions, in which *all* free occurrences of x are replaced by y.

> DEFINITION 3.1 (SUBSTITUTION)
> If A is any expression and y and x are variables, then $[y/x]A$ is the expression A with all free occurrences of x replaced by y.

A complete substitution of a y for x in A can never increase the number of distinct variables in A. Very loosely speaking, if x and y co-refer then $\Box A(x, y)$ is a stronger claim than $\Box A(y, y)$. Substituting y for x thus preserves truth. But $\neg\Box A(x, y)$ is weaker than $\neg\Box A(y, y)$. So here we can't substitute y for x.

The following restriction on substitution principles turns out to do the job.

Definition 3.2 (Modal separation and modal freedom)
Let A be a formula and x, y variables.

y is *quantificationally free (to be substituted) for x in A* if no free occurrence of x in A lies in the scope of some occurrence of $\forall y$.

x and y are *modally separated in A* if no occurrence of $\Box$ in A has free occurrences of both x and y in its scope.

y is *(modally) free (to be substituted) for x in A* if (i) y is quantificationally free for x in A and (ii) *either* x and y are modally separated in A *or* A has the form $\Box B$ and y is modally free for x in B.

[FIXME:CHECK: do I need to allow that x is free for x itself? at the moment I don't, but I used to have a special clause for this.]

For example, x and y are modally separated in $\Box Fx \supset \Diamond Fy$ and in $\forall x \Box Gxy$. y is modally free for x in $\Box x = y$ and $\Box\Box\neg Gxy$, but not in $\Box\Diamond\neg Gxy$. Correspondingly,

$$x = y \supset (\Box x = y \supset \Box y = y) \text{ and}$$
$$x = y \supset (\Box\Box\neg Gxy \supset \Box\Box\neg Gyy) \text{ and}$$
$$x = y \supset (\Box\Box\neg\exists x Gxy \supset \Box\Box\neg\exists z Gyz)$$

are valid, but

$$x = y \supset (\Box\Diamond\neg Gxy \supset \Box\Diamond\neg Gyy)$$

is invalid.

[**TODO:** Is there a more perspicuous constraint we could use? What do Ghilardi and Shehtman etc. say?]

In classical first-order logic, one can prove that if y is (quantificationally) free for x in A then

$$g^{[y/x]} \models A \text{ iff } g \models [y/x]A,$$

where $g^{[y/x]}$ is the assignment that maps x to $g(y)$. This is often called the "substitution lemma". In counterpart semantics, the situation is a little more complicated.

Definition 3.3 (Assignments under substitution)
If g is a variable assignment on some domain U_w, and x, y are variables, then $g^{[y/x]}$ is the x-variant of g on U_w with $g'(x) = g(y)$.

We can prove the substitution lemma under the condition of modal separation and quantificational freedom:

Lemma 3.4 (Separation lemma)
Let A be an $\mathcal{L}$-formula, $\mathcal{M} = \langle W, R, U, D, K, I\rangle$ a counterpart model, w a world in W, and g an assignment on U_w. Then

$$\mathcal{M}, w, g^{[y/x]} \Vdash A \text{ iff } \mathcal{M}, w, g \Vdash [y/x]A,$$

provided that y is quantificationally free for x in A and x and y are modally separated in A.

Proof FIXME If y and x are the same variable, then $g^{[y/x]}$ is g, and $[y/x]A$ is A; and so $\mathcal{M}, w, g^{[y/x]} \Vdash A$ iff $\mathcal{M}, w, g \Vdash [y/x]A$, under any condition. Assume then that y and x are different variables.

The proof is by induction on A. I use 'y is *safe for x in S*' as shorthand for 'y is quantificationally free for x in S and x and y are modally separated in S'.

1. A is $Px_1 \ldots x_n$. Since $g^{[y/x]}(x_i) = g([y/x]x_i)$ for any variable x_i, we have $\mathcal{M}, w, g^{[y/x]} \Vdash Px_1 \ldots x_n$ iff $\mathcal{M}, w, g \Vdash [y/x]Px_1 \ldots x_n$ under any condition.
2. A is $\neg B$. Assume y is safe for x in A. Then y is also safe for x in B. By definition 2.8, $\mathcal{M}, w, g^{[y/x]} \Vdash \neg B$ iff $\mathcal{M}, w, g^{[y/x]} \nVdash B$. By induction hypothesis, $\mathcal{M}, w, g^{[y/x]} \nVdash B$ iff $\mathcal{M}, w, g \nVdash [y/x]B$. And $\mathcal{M}, w, g \nVdash [y/x]B$ iff $\mathcal{M}, w, g \Vdash [y/x]\neg B$, by definition 2.8 and the fact that $[y/x]\neg B = \neg[y/x]B$. So $\mathcal{M}, w, g^{[y/x]} \Vdash \neg B$ iff $\mathcal{M}, w, g \Vdash [y/x]\neg B$.
3. A is $B \supset C$. Parallel to the previous case.
4. A is $\forall zB$. We need to distinguish three cases.
   First, $z = x$. In this case, there are no free occurrences of x in A. So $[y/x]A = A$. Also, by the locality lemma (lemma 2.10), the truth-value of A does not depend on the value of x. Thus $\mathcal{M}, w, g^{[y/x]} \Vdash A$ iff $\mathcal{M}, w, g \Vdash A$ iff $\mathcal{M}, w, g \Vdash [y/x]A$.
   Second, $z = y$. In this case, the assumption that y is quantificationally free for x in A entails that there are again no free occurrences of x in A, and the same reasoning applies.
   Finally, assume $z \notin \{x, y\}$. Assume also that y is safe for x in A, and therefore in B. By definition 2.8, $\mathcal{M}, w, g^{[y/x]} \Vdash forallzB$ iff $\mathcal{M}, w, g^{[y/x]\prime} \Vdash B$ for every z-variant $g^{[y/x]\prime}$ of $g^{[y/x]}$ on D_w. Let $g^{[y/x][z\to d]}$ be the z-variant of $g^{[y/x]}$ on D_w that maps z to d, where d is some element of D_w. By definition 2.8, $\mathcal{M}, w, g^{[y/x]} \Vdash forallzB$ iff $\mathcal{M}, w, g^{[y/x][z\to d]} \Vdash B$ for every $d \in D_w$. Since $z \neq x$, $g^{[y/x][z\to d]}$ is $g^{[z\to d][y/x]}$. So $\mathcal{M}, w, g^{[y/x]} \Vdash forallzB$ iff $\mathcal{M}, w, g^{[z\to d][y/x]} \Vdash B$ for every $d \in D_w$. By induction hypothesis, $\mathcal{M}, w, g^{[z\to d][y/x]} \Vdash B$ iff $\mathcal{M}, w, g^{[z\to d]} \Vdash [y/x]B$. So $\mathcal{M}, w, g^{[y/x]} \Vdash forallzB$ iff $\mathcal{M}, w, g^{[z\to d]} \Vdash [y/x]B$ for every $d \in D_w$, iff $\mathcal{M}, w, g \Vdash [y/x]\forall zB$ by definition 2.8.
5. A is $\Box B$. Assume y is safe for x in A, and therefore in B. This means that either x or y has no free occurrence in A.
   If x has no free occurrence in A then $[y/x]A = A$ and $\mathcal{M}, w, g^{[y/x]} \Vdash A$ iff $\mathcal{M}, w, g \Vdash A$ iff $\mathcal{M}, w, g \Vdash [y/x]A$ by the locality lemma.
   Assume that y has no free occurrence in A. By definition 2.8, $\mathcal{M}, w, g^{[y/x]} \Vdash \Box B$ iff $w', g^{[y/x]\prime} \Vdash B$ for all $w', g^{[y/x]\prime}$ with $w, g^{[y/x]} \triangleright w', g^{[y/x]\prime}$.

FIXME Consider any w'-image $g^{[y/x]\prime}$ of $g^{[y/x]}$ at w; i.e. for some counterpart relation $C \in K_{w,w'}$ and all variables z, $g^{[y/x]\prime}_{w'}(z)$ is a C-counterpart of $g^{[y/x]}_{w}$ (or undefined if there is none). Let g^* be like $g^{[y/x]\prime}$ except that $g^*_{w'}(y) = g^{[y/x]\prime}_{w'}(x)$. Let g' be like g^* except that $g'_{w'}(x)$ is some C-counterpart of $g_w(x)$, or undefined if there is none. Then $g_w \triangleright g'_{w'}$. (In particular, $g'_{w'}(y) = g^*_{w'}(y) = g^{[y/x]\prime}(x)$ is some C-counterpart of $g^{[y/x]}_{w}(x) = g_w(y)$, or undefined if there is none.) So by (2), $w', g'^{[y/x]} \Vdash B$. But $g'^{[y/x]} = g^*$ (since $g^*_{w'}(x) = g^*_{w'}(y)$). So $w', g^* \Vdash B$. And since $y \notin \mathit{Var}(B)$ and g^* is a y-variant of $g^{[y/x]\prime}$ on w', by the coincidence lemma **??**, $w', g^* \Vdash B$ iff $w', g^{[y/x]\prime} \Vdash B$. ■
(i). By definition 2.8, $\mathcal{M}, w, g^{[y/x]} \Vdash \Box B$ iff $w', g^{[y/x]\prime} \Vdash B$ for all $w', g^{[y/x]\prime}$ with wRw' and $g^{[y/x]}_{w} \triangleright g^{[y/x]\prime}_{w'}$. On the other hand, $\mathcal{M}, w, g \Vdash [y/x]\Box B$ iff $\mathcal{M}, w, g \Vdash \Box[y/x]B$ (by definition **??**), iff $w', g' \Vdash [y/x]B$ for all w', g' with wRw' and $g_w \triangleright g'_{w'}$. Since the provisos of (i) carry over from $\Box B$ to B, by induction hypothesis, $w', g'^{[y/x]} \Vdash B$ iff $w', g' \Vdash [y/x]B$. So we have to show that

$$w', g^{[y/x]\prime} \Vdash B \text{ for all } w', g^{[y/x]\prime} \text{ such that } wRw' \text{ and } g^{[y/x]}_{w} \triangleright g^{[y/x]\prime}_{w'} \tag{1}$$

iff

$$w', g'^{[y/x]} \Vdash B \text{ for all } w', g' \text{ such that } wRw' \text{ and } g_w \triangleright g'_{w'}. \tag{2}$$

(1) implies (2) because every interpretation $g'^{[y/x]}$ with $g_w \triangleright g'_{w'}$ is also an interpretation $g^{[y/x]\prime}$ with $g^{[y/x]}_{w} \triangleright g^{[y/x]\prime}_{w'}$. The converse, however, may fail: both $g'^{[y/x]}_{w'}$ and $g^{[y/x]\prime}_{w'}$ assign to x and y some counterpart of $g_w(y)$ (if there is any). But while $g'^{[y/x]}_{w'}$ assigns *the same* counterpart to x and y, $g^{[y/x]\prime}_{w'}$ may choose different counterparts for x and y relative to the same counterpart relation.
If there is no counterpart relation relative to which $g_w(y)$ has multiple counterparts, then this cannot happen. Thus under proviso (b), each $g^{[y/x]\prime}$ with $g^{[y/x]}_{w} \triangleright g^{[y/x]\prime}_{w'}$ is also a $g'^{[y/x]}$ with $g'_w \triangleright g'^{[y/x]}_{w'}$, and so (2) implies (1). ■

If we weaken the condition to modal freedom, we only get one direction of the lemma:

Lemma 3.5 (Restricted substitution lemma)
Let A be an $\mathcal{L}$-formula, $\mathcal{M} = \langle W, R, U, D, K, I \rangle$ a counterpart model, w a world in W, and g an assignment on U_w. If y is (modally) free for x in A, then

$$\text{if } \mathcal{M}, w, g^{[y/x]} \Vdash A \text{ then } \mathcal{M}, w, g \Vdash [y/x]A.$$

Proof FIXME
As in the proof of the previous lemma, we can assume that x and y are different variables. The proof is by induction on A, but we don't have to go through all cases separately.

Assume A is not of the form $\Box B$, and that y is modally free for x in A. By definition of modal freedom, it follows that y is quantificationally free for x in A and x and y are modally separated in A. The target claim then follows by the separation lemma.

Assume A has the form $\Box B$.

(ii). Assume $\mathcal{M}, w, g^{[y/x]} \Vdash \Box B$. By definition 2.8, then $w', g^{[y/x]\prime} \Vdash B$ for all $w', g^{[y/x]\prime}$ with wRw' and $g^{[y/x]}_w \triangleright g^{[y/x]\prime}_{w'}$. As before, every interpretation $g'^{[y/x]}$ with $g_w \triangleright g'_{w'}$ is also an interpretation $g^{[y/x]\prime}$ with $g^{[y/x]}_w \triangleright g^{[y/x]\prime}_{w'}$. So $w', g'^{[y/x]} \Vdash B$ for all $w', g'^{[y/x]}$ with wRw' and $g_w \triangleright g'_{w'}$.

If y is modally free for x in $\Box B$, then y is modally free for x in B. Then by induction hypothesis, $w', g' \Vdash [y/x]B$ *if* $w', g'^{[y/x]} \Vdash B$. So $w', g' \Vdash [y/x]B$ for all w', g' with wRw' and $g_w \triangleright g'_{w'}$. By definition 2.8, this means that $\mathcal{M}, w, g \Vdash \Box[y/x]B$, and so $\mathcal{M}, w, g \Vdash [y/x]\Box B$ by definition **??**. ■

The converse of is not true. For example, $\mathcal{M}, w, g \Vdash [y/x]\Box x = y$ does not imply $\mathcal{M}, w, g^{[y/x]} \Vdash \Box x = y$. But the present direction is enough for substitution principles like Leibniz' Law or Universal Instantiation. Loosely speaking, these principles allow substituting y for x under conditions that ensure that the LHS of lemma 3.5 is satisfied.

For future reference, we may note that the problem of modal capturing only arises if individuals can have multiple counterparts relative to the same counterpart relation.

LEMMA 3.6 (FUNCTIONAL SUBSTITUTION LEMMA)
Let A be an $\mathcal{L}$-formula, $\mathcal{M} = \langle W, R, U, D, K, I \rangle$ a counterpart model, w a world in W, and g an assignment on U_w. Then

$$\mathcal{M}, w, g^{[y/x]} \Vdash A \text{ iff } \mathcal{M}, w, g \Vdash [y/x]A,$$

provided that y is quantificationally free for x in A and all counterpart relations C in K are *functional*, meaning that if $d_1 C d_2$ then there is no $d_3 \neq d_2$ for which $d_1 C d_2$.

PROOF FIXME

As in the proof of the previous lemma, we can assume that x and y are different variables. The proof is by induction on A. All cases except the one for $\Box A$ are parallel to those in the proof of the previous lemma.

Assume A has the form $\Box B$.

(i). By definition 2.8, $\mathcal{M}, w, g^{[y/x]} \Vdash \Box B$ iff $w', g^{[y/x]\prime} \Vdash B$ for all $w', g^{[y/x]\prime}$ with wRw' and $g^{[y/x]}_w \triangleright g^{[y/x]\prime}_{w'}$. On the other hand, $\mathcal{M}, w, g \Vdash [y/x]\Box B$ iff $\mathcal{M}, w, g \Vdash \Box[y/x]B$ (by definition **??**), iff $w', g' \Vdash [y/x]B$ for all w', g' with wRw' and $g_w \triangleright g'_{w'}$. Since the provisos of (i) carry over from $\Box B$ to B, by induction hypothesis, $w', g'^{[y/x]} \Vdash B$ iff $w', g' \Vdash [y/x]B$. So we have to show that

$$w', g^{[y/x]\prime} \Vdash B \text{ for all } w', g^{[y/x]\prime} \text{ such that } wRw' \text{ and } g^{[y/x]}_w \triangleright g^{[y/x]\prime}_{w'} \tag{1}$$

iff

$$w', g'^{[y/x]} \Vdash B \text{ for all } w', g' \text{ such that } wRw' \text{ and } g_w \triangleright g'_{w'}. \tag{2}$$

(1) implies (2) because every interpretation $g'^{[y/x]}$ with $g_w \triangleright g'_{w'}$ is also an interpretation $g^{[y/x]\prime}$ with $g^{[y/x]}_w \triangleright g^{[y/x]\prime}_{w'}$. The converse, however, may fail: both $g'^{[y/x]}_{w'}$ and $g^{[y/x]\prime}_{w'}$ assign to x and y some counterpart of $g_w(y)$ (if there is any). But while $g'^{[y/x]}_{w'}$ assigns *the same* counterpart to x and y, $g^{[y/x]\prime}_{w'}$ may choose different counterparts for x and y relative to the same counterpart relation.

If there is no counterpart relation relative to which $g_w(y)$ has multiple counterparts, then this cannot happen. Thus under proviso (b), each $g^{[y/x]\prime}$ with $g^{[y/x]}_w \triangleright g^{[y/x]\prime}_{w'}$ is also a $g'^{[y/x]}$ with $g'_w \triangleright g'^{[y/x]}_{w'}$, and so (2) implies (1). ■

It will be useful to have a notion of substitution that applies to several variables at once.

Definition 3.7 (Substitution)
A *substitution* is a total function $\sigma : \mathit{Var} \to \mathit{Var}$. If σ is injective, it is called a *transformation*. We write $[y_1, \ldots, y_n/x_1, \ldots, x_n]$ for the substitution that maps x_1 to y_1, …, x_n to y_n, and every other variable to itself.

For any expression A, $\sigma(A)$ is the expression that results from A by simultaneously replacing every free variable x by $\sigma(x)$. I will sometimes write σA or A^σ instead of $\sigma(A)$.

If Γ is a set of formulas, then $\sigma(\Gamma)$ or Γ^σ is $\{C^\tau : C \in \Gamma\}$.

Here is the corresponding generalisation of $g^{[y/x]}$.

Definition 3.8 (Assignments under substitution)
If g is an assignment and σ a substitution, then g^σ is the assignment that maps every variable x to $g(\sigma(x))$.

Substitutions can be composed. If σ and τ are substitutions, then $\tau \cdot \sigma$ is the substitution that maps each variable x to $\tau(\sigma(x))$. Observe that composition appears to behave differently in superscripts of formulas than in superscripts of assignments: for formulas A,

$$(A^\sigma)^\tau = \tau(\sigma(A)) = A^{\tau \cdot \sigma},$$

but for assignments g,

$$(g^\sigma)^\tau = Vg^{\sigma \cdot \tau}.$$

That's because $(g^\sigma)^\tau(x) = g^\sigma(\tau(x)) = g(\sigma(\tau(x))) = g(\sigma \cdot \tau(x)) = g^{\sigma \cdot \tau}(x)$.

Definition 3.7 draws attention to the class of injective substitutions, or transformations. These will play a prominent role in canonical models.

A transformation never substitutes two distinct variables by the same variable. For instance, the identity substitution $[x/x]$ or the swapping operation $[x, y/y, x]$ are transformations.

For transformations τ, it proves convenient to define $\tau(A)$ as a replacement of *all* variables x – free and bound – in A by $\tau(x)$. This operation makes capturing impossible. For the free variable y in $\forall x A(y)$ to be captured by $\forall x$ after substitution, x and y have to be replaced by the same variable. Modal capturing also becomes impossible, as the following lemma shows.

FIXME: I shouldn't use different definitions of $\sigma(A)$ depending on whether σ is injective or not. But if I always allow substituting bound variables, then my definition of $[y/x]A$ is different from how everyone else defines it, and we need to strengthen quantificational freedom to avoid renaming x in $\forall x Gxy$ to y. I then can't easily compare my logics to the standard logics. So I should probably not allow substituting bound variables for transformations, and make the Transformation Lemma subject to the condition that y is quantificationally free for x in A. Need to check how this affects the canonical model stuff.

Lemma 3.9 (Transformation Lemma)
For any world w in any model $\mathcal{M}$, any variable assignment g on U_w, any formula A and transformation τ, $\mathcal{M}, w, g^\tau \models A$ iff $\mathcal{M}, w, g \models A^\tau$.

Proof FIXME by induction on A.

Let's look at the clause for the box. To show: $\mathcal{M}, w, g^\tau \models \Box B$ iff $\mathcal{M}, w, g \models \Box B^\tau$.

Assume $\mathcal{M}, w, g \not\Vdash \Box B^\tau$. Then $w', g' \not\Vdash B^\tau$ for some w', g' with $w, g \triangleright w', g'$. This means that there is a counterpart relation $C \in K_{w,w'}$ such that for all variables x, $g'(x)$ is some C-counterpart of $g(x)$ (or undefined if $g'(x)$ has no C-counterpart). By induction hypothesis, $w', g'^\tau \not\Vdash B$. For any variable x, $g'^\tau(x) = g'(x^\tau)$ and $g^\tau(x) = g(x^\tau)$. So $g'^\tau(x)$ is a C-counterpart of $g^\tau(x)$ (or undefined if $g^\tau(x)$ has no C-counterpart). So $w, g^\tau \triangleright w', g'^\tau$. Hence $w', g'^\tau \not\Vdash B$ for some w', g'^τ with $w, g^\tau \triangleright w', g'^\tau$. So $\mathcal{M}, w, g^\tau \not\Vdash \Box B$.

In the other direction, assume $\mathcal{M}, w, g^\tau \not\Vdash \Box B$. Then $w', g^* \not\Vdash B$ for some w', g^* with $w, g^\tau \triangleright w', g^*$. This means that there is a counterpart relation $C \in K_{w,w'}$ such that for all variables x, $g^*(x)$ is some C-counterpart at w' of $g^\tau(x)$ (if any, else undefined). Let g' be such that for all variables x, $g'(x^\tau) = g^*(x)$, and for all $x \notin \mathrm{Ran}(\tau)$, $g'(x)$ is an arbitrary C-counterpart of $g(x)$, or undefined if there is none. Then $w, g \triangleright w', g'$. Moreover, g^* is g'^τ. By induction hypothesis, $w', g' \not\Vdash B^\tau$. So $w', g' \not\Vdash B^\tau$ for some w', g' with $w, g \triangleright w', g'$. So $\mathcal{M}, w, g \not\Vdash (\Box B)^\tau$.

1. $A = Px_1 \ldots x_n$. $\mathcal{M}, w, V^\tau \Vdash Px_1 \ldots x_n$ iff $\langle V^\tau_w(x_1), \ldots, V^\tau_w(x_n)\rangle \in V^\tau_w(P)$, iff $\langle V_w(x^\tau_1), \ldots, V_w(x^\tau_n)\rangle \in V_w(P)$, iff $\mathcal{M}, w, V \Vdash (Px_1 \ldots x_n)^\tau$.

2. $A = \neg B$. $\mathcal{M}, w, V^\tau \Vdash \neg B$ iff $\mathcal{M}, w, V^\tau \nVdash B$, iff $\mathcal{M}, w, V \nVdash B^\tau$ by induction hypothesis, iff $\mathcal{M}, w, V \Vdash (\neg B)^\tau$.
3. $A = B \supset C$. $\mathcal{M}, w, V^\tau \Vdash B \supset C$ iff $\mathcal{M}, w, V^\tau \nVdash B$ or $\mathcal{M}, w, V^\tau \Vdash C$, iff $\mathcal{M}, w, V \nVdash B^\tau$ or $\mathcal{M}, w, V \Vdash C^\tau$ by induction hypothesis, iff $\mathcal{M}, w, V \Vdash (B \supset C)^\tau$.
4. $A = \langle y : x \rangle B$. By definition 5.2, $\mathcal{M}, w, V^\tau \Vdash \langle y : x \rangle B$ iff $\mathcal{M}, w, (V^\tau)^{[y/x]} \Vdash B$. Now $(V^\tau)^{[y/x]}_w(x) = V^\tau_w(y) = V_w(y^\tau) = V^{[y^\tau/x^\tau]}_w(x^\tau) = (V^{[y^\tau/x^\tau]})^\tau_w(x)$. And for any variable $z \neq x$, $(V^\tau)^{[y/x]}_w(z) = V^\tau_w(z) = V_w(z^\tau) = V^{[y^\tau/x^\tau]}_w(z^\tau)$ (because $z^\tau \neq x^\tau$, by injectivity of τ) $= (V^{[y^\tau/x^\tau]})^\tau_w(z)$. So $(V^\tau)^{[y/x]}$ coincides with $(V^{[y^\tau/x^\tau]})^\tau$ at w. By the locality lemma 2.10, $\mathcal{M}, w, (V^\tau)^{[y/x]} \Vdash B$ iff $\mathcal{M}, w, (V^{[y^\tau/x^\tau]})^\tau \Vdash B$. By induction hypothesis, the latter holds iff $\mathcal{M}, w, V^{[y^\tau/x^\tau]} \Vdash B^\tau$, iff $\mathcal{M}, w, V \Vdash \langle y^\tau : x^\tau \rangle B^\tau$ by definition 5.2, iff $\mathcal{M}, w, V \Vdash (\langle y : x \rangle B)^\tau$ by definition 3.7.
5. $A = \forall x B$. Assume $\mathcal{M}, w, V^\tau \nVdash \forall x B$. Then $\mathcal{M}, w, V^* \nVdash B$ for some existential x-variant V^* of V^τ on w. Let V' be the (existential) x^τ-variant of V on w with $V'_w(x^\tau) = V^*_w(x)$. Then ${V'}^\tau_w(x) = V^*_w(x)$, and for any variable $z \neq x$, ${V'}^\tau_w(z) = V'_w(z^\tau) = V_w(z^\tau)$ (because $z^\tau \neq x^\tau$, by injectivity of τ) $= V^\tau_w(z) = V^*_w(z)$. So ${V'}^\tau$ coincides with V^* on w, and by locality (lemma 2.10), $\mathcal{M}, w, {V'}^\tau \nVdash B$. By induction hypothesis, then $\mathcal{M}, w, V' \nVdash B^\tau$. So there is an existential x^τ-variant V' of V on w such that $\mathcal{M}, w, V' \nVdash B^\tau$. By definition 2.8, this means that $\mathcal{M}, w, V \nVdash \forall x^\tau B^\tau$, and hence $\mathcal{M}, w, V \nVdash (\forall x B)^\tau$ by definition 3.7.

   In the other direction, assume $\mathcal{M}, w, V \nVdash (\forall x B)^\tau$, and thus $\mathcal{M}, w, V \nVdash \forall x^\tau B^\tau$. Then $\mathcal{M}, w, V' \nVdash B^\tau$ for some existential x^τ-variant V' of V on w, and by induction hypothesis $\mathcal{M}, w, {V'}^\tau \nVdash B$. Let V^* be the (existential) x-variant of V^τ on w with $V^*_w(x) = V'_w(x^\tau)$. Then $V^*_w(x) = {V'}^\tau_w(x)$, and for any variable $z \neq x$, $V^*_w(z) = V^\tau_w(z) = V_w(z^\tau) = V'_w(z^\tau)$ (because $z^\tau \neq x^\tau$, by injectivity of τ) $= {V'}^\tau_w(z)$. So V^* coincides with ${V'}^\tau$ on w, and by locality (lemma 2.10), $\mathcal{M}, w, V^* \nVdash B$. So there is an existential x-variant V^* of V^τ on w such that $\mathcal{M}, w, V^* \nVdash B$. By definition 2.8, this means that $\mathcal{M}, w, V^\tau \nVdash \forall x B$.
6. $A = \Box B$. Assume $\mathcal{M}, w, V \nVdash \Box B^\tau$. Then $w', V' \nVdash B^\tau$ for some w', V' with wRw' and V' a w' image of V at w. This means that there is a counterpart relation $C \in K_{w,w'}$ such that for all variables x, $V'_{w'}(x)$ is some C-counterpart at w' of $V_w(x)$ at w (if any, else undefined). By induction hypothesis, $w', {V'}^\tau \nVdash B$. Since for all x, ${V'}^\tau_{w'}(x) = V'_{w'}(x^\tau)$ and $V^\tau_w(x) = V_w(x^\tau)$, it follows that ${V'}^\tau_{w'}(x)$ is a C-counterpart of $V^\tau_w(x)$ (if any, else undefined). So ${V'}^\tau$ is a w'-image of V^τ at w. Hence $w', {V'}^\tau \nVdash B$ for some $w', {V'}^\tau$ with wRw' and ${V'}^\tau$ a w'-image of V^τ at w. So $\mathcal{M}, w, V^\tau \nVdash \Box B$.

   In the other direction, assume $\mathcal{M}, w, V^\tau \nVdash \Box B$. Then $w', V^* \nVdash B$ for some w', V^* with wRw' and V^* a w' image of V^τ at w. This means that there is a counterpart relation $C \in K_{w,w'}$ such that for all variables x, $V^*_{w'}(x)$ is some C-counterpart at w' of $V^\tau_w(x)$ at w (if any, else undefined). Let V' be like V except that for all variables x, $V'_{w'}(x^\tau) = V^*_{w'}(x)$, and for all $x \notin \mathrm{Ran}(\tau)$, $V'_{w'}(x)$ is an arbitrary C-counterpart of $V_w(x)$, or undefined if there is none. V' is a w' image of V at w. Moreover, V^* is ${V'}^\tau$. By induction hypothesis, $w', V' \nVdash B^\tau$. So $w', V' \nVdash B^\tau$ for some w', V' with wRw' and V' a w' image of V at w. So $\mathcal{M}, w, V \nVdash (\Box B)^\tau$. ∎

4 Logics

I now want to describe the minimal logics that are characterised by the semantics from section 2. Following tradition, a *logic* (or *system*) in this context is simply a set of formulas, and I will describe such sets by recursive clauses corresponding to the axioms and rules of a Hilbert-style calculus.

Recall that we have two kinds of models: positive models with two domains, and negative models with a single domain. Accordingly we have two minimal logics.

The logic of all positive models is essentially the combination of standard positive free logic with the propositional modal logic K. The only place to be careful is with substitution principles like Leibniz' Law, which have to be restricted as explained in the previous section. (If we add the unrestricted principles, we get logics for functional structures.)

Standard (non-modal) positive free logic can be defined as the smallest set of formulas L that contains

(Taut) all propositional tautologies,

as well as all instances of

(UD) $\forall xA \supset (\forall x(A \supset B) \supset \forall xB)$,

(VQ) $A \supset \forall xA$, provided x is not free in A,

(FUI) $\forall xA \supset (Ey \supset [y/x]A)$, provided y is quantificationally free for x in A,

($\forall$Ex) $\forall xEx$,

(=R) $x = x$,

(LL) $x = y \supset A \supset [y/x]A$, provided y is quantificationally free for x in A,

and that is closed under modus ponens, universal generalisation, and variable substitution:

(MP) if $\vdash_L A$ and $\vdash_L A \supset B$, then $\vdash_L B$,

(UG) if $\vdash_L A$, then $\vdash_L \forall xA$,

(Sub) if $\vdash_L A$, then $\vdash_L [y/x]A$, provided y is quantificationally free for x in A.

Here, as always, $\vdash_L A$ means $A \in L$.[4]

In the logic of counterpart structures, (LL), (FUI) and (Sub) are restricted to cases where y is modally free for x in A:

4 [**?**] use $\forall xA \leftrightarrow A$ in place of (VQ), which precludes empty inner domains (as [**?**: 38] points out). Neither the positive nor the negative semantics I have presented validates the claim that something exists. If we ruled out empty inner domains in positive or negative models, $\exists xEx$ would be needed as extra axiom.

(FUI*) $\forall x A \supset (Ey \supset [y/x]A)$, provided y is modally free for x in A,

(LL*) $x = y \supset A \supset [y/x]A$, provided y is modally free for x in A,

(Sub*) if $\vdash_L A$, then $\vdash_L [y/x]A$, provided y is modally free for x in A.

In addition, we have the modal schema

(K) $\Box A \supset (\Box(A \supset B) \supset \Box B)$

and closure under necessitation,

(Nec) if $\vdash_L A$, then $\vdash_L \Box A$.

> Definition 4.1 (Minimal positive (quantified modal) logic)
> The *minimal (positive) quantified modal logic* P is the smallest set $L \subseteq \mathcal{L}$ that contains all $\mathcal{L}$-instances of (Taut), (UD), (VQ), (FUI*), ($\forall$Ex), (=R), (LL*), (K), and that is closed under (MP), (UG), (Nec) and (Sub*).

FIXME: I should use a better naming convention. Maybe PQK or PK, analogously NQK or NK?

We shall also be interested in stronger logics adequate for various classes of counterpart structures. As a first stab, I will adopt the following definition.

> Definition 4.2 (Positive logics)
> A *positive quantified modal logic* is a set $L \supseteq \mathsf{P}$ that is closed under (MP), (UG), (Nec) and (Sub*).

As it stands, definition 4.2 allows for "logics" in which (say) $F_1 x$ is a theorem but not $F_2 x$. This clashes with the idea that logical truths should be independent of the interpretation of non-logical terms. A more adequate definition would add a second-order closure condition to the effect that, roughly, whenever $\vdash_L A$ then $\vdash_L [B/Px_1 \dots x_n]A$, where $[B/Px_1 \dots x_n]A$ is A with all occurrences of the atomic formula $Px_1 \dots x_n$ replaced by (the arbitrary formula) B. Making this precise requires some care, especially once we look at negative logics where $Fx \supset Ex$ is valid, but $\neg Fx \supset Ex$ is not. Definition 4.2 will do as long as we start off with a class of counterpart structures and look for a corresponding positive logic; that logic will always satisfy definition 4.2.

The first-order closure condition (Sub*) excludes logics in which, for example, Fx is a theorem but not Fy. To see why (Sub*) needs the proviso ('y is modally free for x in A'), note that we could otherwise move from the (FUI*) instance $\vdash_L \forall x \Diamond Gxy \supset \Diamond Gzy$ to $\vdash_L \forall x \Diamond Gx \supset \Diamond Gyy$, which is invalid as long as individuals can have multiple counterparts.

TODO: mention [?].

You may wonder whether (Sub*) is really needed in definition 4.1, given that the axioms are stated as schemas: doesn't this mean that every substitution instance of an axiom is itself an axiom, and isn't this property of closure under substitution preserved by (MP), (UG) and (Nec)? Not quite. For example,

$$v = y \supset (v = z \supset y = x) \tag{2}$$

is an instance of (LL*), and

$$x = y \supset (x = x \supset y = x) \tag{3}$$

follows from (2) by (Sub*), but (3) is not itself an instance of (LL*). Of course it is possible to axiomatise P without (Sub*), and nothing really hangs on it. I have chosen the above axiomatisation just because I find it comparatively intuitive and convenient for the purposes of this paper.

> Theorem 4.3 (Soundness of P)
> Every member of P is valid in every positive counterpart model.

Proof We show that all P axioms are valid in every positive model, and that validity is closed under (MP), (UG), (Nec) and (Sub*).

FIXME

1. (Taut). Propositional tautologies are valid in every model by the standard satisfaction rules for the connectives.
2. (UD). Assume $w, V \Vdash \forall x(A \supset B)$ and $w, V \Vdash \forall x A$ in some model. By definition 2.8, then $w, V' \Vdash A \supset B$ and $w, V' \Vdash A$ for every existential x-variant V' of V on w, and so $w, V' \Vdash B$ for every such V'. Hence $w, V \Vdash \forall x B$.
3. (VQ). Suppose $w, V \nVdash A \supset \forall x A$ in some model. Then $w, V \Vdash A$ and $w, V \nVdash \forall x A$. If x is not free in A, then by the coincidence lemma **??**, $w, V' \Vdash A$ for every x-variant V' of V on D_w; so $w, V \Vdash \forall x A$. Contradiction. So if x is not free in A, then $A \supset \forall x A$ is valid in every model.
4. (FUI*). Assume $w, V \Vdash \forall x A$ and $w, V \Vdash Ey$ in some model. By definition 2.8, then $w, V' \Vdash A$ for all existential x-variants V' of V on w. So in particular, $w, V^{[y:x]} \Vdash A$. If y is modally free for x in A, then by lemma 3.5, $w, V \Vdash [y/x]A$.
5. ($\forall$Ex). By definition 2.8, $w, V \Vdash \forall x Ex$ iff $w, V' \Vdash Ex$ for all existential x-variants V' of V on w, iff for all existential x-variants V' of V on w there is an existential y-variant V'' of V' on w such that $w, V'' \Vdash x = y$. But this is always the case: for any V', let V'' be $V'^{[x/y]}$.
6. (=R). By definition 2.3, $V_w(=) = \{\langle d, d\rangle : d \in U_w\}$, and by definition 2.3, $V_w(x) \in U_w$ in every positive model. So $w, V \Vdash x = x$ in every such model, by definition 2.8.

7. (LL*). Assume $w, V \Vdash x = y$, $w, V \Vdash A$, and y is modally free for x in A. Since $V_w(x) = V_w(y)$, V coincides with $V^{[y/x]}$ at w. So $w, V^{[y/x]} \Vdash A$ by the coincidence lemma **??**. By lemma 3.5, $w, V^{[y/x]} \Vdash A$ only if $w, V \Vdash [y/x]A$. So $w, V \Vdash [y/x]A$.
8. (K). Assume $w, V \Vdash \Box(A \supset B)$ and $w, V \Vdash \Box A$. Then $w', V' \Vdash A \supset B$ and $w', V' \Vdash A$ for every w', V' such that wRw' and V' is a w'-image of V at w. Then $w', V' \Vdash B$ for any such w', V', and so $w, V \Vdash \Box B$.
9. (MP). Assume $w, V \Vdash A \supset B$ and $w, V \Vdash A$ in some model. By definition 2.8, then $w, V \Vdash B$ as well. So for any world w in any model, (MP) preserves truth at w.
10. (UG). Assume $w, V \nVdash \forall x A$ in some model $\mathcal{M}$. Then $w, V' \nVdash A$ for some existential x-variant V' of V on w. So A is invalid in a model like $\mathcal{M}$ but with V' as the interpretation function in place of V. Hence if A is valid in all positive models, then so is $\forall x A$.
11. (Nec). Assume $w, V \nVdash_{\mathcal{M}} \Box A$ in some model $\mathcal{M}$. Then $w', V' \nVdash A$ for some w' with wRw' and V' some w'-image of V at w. Let $\mathcal{M}^*$ be like $\mathcal{M}$ except with V' in place of V. $\mathcal{M}^*$ is a positive model. Since A is not valid in $\mathcal{M}^*$, it follows contrapositively that whenever A is valid in all positive models, then so is $\Box A$.
12. (Sub*). Assume $w, V \nVdash [y/x]A$ in some model $\langle \mathcal{S}, V \rangle$, and y is modally free for x in A. By lemma 3.5, then $w, V^{[y/x]} \nVdash A$. So A is invalid in the model $\langle \mathcal{S}, V^{[y/x]} \rangle$. Hence if A is valid in all positive models, then so is $[y/x]A$. ■

Let's move on to negative logics. Standard negative free logic replaces (=R) and (∀Ex) by

(∀=R) $\forall x(x{=}x)$,

(Neg) $Px_1 \ldots x_n \supset Ex_1 \wedge \ldots \wedge Ex_n$.

In the logic of all negative counterpart structures, however, we need two further axioms:

(NA) $\neg Ex \supset \Box \neg Ex$,

(TE) $x{=}y \supset \Box(Ex \supset Ey)$.

Here Ex abbreviates $\exists y(x{=}y)$.

In negative counterpart models, a term x can go empty, in the sense that $g(x)$ is undefined. (NA) reflects the fact that when then shift the point of evaluation to another world, $g(x)$ will still be undefined: non-existent objects don't have any counterparts.

(TE) says that if x is identical to y, and x has a counterpart at some accessible world, then y also has a counterpart at that world.

We can, of course, offer counterpart models for negative modal predicate logics without (NA) and (TE). These are dual-domain models in which the extension of all predicates, including identity, is restricted to the inner domain. (NA) then requires that individuals which only figure in the outer domain of a world never have counterparts in the inner domain of another world. (TE) requires that if an individual in the inner domain of a world has a counterpart in the inner domain of another world, then all its counterparts at that world are in the inner domain. The two requirements are obviously independent and non-trivial. Hence the axioms (NA) and (TE) are independent of one another and of the standard principles of basic negative free logic combined with K.

(NA) should not be confused with the claim that no individual exists at an accessible world that isn't a counterpart of something at the present world. This is rather expressed by the Barcan Formula,

$$\forall x \Box A \supset \Box \forall x A, \tag{BF}$$

which is not valid in the class of negative models. For example, if $W = \{w, w'\}$, wRw', $D_w = \emptyset$ and $D_{w'} = \{0\}$, then $w, V \Vdash \forall x \Box x \neq x$, but $w, V \nVdash \Box \forall x\, x \neq x$.

Definition 4.4 (Minimal (strongly) negative (quantified modal) logic)
The *minimal (strongly) negative (quantified modal) logic* N in $\mathcal{L}$ is the smallest set $L \subseteq \mathcal{L}$ that contains all $\mathcal{L}$-instances of (Taut), (UD), (VQ), (FUI*), (Neg), (LL*), ($\forall$=R), (K), (NA), (TE), and that is closed under (MP), (UG), (Nec) and (Sub*).

Definition 4.5 (Negative logics)
A *negative (quantified modal) logic* in $\mathcal{L}$ is a an extension $L \supseteq \mathsf{N}$ of the minimal negative logic N in $\mathcal{L}$ such that L is closed under (MP), (UG), (Nec) and (Sub*).

Theorem 4.6 (Soundness of N)
Every member of N is valid in every negative counterpart model.

Proof FIXME
We show that all N axioms are valid in every negative model, and that validity is closed under (MP), (UG), (Nec) and (Sub*). The proofs for (Taut), (UD), (VQ), (FUI*), (LL*), (K), (MP), (UG), (Nec) and (Sub*) are just as in theorem 4.3. The remaining cases are

1. (Neg). Assume $w, V \Vdash Px_1 \ldots x_n$ in some model. By definition 2.3, $V_w(P) \subseteq U_w^n$, and by definition 2.4, $U_w = D_w$ in negative models. So $V_w(P) \subseteq D_w^n$. By definition 2.8, $w, V \Vdash Px_1 \ldots x_n$ therefore entails that $V_w(x_i) \in D_w$ for all $x_i \in x_1, \ldots, x_n$, and that $w, V \Vdash Ex_i$ for all such x_i.
2. ($\forall$=R). $w, V \Vdash \forall x(x{=}x)$ iff $w, V' \Vdash x{=}x$ for all existential x-variants V' of V on w. This is always the case, since by definition 2.3 and 2.3, $V_w(=) = \{\langle d, d\rangle : d \in D_w\}$ in negative models.
3. (NA). Assume $w, V \Vdash \neg Ex$. By definition 2.8, this means that $V_w(x) \notin D_w$, and therefore that $V_w(x)$ is undefined if the model is negative. But if $V_w(x)$ is undefined, then there is no world w', individual d and counterpart relation $C \in K_{w,w'}$ such that

$\langle V_w(x), w\rangle C\langle d, w'\rangle$. By definitions 2.8 and 2.7, it follows that there is no world w' and interpretation V' with wRw' and $V_w \triangleright V'_{w'}$ such that $w', V' \Vdash Ex$. So then $w, V \Vdash \Box\neg Ex$ by definition 2.8. Thus $w, V \Vdash \neg Ex \supset \Box\neg Ex$.

4. (TE). Assume $w, V \Vdash x = y$. Then $V_w(x) = V_w(y)$ by definitions 2.3 and 2.8. Let w', V' be such that wRw' and $V_w \triangleright V'_{w'}$, and $w', V' \Vdash Ex$. By definition 2.8, the latter means that $V'_{w'}(x)$ is some member of D_w. Moreover, $V_w \triangleright V'_{w'}$ means that there is a $C \in K_{w,w'}$ such that this $V'_{w'}(x) \in D_w$ is a C-counterpart of $V_w(x)$. It follows that $V_w(y) = V_w(x)$ has at least one C-counterpart at w', so $V'_{w'}(y)$ must be some such counterpart, which can only be in D_w. So $w', V' \Vdash Ey$. So if $w, V \Vdash x = y$, then $w, V \Vdash \Box(Ex \supset Ey)$, by definition 2.8, and so $w, V \Vdash x = y \supset \Box(Ex \supset Ey)$. ■

In the remainder of this section, I will prove a few properties derivable from the above axiomatisation. (Some of these will be needed later in the completeness proof.) To this end, let L be an arbitrary positive or negative quantified modal logic.

Lemma 4.7 (Closure under propositional consequence)
For all $\mathcal{L}$-formulas $A_1, \ldots, A_n, B$,

(PC) if $\vdash_L A_1, \ldots, \vdash_L A_n$, and B is a propositional consequence of $A_1, \ldots, A_n$, then $\vdash_L B$.

Proof If B is a propositional consequence of $A_1, \ldots, A_n$, then $A_1 \supset (\ldots \supset (A_n \supset B)\ldots)$ is a tautology. So by (Taut), $\vdash_L A_1 \supset (\ldots \supset (A_n \supset B)\ldots)$. If $\vdash_L A_1, \ldots, \vdash_L A_n$, then by n applications of (MP), $\vdash_L B$. ■

When giving proofs, I will usually omit reference to (PC).

Lemma 4.8 (Redundant axioms)
For any $\mathcal{L}$-formulas A and variables x,

(∀Ex) $\vdash_L \forall x Ex$,

(∀=R) $\vdash_L \forall x(x = x)$.

Proof If L is positive, then (∀Ex) is an axiom. In N, we have $\vdash_L x = x \supset Ex$ by (Neg); so by (UG) and (UD), $\vdash_L \forall x(x = x) \supset \forall x Ex$. Since $\vdash_L \forall x(x = x)$ by (=R), $\vdash_L \forall x Ex$.

If L is negative, then (∀=R) is an axiom. In P, we have $\vdash_L x = x$ by (=R), and so (∀=R) by (UG). ■

Lemma 4.9 (Existence and Self-Identity)
If L is *negative*, then for any $\mathcal{L}$-variable x,

(EI) $\vdash_L Ex \leftrightarrow x{=}x$;

Proof By (FUI*), $\vdash_L \forall x(x{=}x) \supset (Ex \supset x{=}x)$. By ($\forall{=}$R), $\vdash_L \forall x(x{=}x)$. So $\vdash_L Ex \supset x{=}x$. Conversely, $x{=}x \supset Ex$ by (Neg). ∎

Lemma 4.10 (Symmetry and transitivity of identity)
For any $\mathcal{L}$-variables x, y, z,

(=S) $\vdash_L x{=}y \supset y{=}x$;

(=T) $\vdash_L x{=}y \supset y{=}z \supset x{=}z$.

Proof For (= S), let v be some variable $\notin \{x, y\}$. Then

1. $\vdash_L v{=}y \supset (v{=}x \supset y{=}x)$. (LL*)
2. $\vdash_L x{=}y \supset (x{=}x \supset y{=}x)$. (1, (Sub*))
3. $\vdash_L x{=}y \supset x{=}x$. ((=R), or (Neg) and ($\forall$=R))
4. $\vdash_L x{=}y \supset y{=}x$. (2, 3)

For (= T),

1. $\vdash_L x{=}y \supset y{=}x$. (=S)
2. $\vdash_L y{=}x \supset (y{=}z \supset x{=}z)$. (LL*)
3. $\vdash_L x{=}y \supset (y{=}z \supset x{=}z)$. (1, 2)

∎

Lemma 4.11 (Syntactic alpha-conversion)
If A, A' are $\mathcal{L}$-formulas, and A' is an alphabetic variant of A, then

(AC) $\vdash_L A \leftrightarrow A'$.

An *alphabetic variant* of a formula A is a formula A' that results from A by renaming bound variables.

PROOF by induction on A.

1. A is atomic. Then $A = A'$ and $A \leftrightarrow A'$ is a propositional tautology.
2. A is $\neg B$. Then A' is $\neg B'$, where B' is an alphabetic variant of A'. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (PC), $\vdash_L \neg B \leftrightarrow \neg B'$.
3. A is $B \supset C$. Then A' is $B' \supset C'$, where B', C' are alphabetic variants of B, C, respectively. By induction hypothesis, $\vdash_L B \leftrightarrow B'$ and $\vdash_L C \leftrightarrow C'$. So by (PC), $\vdash_L (B \supset C) \leftrightarrow (B' \supset C')$.
4. A is $\forall x B$. Then A' is either $\forall x B'$ or $\forall z [z/x] B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B')$. Assume first that A' is $\forall x B'$. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (UG) and (UD), $\vdash_L \forall x B \leftrightarrow \forall x B'$.
   Alternatively, assume B is $\forall z[z/x]B'$ and $z \notin \mathit{Var}(B')$. Since B' differs from B at most in renaming bound variables, if z were free in B, then $z \in \mathit{Var}(B')$. So z is not free in B. Then

1.	$\vdash_L B \leftrightarrow B'$	(induction hypothesis)
2.	$\vdash_L [z/x]B \leftrightarrow [z/x]B'$	(1, (Sub*))
3.	$\vdash_L \forall x B \supset Ez \supset [z/x]B$	(FUI*)
4.	$\vdash_L \forall x B \supset Ez \supset [z/x]B'$	(2, 3)
5.	$\vdash_L \forall z \forall x B \supset \forall z Ez \supset \forall z[z/x]B'$	(4, (UG), (UD))
6.	$\vdash_L \forall z Ez$	($\forall$Ex)
7.	$\vdash_L \forall z \forall x B \supset \forall z[z/x]B'$	(5, 6)
8.	$\vdash_L \forall x B \supset \forall z \forall x B$	((VQ), z not free in B)
9.	$\vdash_L \forall x B \supset \forall z[z/x]B'$.	(7, 8)

   Conversely,

10.	$\vdash_L \forall z[z/x]B' \supset Ex \supset [x/z][z/x]B'$	(FUI*)
11.	$\vdash_L \forall z[z/x]B' \supset Ex \supset B$	(1, 10, $z \notin \mathit{Var}(B')$)
12.	$\vdash_L \forall x \forall z[z/x]B' \supset \forall x B$	(11, (UG), (UD), ($\forall$Ex))
13.	$\vdash_L \forall z[z/x]B' \supset \forall x \forall z[z/x]B'$	(VQ)
14.	$\vdash_L \forall z[z/x]B' \supset \forall x B$	(12, 13)

5. A is $\Box B$. Then A' is $\Box B'$, where B' is an alphabetic variant of B. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. Then by (Nec), $\vdash_L \Box(B \leftrightarrow B')$, and by (K) and (PC), $\vdash_L \Box B \leftrightarrow \Box B'$. ■

LEMMA 4.12 (CLOSURE UNDER TRANSFORMATIONS)
For any $\mathcal{L}$-formula A and transformation τ on $\mathcal{L}$,

(Sub$^\tau$) $\vdash_L A$ iff $\vdash_L A^\tau$.

Proof Assume $\vdash_L A$. Let $x_1, \dots, x_n$ be the variables in A. If $n = 0$, then $A = A^\tau$ and the result is trivial. If $n = 1$, then A^τ is $[x_1^\tau/x_1]A$, and x_1^τ is either x_1 itself or does not occur in A. In the first case, $[x_1^\tau/x_1]A = A$ and the result is again trivial. In the second case, x_1^τ is modally free for x_1 in A, and thus $\vdash_L [x_1^\tau/x_1]A$ by (Sub*).

Assume then that $n > 1$. Note first that $A^\tau = [x_n^\tau/v_n] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$, where $v_2, \dots, v_n$ are distinct variables not in A or A^τ. This is easily shown by induction on the subformulas B of A (ordered by complexity). To keep things short, let Σ abbreviate $[x_n^\tau/v_n] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]$.

1. If B is $Px_j \dots x_k$, then $x_j, \dots, x_k$ are variables from $x_1, \dots, x_n$, and $\Sigma B = Px_j^\tau \dots x_k^\tau = B^\tau$, by definitions **??** and 3.7.
2. If B is $\neg C$, then by induction hypothesis, $\Sigma C = C^\tau$, and hence $\neg\Sigma C = \neg C^\tau$. But $\Sigma\neg C$ is $\neg\Sigma C$ by definition **??**, and $(\neg C)^\tau$ is $\neg C^\tau$ by definition 3.7.
3. The case for $C \supset D$ is analogous.
4. If B is $\forall zC$, then by induction hypothesis, $\Sigma C = C^\tau$. Since τ is injective, $\Sigma\forall zC$ is $\forall\Sigma z\Sigma C$ by definition **??**, and $(\forall zC)^\tau$ is $\forall z^\tau C^\tau$ by definition 3.7. Moreover, since z is one of $x_1, \dots, x_n$, $\Sigma z = z^\tau$.
5. If B is $\Box C$, then by induction hypothesis, ΣC is C^τ, and hence $\Box\Sigma C$ is $\Box C^\tau$. But $\Sigma\Box C$ is $\Box\Sigma C$ by definition **??**, and $(\Box C)^\tau$ is $\Box C^\tau$ by definition 3.7.

Now we show that L contains all "segments" of $[x_n^\tau/v_n] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$, beginning with the rightmost substitution, $[v_n/x_n]A$. Since v_n is modally free for x_n in A, by (Sub*), $\vdash_L [v_n/x_n]A$. Likewise, for each $1 < i < n$, v_i is modally free for x_i in $[v_{i+1}/x_{i+1}] \dots [v_n/x_n]A$. So $\vdash_L [v_2/x_2] \dots [v_n/x_n]A$.

With respect to $[x_1^\tau/x_1]$, we distinguish three cases. First, if $x_1 = x_1^\tau$, then $\vdash_L [x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$, because $[x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$ is $[v_2/x_2] \dots [v_n/x_n]A$. Second, if $x_1 \neq x_1^\tau$ and $x_1^\tau \notin \mathit{Var}(A)$, then $x_1^\tau \notin \mathit{Var}([v_2/x_2] \dots [v_n/x_n]A)$, since the $v_1, \dots, v_n$ are not in $\mathit{Var}(A)$ or $\mathit{Var}(A^\tau)$. So x_1^τ is modally free for x_1 in $[v_2/x_2] \dots [v_n/x_n]A$, and by (Sub*), $\vdash_L [x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$. Third, if $x_1 \neq x_1^\tau$ and $x_1^\tau \in \mathit{Var}(A)$, then x_1^τ must be one of $x_2, \dots, x_n$. Then again $x_1^\tau \notin \mathit{Var}([v_2/x_2] \dots [v_n/x_n]A)$, and so $\vdash_L [x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$ by (Sub*).

Next, x_2^τ is modally free for v_2 in $[x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$, because τ is injective and hence $x_2^\tau \neq x_1^\tau$, so x_2^τ does not occur in $[x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$. Hence $\vdash_L [x_2^\tau/v_2][x_1^\tau/x_1]$ $[v_2/x_2] \dots [v_n/x_n]A$. By the same reasoning, for each $2 < i \leq n$, x_i^τ is modally free for v_i in $[x_{i-1}^\tau/v_{i-1}] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$. So $\vdash_L [x_n^\tau/v_n] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots$ $[v_n/x_n]A$, i.e. $\vdash_L A^\tau$.

This proves the left-to-right direction of (Sub$^\tau$). The other direction immediately follows. Let $x_1^\tau, \dots, x_n^\tau$ be the variables in A^τ, and let σ be an arbitrary transformation that maps each x_i^τ back to x_i (i.e., to $(x_i^\tau)^{\tau^{-1}}$). By the left-to-right direction of (Sub$^\tau$), $\vdash_L A^\tau$ entails $\vdash_L (A^\tau)^\sigma$, and $(A^\tau)^\sigma$ is simply A. ∎

Lemma 4.13 (Leibniz' Law with partial substitution)
FIXME Let A be a formula of $\mathcal{L}$, and x, y variables of $\mathcal{L}$. Let $[y//x]A$ be A with *one or more* free occurrences of x replaced by y.

(LL$_\text{p}^*$) $\vdash_L x = y \supset A \supset [y//x]A$, provided the following conditions are satisfied.

(i) y is quantificationally free for x in A.

(ii) Either y is modally free for x in A, or $[y//x]A$ does not replace any occurrence of x in the scope of a modal operator in A that also contains y.

(iii) In the scope of any modal operator in A, $[y//x]A$ either replaces all or no occurrences of x by y.

Proof FIXME Let $v \neq y$ be a variable not in $Var(A)$, and let $[v//x]A$ be like $[y//x]A$ except that all new occurrences of y are replaced by v: if $[y//x]A$ satisfies (i)–(iii), then so does $[y//x]A$ with all new occurrences of y replaced by v. Moreover, in the resulting formula $[v//x]A$ all occurrences of v are free and free for y, by clause (i); so $[y/v][v//x]A = [y//x]A$ by definition **??**. By (LL*),

$$\vdash_L v = y \supset [v//x]A \supset [y/v][v//x]A, \tag{1}$$

provided that y is modally free for v in $[v//x]A$, i.e. provided that either y is modally free for x in A, or $[v//x]A$ (and thus $[y//x]A$) does not replace any occurrence of x in the scope of a modal operator in A that also contains y. This is guaranteed by condition (ii). Since $[y/v][v//x]A$ is $[y//x]A$, (1) can be shortened to

$$\vdash_L v = y \supset [v//x]A \supset [y//x]A. \tag{2}$$

By (Sub*), it follows that

$$\vdash_L [x/v](v = y \supset [v//x]A \supset [y//x]A), \tag{3}$$

provided that x is modally free for v in $v = y \supset [v//x]A \supset [y//x]A$. Since this isn't a formula of the form $\Box B$, x is modally free for v here iff no free occurrences of x and v lie in the scope of the same modal operator in $[v//x]A$. So whenever $[v//x]A$ (and thus $[y//x]A$) replaces some occurrences of x in the scope of a modal operator in A, then it must replace all occurrences of x in the scope of that operator. This is guaranteed by condition (iii). By definition **??**, (3) can be simplified to

$$\vdash_L x = y \supset A \supset [y//x]A. \tag{4}$$

■

FIXME I will never actually use (LL^*_p). I mention it only because Leibniz' Law is often stated for partial substitutions, and you may have wondered what that would look like in our systems. Now you know. We could indeed have used (LL^*_p) as basic axiom instead of (LL*); (LL*) would then be derivable, because every formula A has an alphabetic variant A' such that $[y/x]A$ is an instance of $[y//x]A'$ that satisfies (i)–(iii) iff y is modally free for x in A, and because (LL*) is not used in the proof of lemma 4.11. I have chosen

(LL*) as basic due to its greater simplicity.[5]

LEMMA 4.14 (LEIBNIZ' LAW WITH SEQUENCES)
For any $\mathcal{L}$-formula A and variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct,

(LL$^*_\text{n}$) $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A$, provided each y_i is modally free for x_i in $[y_1, \ldots, y_{i-1}/x_1, \ldots, x_{n-1}]A$.

For $i = 1$, the proviso is meant to say that y_1 is modally free for x_1 in A.

PROOF By induction on n. For $n = 1$, (LL$^*_\text{n}$) is FIXME:CHECK
(LL*). Assume then that $n > 1$ and that each y_i in $y_1, \ldots, y_n$ is modally free for x_i in $[y_1, \ldots, y_{i-1}/x_1, \ldots, x_{n-1}]A$. Let z be some variable not in A, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$. So z is modally free for x_n in A. By (LL*),

$$\vdash_L x_n = z \supset A \supset [z/x_n]A. \tag{1}$$

By induction hypothesis,

$$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_{n-1} = y_{n-1} \supset [z/x_n]A \supset [y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A. \tag{2}$$

By assumption, y_n is modally free for x_n in $[y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}]A$. Then y_n is also modally free for z in $[y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$. So by (LL*),

$$\vdash_L z = y_n \supset [y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A \supset [y_n/z][y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A. \tag{3}$$

But $[y_n/z][y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$ is $[y_1, \ldots, y_n/x_1, \ldots, x_n]A$. Combining (1)–(3), we therefore have

$$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_{n-1} = y_{n-1} \supset x_n = z \wedge z = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A. \tag{4}$$

5 Kutz's system uses the following version of (LL$^*_\text{p}$) ([**?**: 43]):

(LL$^\text{K}_\text{p}$) $\vdash x = y \supset A \supset [y//x]A$, provided that
(i) y is quantificationally free for x in A,
(ii) y is not free in the scope of a modal operator in A, and
(iii) in the scope of any modal operator in A, $[y//x]A$ either replaces all or no occurrences of x by y.

Evidently, this is a lot more restrictive than (LL$^*_\text{p}$). For example, (LL$^*_\text{p}$) validates

$$\vdash x = y \supset \Box Gxy \supset \Box Gyy \quad \text{and}$$
$$\vdash x = y \supset (Fx \vee \Diamond Gxy) \supset (Fy \vee \Diamond Gxy),$$

which can't be derived in Kutz's system (which is therefore incomplete).

So by (Sub*),

$$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_{n-1} = y_{n-1} \supset x_n = x_n \wedge x_n = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A. \tag{5}$$

Since $\vdash_L x_n = y_n \supset x_n = x_n$ (by either (=R) or (Neg) and (∀=R)), it follows that

$$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A. \tag{6}$$

■

LEMMA 4.15 (CROSS-SUBSTITUTION)
For any $\mathcal{L}$-formula A and variables x, y,

(CS) $\vdash_L x = y \supset \Box A \supset \Box(y = z \supset [z/x]A)$, provided z is not free in A.

More generally, for any variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct,

(CS_n) $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset \Box A \supset \Box(y_1 = z_1 \wedge \ldots \wedge y_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$, provided none of $z_1, \ldots, z_n$ is free in A.

PROOF FIXME:CHECK For (CS), assume z is not free in A. Then

1.	$\vdash_L x = z \supset A \supset [z/x]A$.	(LL*)
2.	$\vdash_L A \supset (x = z \supset [z/x]A)$.	(1)
3.	$\vdash_L \Box A \supset \Box(x = z \supset [z/x]A)$.	(2, (Nec), (K))
4.	$\vdash_L x = y \supset \Box(x = z \supset [z/x]A) \supset \Box(y = z \supset [z/x]A)$.	(LL*)
5.	$\vdash_L x = y \supset \Box A \supset \Box(y = z \supset [z/x]A)$.	(3, 4)

Step 4 is justified by the fact that x is not free in $[z/x]A$ and so x and y are modally separated in $x = z \supset [z/x]A$.

The proof for (CS_n) is analogous. Assume none of $z_1, \ldots, z_n$ is free in A. Then

1.	$\vdash_L x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset A \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A$.	(LL^*_n)
2.	$\vdash_L A \supset (x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$.	(1)
3.	$\vdash_L \Box A \supset \Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$.	(2, (Nec), (K))
4.	$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset$ $\Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A) \supset$ $\Box(y_1 = z_1 \wedge \ldots \wedge y_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$.	(LL^*_n)
5.	$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset \Box A \supset$ $\Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$.	(3, 4)

Step 4 is justified by the fact that none of $x_1, \ldots, x_n$ is free in $[z_1, \ldots, z_n/x_1, \ldots, x_n]A$, and each y_i is modally free for x_i in $[y_1, \ldots, y_{i-1}/x_1, \ldots, x_{i-1}]\Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$, i.e. in $\Box(y_1 = z_1 \wedge \ldots \wedge y_{i-1} = z_{i-1} \wedge x_i = z_i \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$, because x_i and y_i are modally separated in $y_1 = z_1 \wedge \ldots \wedge y_{i-1} = z_{i-1} \wedge x_i = z_i \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A$. ■

LEMMA 4.16 (SUBSTITUTION-FREE UNIVERSAL INSTANTIATION)
For any $\mathcal{L}$-formula A and variables x, y,

(FUI**) $\vdash_L \forall x A \supset (Ey \supset \exists x(x = y \wedge A))$.

PROOF FIXME:CHECK Let z be a variable not in $Var(A), x, y$.

1.	$\vdash_L z = y \supset Ey \supset Ez$	(LL*)
2.	$\vdash_L \forall x A \supset Ez \supset [z/x]A$	((FUI*), $z \notin Var(A)$)
3.	$\vdash_L \forall x A \wedge Ey \supset z = y \supset [z/x]A$	(1, 2)
4.	$\vdash_L \forall x(x = z \supset \neg A) \supset Ez \supset (z = z \supset [z/x]\neg A)$	((FUI*), $z \notin Var(A)$)
5.	$\vdash_L Ez \supset z = z$	((=R), or (∀=R), (FUI*))
6.	$\vdash_L \forall x(x = z \supset \neg A) \supset Ez \supset [z/x]\neg A$	(4, 5)
7.	$\vdash_L Ez \supset [z/x]A \supset \exists x(x = z \wedge A)$	(6)
8.	$\vdash_L \forall x A \wedge Ey \supset z = y \supset \exists x(x = z \wedge A)$	(1, 3, 7)
9.	$\vdash_L z = y \supset \exists x(x = z \wedge A) \supset \exists x(x = y \wedge A)$	((LL*), $z \notin Var(A)$)
10.	$\vdash_L \forall x A \wedge Ey \supset z = y \supset \exists x(x = y \wedge A)$	(8, 9)
11.	$\vdash_L \forall z(\forall x A \wedge Ey) \supset \forall z(z = y \supset \exists x(x = y \wedge A))$	(10, (UG), (UD))
12.	$\vdash_L \forall x A \wedge Ey \supset \forall z(z = y \supset \exists x(x = y \wedge A))$	(11, (VQ))
13.	$\vdash_L \forall z(z = y \supset \exists x(x = y \wedge A)) \supset y = y \supset \exists x(x = y \wedge A)$	((FUI*), $z \notin Var(A)$)
14.	$\vdash_L Ey \supset y = y$	((=R), or (∀=R), (FUI*))
15.	$\vdash_L \forall z(z = y \supset \exists x(x = y \wedge A)) \supset Ey \supset \exists x(x = y \wedge A)$	(13, 14)
16.	$\vdash_L \forall x A \supset Ey \supset \exists x(x = y \wedge A)$	(12, 15)

■

(FUI*) can also be derived from (FUI**), so we could just as well have used (FUI**) as basic axiom instead of (FUI*).

5 Object-language substitution

FIXME: This whole section.

We had to give weird restrictions on substitution principles. This is because modal operators are *slightly opaque*: even if $x = y$, $\Box Gxy$ and $\Box Gxx$ say different things. Informally, $\Box Gxy$ says that all counterparts of x (and therefore of y) are G-related to one another, while $\Box Gxx$ merely says that all counterparts of x are G-related to themselves.

If an individual has multiple counterparts at a certain world, and the point of evaluation shifts to this world, we may think of the corresponding terms as becoming "ambiguous", denoting all the counterparts at the same time. To verify $\Box\phi(x)$, we require that $\phi(x)$ is true at all accessible worlds under all "disambiguations". In principle, we could allow for "mixed disambiguations" under which different occurrences of x can pick out different counterparts. On this interpretation $x = y \wedge \Box Gxy$ and $x = y \wedge \Box Gxx$ would be equivalent. $\Box x = x$ and $\Box(Fx \vee \neg Fx)$ would become invalid (even in positive models). I have assumed "uniform disambiguations" mainly because it simplifies the semantics. Mixed disambiguations can't be represented by a classical assignment function. We would have to allow for the possibility that $g(x)$ is a *set* of individuals.

The present issue might remind you of the old observation that a sentence like 'Brutus killed himself' can be understood either as an application of a *monadic* predicate 'killing himself' to the subject Brutus, or as an application of the binary 'killing' to Brutus and Brutus. Peter Geach once suggested a syntactic mechanism for distinguishing these readings, by introducing an operator $\langle z : x, y\rangle$ that turns a binary expression into a unary expression: while Gxy is satisfied by pairs of individuals, $\langle z : x, y\rangle Gxy$ is satisfied by a single individual. The operator $\langle z : x, y\rangle$, which might be read 'z is an x and a y such that' acts as a quantifier that binds both x and y.

A similar trick can be used in our modal context. On our uniform reading, $\Box x = x$ says that all counterparts of x are self-identical at all accessible worlds. To say that at all accessible worlds (and under all counterpart relations), all x counterparts are identical to all x counterparts we could instead say $\langle x : y, z\rangle\Box y = z$. The effect of $\langle x : y, z\rangle$ is to introduce two variables y and z that co-refer with x. By using distinct but co-refering variables in a modal context, we can express relations between possibly distinct counterparts; by using the same variable, we make sure that the same counterpart must be assigned to every occurrence.

With $\langle x : y, z\rangle\Box y = z$, we actually end up with *three* co-referring variables: y and z are made to co-refer with x, but we also have x itself. The job can also be done with $\langle x : y\rangle\Box x = y$ – read: 'x is a y such that ...'.

This operator is a close cousins of lambda abstraction and application, as introduced to modal logic in [**?**]. Lambda abstraction converts a formula A into a predicate $(\lambda x.A)$, which can then be applied to a singular term y to form a new formula $(\lambda x.A)y$. Semantically, $(\lambda x.A)y$ is true under an interpretation V at a world w iff A is true under the x-variant V' of V on w with $V'_w(y) = V_w(x)$. So $(\lambda x.A)y$ is another way of writing $\langle y : x\rangle A$.

Lambda abstraction is useful to ensure transparency. FIXME expand.

To avoid accidental "capturing" of y in the consequent of (**??**), we can use the Geach quantifier:

$$\forall x \Diamond Gxy \supset \langle y : x \rangle \Diamond Gxy \tag{??'}$$

$$x = y \supset \Diamond Gxy \supset \langle y : x \rangle \Diamond Gxy \tag{??'}$$

The Geach quantifier $\langle y : x \rangle$ functions as an *object-language substitution operator*.

In some contexts, it may be useful to add this operator to our language.

> DEFINITION 5.1 (LANGUAGES OF QML WITH SUBSTITUTION)
> A *language of quantified modal logic with substitution* is the standard language of quantified modal logic (definition 2.2) with an added construct $\langle \, : \, \rangle$ and the rule that whenever x, y are variables and A is a formula, then $\langle y : x \rangle A$ is a formula.

As for the semantics: just as $\forall x A$ is true relative to an interpretation V iff A is true relative to all x-variants of V (on the relevant domain), $\langle y : x \rangle A$ is true relative to V iff A is true relative to the x-variant of V that maps x to $V(y)$. In our modal framework:

> DEFINITION 5.2 (SEMANTICS FOR THE SUBSTITUTION OPERATOR)
> $w, V \Vdash \langle y : x \rangle A$ iff $w, V' \Vdash A$, where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$.

Note that V' need not be an *existential* x-variant of V on w.

The locality lemma 2.10 is easily adjusted to languages with substitution. Here is the only new step in the induction:

A is $\langle y : x \rangle B$. $w, V \Vdash \langle y : x \rangle B$ iff $w, V^* \Vdash B$ where V^* is the x-variant of V on w with $V^*_w(x) = V_w(y)$. Let V'^* be the x-variant of V' on w with $V'^*_w(x) = V'_w(y)$. Then V^* and V'^* agree at w on all variables in B, so by induction hypothesis, $w, V^* \Vdash B$ iff $V'^*, w \Vdash B$. And this holds iff $w, V' \Vdash \langle y : x \rangle B$ by the semantics of $\langle y : x \rangle$. ■

Substitution operators turn out to have significant expressive power. As [**?**] shows (in effect), if a language has substitution operators, it no longer needs variables or individual constants in its atomic formulas: instead of Fx, we can simply say F, with the convention that the implicit variable is always x (for binary predicates, the first variable is x, the second y, etc.); Fy turns into $\langle y : x \rangle F$, Gyz into $\langle y : x \rangle \langle z : y \rangle G$. Similarly, $\forall x Fx$ can be replaced by $\forall F$, and $\forall y Gxy$ by $\forall \langle y : z \rangle \langle x : y \rangle \langle z : x \rangle G$. So we also don't need different quantifiers for different variables. In this essay, I will not exploit the full power of substitution operators – mainly for the sake of familiarity. Our languages with substitution operators will still have ordinary formulas $Px_1 \dots x_n$ and quantifiers $\forall x, \forall y$, etc.

There are distinctions one can draw with $\langle y : x \rangle$ that cannot be drawn without it. For example, the substitution quantifier allows us to say that an individual y has multiple counterparts at some accessible world (under the same counterpart relation): $\langle y : x \rangle \Diamond y \neq x$. This can't be expressed without the operator – at least not in positive models.

It is clear that $\Diamond y \neq y$ is not an adequate translation of $\langle y : x \rangle \Diamond x \neq y$. Before substituting y for x in $\Diamond x \neq y$, we would have to make x free for y by renaming the modally bound occurrence of y. However, the diamond, unlike the quantifier $\forall y$, binds y in such a way that the domain over which it ranges (the counterparts of y's original referent) depends on the previous reference of y. So we can't just replace y by some other variable z, translating $\langle y : x \rangle \Diamond x \neq y$ as $\Diamond y \neq z$. This only works if z happens to corefer with y. Since we can't presuppose that there is always another name available for any given individual, we would somehow have to introduce a name z that corefers with y. For instance, if we could transform $\Diamond x \neq y$ into $\exists z(y = z \land \Diamond x \neq z)$, the variable x would have become free for y in the scope of the diamond, so we could translate $\langle y : x \rangle \Diamond x \neq y$ as $\exists z(y = z \land \Diamond x \neq y)$. The problem is that the quantifier $\exists$ ranges only over existing objects, while $\langle y : x \rangle$ bears no such restriction. In positive models, $V_w(y)$ can have multiple counterparts even if it lies outside D_w, so that $\exists z(y = z \land \Diamond x \neq y)$ is false. (One would need an "outer quantifier" in place of $\exists$.)

This, incidentally, shows that there is no way of defining a substitution operator $[y/x]A$ that satisfies the unrestricted "substitution lemma"

$$\mathcal{M}, g \models [y/x]A \text{ iff } \mathcal{M}, g^{[y/x]} \Vdash A.$$

Here is the full proof in more detail.

Theorem 5.3 (Undefinability of substitution)
There is no operation Φ on formulas A such that for any world w in a counterpart models $\mathcal{M}$ and assignment g on U_w, $\mathcal{M}, w, g \Vdash \Phi(A)$ iff $\mathcal{M}, w, g^{[y/x]} \Vdash A$.

Proof FIXME Let $\mathcal{M}_1 = \langle \mathcal{S}_1, V \rangle$ be a positive counterpart model with $W = \{w\}$, $R = \{\langle w, w \rangle\}$, $U_w = \{x, y, y^*\}$, $D_w = \{x\}$, $K_{w,w} = \{\{\langle d, d \rangle : d \in U_w\}\}, V_w(y) = y$, $V_w(z) = x$ for every variable $z \neq y$, and $V_w(P) = \emptyset$ for all non-logical predicates P. Let $\mathcal{M}_2 = \langle \mathcal{S}_2, V \rangle$ be like $\mathcal{M}_1$ except that y^* is also a counterpart of y, i.e. $K_{w,w'} = \{\{\langle x, x \rangle, \langle y, y \rangle, \langle y^*, y^* \rangle, \langle y, y^* \rangle\}\}$. Then $w, V^{[y/x]} \Vdash_{\mathcal{S}_2} \Diamond y \neq x$, but $w, V^{[y/x]} \nVdash_{\mathcal{S}_1} \Diamond y \neq x$.

On the other hand, every $\mathcal{L}$-sentence has the same truth-value at w under V in both models. We prove this by showing that for every $\mathcal{L}$-sentence A, $w, V \Vdash_{\mathcal{S}_1} A$ iff $w, V \Vdash_{\mathcal{S}_2} A$ iff $w, V^* \Vdash_{\mathcal{S}_2} A$, where V^* is the y-variant of V on w with $V^*_w(y) = V_w(y^*)$.

1. A is $Px_1 \dots x_n$. It is clear that $w, V \Vdash_{\mathcal{S}_1} Px_1 \dots x_n$ iff $w, V \Vdash_{\mathcal{S}_2} Px_1 \dots x_n$ because counterpart relations do not figure in the evaluation of atomic formulas. Moreover, for non-logical P, $w, V \nVdash_{\mathcal{S}_2} Px_1 \dots x_n$ and $w, V^* \nVdash_{\mathcal{S}_2} Px_1 \dots x_n$, because $V_w(P) = V^*_w(P) = \emptyset$. For the identity predicate, observe that $w, V \nVdash_{\mathcal{S}_2} u{=}v$ iff exactly one of u, v is y, since $V_w(z) = x$ for all terms $z \neq y$. For the same reason, $w, V^* \nVdash_{\mathcal{S}_2} u{=}v$ iff exactly one of u, v is y. So $w, V \Vdash_{\mathcal{S}_2} u{=}v$ iff $w, V^* \Vdash_{\mathcal{S}_2} u{=}v$.
2. A is $\neg B$. $w, V \Vdash_{\mathcal{S}_1} \neg B$ iff $w, V \nVdash_{\mathcal{S}_1} B$ by definition 2.8, iff $w, V \nVdash_{\mathcal{S}_2} B$ by induction hypothesis, iff $w, V \Vdash_{\mathcal{S}_2} \neg B$ by definition 2.8. Moreover, $w, V \nVdash_{\mathcal{S}_2} B$ iff $w, V^* \nVdash_{\mathcal{S}_2} B$ by induction hypothesis, iff $w, V^* \Vdash_{\mathcal{S}_2} \neg B$ by definition 2.8.
3. A is $B \supset C$. $w, V \Vdash_{\mathcal{S}_1} B \supset C$ iff $w, V \nVdash_{\mathcal{S}_1} B$ or $w, V \Vdash_{\mathcal{S}_1} C$ by definition 2.8, iff $w, V \nVdash_{\mathcal{S}_2} B$ or $w, V \Vdash_{\mathcal{S}_2} C$ by induction hypothesis, iff $w, V \Vdash_{\mathcal{S}_2} B \supset C$ by definition 2.8. Moreover, $w, V \nVdash_{\mathcal{S}_2} B$ or $w, V \Vdash_{\mathcal{S}_2} C$, iff $w, V^* \nVdash_{\mathcal{S}_2} B$ or $w, V^* \Vdash_{\mathcal{S}_2} C$ by induction hypothesis, iff $w, V^* \Vdash_{\mathcal{S}_2} B \supset C$ by definition 2.8.
4. A is $\forall z B$. Let v be a variable not in $\mathit{Var}(B) \cup \{y\}$. By lemma **??**, $w, V \Vdash_{\mathcal{S}_1} \forall z B$ iff $w, V \Vdash_{\mathcal{S}_1} \forall v [v/z] B$. By definition 2.8, $w, V \Vdash_{\mathcal{S}_1} \forall v [v/z] B$ iff $w, V' \Vdash_{\mathcal{S}_1} [v/z] B$ for all existential v-variants V' of V on w. As $D_w = \{x\}$ and $V(v) = x$, the only such v-variant is V itself. So $w, V \Vdash_{\mathcal{S}_1} \forall z B$ iff $w, V \Vdash_{\mathcal{S}_1} [v/z] B$. By the same reasoning, $w, V \Vdash_{\mathcal{S}_2} \forall z B$ iff $w, V \Vdash_{\mathcal{S}_2} [v/z] B$. But by induction hypothesis, $w, V \Vdash_{\mathcal{S}_1} [v/z] B$ iff $w, V \Vdash_{\mathcal{S}_2} [v/z] B$. So $w, V \Vdash_{\mathcal{S}_1} \forall z B$ iff $w, V \Vdash_{\mathcal{S}_2} \forall z B$. Moreover, by induction hypothesis, $w, V \Vdash_{\mathcal{S}_2} [v/z] B$ iff $w, V^* \Vdash_{\mathcal{S}_2} [v/z] B$, iff $w, V^* \Vdash_{\mathcal{S}_2} \forall v [v/z] B$ because V^* is the only existential v-variant of V^* on w, iff $w, V^* \Vdash_{\mathcal{S}_2} \forall z B$ by lemma **??**.
5. A is $\Box B$. In both structures, the only world accessible from w is w itself. Also in $\mathcal{S}_1$, V is the only w-image of V at w. So by definition 2.8, $w, V \Vdash_{\mathcal{S}_1} \Box B$ iff $w, V \Vdash_{\mathcal{S}_1} B$. In $\mathcal{S}_2$, there are two w-images of V at w: V and V^*. So $w, V \Vdash_{\mathcal{S}_2} \Box B$ iff both $w, V \Vdash_{\mathcal{S}_2} B$ and $w, V^* \Vdash_{\mathcal{S}_2} B$. By induction hypothesis, $w, V \Vdash_{\mathcal{S}_1} B$ iff both $w, V \Vdash_{\mathcal{S}_2} B$ and $w, V^* \Vdash_{\mathcal{S}_2} B$. So $w, V \Vdash_{\mathcal{S}_1} \Box B$ iff $w, V \Vdash_{\mathcal{S}_2} \Box B$. Moreover, in $\mathcal{S}_2$, V^* is the only w-image of V^* at w. So $w, V^* \Vdash_{\mathcal{S}_2} \Box B$ iff $w, V^* \Vdash_{\mathcal{S}_2} B$. By induction hypothesis, $w, V^* \Vdash_{\mathcal{S}_2} B$ iff $w, V \Vdash_{\mathcal{S}_2} B$. So $w, V^* \Vdash_{\mathcal{S}_2} \Box B$ iff both $w, V^* \Vdash_{\mathcal{S}_2} B$ and $w, V \Vdash_{\mathcal{S}_2} B$, which as we just saw holds iff $w, V \Vdash_{\mathcal{S}_2} \Box B$. ■

(In negative models, $\langle y : x \rangle A$ can be translated into $\exists x (x{=}y \land A) \lor (\neg E y \land [y/x] A)$, which still has the downside of being very impractical, since the result of the substitution has much greater syntactic complexity than the original sentence.)

What is the logic for languages with object-language substitution?

We first have to lay down some axioms governing the substitution operator. An obvious suggestion would be the lambda-conversion principle

$$\langle y : x \rangle A \leftrightarrow [y/x] A,$$

which would allow us to move back and forth between e.g. $\langle y : x \rangle F x$ and $F y$. But we've seen in lemma 3.5 that if things can have multiple counterparts, then these transitions are sound only under certain conditions: the move from $\langle y : x\rangle A$ to $[y/x]A$ requires that y is modally free for x in A, the other direction requires that y and x are modally separated in A. So we have the following somewhat more complex principles:

(SC1) $\langle y : x\rangle A \leftrightarrow [y/x]A$, provided y and x are modally separated in A.

(SC2) $\langle y : x\rangle A \supset [y/x]A$, provided y is modally free for x in A.

But now we need further principles telling us how $\langle y : x\rangle$ behaves when y is not modally free for x. For example, $\langle y : x\rangle\neg A$ should always entail $\neg\langle y : x\rangle A$, even if y is not modally free for x in A. More generally, the substitution operator commutes with every non-modal operator as long as there is no clash of bound variables:

(S¬) $\langle y : x\rangle\neg A \leftrightarrow \neg\langle y : x\rangle A$,

(S⊃) $\langle y : x\rangle(A \supset B) \leftrightarrow (\langle y : x\rangle A \supset \langle y : x\rangle B)$,

(S∀) $\langle y : x\rangle\forall zA \leftrightarrow \forall z\langle y : x\rangle A$, provided $z \notin \{x, y\}$,

(SS1) $\langle y : x\rangle\langle y_2 : z\rangle A \leftrightarrow \langle y_2 : z\rangle\langle y : x\rangle A$, provided $z \notin \{x, y\}$ and $y_2 \neq x$.

Substitution does not commute with the box. Roughly speaking, this is because $\langle y : x\rangle\Box A(x, y)$ says that at all accessible worlds, all counterparts x' and y' of y are $A(x', y')$, while $\Box\langle y : x\rangle A(x, y)$ says that at all accessible worlds, every counterpart $x' = y'$ of y is such that $A(x', y')$. In the first case, x' and y' may be different counterparts of y, while in the second case, they must be the same. Thus $\langle y : x\rangle\Box A$ entails $\Box\langle y : x\rangle A$, but the other direction holds only if either y does not have multiple counterparts at accessible worlds (relative to the same counterpart relation), or at most one of x and y occurs freely in A (including the special case where x and y are the same variable).

(S□) $\langle y : x\rangle\Box A \supset \Box\langle y : x\rangle A$,

(S◇) $\langle y : x\rangle\Diamond A \supset \Diamond\langle y : x\rangle A$, provided at most one of x, y is free in A.

These principles largely make (SC1) and (SC2) redundant. We only need to add the special case for substituting free variables in atomic formulas and in substitution operators, as well as a principle for vacuous substitutions:

(SAt) $\langle y : x\rangle Px_1 \ldots x_n \leftrightarrow P[y/x]x_1 \ldots [y/x]x_n$.

(SS2) $\langle y : x\rangle\langle x : z\rangle A \leftrightarrow \langle y : z\rangle\langle y : x\rangle A$.

(VS) $A \leftrightarrow \langle y : x\rangle A$, provided x is not free in A.

> Lemma 5.4 (Soundness of the substitution axioms)
> If $\mathcal{L}_s$ is a language of quantified modal logic with substitution, then every $\mathcal{L}_s$-instance of (S¬), (S⊃), (S∀), (SS1), (S□), (S◇), (SAt), (SS2), and (VS) is valid in every (positive or negative) counterpart model.

PROOF

1. (S¬). $w, V \Vdash \langle y : x\rangle\neg A$ iff $w, V^{[y/x]} \Vdash \neg A$ by definition 5.2, iff $w, V^{[y/x]} \nVdash A$ by definition 2.8, iff $w, V \nVdash \langle y : x\rangle A$ by definition 5.2, iff $w, V \Vdash \neg\langle y : x\rangle A$ by definition 2.8.
2. (S⊃). $w, V \Vdash \langle y : x\rangle(A \supset B)$ iff $w, V^{[y/x]} \Vdash A \supset B$ by definition 5.2, iff $w, V^{[y/x]} \nVdash A$ or $w, V^{[y/x]} \Vdash B$ by definition 2.8, iff $w, V \nVdash \langle y : x\rangle A$ or $w, V \Vdash \langle y : x\rangle B$ by definition 5.2, iff $w, V \Vdash \langle y : x\rangle A \supset \langle y : x\rangle B$ by definition 2.8.
3. (S∀). Assume $z \notin \{x, y\}$. Then the existential z-variants V' of $V^{[y/x]}$ on w coincide at w with the functions $(V^*)^{[y/x]}$ where V^* is an existential z-variant V^* of V on w. And so $w, V \Vdash \langle y : x\rangle\forall zA$ iff $w, V^{[y/x]} \Vdash \forall zA$ by definition 5.2, iff $w, V' \Vdash A$ for all existential z-variants V' of $V^{[y/x]}$ on w by definition 2.8, iff $w, (V^*)^{[y/x]} \Vdash A$ for all existential z-variants V^* of V on w, iff $w, V^* \Vdash \langle y : x\rangle A$ for all existential z-variants V^* of V on w by definition 5.2, iff $w, V \Vdash \forall z\langle y : x\rangle A$ by definition 2.8.
4. (SS1). Assume $z \notin \{x, y\}$ and $y_2 \neq x$. Then the function $[y/x] \cdot [y_2/z]$ is identical to the function $[y_2/z] \cdot [y/x]$. So $w, V \Vdash \langle y : x\rangle\langle y_2 : z\rangle A$ iff $w, V^{[y/x]\cdot[y_2/z]} \Vdash A$ by definition 5.2, iff $w, V^{[y_2/z]\cdot[y/x]} \Vdash A$, iff $w, V \Vdash \langle y_2 : z\rangle\langle y : x\rangle A$ by definition 5.2.
5. (S□). Assume $w, V \nVdash \Box\langle y : x\rangle A$. By definitions 2.8 and 5.2, this means that $w', V'^{[y/x]} \nVdash A$ for some w', V' such that wRw' and $V_w \triangleright V'_{w'}$, i.e. there is a $C \in K_{w,w'}$ for which $V'_{w'}$ assigns to every variable z a C-counterpart of its value under V_w (or nothing if there is none). Then for all z, $V'^{[y/x]}_{w'}(z)$ is a C-counterpart of $V^{[y/x]}_w(z)$ (or undefined if there is none), since $V'^{[y/x]}_{w'}(x) = V'_{w'}(y)$ is a C-counterpart of $V_w(y) = V^{[y/x]}_w(x)$ (or undefined if there is none). So $V^{[y/x]}_w \triangleright V'^{[y/x]}_{w'}$. And so $w', V^* \nVdash A$ for some w', V^* such that wRw' and $V^{[y/x]}_w \triangleright V^*_{w'}$. So $w, V \Vdash \langle y : x\rangle\Box A$ by definitions 2.8 and 5.2.
6. (S◇). Assume $w, V \Vdash \langle y : x\rangle\Diamond A$ and at most one of x, y is free in A. By definitions 2.8 and 5.2, $w', V^* \Vdash A$ for some w', V^* such that wRw' and $V^{[y/x]}_w \triangleright V^*_{w'}$, i.e. there is a $C \in K_{w,w'}$ for which $V^*_{w'}$ assigns to every variable z a C-counterpart of its value under V_w (or nothing if there is none). We have to show that there is a w'-image V' of V at w such that $w, V'^{[y/x]} \Vdash A$, since then $w, V \Vdash \Diamond\langle y : x\rangle A$.
   If x is the same variable as y, then $V^*_{w'}(x) = V^*_{w'}(y)$ is a C-counterpart at w' of $V^{[y/x]}_w(x) = V^{[y/x]}_w(y) = V_w(x) = V_w(y)$ at w (or undefined if there is none), so we can choose V^* itself as V'. We then have $w, V'^{[y/x]} \Vdash A$ because $V'^{[y/x]} = V'$.
   Else if x is not free in A, let V' be some x-variant of V^* at w' such that $V^*_{w'}(x)$ is some C-counterpart at w' of $V_w(x)$ at w (or undefined if there is none). Since $V^*_{w'}(y)$ is a C-counterpart at w' of $V^{[y/x]}_w(y) = V_w(y)$ at w (or undefined if there is none), V' is a w'-image of V at w. Moreover, $V'^{[y/x]}$ and V^* agree at w' about all variables other than x; so by the coincidence lemma **??**, $w', V'^{[y/x]} \Vdash A$.
   Else if y is not free in A, let V' be like V^* except that $V'_{w'}(y) = V^*_{w'}(x)$ and $V'_{w'}(x)$ is some C-counterpart at w' of $V_w(x)$ at w (or undefined if there is none). Since $V'_{w'}(y) = V^*_{w'}(x)$ is a C-counterpart at w' of $V^{[y/x]}_w(x) = V_w(y)$ at w (or undefined if there is none), V' is a w'-image of V at w. Moreover, $V'^{[y/x]}$ and V^* agree at w' about all variables other than y; in particular, $V'^{[y/x]}_{w'}(x) = V'_{w'}(y) = V^*_{w'}(x)$. So by the coincidence lemma **??**, $w', V'^{[y/x]} \Vdash A$.
7. (SAt). $w, V \Vdash \langle y : x\rangle Px_1 \dots x_n$ iff $w, V^{[y/x]} \Vdash Px_1 \dots x_n$ by definition 5.2, iff $w, V \Vdash [y/x]Px_1 \dots x_n$ by lemma 3.5.

8. (SS2). $w, V \Vdash \langle y : x \rangle \langle x : z \rangle A$ iff $w, V^{[y/x]\cdot[x/z]} \Vdash A$ by definition 5.2, iff $w, V^{[y/z]\cdot[y/x]} \Vdash A$ because $[y/x] \cdot [x/z] = [y/z] \cdot [y/x]$, iff $w, V \Vdash \langle y : z \rangle \langle y : x \rangle A$ by definition 5.2.
9. (VS). By definition 5.2, $w, V \Vdash \langle y : x \rangle A$ iff $w, V^{[y/x]} \Vdash A$. If x is not free in A, then $V^{[y/x]}$ agrees with V at w about all free variables in A. So by the coincidence lemma **??**, $w, V^{[y/x]} \Vdash A$ iff $w, V \Vdash A$. So then $w, V \Vdash \langle y : x \rangle A$ iff $w, V \Vdash A$. ■

Definition 5.5 (Positive logics with substitution)
Given a language $\mathcal{L}_s$ with substitution, a *positive (quantified modal) logic with substitution* in $\mathcal{L}_s$ is a set of formulas $L \subseteq \mathcal{L}_s$ that contains all $\mathcal{L}_s$-instances of the substitution axioms (S¬), (S⊃), (S∀), (SS1), (S□), (S◇), (SAt), (SS2), (VS), as well as (Taut), (UD), (VQ), (∀Ex), (=R), (K),

(FUI$_s$) $\forall x A \supset (Ey \supset \langle y : x \rangle A)$,

(LL$_s$) $x = y \supset (A \supset \langle y : x \rangle A)$,

and that is closed under (MP), (UG), (Nec) and

(Sub$_s$) if $\vdash_L A$, then $\vdash_L \langle y : x \rangle A$.

Let the smallest such logic be called P_s.

Definition 5.6 (Negative logics with substitution)
Given a language $\mathcal{L}_s$ with substitution, a *negative (quantified modal) logic with substitution* in $\mathcal{L}_s$ is a set $L \subseteq \mathcal{L}_s$ that contains all $\mathcal{L}_s$-instances of the substitution axioms (S¬), (S⊃), (S∀), (SS1), (S□), (S◇), (SAt), (SS2), (VS), as well as (Taut), (UD), (VQ), (Neg), (NA), (∀=R), (K), (FUI$_s$), (LL$_s$), and that is closed under (MP), (UG), (Nec) and (Sub$_s$). Let the smallest such logic be called N_s.

Theorem 5.7 (Soundness of P_s)
Every member of P_s is valid in every positive counterpart model.

Proof We have to show that all P_s axioms are valid in every model, and that validity is closed under (MP), (UG), (Nec) and (Sub$_s$). For (Taut), (UD), (VQ), (∀Ex), (=R), (K), (MP), (UG), (Nec), see the proof of theorem 4.3. For the substitution axioms, see lemma 5.4. The remaining cases are (FUI$_s$), (LL$_s$), and (Sub$_s$).

1. (FUI$_s$). Assume $w, V \Vdash \forall x A$ and $w, V \Vdash Ey$ in some model. By definition 2.8, the latter means that $V_w(y) \in D_w$, and the former means that $w, V' \Vdash A$ for all existential x-variants V' of V on w. So in particular, $w, V' \Vdash A$, where V' is the x-variant of V on w with $V_w(x) = V_w(y)$. So $w, V \Vdash \langle y : x\rangle A$ by definition 5.2.
2. (LL$_s$). Assume $w, V \Vdash x = y$ and $w, V \Vdash A$. By definitions 2.8 and 2.3, then $V_w(x) = V_w(y)$. So $w, V \Vdash \langle y : x\rangle A$ by definition 5.2.
3. (Sub$_s$). Assume $w, V \nVdash \langle y : x\rangle A$ in some model $\mathcal{M} = \langle \mathcal{S}, V\rangle$. By definition 5.2, then $w, V' \nVdash A$, where V' is the x-variant of V on w with $V'(x) = V(y)$. So A is invalid in the model $\langle \mathcal{S}, V'\rangle$. Hence if A is valid in all positive models, then so is $\langle y : x\rangle A$. ■

Theorem 5.8 (Soundness of N_s)
Every member of N_s is valid in every negative counterpart model.

Proof All the cases needed here are covered in the proofs of theorem 4.6 and 5.7. ■

To derive some further properties of these systems, let $\mathcal{L}$ range over languages of quantified modal logic with substitution, and L over positive or negative logics in $\mathcal{L}$.

Closure under propositional consequence and the validity of ($\forall$Ex) and ($\forall$=R) are proved just as for substitution-free logics (see lemmas 4.7 and 4.8). So we move on immediately to more interesting properties.

Lemma 5.9 (Substitution expansion)
If A is an $\mathcal{L}$-formula and x, y, z $\mathcal{L}$-variables, then

(SE1) $\vdash_L A \leftrightarrow \langle x : x\rangle A$;

(SE2) $\vdash_L \langle y : x\rangle A \leftrightarrow \langle y : z\rangle\langle z : x\rangle A$, provided z is not free in A.

Proof (SE1) is proved by induction on A.
1. A is atomic. Then $\vdash_L \langle x : x\rangle A \leftrightarrow [x/x]A$ by (SAt), and so $\vdash_L \langle x : x\rangle A \leftrightarrow A$ because $[x/x]A = A$.
2. A is $\neg B$. By induction hypothesis, $\vdash_L B \leftrightarrow \langle x : x\rangle B$. So by (PC), $\vdash_L \neg B \leftrightarrow \neg\langle x : x\rangle B$. And by $\langle S\neg\rangle$, $\vdash_L \langle x : x\rangle\neg B \leftrightarrow \neg\langle x : x\rangle B$.
3. A is $B \supset C$. By induction hypothesis, $\vdash_L B \leftrightarrow \langle x : x\rangle B$ and $\vdash_L C \leftrightarrow \langle x : x\rangle C$. So $\vdash_L (B \supset C) \leftrightarrow (\langle x : x\rangle B \supset \langle x : x\rangle C)$. And by $\langle S \supset\ \rangle$, $\vdash_L \langle x : x\rangle(B \supset C) \leftrightarrow (\langle x : x\rangle B \supset \langle x : x\rangle C)$.
4. A is $\forall z B$. If $z = x$, then $\vdash_L \forall x B \leftrightarrow \langle x : x\rangle\forall x B$ by (VS). If $z \neq x$, then by induction hypothesis, $\vdash_L B \leftrightarrow \langle x : x\rangle B$; by (UG) and (UD), $\vdash_L \forall z B \leftrightarrow \forall z\langle x : x\rangle B$; and $\vdash_L \langle x : x\rangle\forall z B \leftrightarrow \forall z\langle x : x\rangle B$ by (S$\forall$).

5. A is $\langle y : z\rangle B$. If $z = x$, then $\vdash_L \langle y : x\rangle B \leftrightarrow \langle x : x\rangle\langle y : x\rangle B$ by (VS). If $z \neq x$, then by induction hypothesis, $\vdash_L B \leftrightarrow \langle x : x\rangle B$; by ($\mathrm{Sub_s}$) and (S$\supset$), $\vdash_L \langle y : z\rangle B \leftrightarrow \langle y : z\rangle\langle x : x\rangle B$; and $\vdash_L \langle x : x\rangle\langle y : z\rangle B \leftrightarrow \langle y : z\rangle\langle x : x\rangle B$ by (SS1) (if $y \neq x$) or (SS2) (if $y = x$).
6. A is $\Box B$. By (S$\Box$), $\vdash_L \langle x : x\rangle\Box B \supset \Box\langle x : x\rangle B$. Conversely, since at most one of x, x is free in $\neg B$, by (S$\Diamond$), $\vdash_L \langle x : x\rangle\Diamond\neg B \supset \Diamond\langle x : x\rangle\neg B$. Contraposing and unraveling the definition of the diamond, we have $\vdash_L \Box\neg\langle x : x\rangle\neg B \supset \neg\langle x : x\rangle\neg\Box\neg\neg B$. Since $\vdash_L \Box\neg\langle x : x\rangle\neg B \leftrightarrow \Box\langle x : x\rangle B$ and $\vdash_L \neg\langle x : x\rangle\neg\Box\neg\neg B \leftrightarrow \langle x : x\rangle B$ (by (S$\neg$), ($\mathrm{Sub_s}$), (S$\supset$), (Nec) and (K)), this means that $\vdash_L \Box\langle x : x\rangle B \supset \langle x : x\rangle\Box B$.

As for (SE2): by (VQ), $\vdash_L \langle y : x\rangle A \leftrightarrow \langle y : z\rangle\langle y : x\rangle A$. And $\vdash_L \langle y : x\rangle\langle y : z\rangle A \leftrightarrow \langle y : z\rangle\langle y : x\rangle A$ by (SS1) (if $y \neq x$) or (SS2) (if $y = x$). Moreover, by (SS2), $\vdash_L \langle y : z\rangle\langle z : x\rangle A \leftrightarrow \langle y : x\rangle\langle y : z\rangle A$. So by (PC), $\vdash_L \langle y : x\rangle A \leftrightarrow \langle y : z\rangle\langle z : x\rangle A$. ∎

Lemma 5.10 (Substituting bound variables)
For any $\mathcal{L}$-sentence A and variables x, y,

(SBV) $\vdash_L \forall x A \leftrightarrow \forall y\langle y : x\rangle A$, provided y is not free in A.

Proof

1.	$\vdash_L \forall y\langle y : x\rangle A \supset Ex \supset \langle x : y\rangle\langle y : x\rangle A$.	($\mathrm{FUI_s}$)
2.	$\vdash_L \langle x : y\rangle\langle y : x\rangle A \leftrightarrow A$.	((SE1), (SE2))
3.	$\vdash_L \forall x\forall y\langle y : x\rangle A \supset \forall x Ex \supset \forall x A$.	(1, 2, (UG), (UD))
4.	$\vdash_L \forall x\forall y\langle y : x\rangle A \supset \forall x A$.	(3, ($\forall$Ex))
5.	$\vdash_L \forall y\langle y : x\rangle A \supset \forall x\forall y\langle y : x\rangle A$.	(VQ)
6.	$\vdash_L \forall y\langle y : x\rangle A \supset \forall x A$.	(4, 5)
7.	$\vdash_L \forall x A \supset Ey \supset \langle y : x\rangle A$.	($\mathrm{FUI_s}$)
8.	$\vdash_L \forall y\forall x A \supset \forall y\langle y : x\rangle A$.	(7, (UG), (UD), ($\forall$Ex))
9.	$\vdash_L \forall x A \supset \forall y\forall x A$.	((VQ), y not free in A)
10.	$\vdash_L \forall x A \supset \forall y\langle y : x\rangle A$.	(8, 9)
11.	$\vdash_L \forall x A \leftrightarrow \forall y\langle y : x\rangle A$.	(6, 10)

∎

Lemma 5.11 (Substituting empty variables)
For any $\mathcal{L}$-sentence A and variables x, y,

(SEV) $\vdash_L x \neq x \wedge y \neq y \supset (A \leftrightarrow \langle y : x\rangle A)$.

Proof (SEV) is trivial if L is positive, in which case $\vdash_L x=x$. For negative L, it is proved by induction on A.

1. A is atomic. If $x \notin \mathit{Var}(A)$, then $\vdash_L A \leftrightarrow \langle y : x\rangle A$ by (VS), and so $\vdash_L x \neq x \wedge y \neq y \supset (A \leftrightarrow \langle y : x\rangle A)$ by (PC). If $x \in \mathit{Var}(A)$, then by (Neg)
$$\vdash_L x \neq x \wedge y \neq y \supset \neg A. \tag{1}$$
Also by (Neg), $\vdash_L x \neq x \wedge y \neq y \supset \neg[y/x]A$. By (SAt), $\vdash_L [y/x]A \leftrightarrow \langle y : x\rangle A$, and so $\vdash_L \neg[y/x]A \leftrightarrow \neg\langle y : x\rangle A$. So
$$\vdash_L x \neq x \wedge y \neq y \supset \neg\langle y : x\rangle A. \tag{2}$$
Combining (1) and (2) yields $\vdash_L x \neq x \wedge y \neq y \supset (A \leftrightarrow \langle y : x\rangle A)$.
2. A is $\neg B$. By induction hypothesis, $\vdash_L x \neq x \wedge y \neq y \supset (B \leftrightarrow \langle y : x\rangle B)$. So by (PC), $\vdash_L x \neq x \wedge y \neq y \supset (\neg B \leftrightarrow \neg\langle y : x\rangle B)$, and by (S$\neg$), $\vdash_L x \neq x \wedge y \neq y \supset (\neg B \leftrightarrow \langle y : x\rangle \neg B)$.
3. A is $B \supset C$. By induction hypothesis, $\vdash_L x \neq x \wedge y \neq y \supset (B \leftrightarrow \langle y : x\rangle B)$ and $\vdash_L x \neq x \wedge y \neq y \supset (C \leftrightarrow \langle y : x\rangle C)$. So by (PC), $\vdash_L x \neq x \wedge y \neq y \supset ((B \supset C) \leftrightarrow (\langle y : x\rangle B \supset \langle y : x\rangle C))$, and by (S$\supset$), $\vdash_L x \neq x \wedge y \neq y \supset ((B \supset C) \leftrightarrow \langle y : x\rangle(B \supset C))$.
4. A is $\forall zB$. We distinguish three cases.
   a) $z \notin \{x, y\}$. Then

1.	$\vdash_L x \neq x \wedge y \neq y \supset (B \leftrightarrow \langle y : x\rangle B)$	(ind. hyp.)
2.	$\vdash_L \forall z\, x \neq x \wedge \forall z\, y \neq y \supset (\forall zB \leftrightarrow \forall z\langle y : x\rangle B)$	(1, UG, UD)
3.	$\vdash_L x \neq x \wedge y \neq y \supset (\forall zB \leftrightarrow \forall z\langle y : x\rangle B)$	(2, VQ)
4.	$\vdash_L x \neq x \wedge y \neq y \supset (\forall zB \leftrightarrow \langle y : x\rangle \forall zB)$.	(3, (S$\forall$))

   b) $z = x$. Then A is $\forall xB$, and $\vdash_L \forall xB \leftrightarrow \langle y : x\rangle \forall xB$ by (VS). So $\vdash_L x \neq x \wedge y \neq y \supset (\forall xB \leftrightarrow \langle y : x\rangle \forall xB)$ by (PC).
   c) $z = y \neq x$. Then A is $\forall yB$. Let v be a variable not in $\mathit{Var}(A), x, y$.

1.	$\vdash_L x \neq x \wedge v \neq v \supset (B \leftrightarrow \langle v : x\rangle B)$.	(ind. hyp.)
2.	$\vdash_L \forall y x \neq x \wedge \forall y v \neq v \supset (\forall yB \leftrightarrow \forall y\langle v : x\rangle B)$.	(1, UG, UD)
3.	$\vdash_L x \neq x \wedge v \neq v \supset (\forall yB \leftrightarrow \forall y\langle v : x\rangle B)$.	(2, VQ)
4.	$\vdash_L x \neq x \wedge v \neq v \supset (\forall yB \leftrightarrow \langle v : x\rangle \forall yB)$.	(3, (S$\forall$))
5.	$\vdash_L \langle y : v\rangle x \neq x \wedge \langle y : v\rangle v \neq v \supset (\langle y : v\rangle \forall yB \leftrightarrow \langle y : v\rangle\langle v : x\rangle \forall yB)$.	(4, (Sub$_s$), (S$\supset$))
6.	$\vdash_L x \neq x \wedge y \neq y \supset (\langle y : v\rangle \forall yB \leftrightarrow \langle y : v\rangle\langle v : x\rangle \forall yB)$.	(5, (VS), (SAt))
7.	$\vdash_L x \neq x \wedge y \neq y \supset (\forall yB \leftrightarrow \langle y : v\rangle\langle v : x\rangle \forall yB)$.	(6, (VS))
8.	$\vdash_L x \neq x \wedge y \neq y \supset (\forall yB \leftrightarrow \langle y : x\rangle \forall yB)$.	(7, (SE2))

5. A is $\langle y_2 : z\rangle B$. We have four cases.

a) $z \notin \{x, y\}$ and $y_2 \neq x$. Then

1.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (B \leftrightarrow \langle y:x\rangle B)$	(ind. hyp.)
2.	$\vdash_L \langle y_2:z\rangle x{\neq}x \wedge \langle y_2:z\rangle y{\neq}y \supset (\langle y_2:z\rangle B \leftrightarrow \langle y_2:z\rangle\langle y:x\rangle B)$	(1, (Sub$_\mathrm{s}$), (S$\supset$))
3.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2:z\rangle B \leftrightarrow \langle y_2:z\rangle\langle y:x\rangle B)$	(2, (VS))
4.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2:z\rangle B \leftrightarrow \langle y:x\rangle\langle y_2:z\rangle B)$.	(3, (SS1))

b) $z \neq x$ and $y_2 = x$. Then A is $\langle x:z\rangle B$.

1.	$\vdash_L x{\neq}x \wedge z{\neq}z \supset (B \leftrightarrow \langle x:z\rangle B)$	(ind. hyp.)
2.	$\vdash_L \langle y:z\rangle x{\neq}x \wedge \langle y:z\rangle z{\neq}z \supset (\langle y:z\rangle B \leftrightarrow \langle y:z\rangle\langle x:z\rangle B)$	(1, (Sub$_\mathrm{s}$), (S$\supset$))
3.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y:z\rangle B \leftrightarrow \langle y:z\rangle\langle x:z\rangle B)$	(2, (SAt), $z \neq x$)
4.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y:z\rangle B \leftrightarrow \langle x:z\rangle B)$	(3, (VS), $z \neq x$)
5.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (B \leftrightarrow \langle y:x\rangle B)$	(ind. hyp.)
6.	$\vdash_L \langle y:z\rangle x{\neq}x \wedge \langle y:z\rangle y{\neq}y \supset (\langle y:z\rangle B \leftrightarrow \langle y:z\rangle\langle y:x\rangle B)$	(5, (Sub$_\mathrm{s}$),(S$\supset$))
7.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y:z\rangle B \leftrightarrow \langle y:z\rangle\langle y:x\rangle B)$	(6, (SAt), $z \neq x$)
8.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle x:z\rangle B \leftrightarrow \langle y:z\rangle\langle y:x\rangle B)$	(4, 7)
9.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle x:z\rangle B \leftrightarrow \langle y:x\rangle\langle x:z\rangle B)$.	(8, (SS2))

c) $z = x$. Then A is $\langle y_2:x\rangle B$, and $\vdash_L \langle y_2:x\rangle B \leftrightarrow \langle y:x\rangle\langle y_2:x\rangle B$ by (VS). So $\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2:x\rangle B \leftrightarrow \langle y:x\rangle\langle y_2:x\rangle B)$ by (PC).

d) $z = y \neq x$ and $y_2 \neq x$. Then A is $\langle y_2:y\rangle B$. Let v be a variable not in $\mathit{Var}(A), x, y, y_2$.

1.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset (B \leftrightarrow \langle v:x\rangle B)$.	(ind. hyp.)
2.	$\vdash_L \langle y_2:y\rangle x{\neq}x \wedge \langle y_2:y\rangle v{\neq}v \supset (\langle y_2:y\rangle B \leftrightarrow \langle y_2:y\rangle\langle v:x\rangle B)$.	(1, (Sub$_\mathrm{s}$), (S$\supset$))
3.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset (\langle y_2:y\rangle B \leftrightarrow \langle y_2:y\rangle\langle v:x\rangle B)$.	(2, (VS))
4.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset (\langle y_2:y\rangle B \leftrightarrow \langle v:x\rangle\langle y_2:y\rangle B)$.	(3, (SS1), $y_2 \neq x$)
5.	$\vdash_L \langle y:v\rangle x{\neq}x \wedge \langle y:v\rangle v{\neq}v \supset (\langle y:v\rangle\langle y_2:y\rangle B \leftrightarrow \langle y:v\rangle\langle v:x\rangle\langle y_2:y\rangle B)$.	(4, (Sub$_\mathrm{s}$), (S$\supset$))
6.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y:v\rangle\langle y_2:y\rangle B \leftrightarrow \langle y:v\rangle\langle v:x\rangle\langle y_2:y\rangle B)$.	(5, (VS), (SAt))
7.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2:y\rangle B \leftrightarrow \langle y:v\rangle\langle v:x\rangle\langle y_2:y\rangle B)$.	(6, (VS))
8.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2:y\rangle B \leftrightarrow \langle y:x\rangle\langle y_2:y\rangle B)$.	(7, (SE2))

6. A is $\Box B$. Let v be a variable not in $\mathit{Var}(B)$.

1.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset (B \leftrightarrow \langle v:x\rangle B)$.	(ind. hyp.)
2.	$\vdash_L \Box(x{\neq}x \wedge v{\neq}v) \supset (\Box B \leftrightarrow \Box\langle v:x\rangle B)$.	(1, (Nec), (K))
3.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset \Box(x{\neq}x \wedge v{\neq}v)$	((NA), (EI), (Nec), (K))
4.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset (\Box B \leftrightarrow \Box\langle v:x\rangle B)$.	(2, 3)
5.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset (\Box B \leftrightarrow \langle v:x\rangle \Box B)$.	(4, (S$\Box$), (S$\Diamond$), $v \notin \mathit{Var}(B)$)
6.	$\vdash_L \langle y:v\rangle x{\neq}x \wedge \langle y:v\rangle v{\neq}v \supset (\langle y:v\rangle\Box B \leftrightarrow \langle y:v\rangle\langle v:x\rangle\Box B)$.	(5, (Sub$_\mathrm{s}$), (S$\supset$))
7.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y:v\rangle\Box B \leftrightarrow \langle y:v\rangle\langle v:x\rangle\Box B)$.	(6, (SAt))
8.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\Box B \leftrightarrow \langle y:v\rangle\langle v:x\rangle\Box B)$.	(7, (VS))
9.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\Box B \leftrightarrow \langle y:x\rangle\Box B)$.	(8, (SE2))

Now we can prove (SC1) and (SC2). I will also prove that $\langle y:x\rangle A$ and $[y/x]A$ are provably equivalent conditional on $y{\neq}y$. Compare lemma 3.5 for a (slightly stronger) semantic version of this lemma.

Lemma 5.12 (Substitution conversion)
For any $\mathcal{L}$-formula A and variables x, y,

(SC1) $\vdash_L \langle y:x\rangle A \leftrightarrow [y/x]A$, provided y and x are modally separated in A.

(SC2) $\vdash_L \langle y:x\rangle A \supset [y/x]A$, provided y is modally free for x in A.

(SCN) $\vdash_L y{\neq}y \supset (\langle y:x\rangle A \leftrightarrow [y/x]A)$.

Proof If x and y are the same variable, then by (SE1), $\vdash_L \langle x:x\rangle A \leftrightarrow [x/x]A$. Assume then that x and y are different variables. We first prove (SC1) and (SC2), by induction on A. Observe that if A is not a box formula $\Box B$, then by definition 3.2, y is modally free for x in A iff y and x are modally separated in A, in which case y and x are also modally separated in any subformula of A.

1. A is atomic. By (SAt), $\vdash_L \langle y:x\rangle A \leftrightarrow [y/x]A$ holds without any restrictions.
2. A is $\neg B$. If y and x are modally separated in A, then by induction hypothesis, $\vdash_L \langle y:x\rangle B \leftrightarrow [y/x]B$. So by (PC), $\vdash_L \neg\langle y:x\rangle B \leftrightarrow \neg[y/x]B$. By (S$\neg$) and definition **??**, it follows that $\vdash_L \langle y:x\rangle\neg B \leftrightarrow [y/x]\neg B$.
3. A is $B \supset C$. If y and x are modally separated in A, then by induction hypothesis, $\vdash_L \langle y:x\rangle B \leftrightarrow [y/x]B$ and $\vdash_L \langle y:x\rangle C \leftrightarrow [y/x]C$. By (S$\supset$), $\vdash_L \langle y:x\rangle(B \supset C) \leftrightarrow (\langle y:x\rangle B \supset \langle y:x\rangle C)$. So $\vdash_L \langle y:x\rangle(B \supset C) \leftrightarrow ([y/x]B \supset [y/x]C)$, and so $\vdash_L \langle y:x\rangle(B \supset C) \leftrightarrow [y/x](B \supset C)$ by definition **??**.
4. A is $\forall zB$. We have to distinguish four cases, assuming each time that y and x are modally separated in A.
   a) $z \notin \{x, y\}$. By induction hypothesis, $\vdash_L \langle y:x\rangle B \leftrightarrow [y/x]B$. So by (UG) and (UD), $\vdash_L \forall z\langle y:x\rangle B \leftrightarrow \forall z[y/x]B$. Since $z \notin \{x, y\}$, $\vdash_L \langle y:x\rangle\forall zB \leftrightarrow \forall z\langle y:x\rangle B$ by (S$\forall$), and $\forall z[y/x]B$ is $[y/x]\forall zB$ by definition **??**; so $\vdash_L \langle y:x\rangle\forall zB \leftrightarrow [y/x]\forall zB$.

b) $z = y$ and $x \notin \mathit{Varf}(B)$. By definition **??**, then $[y/x]\forall zB$ is $\forall y[y/x]B$.

1.	$\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$.	(induction hypothesis)
2.	$\vdash_L \forall y\langle y : x\rangle B \leftrightarrow \forall y[y/x]B$.	(1, (UG), (UD))
3.	$\vdash_L B \leftrightarrow \langle y : x\rangle B$.	((VS), $x \notin \mathit{Varf}(B)$)
4.	$\vdash_L \forall yB \leftrightarrow \forall y\langle y : x\rangle B$.	(3, (UG), (UD))
5.	$\vdash_L \forall yB \leftrightarrow \langle y : x\rangle\forall yB$.	((VS), $x \notin \mathit{Varf}(B)$)
6.	$\vdash_L \langle y : x\rangle\forall yB \leftrightarrow \forall y[y/x]B$.	(2, 4, 5)

c) $z = x$ and $y \notin \mathit{Varf}(B)$. By definition **??**, then $[y/x]\forall zB$ is $\forall y[y/x]B$.

1.	$\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$.	(induction hypothesis)
2.	$\vdash_L \forall y\langle y : x\rangle B \leftrightarrow \forall y[y/x]B$.	(1, (UG), (UD))
3.	$\vdash_L \forall xB \leftrightarrow \forall y\langle y : x\rangle B$.	((SBV), $y \notin \mathit{Varf}(B)$)
4.	$\vdash_L \forall xB \leftrightarrow \langle y : x\rangle\forall xB$.	(VS)
5.	$\vdash_L \langle y : x\rangle\forall xB \leftrightarrow \forall y[y/x]B$.	(2, 3, 4)

d) $z = x$ and $y \in \mathit{Varf}(B)$, or $z = y$ and $x \in \mathit{Varf}(B)$. By definition **??**, then $[y/x]\forall zB$ is $\forall v[y/x][v/z]B$ for some variable $v \notin \mathit{Var}(B) \cup \{x, y\}$. Since v and z are modally separated in B, by induction hypothesis $\vdash_L \langle v : z\rangle B \leftrightarrow [v/z]B$. So by (UG) and (UD), $\vdash_L \forall v\langle v : z\rangle B \leftrightarrow \forall v[v/z]B$. By (SBV), $\vdash_L \forall zB \leftrightarrow \forall v\langle v : z\rangle B$. So $\vdash_L \forall zB \leftrightarrow \forall v[v/z]B$. Moreover, as $z \in \{x, y\}$, y and x are modally separated in $[v/z]B$. So by induction hypothesis, $\vdash_L \langle y : x\rangle[v/z]B \leftrightarrow [y/x][v/z]B$. Then

1.	$\vdash_L \forall zB \leftrightarrow \forall v[v/z]B$	(as just shown)
2.	$\vdash_L \langle y : x\rangle\forall zB \leftrightarrow \langle y : x\rangle\forall v[v/z]B$	(1, (Sub$^{\mathrm{s}}$), (S$\neg$), (S$\supset$))
3.	$\vdash_L \langle y : x\rangle\forall v[v/z]B \leftrightarrow \forall v\langle y : x\rangle[v/z]B$.	(S$\forall$)
4.	$\vdash_L \langle y : x\rangle\forall zB \leftrightarrow \forall v\langle y : x\rangle[v/z]B$.	(2, 3)
5.	$\vdash_L \langle y : x\rangle[v/z]B \leftrightarrow [y/x][v/z]B$.	(induction hypothesis)
6.	$\vdash_L \forall v\langle y : x\rangle[v/z]B \leftrightarrow \forall v[y/x][v/z]B$.	(5, (UG), (UD))
7.	$\vdash_L \langle y : x\rangle\forall zB \leftrightarrow \forall v[y/x][v/z]B$.	(4, 6)

5. A is $\langle y_2 : z\rangle B$. Again we have four cases, assuming x and y are modally separated in A.

a) $z \notin \{x, y\}$. By definition **??**, then $[y/x]\langle y_2 : z\rangle B$ is $\langle[y/x]y_2 : z\rangle[y/x]B$.

1.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle[y/x]y_2 : z\rangle\langle y : x\rangle B$	((SS1) or (SS2))
2.	$\vdash \langle y : x\rangle B \leftrightarrow [y/x]B$	(induction hypothesis)
3.	$\vdash \langle[y/x]y_2 : z\rangle(\langle y : x\rangle B \leftrightarrow [y/x]B)$	(2, (Sub$_{\mathrm{s}}$))
4.	$\vdash \langle[y/x]y_2 : z\rangle\langle y : x\rangle B \leftrightarrow \langle[y/x]y_2 : z\rangle[y/x]B$	(3, (S$\supset$), (S$\neg$))
5.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle[y/x]y_2 : z\rangle[y/x]B$.	(1, 4)

b) $z = y$ and $x \notin \mathit{Varf}(B)$. By definition **??**, then $[y/x]\langle y_2 : z\rangle B$ is $\langle [y/x]y_2 : y\rangle [y/x]B$. By induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$. So by (Sub$_\mathrm{s}$) and (S$\supset$), $\vdash_L \langle [y/x]y_2 : y\rangle\langle y : x\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle [y/x]B$. If $y_2 = x$, then $\vdash_L \langle y : x\rangle\langle y_2 : y\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle\langle y : x\rangle B$ by (SS2). If $y_2 \neq x$, then

1.	$\vdash_L \langle y_2 : y\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : y\rangle B$	((VS), $x \notin \mathit{Varf}(\langle y_2 : y\rangle B)$)
2.	$\vdash_L B \leftrightarrow \langle y : x\rangle B$	((VS), $x \notin \mathit{Varf}(B)$)
3.	$\vdash_L \langle y_2 : y\rangle B \leftrightarrow \langle y_2 : y\rangle\langle y : x\rangle B$	(1, (Sub$_\mathrm{s}$), (S$\supset$))
4.	$\vdash_L \langle y : x\rangle\langle y_2 : y\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle\langle y : x\rangle B$	(1, 3)

So either way $\vdash_L \langle y : x\rangle\langle y_2 : y\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle\langle y : x\rangle B$. So $\vdash_L \langle y : x\rangle\langle y_2 : y\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle [y/x]B$.

c) $z = x$ and $y \notin \mathit{Varf}(B)$. By definition **??**, then $[y/x]\langle y_2 : z\rangle B$ is ([y/x]y$_2$: y)$[y/x]B$. By induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$. So by (Sub$_\mathrm{s}$) and (S$\supset$), $\vdash_L \langle [y/x]y_2 : y\rangle\langle y : x\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle [y/x]B$. Since $y \notin \mathit{Varf}(B)$, by (SE2), $\vdash_L \langle [y/x]y_2 : y\rangle\langle y : x\rangle B \leftrightarrow \langle [y/x]y_2 : x\rangle B$. Moreover, $\vdash_L \langle [y/x]y_2 : x\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : x\rangle B$ by either (VS) (if $x \neq y_2$) or by (SE1), (Sub$_\mathrm{s}$) and (S$\supset$) (if $x = y_2$). So $\vdash_L \langle y : x\rangle\langle y_2 : x\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle\langle y : x\rangle B$.

d) $z = x$ and $y \in \mathit{Varf}(B)$, or $z = y$ and $x \in \mathit{Varf}(B)$. By definition **??**, then $[y/x]\langle y_2 : z\rangle B$ is $\langle [y/x]y_2 : v\rangle [y/x][v/z]B$, where $v \notin \mathit{Var}(B) \cup \{x, y, y_2\}$.

1.	$\vdash \langle v : z\rangle B \leftrightarrow [v/z]B$	(induction hypothesis)
2.	$\vdash \langle y_2 : v\rangle\langle v : z\rangle B \leftrightarrow \langle y_2 : v\rangle [v/z]B$	(1, (Sub$_\mathrm{s}$), (S$\supset$), (S$\neg$))
3.	$\vdash \langle y_2 : z\rangle B \leftrightarrow \langle y_2 : v\rangle\langle v : z\rangle B$	(SE2)
4.	$\vdash \langle y_2 : z\rangle B \leftrightarrow \langle y_2 : v\rangle [v/z]B$	(2, 3)

Since $z \in \{x, y\}$, x and y are modally separated in $[v/z]B$. So:

5.	$\vdash \langle y : x\rangle [v/z]B \leftrightarrow [y/x][v/z]B$	(ind. hyp.)
6.	$\vdash \langle [y/x]y_2 : v\rangle\langle y : x\rangle [v/z]B \leftrightarrow \langle [y/x]y_2 : v\rangle [y/x][v/z]B$	(5, (Sub$_\mathrm{s}$), (S$\supset$))
7.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : v\rangle [v/z]B$	(4, (Sub$_\mathrm{s}$), (S$\supset$))
8.	$\vdash \langle y : x\rangle\langle y_2 : v\rangle [v/z]B \leftrightarrow \langle [y/x]y_2 : v\rangle\langle y : x\rangle [v/z]B$	((SS1) or (SS2))
9.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : v\rangle\langle y : x\rangle [v/z]B$	(7, 8)
10.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : v\rangle [y/x][v/z]B$	(6, 9)

6. A is $\Box B$. For (SC1), assume x and y are modally separated in A. Then they are also modally separated in B, so by induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$. By (Nec) and (K), then $\vdash_L \Box\langle y : x\rangle B \leftrightarrow \Box[y/x]B$. By (S$\Box$), $\vdash_L \langle y : x\rangle\Box B \supset \Box\langle y : x\rangle B$. Since at most one of x, y is free in B, by (S$\Diamond$), $\vdash_L \langle y : x\rangle\Diamond\neg B \supset \Diamond\langle y : x\rangle\neg B$; so $\vdash_L \Box\langle y : x\rangle B \supset \langle y : x\rangle\Box B$ (by (S$\neg$), (Sub$_\mathrm{s}$), (S$\supset$), (Nec), (K)). So $\vdash_L \langle y : x\rangle\Box B \leftrightarrow \Box[y/x]B$. Since $\Box[y/x]B$ is $[y/x]\Box B$ by definition **??**, this means that $\vdash_L \langle y : x\rangle\Box B \leftrightarrow [y/x]\Box B$. For (SC2), assume y is modally free for x in $\Box B$. Then y is modally free for x in B, so by induction hypothesis, $\vdash \langle y : x\rangle B \supset [y/x]B$. By (Nec) and (K), then $\vdash \Box\langle y : x\rangle B \supset$

$\Box[y/x]B$. By (S$\Box$), $\vdash \langle y:x\rangle\Box B \supset \Box\langle y:x\rangle B$. So $\vdash \langle y:x\rangle\Box B \supset \Box[y/x]B$.

Here is the proof for (SCN). The first three clauses are very similar.

1. A is atomic. Then $\vdash_L \langle y:x\rangle A \leftrightarrow [y/x]A$ as we've seen above, and so $\vdash_L y{\neq}y \supset (\langle y:x\rangle A \leftrightarrow [y/x]A)$ by (PC).
2. A is $\neg B$. By induction hypothesis, $\vdash_L y\neq y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$. So by (PC), $\vdash_L y\neq y \supset (\neg\langle y:x\rangle B \leftrightarrow \neg[y/x]B)$. By (S$\neg$) and definition **??**, it follows that $\vdash_L y{\neq}y \supset (\langle y:x\rangle\neg B \leftrightarrow [y/x]\neg B)$.
3. A is $B \supset C$. By induction hypothesis, $\vdash_L y\neq y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$ and $\vdash_L y\neq y \supset (\langle y:x\rangle C \leftrightarrow [y/x]C)$. By (S$\supset$), $\vdash_L y\neq y \supset (\langle y:x\rangle(B\supset C) \leftrightarrow (\langle y:x\rangle B \supset \langle y:x\rangle C))$. So $\vdash_L y\neq y \supset (\langle y:x\rangle(B\supset C) \leftrightarrow ([y/x]B \supset [y/x]C))$, and so $\vdash_L y{\neq}y \supset (\langle y:x\rangle(B\supset C) \leftrightarrow [y/x](B\supset C))$ by definition **??**.
4. A is $\forall zB$. If $z\notin\{x,y\}$, then by induction hypothesis, $\vdash_L y{\neq}y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$. So by (UG) and (UD), $\vdash_L \forall z\, y\neq y \supset (\forall z\langle y:x\rangle B \leftrightarrow \forall z[y/x]B)$. Since $z\notin\{x,y\}$, $\vdash_L \langle y:x\rangle\forall zB \leftrightarrow \forall z\langle y:x\rangle B$ by (S$\forall$), and $\vdash_L y{\neq}y \supset \forall z\, y{\neq}y$ by (VQ), and $\forall z[y/x]B$ is $[y/x]\forall zB$ by definition **??**; so $\vdash_L y{\neq}y \supset (\langle y:x\rangle\forall zB \leftrightarrow [y/x]\forall zB)$.
   Alternatively, if $z\in\{x,y\}$, then either x or y is not free in A, and thus x and y are modally separated in A. By (SC2), then $\vdash_L \langle y:x\rangle\forall zB \leftrightarrow [y/x]\forall zB$, and so by (PC), $\vdash_L y{\neq}y \supset (\langle y:x\rangle\forall zB \leftrightarrow [y/x]\forall zB)$.
5. A is $\langle y_2:z\rangle B$. If $z\notin\{x,y\}$, then by induction hypothesis, $\vdash_L y\neq y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$. So by (Sub$_\mathrm{s}$) and (S$\supset$), $\vdash_L \langle [y/x]y_2 : z\rangle y\neq y \supset (\langle [y/x]y_2 : z\rangle\langle y:x\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle[y/x]B)$. By (VS), $\langle [y/x]y_2 : z\rangle y\neq y \leftrightarrow y\neq y$. And by (SS1) or (SS2), $\langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle\langle y:x\rangle B$. So $\vdash_L y{\neq}y \supset (\langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle[y/x]B)$. But by definition **??**, $[y/x]\langle y_2:z\rangle B$ is $\langle [y/x]y_2 : y\rangle[y/x]B$.
   Alternatively, if $z\in\{x,y\}$, then either x or y is not free in A, and thus x and y are modally separated in A. By (SC2), then $\vdash_L \langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow [y/x]\langle y_2:z\rangle B$, and so by (PC), $\vdash_L y{\neq}y \supset (\langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow [y/x]\langle y_2:z\rangle B)$.
6. A is $\Box B$. Then

1.	$\vdash_L y{\neq}y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$.	(ind. hyp.)
2.	$\vdash_L \Box y{\neq}y \supset (\Box\langle y:x\rangle B \leftrightarrow \Box[y/x]B)$.	(1, (Nec), (K))
3.	$\vdash_L y{\neq}y \supset \Box y{\neq}y$.	((=R) or (NA), (EI) and (Nec))
4.	$\vdash_L y{\neq}y \supset (\Box\langle y:x\rangle B \leftrightarrow \Box[y/x]B)$.	(2, 3)
5.	$\vdash_L y{\neq}y \supset \langle y:x\rangle(x{\neq}x \wedge y{\neq}y)$	((SAt), (S$\supset$), (S$\neg$))
6.	$\vdash_L (x{\neq}x \wedge y{\neq}y) \supset \Box(x{\neq}x \wedge y{\neq}y)$.	((=R) or (NA), (EI), (Nec) and (K))
7.	$\vdash_L \Box(x{\neq}x \wedge y{\neq}y) \supset (\Box B \leftrightarrow \Box\langle y:x\rangle B)$.	((SEV), (Nec), (K))
8.	$\vdash_L (x{\neq}x \wedge y{\neq}y) \supset (\Box B \leftrightarrow \Box\langle y:x\rangle B)$.	(6, 7)
9.	$\vdash_L \langle y:x\rangle(x{\neq}x \wedge y{\neq}y) \supset (\langle y:x\rangle\Box B \leftrightarrow \langle y:x\rangle\Box\langle y:x\rangle B)$.	(8, (Sub$_\mathrm{s}$), (S$\supset$))
10.	$\vdash_L \langle y:x\rangle(x{\neq}x \wedge y{\neq}y) \supset (\langle y:x\rangle\Box B \leftrightarrow \Box\langle y:x\rangle B)$.	(9, (VS))
11.	$\vdash_L y{\neq}y \supset (\langle y:x\rangle\Box B \leftrightarrow \Box\langle y:x\rangle B)$.	(7, 10)
12.	$\vdash_L y{\neq}y \supset (\langle y:x\rangle\Box B \leftrightarrow [y/x]\Box B)$.	(4, 13, def. **??**)

■

Lemma 5.13 (Syntactic alpha-conversion)
If A, A' are $\mathcal{L}$-formulas, and A' is an alphabetic variant of A, then

(AC) $\vdash_L A \leftrightarrow A'$.

Proof by induction on A.

1. A is atomic. Then $A = A'$ and $\vdash_L A \leftrightarrow A'$ by (Taut).
2. A is $\neg B$. Then A' is $\neg B'$ with B' an alphabetic variant of B. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. By (PC), $\vdash_L \neg B \leftrightarrow \neg B'$.
3. A is $B \supset C$. Then A' is $B' \supset C'$ with B', C' alphabetic variants of B, C, respectively. By induction hypothesis, $\vdash_L B \leftrightarrow B'$ and $\vdash_{sC} C \leftrightarrow C'$. By (PC), then $\vdash_L (B \supset C) \leftrightarrow (B' \supset C')$.
4. A is $\forall x B$. Then A' is either $\forall x B'$ or $\forall z[z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B')$. Assume first that A' is $\forall x B'$. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (UG) and (UD), $\vdash_L \forall x B \leftrightarrow \forall x B'$.
   Alternatively, assume A' is $\forall z[z/x]B'$ and $z \notin \mathit{Var}(B')$. Since B' differs from B at most in renaming bound variables, if z were free in B, then $z \in \mathit{Var}(B')$. So z is not free in B. Then

1.	$\vdash_L B \leftrightarrow B'$.	induction hypothesis
2.	$\vdash_L \langle z : x\rangle B \leftrightarrow \langle z : x\rangle B'$.	(1, (Sub$_\mathrm{s}$), (S$\neg$))
3.	$\vdash_L \langle z : x\rangle B' \leftrightarrow [z/x]B'$.	((SC1), $z \notin \mathit{Var}(B')$)
4.	$\vdash_L \langle z : x\rangle B \leftrightarrow [z/x]B'$.	(2, 3)
5.	$\vdash_L \forall z\langle z : x\rangle B \leftrightarrow \forall z[z/x]B'$.	(4, (UG), (UD))
6.	$\vdash_L \forall x B \leftrightarrow \forall z\langle z : x\rangle B$.	((SBV), z not free in B)
7.	$\vdash_L \forall x B \leftrightarrow \forall z[z/x]B'$.	(5, 6)

5. A is $\langle y : x\rangle B$. Then A' is either $\langle y : x\rangle B'$ or $\langle y : z\rangle[z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B)$. Assume first that A' is $\langle y : x\rangle B'$. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (Sub$_\mathrm{s}$) and (S$\supset$), $\vdash_L \langle y : x\rangle B \leftrightarrow \langle y : x\rangle B'$.
   Alternatively, assume A' is $\langle y : z\rangle[z/x]B'$ and $z \notin \mathit{Var}(B')$. Again, it follows that z is not free in B. So

1.	$\vdash_L B \leftrightarrow B'$.	induction hypothesis
2.	$\vdash_L \langle z : x\rangle B \leftrightarrow \langle z : x\rangle B'$.	(1, (Sub$_\mathrm{s}$), (S$\supset$))
3.	$\vdash_L \langle z : x\rangle B' \leftrightarrow [z/x]B'$.	((SC1), $z \notin \mathit{Var}(B')$)
4.	$\vdash_L \langle z : x\rangle B \leftrightarrow [z/x]B'$.	(2, 3)
5.	$\vdash_L \langle y : z\rangle\langle z : x\rangle B \leftrightarrow \langle y : z\rangle[z/x]B'$.	(4, (Sub$_\mathrm{s}$), (S$\supset$))
6.	$\vdash_L \langle y : z\rangle\langle z : x\rangle B \leftrightarrow \langle y : x\rangle B$.	((SE2), z not free in B)
7.	$\vdash_L \langle y : x\rangle B \leftrightarrow \langle y : z\rangle[z/x]B'$.	(5, 6)

6. A is $\Box A'$. Then B is $\Box B'$ with B' an alphabetic variant of A'. By induction hypothesis, $\vdash_L A' \leftrightarrow B'$. Then by (Nec), $\vdash_L \Box(A' \leftrightarrow B')$, and by (K), $\vdash_L \Box A' \leftrightarrow \Box B'$. ∎

Theorem 5.14 (Substitution and non-substitution logics)
For any $\mathcal{L}$-formula A and variables x, y,

(FUI*) $\vdash_L \forall x A \supset (Ey \supset [y/x]A)$, provided y is modally free for x in A,

(LL*) $\vdash_L x = y \supset A \supset [y/x]A$, provided y is modally free for x in A,

(Sub*) if $\vdash_L A$, then $\vdash_L [y/x]A$, provided y is modally free for x in A.

It follows that $\mathsf{P} \subseteq \mathsf{P}_s$ and $\mathsf{N} \subseteq \mathsf{N}_s$.

Proof Assume y is modally free for x in A. Then by (SC2), $\vdash_L \langle y : x\rangle A \supset [y/x]A$. By (FUI$_\mathrm{s}$), $\vdash_L \forall x A \supset (Ey \supset \langle y : x\rangle A)$, so by (PC), $\vdash_L \forall x A \supset (Ey \supset [y/x]A)$. Similarly, by (LL$_\mathrm{s}$), $\vdash_L x = y \supset A \supset \langle y : x\rangle A$, so by (PC), $\vdash_L x = y \supset A \supset [y/x]A$. Finally, by (Sub$_\mathrm{s}$), if $\vdash_L A$, then $\vdash_L \langle y : x\rangle A$, so then $\vdash_L [y/x]A$ by (PC). ∎

Lemma 5.15 (Symmetry and transitivity of identity)
For any $\mathcal{L}$-variables x, y, z,

(=S) $\vdash_L x = y \supset y = x$;

(=T) $\vdash_L x = y \supset y = z \supset x = z$.

Proof Immediate from lemma 5.14 and lemma 4.10. ∎

Lemma 5.16 (Variations on Leibniz' Law)
If A is an $\mathcal{L}$-formula and x, y, y' are $\mathcal{L}$-variables, then

(LV1) $\vdash_L x = y \supset \langle y : x\rangle A \supset A$.

(LV2) $\vdash_L y = y' \supset \langle y : x\rangle A \supset [y'/x]A$, provided y' is modally free for x in A.

PROOF (LV1). Let z be an $\mathcal{L}$-variable not in $Var(A)$. Then

1.	$\vdash_L x{=}z \supset \langle z : x\rangle A \supset \langle x : z\rangle\langle z : x\rangle A.$	(LL_s)
2.	$\vdash_L x{=}z \supset \langle z : x\rangle A \supset \langle x : x\rangle A.$	(1, (SE2), $z \notin Var(A)$)
3.	$\vdash_L x{=}z \supset \langle z : x\rangle A \supset A.$	(2, (SE1))
4.	$\vdash_L \langle y : z\rangle x{=}z \supset \langle y : z\rangle\langle z : x\rangle A \supset \langle y : z\rangle A.$	(3, (VS), (S$\supset$))
5.	$\vdash_L x{=}z \supset \langle y : z\rangle\langle z : x\rangle A \supset \langle y : z\rangle A.$	(4, (SAt))
6.	$\vdash_L x{=}z \supset \langle y : x\rangle A \supset \langle y : z\rangle A.$	(5, (SE2), $z \notin Var(A)$)
7.	$\vdash_L x{=}z \supset \langle y : x\rangle A \supset A.$	(6, (VS), $z \notin Var(A)$).

(LV2).

1.	$\vdash_L x{=}y \wedge y{=}y' \supset x{=}y'.$	(=T)
2.	$\vdash_L A \wedge x{=}y' \supset [y'/x]A.$	((LL*), y' m.f. in A)
3.	$\vdash_L A \wedge x{=}y \wedge y{=}y' \supset [y'/x]A.$	(1, 2)
4.	$\vdash_L \langle y : x\rangle A \wedge \langle y : x\rangle x{=}y \wedge \langle y : x\rangle y{=}y' \supset \langle y : x\rangle[y'/x]A.$	(3, (Sub_s), (S$\neg$), (S$\supset$))
5.	$\vdash_L y{=}y \supset \langle y : x\rangle x{=}y.$	(SAt)
6.	$\vdash_L y{=}y' \supset y{=}y.$	((LL*), (=S))
7.	$\vdash_L y{=}y' \supset \langle y : x\rangle y{=}y'.$	(VS)
8.	$\vdash_L \langle y : x\rangle A \wedge y{=}y' \supset \langle y : x\rangle[y'/x]A.$	(4, 5, 6, 7)
9.	$\vdash_L \langle y : x\rangle[y'/x]A \supset [y'/x]A.$	(VS)
10.	$\vdash_L \langle y : x\rangle A \wedge y{=}y' \supset [y'/x]A.$	(8, 9)

■

LEMMA 5.17 (LEIBNIZ' LAW WITH SEQUENCES)
For any $\mathcal{L}$-formula A and variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct,

$$(\mathrm{LL_n})\ \vdash_L x_1{=}y_1 \wedge \ldots \wedge x_n{=}y_n \supset A \supset \langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A.$$

PROOF For $n = 1$, (LL_n) is (LL_s). Assume then that $n > 1$. To keep formulas in the following proof at a managable length, let $\underline{\phi(i)}$ abbreviate the sequence $\phi(1), \ldots, \phi(n-1)$. For example, $\langle \underline{y_i} : \underline{x_i}\rangle$ is $\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle$. Let z be the alphabetically first variable not in A or

$x_1, \ldots, x_n$. Now

1.	$\vdash_L x_n = y_n \supset \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A.$	($\mathrm{LL_s}$)
2.	$\vdash_L \langle y_n : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : z \rangle \langle z : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A.$	(SE1)
3.	$\vdash_L \langle z : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	((SS1) or (SS2))
4.	$\vdash_L \langle y_n : z \rangle \langle z : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A$ $\supset \langle y_n : z \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	(3, ($\mathrm{Sub_s}$), (S$\supset$))
5.	$\vdash_L x_n = y_n \supset \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : z \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	(1, 2, 4)
6.	$\vdash_L x_n = z \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$ $\supset \langle z : x_n \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	($\mathrm{LL_s}$)
7.	$\vdash_L x_n = z \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$ $\supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A.$	(6, (SS1))
8.	$\vdash_L z = x_n \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A$ $\supset \langle x_n : z \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A.$	($\mathrm{LL_s}$)
9.	$\vdash_L z = x_n \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A$ $\supset \langle \underline{y_i} : \underline{x_i} \rangle \langle x_n : z \rangle \langle z : x_n \rangle \langle z : x_n \rangle A.$	(8, (SS1), (SS2))
10.	$\vdash_L \langle x_n : z \rangle \langle z : x_n \rangle \langle z : x_n \rangle A \leftrightarrow \langle z : x_n \rangle A$	((SE1), (SE2))
11.	$\vdash_L \langle \underline{y_i} : \underline{x_i} \rangle \langle x_n : z \rangle \langle z : x_n \rangle \langle z : x_n \rangle A \supset \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$	(10, ($\mathrm{Sub_s}$), (S$\supset$))
12.	$\vdash_L z = x_n \supset x_n = z$	(=S)
13.	$\vdash_L z = x_n \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A \supset \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	(7, 9, 11, 12)
14.	$\vdash_L x_n = y_n \supset \langle y_n : z \rangle z = x_n$	((=S), (SAt))
15.	$\vdash_L x_n = y_n \supset \langle y_n : z \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$ $\supset \langle y_n : z \rangle \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	13, 14, ($\mathrm{Sub_s}$), (S$\supset$)
16.	$\vdash_L x_n = y_n \supset \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : z \rangle \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	5, 15
17.	$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_{n-1} = y_{n-1} \supset A \supset \langle \underline{y_i} : \underline{x_i} \rangle A.$	(induction hypothesis)
18.	$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset \langle y_n : z \rangle \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	(16, 17)
19.	$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset \langle y_1, \ldots, y_n : x_1, \ldots, x_n \rangle A.$	(18, def. **??**)

■

Lemma 5.18 (Closure under transformations)
For any $\mathcal{L}$-formula A and transformation τ on $\mathcal{L}$,

(Sub^τ) $\vdash_L A$ iff $\vdash_L A^\tau$.

Proof The proof is exactly as in lemma 4.12. ■

6 Correspondence

A well-known feature of Kripke semantics for propositional modal logic is that various modal schemas correspond to conditions on the accessibility relation, in the sense that the schema is valid in all and only those Kripke frames whose accessibility relation satisfies the condition: $\Box A \supset A$ *corresponds to* (or *defines*) the class of reflexive frames, $A \supset \Box\Diamond A$ the class of symmetrical frames, and so on.

In our counterpart semantics, (i) is not enough to ensure the validity of $\Box A \supset A$. Consider $\Box Fx \supset Fx$. Loosely speaking, the antecedent $\Box Fx$ is true at w iff all counterparts of x are F at all accessible worlds. This does not entail that x is F at w unless (i) w can see itself and (ii) x is its own counterpart at w.

How does this observation generalise?

Let's stick to positive logics for a while.

First a brief review of some definitions from propositional modal logic.

> Definition 6.1 (Languages of propositional modal logic)
> A set of formulas $\mathcal{L}_0$ is a *(unimodal) propositional language* if there is a denumerable set of expressions *Prop* (the *sentence letters* of $\mathcal{L}_0$) distinct from $\{\neg, \supset, \Box\}$ such that $\mathcal{L}_0$ is generated by the rule
>
> $$P \mid \neg A \mid (A \supset B) \mid \Box A,$$
>
> where $P \in$ *Prop*.

Note that the language $\mathcal{L}$ of quantified modal logic is a unimodal propositional language if we define a "sentence letters" as any $\mathcal{L}$-formula that isn't of the form $\neg A$, $A \supset B$ or $\Box A$. (On this usage, $\forall x\Box(Fx \supset Gx)$, for example, is a sentence letter.) Let's call such $\mathcal{L}$-formulas *quasi-atomic*.

> Definition 6.2 (Frames and valuations)
> A *frame* is a pair consisting of a non-empty set W and a relation $R \subseteq W^2$.
>
> A *valuation of* a unimodal propositional language $\mathcal{L}_0$ *on* a frame $\mathcal{F} = \langle W, R\rangle$ is a function V that maps every sentence letter of $\mathcal{L}_0$ to a subset of W.

> Definition 6.3 (Propositional truth)
> For any frame $\mathcal{F} = \langle W, R\rangle$, point $w \in W$, valuation V on $\mathcal{F}$, sentence letter P and $\mathcal{L}_0$-sentences A and B,

$$\begin{aligned}
\mathcal{F}, V, w \Vdash_0 P \quad & \text{iff } w \in V(P), \\
\mathcal{F}, V, w \Vdash_0 \neg A \quad & \text{iff } \mathcal{F}, V, w \nVdash_0 A, \\
\mathcal{F}, V, w \Vdash_0 A \supset B \quad & \text{iff } \mathcal{F}, V, w \nVdash_0 A \text{ or } \mathcal{F}, V, w \Vdash_0 B, \\
\mathcal{F}, V, w \Vdash_0 \Box A \quad & \text{iff } \mathcal{F}, V, w' \Vdash_0 A \text{ for all } w' \text{ with } wRw'.
\end{aligned}$$

Definition 6.4 (Frame validity)
A formula A of a unimodal propositional language is *valid on a frame* $\mathcal{F} = \langle W, R\rangle$ if $\mathcal{F}, V, w \Vdash_0 A$ for all $w \in W$ and valuation functions V of the language on $\mathcal{F}$.

A set of formulas $\mathbb{A}$ is *valid on a frame* $\mathcal{F}$ if all members of $\mathbb{A}$ are valid on $\mathcal{F}$.

Definition 6.5 (Frame correspondence)
A set of formulas $\mathbb{A}$ of a unimodal propositional language *defines* or *corresponds to* a class of frames $\mathcal{C}$ iff $\mathcal{C}$ is the class of frames in which all members of A are valid.

In propositional modal logic, formulas are true or false relative to a world; the box shifts the world of evaluation. In counterpart semantics, formulas are true or false relative to a pair $\langle w, g\rangle$ of a world and an assignment function, and the box shifts these points of evaluation: $\Box A$ is true at w, g iff A is true at all w', g' such that $w, g \triangleright w', g'$.

This suggests that if a schema of propositional modal logic corresponds to a certain property of the accessibility relation R of Kripke frames, then it corresponds to the same property of the relation $\triangleright$ in counterpart structures. That is, if a schema A is valid in all and only the Kripke frames whose accessibility relation satisfies a certain condition, then the schema is valid in all and only the counterpart structures whose image relation (on w, g pairs) satisfies this condition.

It is, in fact, easy to show that the schema is valid in *all* those counterpart structures. But I'm not able to show that it is valid in *only* those structures.

The problem is that any $\mathcal{L}$-instance of a propositional schema A only contains finitely many free variables. A formula of $\mathcal{L}$ is true or false only relative to a world w and a *finite fragment* of an assignment g on U_w – a fragment that interprets the free variables in the formula.

[TODO: Give either a counterexample to correspondence transfer for $\triangleright$ or show that it holds.]

On page 8, I mentioned that one could reformulate the definition of satisfaction from section 2 in terms of infinite sequences of individuals in place of the assignment functions.

To account for the fact that $\mathcal{L}$-formulas really only constrain a finite initial segment of such sequences, I will now reformulate the semantics in terms of finite sequences.

Definition 6.6 (Rank)
Let ρ be some fixed "alphabetical" order on the variables of $\mathcal{L}$, i.e. a bijection $\mathit{Var} \to \mathbb{N}^+$. I will use v to denote the inverse of ρ, so that v_1 is the alphabetically first variable, v_2 the second, and so on. The *rank* of an $\mathcal{L}$-formula A is the smallest number $r \in \mathbb{N}$ such that all members of $\mathit{Var}(A)$ have a ρ-value less than r.

Definition 6.7 (n-sequential accessibility)
Given a counterpart structure $\mathcal{S} = \langle W, R, U, D, K \rangle$ and number $n \in \mathbb{N}$, the *n-sequential accessibility relation $R^n_{\mathcal{S}}$ of $\mathcal{S}$* is the binary relation such that $\langle w, d_1, \ldots, d_n \rangle R^n_{\mathcal{S}} \langle w', d'_1, \ldots, d'_n \rangle$ iff wRw' and for some $C \in K_{w,w'}$, $d_1 C d'_1, \ldots, d_n C d'_n$.

(Note that $R^0_{\mathcal{S}} = R$.)
[FIXME: I should use consistent terminology. Isn't this an "image" relation? 'Image' isn't a great label.]

Definition 6.8 (Finitary satisfaction)
Let A be an $\mathcal{L}$-formula and r a number greater or equal to A's rank. Let $\mathcal{M} = \langle W, R, U, D, K, I \rangle$ be a counterpart model, w a member of W, and $d_1, \ldots, d_r$ (not necessarily distinct) elements of U_w. Then

$\mathcal{M}, w, d_1, \ldots, d_r \Vdash Px_1 \ldots x_n$	iff $\langle d_{\rho(x_1)}, \ldots, d_{\rho(x_n)} \rangle \in I_w(P)$.
$\mathcal{M}, w, d_1, \ldots, d_r \Vdash \neg A$	iff $\mathcal{M}, w, d_1, \ldots, d_r \nVdash A$.
$\mathcal{M}, w, d_1, \ldots, d_r \Vdash A \supset B$	iff $\mathcal{M}, w, d_1, \ldots, d_r \nVdash A$ or $\mathcal{M}, w, d_1, \ldots, d_r \Vdash B$.
$\mathcal{M}, w, d_1, \ldots, d_r \Vdash \forall x A$	iff $\mathcal{M}, w, d'_1, \ldots, d'_r \Vdash A$ for all $d'_1, \ldots, d'_r$ such that $d'_{\rho(x)} \in D_w$ and $d'_i = d_i$ for all $i \neq \rho(x)$.
$\mathcal{M}, w, d_1, \ldots, d_r \Vdash \langle y : x \rangle A$	iff $\mathcal{M}, w, d'_1, \ldots, d'_r \Vdash A$ for all $d'_1, \ldots, d'_r$ such that $d'_{\rho(x)} = d_{\rho(y)}$ and $d'_i = d_i$ for all $i \neq \rho(x)$.
$\mathcal{M}, w, d_1, \ldots, d_r \Vdash \Box A$	iff $w', d'_1, \ldots, d'_r \Vdash A$ for all $w, d'_1, \ldots, d'_r$ such that $\langle w, d_1, \ldots, d_r \rangle R^n_{\mathcal{S}} \langle w', d'_1, \ldots, d'_n \rangle$,

where $R^n_{\mathcal{S}}$ is the n-sequential counterpart relation of $\mathcal{S}$.

It is obvious that this definition is equivalent to that of section 2, in the sense that $\mathcal{M}, w, d_1, \ldots, d_r \Vdash A$ iff $\mathcal{M}, w, g \Vdash A$ whenever $g(v_i) = d_i$ for $1 \leq i \leq r$.

LEMMA 6.9
For any counterpart model $\mathcal{M} = \langle W, R, U, D, K, I \rangle$, world $w \in W$, assignment g on U_w, and formula A of rank $\leq r$,

$$\mathcal{M}, w, g \Vdash A \text{ iff } \mathcal{M}, w, g(v_1), \ldots, g(v_r) \Vdash A.$$

PROOF by induction on A. FIXME

(i) A is $Px_1 \ldots x_n$. $w, V \Vdash_{\mathcal{S}} Px_1 \ldots x_n$ iff $\langle V_w(x_1), \ldots, V_w(x_n) \rangle \in V_w(P)$ by definition 2.8, iff $\langle d_{\rho(x_1)}, \ldots, d_{\rho(x_n)} \rangle \in I_w(P)$, iff $x, d_1, \ldots, d_r \Vdash_{\mathcal{S},I} Px_1 \ldots x_n$ by definition **??**.

(ii) A is $\neg B$. $w, V \Vdash_{\mathcal{S}} \neg B$ iff $w, V \nVdash_{\mathcal{S}} B$ by definition 2.8, iff $w, d_1, \ldots, d_r \nVdash_{\mathcal{S},I} B$ by induction hypothesis (since B has rank $\leq r$), iff $w, d_1, \ldots, d_r \Vdash_{\mathcal{S},I} \neg B$ by definition **??**.

(iii) A is $B \supset C$. $w, V \Vdash_{\mathcal{S}} B \supset C$ iff $w, V \nVdash_{\mathcal{S}} B$ or $w, V \Vdash_{\mathcal{S}} C$ by definition 2.8, iff $w, d_1, \ldots, d_r \nVdash_{\mathcal{S},I} B$ or $w, d_1, \ldots, d_r \Vdash_{\mathcal{S},I} C$ by induction hypothesis (since B and C have rank $\leq r$), iff $w, d_1, \ldots, d_r \Vdash_{\mathcal{S},I} B \supset C$ by definition **??**.

(iv) A is $\forall x B$. By definition 2.8, $w, V \Vdash_{\mathcal{S}} \forall x B$ iff $w, V' \Vdash_{\mathcal{S}} B$ for all existential x-variants V' of V on w. By definition 2.6, V' is an existential x-variant of V on w iff V and V' agree on all predicates, $V'_w(x) \in D_w$ and $V'_w(y) = V_w(y)$ for all $y \neq x$. Take any such V'. By induction hypothesis (since B has rank $\leq r$), $w, V' \Vdash_{\mathcal{S}} B$ iff $w, V'_w(v_1), \ldots, V'_w(v_r) \Vdash_{\mathcal{S},I} B$. So $w, V \Vdash_{\mathcal{S}} \forall x B$ iff $w, V'_w(v_1), \ldots, V'_w(v_r) \Vdash_{\mathcal{S},I} B$ for all V' such that $V'_w(x) \in D_w$ and $V'_w(y) = V_w(y)$ for all $y \neq x$. In other words, $w, V \Vdash_{\mathcal{S}} \forall x B$ iff $w, d'_1, \ldots, d'_r \Vdash_{\mathcal{S},I} B$ for all $d'_1, \ldots, d'_r$ such that $d'_{\rho(x)} \in D_w$ and $d'_i = d_i$ for all $i \neq \rho(x)$, iff $w, d_1, \ldots, d_r \Vdash_{\mathcal{S},I} \forall x B$ by definition **??**.

(v) A is $\langle y : x \rangle B$. By definition 2.8, $w, V \Vdash_{\mathcal{S}} \langle y : x \rangle B$ iff $w, V' \Vdash_{\mathcal{S}} B$ where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$. By induction hypothesis (since B has rank $\leq r$), $w, V' \Vdash_{\mathcal{S}} B$ iff $w, V'_w(v_1), \ldots, V'_w(v_r) \Vdash_{\mathcal{S},I} B$. So $w, V \Vdash_{\mathcal{S}} \langle y : x \rangle B$ iff $w, d'_1, \ldots, d'_r \Vdash_{\mathcal{S},I} B$ for all $d'_1, \ldots, d'_r$ such that $d'_{\rho(x)} = d_{\rho(y)}$ and $d'_i = d_i$ for all $i \neq \rho(x)$, iff $w, d_1, \ldots, d_r \Vdash_{\mathcal{S},I} \langle y : x \rangle B$ by definition **??**.

(vi) A is $\Box B$. By definition 2.8, $w, V \Vdash_{\mathcal{S}} \Box B$ iff $w', V' \Vdash_{\mathcal{S}} B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. By definition 2.7 and totality of $\mathcal{S}$ and V, the latter holds iff V' and V agree on all predicates and for some $C \in K_{w,w'}$ and all variables x, $V_w(x) C V'_{w'}(x)$. Take any such w', V'. By induction hypothesis (since B has rank $\leq r$), $w', V' \Vdash_{\mathcal{S}} B$ iff $w', V'_w(v_1), \ldots, V'_w(v_r) \Vdash_{\mathcal{S},I} B$. So $w, V \Vdash_{\mathcal{S}} \Box B$ iff $w', V'_w(v_1), \ldots, V'_w(v_r) \Vdash_{\mathcal{S},I} B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$, iff $w', d'_1, \ldots, d'_r \Vdash_{\mathcal{S},I} B$ for all $w', d'_1, \ldots, d'_r$ such that $\langle w, d_1, \ldots, d_r \rangle R^n_{\mathcal{S}} \langle w', d'_1, \ldots, d'_n \rangle$ by definition 6.7, iff $w, d_1, \ldots, d_r \Vdash_{\mathcal{S},I} \Box B$ by definition **??**. ■

Now recall that $\mathcal{L}$-formulas can be regarded as formulas of propositional modal logic, with strangely complicated "sentence letters". In a similar way, counterpart models for QML can be regarded as disguised Kripke models for propositional modal logic. The "worlds" in these Kripke models are finite sequences of worlds and individuals from the counterpart model.

Definition 6.10 (N-ary Opaque Propositional Guise)
The *n-ary opaque propositional guise of a counterpart structure* $\mathcal{S} = \langle W, R, U, D, K \rangle$ is the Kripke frame $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle$ where $W^n_\mathcal{S}$ is the set of points $\langle w, d_1, \ldots, d_n \rangle$ such that $w \in W, d_1 \in U_w, \ldots, d_n \in U_w$, and $R^n_\mathcal{S}$ is the n-ary accessibility relation for $\mathcal{S}$.

The *n-ary opaque propositional guise of a predicate interpretation* I on $\mathcal{S}$ is the valuation function $V^n_\mathcal{S}$ on $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle$ such that for every quasi-atomic formula $A \in \mathcal{L}$, $V^n_\mathcal{S}(A) = \{\langle w, d_1, \ldots, d_n \rangle : w, d_1, \ldots, d_n \Vdash A\}$.

Lemma 6.11 (Truth-preservation under opaque guises)
For any positive counterpart model $\mathcal{M} = \langle W, R, U, D, K, I \rangle$, world $w \in W$, individuals $d_1, \ldots, d_n \in U_w$, and $\mathcal{L}$-formula A with rank $\leq n$,

$$\mathcal{M}, w, d_1, \ldots, d_n \Vdash A \text{ iff } \mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_0 A,$$

where $\mathcal{S}^n$ and V^n are the n-ary opaque propositional guises of $\mathcal{S}$ and I respectively.

Proof by induction on A, where quasi-atomic formulas all have complexity zero. FIXME:CHECK

(i) A is quasi-atomic. By definition 6.16, $V^n(A) = \{\langle w, d_1, \ldots, d_n \rangle : w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} A\}$. So $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} A$ iff $\langle w, d_1, \ldots, d_n \rangle \in V^n(A)$, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_0 A$ by definition 6.3.

(ii) A is $\neg B$. $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} \neg B$ iff $w, d_1, \ldots, d_n \nVdash_{\mathcal{S},I} B$ by definition **??**, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \nVdash_0 B$ by induction hypothesis, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_0 \neg B$ by definition 6.3.

(iii) A is $B \supset C$. $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} B \supset C$ iff $w, d_1, \ldots, d_n \nVdash_{\mathcal{S},I} B$ or $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} C$ by definition **??**, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \nVdash_0 B$ or $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_0 C$ by induction hypothesis, iff $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \Vdash_0 B \supset C$ by definition 6.3.

(iv) A is $\Box B$. $w, d_1, \ldots, d_n \Vdash_{\mathcal{S},I} \Box B$ iff $w', d'_1, \ldots, d'_n \Vdash_{\mathcal{S},I} B$ for all $w, d'_1, \ldots, d'_r$ such that $\langle w, d_1, \ldots, d_r \rangle R^n_\mathcal{S} \langle w', d'_1, \ldots, d'_n \rangle$ by definition **??**, iff $\mathcal{S}^*, V^*, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_0 B$ for all such $w, d'_1, \ldots, d'_r$ by induction hypothesis, iff $\mathcal{S}^*, V^*, \langle w, d_1, \ldots, d_n \rangle \Vdash_0 \Box B$ by definition 6.3. ■

Lemma 6.12 (Finite correspondence transfer)
If A is a formula of (unimodal) propositional modal logic that is valid in all and only the Kripke frames in some class F, and $n \in \mathbb{N}$, then the set of $\mathcal{L}$-formulas that result from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas of rank $\leq n$

is valid in all and only the total counterpart structures $\mathcal{S} = \langle W, R, U, D, K \rangle$ whose n-ary opaque propositional guise is in F.

PROOF FIXME:CHECK

Assume A is valid in all and only the Kripke frames in F, and let $p_1, \ldots, p_k$ be the sentence letters in A. Let $\mathcal{S} = \langle W, R, U, D, K \rangle$ be a total counterpart structure whose n-ary opaque propositional guise $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle$ is in F. Suppose for reductio that some formula A' is not (positively) valid in $\mathcal{S}$ that results from A by uniformly substituting the sentence letters p_i in A by $\mathcal{L}$-formulas $p_i^\mathcal{L}$ of rank $\leq n$. Then there is an interpretation V on $\mathcal{S}$ and a world $w \in W$ such that $w, V \nVdash_\mathcal{S} A'$. By lemma 6.9, this means that $w, d_1, \ldots, d_r \nVdash_{\mathcal{S},I} A'$, where I is V restricted to predicates and $d_1 = V_w(v_1), \ldots, d_r = V_w(v_r)$. By lemma 6.11, it follows that $\mathcal{S}^n, V^n, \langle w, d_1, \ldots, d_n \rangle \nVdash_0 A'$. But then $\mathcal{S}^n, V^{n\prime}, \langle w, d_1, \ldots, d_n \rangle \nVdash_0 A$, where $V^{n\prime}$ is such that for all sentence letters p_i in A, $V^{n\prime}(p_i) = V^n(p_i^\mathcal{L})$. This contradicts the assumption that A is valid in $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle$.

We also have to show that the relevant $\mathcal{L}$-formulas are valid *only* in structures $\mathcal{S}$ whose n-ary opaque propositional guise is in F. So let $\mathcal{S}$ be a structure whose guise $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle$ is not in F. Since A is valid only in frames in F, we know that there is some valuation V on $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle$ and some $\langle w, d_1, \ldots, d_n \rangle \in W^n_\mathcal{S}$ such that $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w, d_1, \ldots, d_n \rangle \nVdash_0 A$. Let A' result from A by uniformly substituting each sentence letter p_i in A by an n-ary predicate P_i followed by the variables $v_1 \ldots v_n$, with distinct predicates for distinct sentence letters. Let I be a predicate interpretation such that for all P_i and $w' \in W$, $I_{w'}(P_i) = \{\langle d'_1, \ldots, d'_n \rangle : \langle w', d'_1, \ldots, d'_n \rangle \in V(p_i)\}$. A simple induction on subformulas B of A shows that for all $\langle w', d'_1, \ldots, d'_n \rangle \in W^n_\mathcal{S}$, $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_0 B$ iff $w', d'_1, \ldots, d'_n \Vdash B'$, where B' is B with all p_i replaced by $P_i v_1 \ldots v_n$. Given that $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w, d_1, \ldots, d_n \rangle \nVdash_0 A$ it follows that $w, d_1, \ldots, d_n \nVdash_{\mathcal{S},I} A'$.

(i) B is a sentence letter p_i. Then B' is $P_i v_1 \ldots v_n$. $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_0 p_i$ iff $\langle w', d'_1, \ldots, d'_n \rangle \in V(p_i)$ by definition 6.3, iff $\langle d'_1, \ldots, d'_n \rangle \in I_w(P_i)$ by construction of I, iff $w', d'_1, \ldots, d'_n \Vdash_{\mathcal{S},I} P_i v_1 \ldots v_n$ by definition **??**.

(ii) B is $\neg C$. Then B' is $\neg C'$, where C' is C with all p_i replaced by $P_i v_1 \ldots v_n$. $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_0 \neg C$ iff $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \nVdash_0 C$ by definition 6.3, iff $w', d'_1, \ldots, d'_n \nVdash_{\mathcal{S},I} C'$ by induction hypothesis, iff $w', d'_1, \ldots, d'_n \Vdash_{\mathcal{S},I} \neg C'$ by definition **??**.

(iii) B is $C \supset D$. Then B' is $C' \supset D'$, where C' and D' are C and D respectively with all p_i replaced by $P_i v_1 \ldots v_n$. $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_0 C \supset D$ iff $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \nVdash_0 C$ or $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_0 D$ by definition 6.3, iff $w', d'_1, \ldots, d'_n \nVdash_{\mathcal{S},I} C'$ or $w', d'_1, \ldots, d'_n \Vdash_{\mathcal{S},I} D'$ by induction hypothesis, iff $w', d'_1, \ldots, d'_n \Vdash_{\mathcal{S},I} C' \supset D'$ by definition **??**.

(iv) B is $\Box C$. Then B' is $\Box C'$, where C' is C with all p_i replaced by $P_i v_1 \ldots v_n$. $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w', d'_1, \ldots, d'_n \rangle \Vdash_0 \Box C$ iff $\langle W^n_\mathcal{S}, R^n_\mathcal{S} \rangle, V, \langle w'', d''_1, \ldots, d''_n \rangle \Vdash_0 C$ for all $\langle w'', d''_1, \ldots, d''_n \rangle$ with $\langle w', d'_1, \ldots, d'_n \rangle R^n_\mathcal{S} \langle w'', d''_1, \ldots, d''_n \rangle$ by definition 6.3, iff $w'', d''_1, \ldots, d''_n \nVdash_{\mathcal{S},I} C'$ for all such $\langle w'', d''_1, \ldots, d''_n \rangle$ by induction hypothesis, iff $w', d'_1, \ldots, d'_n \Vdash_{\mathcal{S},I} \Box C'$ by definition **??**. ■

Given that a modal schema restricted to the variables $v_1, \ldots, v_n$ defines a constraint on the n-sequential accessibility relation of a counterpart structure $\mathcal{S}$, the unrestricted schema defines a constraint on all sequential accessibility relations. Let's fold these into a single entity.

Definition 6.13 (Sequential accessibility relation)
The *sequential accessibility relation* $R^*_{\mathcal{S}}$ of a total counterpart structure $\mathcal{S}$ is the union of the n-sequential accessibility relations $R^n_{\mathcal{S}}$ of $\mathcal{S}$, i.e. $R^*_{\mathcal{S}} = \bigcup_{n \in \mathbb{N}} R^n_{\mathcal{S}}$.

Definition 6.14 (Opaque Propositional Guise)
The *total opaque propositional guise of a counterpart structure* $\mathcal{S} = \langle W, R, U, D, K \rangle$ is the disjoint union of the n-ary opaque propositional guises of $\mathcal{S}$, i.e. the Kripke frame $\langle W^*_{\mathcal{S}}, R^*_{\mathcal{S}} \rangle$ such that $R^*_{\mathcal{S}}$ is the sequential accessibility relation of $\mathcal{S}$ and $W^*_{\mathcal{S}}$ is the set of points w^* such that for some $n \in \mathbb{N}$, world $w \in W$ and individuals $d_1, \ldots, d_n \in U_w$, $w^* = \langle w, d_1, \ldots, d_n \rangle$.

Theorem 6.15 ((Positive) correspondence transfer)
If A is a formula of (unimodal) propositional modal logic that is valid in all and only the Kripke frames in some class F, then the set of $\mathcal{L}$-formulas that result from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas is positively valid in all and only the total counterpart structures $\mathcal{S}$ whose opaque propositional guise is in F.

Proof FIXME:CHECK Since validity in propositional modal logic is preserved under disjoint unions, A is valid in the opaque propositional guise of a structure $\mathcal{S}$ iff A is valid in each n-ary opaque propositional guise of $\mathcal{S}$, with $n \in \mathbb{N}$. (See e.g. [?], p.140, Theorem 3.14.(i).) Thus the opaque propositional guise of $\mathcal{S}$ is in F iff all n-ary opaque propositional guises of $\mathcal{S}$ are in F.

Assume A is valid in all and only the Kripke frames in F. Let A' be an $\mathcal{L}$-formula that results from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas. By lemma 6.12, A' is (positively) valid in all total counterpart structures $\mathcal{S}$ whose n-ary propositional guise is in F, where n is the rank of A'. Any total structure whose propositional guise is in F satisfies this condition.

To show that the A schema is valid *only* in structures $\mathcal{S}$ whose guise is in F, let $\mathcal{S}$ be a structure whose guise is not in F. Then there is some n such that the n-ary guise of $\mathcal{S}$ is not in F. By lemma 6.12, there is an $\mathcal{L}$-substitution instance A' of A with rank n that is not valid in $\mathcal{S}$. ■

As a union of relations of different arity, R^* is a somewhat gerrymandered entity. It may help to understand statements about R^* as universal statements about its members

R^n. For example, the schema $\Box A \supset A$ is valid iff (0) every world can see itself and (1) every individual at every world is its own counterpart (relative to some counterpart relation), (2) every pair of individuals at every world is its own counterpart, and so on. Each clause (i) covers instances of the schema with i free variables.

Thus far, I have set aside negative counterpart models. In negative models, variables can be undefined, so sequential accessibility relations must be redefined to hold between pairs of a world w and a possibly *gappy* sequence of individuals from D_w, i.e. a partial function from numbers to members of D_w. Now we could re-run the above arguments, but we can also cut all this short by using lemma 2.12.

Definition 6.16 (Opaque Propositional Guise of Negative Models)
The *opaque propositional guise of a negative counterpart model* $\mathcal{M}$ is the opaque propositional guise of its positive transpose $\mathcal{M}^+$. Accordingly, the *sequential accessibility relation* $R^*_{\mathcal{M}}$ *of* $\mathcal{M}$ is the sequential accessibility relation of $\mathcal{M}^+$.

Corollary 6.17 (Negative correspondence transfer)
If A is a formula of (unimodal) propositional modal logic that is valid in all and only the Kripke frames in some class F, then the set of $\mathcal{L}$-formulas that result from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas is negatively valid in all and only the single-domain counterpart structures $\mathcal{S}$ whose opaque propositional guise is in F.

Proof FIXME Assume A is valid in all and only the Kripke frames in F. Let A' be an $\mathcal{L}$-formula that results from A by uniformly substituting sentence letters by $\mathcal{L}$-formulas. Let $\mathcal{S}$ be a single-domain counterpart structure whose guise is in F, and let w, V be a world from $\mathcal{S}$ and a partial interpretation on $\mathcal{S}$. By lemma 2.12, $w, V \Vdash_{\mathcal{S}} A'$ iff $w, V^+ \Vdash_{\mathcal{S}^+} A'$. By theorem 6.15, $w, V^+ \Vdash_{\mathcal{S}^+} A'$. So A' is negatively valid in $\mathcal{S}$.

Now let $\mathcal{S}$ be a structure whose guise is not in F. By theorem 6.15, there is a substitution instance A' of A, a world w and a total interpretation V' on $\mathcal{S}^+$ such that $w, V' \nVdash_{\mathcal{S}^+} A'$. Define V so that V and V' agree on all predicates and for all worlds w' and variables x, $V_{w'}(x)$ is $V'_{w'}(x)$ if $V'_{w'}(x) \in D_w$, otherwise $V_{w'}(x)$ is undefined. Then V' is the positive transpose of V. By lemma 2.12, $w, V \Vdash_{\mathcal{S}} A'$ iff $w, V' \Vdash_{\mathcal{S}^+} A'$. So $w, V \nVdash_{\mathcal{S}} A'$. So A' is not negatively valid in $\mathcal{S}$. ∎

There are of course further aspects of structures that can only be captured in quantified modal logic.

Definition 6.18 (Types of structures)
A structure $\mathcal{S} = \langle W, R, U, D, K \rangle$ is

total if every individual at every world has at least one counterpart at every accessible world: whenever wRw' and $d \in U_w$ then there is a $d' \in U_{w'}$ with $\langle d, w \rangle C \langle d', w' \rangle$;

functional if every individual at every world has at most one counterpart at every accessible world;

inversely functional if no two individuals at any world have a common counterpart at some accessible world;

injective if it is both functional and inversely functional.

[TODO: define a correspondence language and show some general correspondence results with it?]

[TODO: Show some correspondence results for quantified schemas. E.g. BF, CBF, NE, etc.]

7 Canonical models

We can use the canonical model technique to prove (strong) completeness. Let me begin with an informal exposition of the key ideas.

From now on, let $\mathcal{L}$ be a language with or without substitution. Recall that a "logic" is a certain kind of set of sentences. For any logic L, '$\vdash_L A$' means $A \in L$. For any set Γ of sentences, we read '$\Gamma \vdash_L A$' as saying that there are 0 or more sentences $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_L B_1 \land \ldots \land B_n \supset A$. (For $n = 0$, $B_1 \land \ldots \land B_n \supset A$ is A.) We say that Γ is *L-consistent* iff there are no members $A_1, \ldots, A_n$ of Γ such that $\vdash_L \neg(A_1 \land \ldots \land A_n)$.

A logic L is *weakly complete* with respect to a class of models $\mathbb{M}$ iff L contains every formula valid in $\mathbb{M}$: whenever A is valid in $\mathbb{M}$, then $\vdash_L A$. Equivalently, every formula $A \notin L$ is false at some world in some model in $\mathbb{M}$. L is *strongly complete* with respect to $\mathbb{M}$ iff whenever A is a semantic consequence of a set of formulas Γ in $\mathbb{M}$, then $\Gamma \vdash_L A$. Since $\Gamma \nvdash_L A$ iff $\Gamma \cup \{\neg A\}$ is L-consistent, and A is a semantic consequence of Γ in $\mathbb{M}$ iff no world in any model in $\mathbb{M}$ verifies all members of $\Gamma \cup \{\neg A\}$, this means that L is strongly complete with respect to $\mathbb{M}$ iff for every L-consistent set of formulas Γ there is a world in some model in $\mathbb{M}$ at which all members of Γ are true. xxx add assignment-relativity

To prove strong completeness, we can associate with each logic L a *canonical model* $\mathcal{M}_L$. The worlds of $\mathcal{M}_L$ are construed as maximal L-consistent sets of formulas. We show that for each such world w there is an assignment g such that a formula A is true at w under g iff $A \in w$. Since every L-consistent set of formulas can be extended to a maximal L-consistent set, it follows that every L-consistent set of formulas is verified at some world in $\mathcal{M}_L$, relative to some assignment. It follows that L is strongly complete with respect to every model class that contains $\mathcal{M}_L$.

How do we ensure that $\mathcal{M}_L, w, g \models A$ iff $A \in w$, for any formula A? The usual approach is to stipulate that for any variable x, $g(x)$ is the class of variables $[x]_w = z : x{=}z \in w$, and that the model's interpretation function I_w assigns to each predicate P at w the set of n-tuples $\langle [x_1]_w, \ldots, [x_n] \rangle$ such that $Px_1 \ldots x_n \in w$.

A well-known problem now arises from the fact that first-order logic does not require every individual to have a name. This means that there are consistent sets Γ that contain $\exists x Fx$ as well as $\neg Fx_i$ for every variable x_i. If we extend Γ to a maximal consistent set w and apply the construction just outlined, then $I_w(F)$ would be the empty set. We would have $\mathcal{M}, w, g \Vdash \neg\exists x Fx$, although $\exists x Fx \in w$. To avoid this, we require that the worlds in a canonical model are all *witnessed*, so that whenever an existential formula $\exists x Fx$ is in w, then some witnessing instance Fy is in w as well. We still want the set Γ to be verified at some world. So the worlds are construed in a larger language $\mathcal{L}^*$ that adds infinitely many new variables to the original language $\mathcal{L}$. The new variables may then serve as witnesses. (In the new language, there are again consistent sets of sentences that are not included in any world, but not so in the old language.)

In modal logic, the problem of witnesses reappears in another form. If we define the worlds of the canonical model as maximally consistent, witnessed sets of $\mathcal{L}^*$-sentences, then a world w might contain $\Diamond\exists x Fx$ but also $\Box\neg Fx_i$ for every $\mathcal{L}^*$-variable x_i. For all these sentences to be true at w relative to the canonical assignment g, Kripke semantics requires that there is a world w' accessible from w that verifies all instances of $\neg Fx_i$ as well as $\exists x Fx$. But then w' isn't witnessed!

In counterpart semantics, the truth of $\Box\neg Fx_i$ at w only requires that $\neg Fx_i$ is true at w' under all w'-*images* g' of g at w – i.e. under assignments g' such that $g'(x_i)$ is some counterpart of $g(x_i) = [x_1]_w$. Suppose, for example, that $[x_1]_w = \{x_1\}$ and each individual $[x_i]_w$ at w has $[x_{i+1}]_{w'}$ as unique counterpart at w'. Then the truth of $\Box\neg Fx_1, \Box\neg Fx_2$, etc. at w only requires that $\neg Fx_1, \neg Fx_2$, etc. are true at w' under an assignment of $[x_2]_{w'}$ to x_1, $[x_3]_{w'}$ to x_2, etc. So $Fx_2, Fx_3, \ldots \in w'$, but the variable x_1 becomes available to serve as a witness for $\Diamond\exists x Fx$.

How, in general, should we define the counterpart relation in our canonical models? Kripke semantics effectively settles the counterpart relation from the outside: if $[x]_w = \{x, y, \ldots\}$, then $[x']_{w'}$ is a counterpart of $[x]_w$ iff $[x']_{w'} = \{x, y, \ldots\}$. To allow for contingent identity, we could relax this clause and say that $[x]_w$ has $[x']_{w'}$ as counterpart iff there is some z that occurs both in $[x]_w$ and $[x']_{w'}$. To get around the problem of modal witnessing, we could fix a different counterpart relation. For example, we could pick an arbitrary transformation τ whose range excludes infinitely many variables and say that $[x]$ always has $[x^\sigma]$ as counterpart (i.e. $[x]_w$ has $[x']_{w'}$ as counterpart iff there is a $z \in [x]_w$

with $z^\sigma \in [x']_{w'}$). With this approach, one can indeed prove completeness for all four basic logics. But we run into problems when we look at stronger logics. For example, it is easy to see that $\mathsf{P}+\mathsf{T}$ is valid in a structure iff every world can see itself and all sequences of individuals are their own counterparts. In order to prove structure completeness for $\mathsf{P}+\mathsf{T}$, we want the canonical model of $\mathsf{P}+\mathsf{T}$ to be "reflexive", in this sense. But it won't be. (Let Γ contain $x_1 \neq x_1^\tau$ as well as all $\mathcal{L}$-instances of $\Box A \supset A$. Γ is $\mathsf{P}+\mathsf{T}$-consistent. So it is part of a world w in the canonical model. If the model is reflexive, then for all w, wRw and for all d, $\langle d, w\rangle C\langle d, w\rangle$. The second condition requires that $[x_1]_w C[x_1]_w$, i.e. there is some $z \in [x_1]_w$ with $z^\tau \in [x_1]_w$. Obviously there are maximally consistent extensions of Γ that contain no identity $x_1 = z^\tau$.)

It is better not to define canonical counterparthood in a fixed, external manner. Compare accessibility: in standard propositional logic, whether w' is accessible from w depends on whether it verifies all formulas A for which w contains $\Box A$. By analogy, we should say that whether $[x']_{w'}$ is a counterpart of $[x]_w$ is determined by whether $[x']_{w'}$ satisfies the modal profile attributed to $[x]_w$ in w. The above proposal ensured that if $\Box A(x) \in w$, then $A(x^\tau) \in w'$ for accessible w', so that w' verifies that the counterpart $[x^\tau]_{w'}$ of $[x]_w$ satisfies condition A. But this doesn't tell us that *everything* that satisfies $A(x)$ for all $\Box A(x) \in w$ qualifies as counterpart of $[x]_w$. If we had this, it would be easy to show that the CM of $\mathsf{P}+\mathsf{T}$ is reflexive: since every w contains $\Box A(x) \supset A(x)$, $[x]_w$ at w must be a counterpart of itself at w.

So we need to define counterparthood in such a way that we can read off whether $\langle [x]_w, w\rangle C\langle [y]_{w'}, w'\rangle$ by comparing what w says about the boxed properties of x and what w' says about y.

Take a concrete example. Suppose w and w' look as follows.

$$w : \{x \neq y, \Box x \neq y, \Box Fx, \Box Fy, \ldots\}$$
$$w' : \{\neg Fx, Fu, Fv, u \neq y, \ldots\}$$

We can tell that $[x]_{w'}$ does not qualify as counterpart of $[x]_w$, since it doesn't satisfy the "modal profile" that w attributes to $[x]_w$: w contains $\Box Fx$, so all counterparts of $[x]_w$ should satisfy F. Both $[u]_{w'}$ and $[v]_{w'}$ meet this condition. So we might say that both of them are counterparts of $[x]_w$. But then they should also be counterparts of $[y]_w$, and we get a violation of the "joint modal profile" expressed by $\Box x \neq y$, which requires that no counterpart of $[x]_w$ is identical to any counterpart of $[y]_w$. Structurally, this is the "problem of internal relations" noted in [**?**]. In response, we assume that there can be multiple counterpart relations linking the individuals at w to those at w'. One relation links $[x]_w$ with $[u]_{w'}$ and $[y]_w$ with $[v]_w$, another $[x]_w$ with $[v]_{w'}$ and $[y]_w$ with $[u]_{w'}$. So $[x]_w$ does have both $[u]_{w'}$ and $[v]_{w'}$ as counterpart, but the pair $\langle [x]_w, [y]_w\rangle$ has only two rather than four counterparts.

It proves convenient to impose a further restriction on canonical counterpart relations. We will assume that every counterpart relation C in a canonical model corresponds to a transformation τ so that $[x]_w$ at w has $[x^\tau]_{w'}$ at w' as counterpart (unless $[x^\tau]_{w'}$ is empty – see below). This means that if an individual $[x]_w$ at w has two counterparts at w' under the same counterpart relation, then there must be at least two variables x, y in $[x]_w$, so that $[x^\tau]_{w'}$ and $[y^\tau]_{w'}$ can serve as the two counterparts.

This is, to a first approximation, how we will construct the accessibility and counterpart relations in canonical model.

Let's say that w' is *accessible from w via* a transformation τ, for short: $w \xrightarrow{\tau} w'$, iff w' contains A^τ whenever w contains $\Box A$. Define wRw' to be true iff $w \xrightarrow{\tau} w'$ for some τ, and let C be a counterpart relation between w and w' iff $C = \{\langle [x]_w, [y]_{w'} \rangle : x^\tau = y\}$ for some τ such that $w \xrightarrow{\tau} w'$.

If $w \xrightarrow{\tau} w'$, then $[x]_w$ has $[x^\tau]_{w'}$ as counterpart, and g^τ is a w'-image of g at w. But it might not be the only w'-image of g at w – and not only because there can be several τ with $w \xrightarrow{\tau} w'$. For example, assume w contains $x = y$ but not $\Box x = y$. Then there is some world w' and transformation τ such that w' contains $x^\tau \neq y^\tau$. That is, the individual $[x]_w = [y]_w = \{x, y, \ldots\}$ at w has two τ-induced counterparts at w', $[x^\tau]_{w'}$ and $[y^\tau]_{w'}$, which g^τ assigns to x and y, respectively. But then there will also be another w'-image of g at w which assigns, for example, $[y^\tau]_{w'}$ to both x and y.

Here is a case where this leads to trouble. Assume we are working with a positive logic without explicit substitution. As before, w contains $x = y$ but not $\Box x = y$, so that for some $w \xrightarrow{\tau} w'$, w' contains $x^\tau \neq y^\tau$. Assume further that w contains $\Box\Diamond x \neq y$. Then w' contains $\Diamond x^\tau \neq y^\tau$. To verify $\Box\Diamond x \neq y$ at w, we need to ensure that $w', g' \Vdash \Diamond x \neq y$ for all w'-images g' of V at w, not just for g^τ. Consider the image g' that assigns $[y^\tau]_{w'}$ to both x and y. To ensure that $w', g' \Vdash \Diamond x \neq y$, there must be some $w' \xrightarrow{\sigma} w''$ and $w', g' \triangleright w'', V''$ with $w'', g'' \Vdash x \neq y$. Here, $w', g' \triangleright w'', g''$ means that there is a σ such that $w' \xrightarrow{\sigma} w''$ and $g'(x) C^\sigma g''(x)$ for all x, where C^σ is the counterpart relation induced by σ. In other words, we need a transformation σ, world w'' and assignment g'' such that $w'', g'' \Vdash x \neq y$, where $w' \xrightarrow{\sigma} w''$ and g'' is such that for all x there is a $z \in g'(x)$ with $z^\sigma \in g''(x)$. Since $g'(x) = g'(y) = [y^\tau]_{w'}$, this means that $[y^\tau]_{w'}$ must have two counterparts at some w'' relative to the same transformation σ. So far, we have no guarantee that this is the case. There has to be a variable z other than y^τ such that w' contains $z = y^\tau$ as well as $\Diamond z \neq y^\tau$. The latter ensures that $z^\sigma \neq (y^\tau)^\sigma \in w''$ for some $w \xrightarrow{\sigma} w''$; $[z^\sigma]_{w''}$ and $[(y^\tau)^\sigma]_{w''}$ are then both counterparts at w'' of $[y^\tau]_{w'}$.

Hence we complicate the definition of $w \xrightarrow{\tau} w'$. We stipulate that if w' does not contain $z = y^\tau$ and $\Diamond z \neq y^\tau$ for some suitable z, then w' is not τ-accessible from w. In general, if w contains $\Box A$ as well as $x = y$, and x is free in A, then for w' to be accessible from w via τ, we require that it must contain not only A^τ, but also $z = y^\tau$ and $[z/x^\tau]A^\tau$, for some z not free in A^τ.

This requirement might be easier to understand if we consider the same situation in a language with substitution. Here $\Box\Diamond x \neq y$ and $x = y$ entail $\Box\langle y : x\rangle\Diamond x \neq y$ (by (LL_s) and ($S\Box$)). By the original, simple definition of $w \xrightarrow{\tau} w'$, each world w' accessible from w via τ must contain $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$. This formula says that $[y^\tau]_{w'}$ has multiple counterparts at some accessible world w''. Before we worry about images other than g^τ, we ought to make sure that $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$ is true at w' under g^τ. This requires that there is a variable z other than y^τ such that w' contains $z = y^\tau$ and $\Diamond z \neq y^\tau$. In effect, z is a kind of witness for the substitution formula $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$. Just as an existential formula $\exists xA$ must be witnessed by an instance $[z/x]A$, a substitution formula $\langle y : x\rangle A$ must be witnessed by $[z/x]A$ together with $z = y$. Loosely speaking, $\langle y : x\rangle A(x)$ says that y is identical to some x such that $A(x)$. In a canonical model, we want a concrete witness z so that y is identical to z and $A(z)$. y itself may not serve that purpose, because $\langle y : x\rangle A(x)$ does not guarantee $A(y)$.

The requirement of substitutional witnessing entails that if w contains $\Box A$, then any τ-accessible w' contains not only A^τ, but also $z = y^\tau$ and $[z/x^\tau]A^\tau$ (for some suitable z). So we don't need to complicate the accessibility relation. In our example, since w' contains A^τ whenever w contains $\Box A$, w' contains $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$, which settles that $[y^\tau]_{w'}$ has two counterparts at some accessible world. Without substitution, $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$ is inexpressible (see lemma 5.3). So we have to limit the accessible worlds by requiring membership of the relevant witnessing formulas in addition to A^τ.

On to the details. Let $\mathcal{L}$ be some language with or without substitution and L a positive or strongly negative quantified modal logic in $\mathcal{L}$. Define the extended language $\mathcal{L}^*$ by adding infinitely many new variables Var^+ to $\mathcal{L}$.

Definition 7.1 (Henkin set)
A *Henkin set* for L is a set H of $\mathcal{L}^*$-formulas that is

1. *L-consistent*: there are no $A_1, \ldots, A_n \in H$ with $\vdash_{L(\mathcal{L}^*)} \neg(A_1 \wedge \ldots \wedge A_n)$,
2. *maximal*: for every $\mathcal{L}^*$-formula A, H contains either A or $\neg A$,
3. *witnessed*: whenever H contains an existential formula $\exists xA$, then there is a variable $y \notin Var(A)$ such that H contains $[y/x]A$ as well as Ey, and
4. *substitutionally witnessed*: whenever H contains a substitution formula $\langle y : x\rangle A$ as well as $y = y$, then there is a variable $z \notin Var(\langle y : x\rangle A)$ such that H contains $y = z$.

I write $\mathcal{H}_L$ for the class of Henkin sets for L in $\mathcal{L}^*$.

If L is without substitution, the fourth clause is trivial.

Above I said that witnessing a substitution formula $\langle y : x\rangle A$ requires $y = z$ as well as $[z/x]A$, but in fact $y = z$ is enough, since $[z/x]A$ follows from $\langle y : x\rangle A$ and $y = z$ by (LV2) (lemma 5.16). I have also added the condition that H contains $y=y$. In negative logics, a Henkin set may contain $y \neq y$ as well as $\langle y : x \rangle A$; adding $y = z$ would render the set inconsistent.

The requirement of substitutional witnessing generalises to substitution sequences: if H contains a substitution formula $\langle y_1, \ldots, y_n : x_1, \ldots, x_n \rangle A$ as well as $y_i = y_i$ for all y_i in $y_1, \ldots, y_n$, then there are (distinct) new variables $z_1, \ldots, z_n$ such that H contains $y_1 = z_1, \ldots, y_n = z_n$ as well as $[z_1, \ldots, z_n / x_1, \ldots, x_n]A$. This is easily proved by induction on n. Suppose H contains $\langle y_1, \ldots, y_n : x_1, \ldots, x_n \rangle A$. By definition **??**, this is $\langle y_n : v \rangle \langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1} \rangle \langle v : x_n \rangle A$, where v is new. Witnessing requires $y_n = z_n \in H$ and (hence) $[z_n/v]\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1} \rangle \langle v : x_n \rangle A = \langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1} \rangle \langle z_n : x_n \rangle A \in H$ for some new z_n. By induction hypothesis, the latter means that there are (distinct) $z_1, \ldots, z_{n-1} \notin \mathit{Var}(\langle z_n : x_n \rangle A)$ such that H contains $y_1 = z_1, \ldots, y_{n-1} = z_{n-1}$ as well as $[z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]\langle z_n : x_n \rangle A$. Since all the x_i and z_i are pairwise distinct, $[z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]\langle z_n : x_n \rangle A$ is $\langle z_n : x_n \rangle [z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]A$. By (SC1), it follows that $[z_n/x_n][z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]A = [z_1, \ldots, z_n/x_1, \ldots, x_n]A \in H$.

Definition 7.2 (Variable classes)
For any Henkin set H, define $\sim_H$ to be the binary relation on the variables of $\mathcal{L}^*$ such that $x \sim_H y$ iff $x = y \in H$. For any variable x, let $[x]_H$ be $\{y : x \sim_H y\}$.

Lemma 7.3 ($\sim$-Lemma)
$\sim_H$ is transitive and symmetrical.

Proof Immediate from lemmas 4.10 and 5.15. ■

Definition 7.4 (Accessibility via transformations)
Let w, w' be Henkin sets and τ a transformation.

If $\mathcal{L}$ is with substitution, then w' *is accessible from w via τ*, for short: $w \xrightarrow{\tau} w'$, iff for every $\mathcal{L}$-formula A, if $\Box A \in w$, then $A^\tau \in w'$.

If $\mathcal{L}$ is without substitution, then $w \xrightarrow{\tau} w'$ iff for every $\mathcal{L}$-formula A and variables $x_1 \ldots x_n$, $y_1, \ldots, y_n$ $(n \geq 0)$ such that the $x_1 \ldots x_n$ are pairwise distinct members of $\mathit{Varf}(A)$, if $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box A \in w$ and $y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \in w'$, then there are variables $z_1 \ldots z_n \notin \mathit{Var}(A^\tau)$ such that $z_1 = y_1^\tau \wedge \ldots \wedge z_n = y_n^\tau \wedge [z_1 \ldots z_n / x_1^\tau \ldots x_n^\tau]A^\tau \in w'$.

This generalises the witnessing requirements on accessible world as explained above to multiple variables and negative logics. (In this case, the generalised version for n variable pairs is not entailed by the requirement for a single pair, unlike in the case of substitutional witnessing.) Note that the $x_1, \ldots, x_n$ need not be *all* the free variables in A. Also recall from p.5 that a conjunction of zero sentences is the tautology $\top$; so for $n = 0$, the accessibility requirement says that if $\top \wedge \Box A \in w$ and $\top \in w'$, then $\top \wedge A^\tau \in w'$ – equivalently: if $\Box A \in w$, then $A^\tau \in w'$.

Definition 7.5 (Canonical model)
The *canonical model* $\langle W, R, U, D, K, V \rangle$ for L is defined as follows.

1. The *worlds* W are the Henkin sets $\mathcal{H}_L$.
2. For each $w \in W$, the *outer domain* U_w comprises the non-empty sets $[x]_w$, where x is a $\mathcal{L}^*$-variable.
3. For each $w \in W$, the *inner domain* D_w comprises the sets $[x]_w$ for which $Ex \in w$.
4. The *accessibility relation* R holds between world w and world w' iff there is some transformation τ such that $w \xrightarrow{\tau} w'$.
5. C is a *counterpart relation* $\in K_{w,w'}$ iff there is a transformation τ such that (i) $w \xrightarrow{\tau} w'$ and (ii) for all $d \in U_w, d' \in U_{w'}$, dCd' iff there is an $x \in d$ such that $x^\tau \in d'$.
6. The *predicate interpretation* I assigns to every non-logical predicate P and world w the set $I_w(P) = \{\langle [x_1]_w, \ldots, [x_n]_w \rangle : Px_1 \ldots x_n \in w\}$.

Definition 7.6 (Canonical Assignment)
If w is a world in a canonical model $\mathcal{M}$ then the *canonical variable assignment on* U_w is the function g such that $g(x)$ is either $[x]_w$ or undefined if $[x]_w = \emptyset$.

This takes into account the fact that in negative logics, $\neg Ex$ entails $x \neq y$ for every variable y. So if $\neg Ex \in w$, then $[x]_w$ is the empty set. However, we don't want to say that empty terms denote the empty set (so that $\emptyset \in D_w$, and $x = x$ would have to be true). Instead, the canonical assignment assigns to each variable x at w the set $[x]_w$, *unless that set is empty*, in which case $g(x)$ remains undefined. Similarly, clause 5 in definition 7.5 ensures that $[x]_w$ at w has $[x^\tau]_{w'}$ as counterpart at w' only if $[x^\tau]_{w'} \neq \emptyset$.

The term '$\{\langle [x_1]_w, \ldots, [x_n]_w \rangle : Px_1 \ldots x_n \in w\}$' in clause 4 is meant to denote the set of n-tuples $\langle d_1, \ldots, d_n \rangle$ for which there are variables $x_1, \ldots, x_n$ such that $d_1 = [x_1]_w$ and ... and $d_n = [x_n]_w$ and $Px_1 \ldots x_n \in w$. These d_i are guaranteed to be non-empty because

$x_i = x_i \in w$ whenever $Px_1 \dots x_n \in w$: if L is positive, then $\vdash_L z_i = z_i$ by (=R); if L is negative, then $\vdash_L Pz_1 \dots z_n \supset Ez_i$ by (Neg) and hence $\vdash_L Pz_1 \dots z_n \supset z_i = z_i$ by ($\forall$=R) and (FUI*).

Lemma 7.7 (Charge of canonical models)
If L is positive, then the canonical model for L is positive. If L is strongly negative, then the canonical model for L is negative.

Proof If L is positive, then for all $\mathcal{L}^*$-variables x, every Henkin set for L contains $x = x$ (by (=R)). So $V_w(x) = [x]_w$ is never empty. Nor is $[x^\tau]_{w'}$, for any world w' with $w \xrightarrow{\tau} w'$. So everything at any world has a counterpart at every accessible world under every counterpart relation. So the canonical model for a positive logic is positive.

If L is strongly negative, then every Henkin set for L contains $x = x \supset Ex$, for all $\mathcal{L}^*$-variables x (by (Neg)). So $V_w(x) = [x]_w \neq \emptyset$ iff $Ex \in w$, which means that $D_w = U_w$ for all worlds w in the model. So the canonical model for a strongly negative logic is negative. ■

Lemma 7.8 (Extensibility Lemma)
If Γ is an L-consistent set of $\mathcal{L}^*$-sentences in which infinitely many $\mathcal{L}^*$-variables do not occur, then there is a Henkin set $H \in \mathcal{H}_L$ such that $\Gamma \subseteq H$.

Proof FIXME:CHECK Let $S_1, S_2, \dots$ be an enumeration of all $\mathcal{L}^*$-sentences, and $z_1, z_2, \dots$ an enumeration of the unused $\mathcal{L}^*$-variables in such a way that $z_i \notin Var(S_1 \wedge \dots \wedge S_i)$. Let $\Gamma_0 = \Gamma$, and define Γ_n for $n \geq 1$ as follows.

(i) If $\Gamma_{n-1} \cup \{S_n\}$ is not L-consistent, then $\Gamma_n = \Gamma_{n-1}$;
(ii) else if S_n is an existential formula $\exists xA$, then $\Gamma_n = \Gamma_{n-1} \cup \{\exists xA, [z_n/x]A, Ez_n\}$;
(iii) else if S_n is a substitution formula $\langle y : x \rangle A$, then $\Gamma_n = \Gamma_{n-1} \cup \{\langle y : x \rangle A, y = y \supset y = z_n\}$;
(iv) else $\Gamma_n = \Gamma_{n-1} \cup \{S_n\}$.

Define w as the union of all Γ_n. We show that w is a Henkin set for L.

1. w is L-consistent. This is shown by proving that Γ_0 is L-consistent and that whenever Γ_{n-1} is L-consistent, then so is Γ_n. It follows that no finite subset of w is L-inconsistent, and hence that w itself is L-consistent. The base step, that Γ_0 is L-consistent is given by assumption. Now assume (for $n > 0$) that Γ_{n-1} is L-consistent. Then Γ_n is constructed by applying one of (i)–(iv).
   a) If case (i) in the construction applies, then $\Gamma_n = \Gamma_{n-1}$, and so Γ_n is also L-consistent.
   b) Assume case (ii) in the construction applies, and suppose that $\Gamma_n = \Gamma_{n-1} \cup \{\exists xA, [z_n/x]A, Ez\}$ is L-inconsistent. Then there is a finite subset $\{C_1, \dots, C_m\} \subseteq \Gamma_{n-1}$ such that

      1. $\vdash_L \neg(C_1 \wedge \dots \wedge C_m \wedge \exists xA \wedge [z_n/x]A \wedge Ez_n)$.

Let $\underline{C}$ abbreviate $C_1 \wedge \ldots \wedge C_m$. Then

2. $\vdash_L \underline{C} \wedge \exists x A \supset (E z_n \supset \neg[z_n/x]A)$ (1)
3. $\vdash_L \forall z_n(\underline{C} \wedge \exists x A) \supset \forall z_n E z_n \supset \forall z_n \neg[z_n/x]A$ (2, (UG), (UD))
4. $\vdash_L \underline{C} \wedge \exists x A \supset \forall z_n(\underline{C} \wedge \exists x A)$ ((VQ), z_n not in Γ_{n-1})
5. $\vdash_L \underline{C} \wedge \exists x A \supset \forall z_n E z_n \supset \forall z_n \neg[z_n/x]A.$ (3, 4)
6. $\vdash_L \underline{C} \wedge \exists x A \supset \forall z_n \neg[z_n/x]A.$ (5, ($\forall$Ex))
7. $\vdash_L \forall z_n \neg[z_n/x]A \leftrightarrow \forall x \neg A$ ((AC), $z_n \notin \mathit{Var}(A)$)
8. $\vdash_L \underline{C} \wedge \exists x A \supset \neg\exists x A.$ (6, 7)

So $\{C_1, \ldots C_m, \exists x A\}$ is not L-consistent, contradicting the assumption that clause (ii) applies.

c) Assume case (iii) in the construction applies (hence L is with substitution), and suppose that $\Gamma_n = \Gamma_{n-1} \cup \{\langle y : x\rangle A, y{=}y \supset y{=}z_n\}$ is L-inconsistent. Then there is a finite subset $\{C_1, \ldots, C_m\} \subseteq \Gamma_{n-1}$ such that

1. $\vdash_L \neg(\underline{C} \wedge \langle y : x\rangle A \wedge (y{=}y \supset y{\neq}z))$.

(As before, $\underline{C}$ is $C_1 \wedge \ldots \wedge C_m$.) But then

2. $\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset y{=}y \wedge y{\neq}z_n$ (1)
3. $\vdash_L \langle y : z_n\rangle(\underline{C} \wedge \langle y : x\rangle A \supset y{=}y \wedge y{\neq}z_n)$ (2, ($\mathrm{Sub_s}$))
4. $\vdash_L \langle y : z_n\rangle(\underline{C} \wedge \langle y : x\rangle A) \supset \langle y : z_n\rangle y{=}y \wedge \langle y : z_n\rangle y{\neq}z_n$ (3, (S$\supset$), (S$\neg$))
5. $\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset \langle y : z_n\rangle(\underline{C} \wedge \langle y : x\rangle A)$ ((VS), z_n not in Γ_{n-1}, S_n)
6. $\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset \langle y : z_n\rangle y{=}y \wedge \langle y : z_n\rangle y{\neq}z_n$ (4, 5)
7. $\vdash_L \langle y : z_n\rangle y{\neq}z_n \leftrightarrow y{\neq}y$ (SAt)
8. $\vdash_L \langle y : z_n\rangle y{=}y \leftrightarrow y{=}y$ (SAt)
9. $\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset (y{=}y \wedge y{\neq}y)$. (6, 7, 8)

So $\{C_1, \ldots, C_m, \langle y : x\rangle A\}$ is L-inconsistent, contradicting the assumption that clause (iii) applies.

d) Assume case (iv) in the construction applies. Then $\Gamma_n = \Gamma_{n-1} \cup \{S_n\}$ is L-consistent, since otherwise (i) would have applied.

2. w is maximal. Assume some formula S_n is not in w. Then case (i) applied to S_n, so $\Gamma_{n-1} \cup \{S_n\}$ is not L-consistent. So there are $C_1, \ldots, C_m \in \Gamma_{n-1}$ such that $\vdash_L C_1 \wedge \ldots C_m \supset \neg S_n$. Similarly, if $S_k = \neg S_n$ is not in w, then there are $D_1, \ldots, D_l \in \Gamma_{k-1}$ such that $\vdash_L D_1 \wedge \ldots D_l \supset \neg S_k$. By (PC), it follows that there are $C_1, \ldots, C_m, D_1, \ldots D_l \in w$ such that
$$\vdash_L C_1 \wedge \ldots \wedge C_m \wedge D_1 \wedge \ldots \wedge D_l \supset (\neg S_n \wedge \neg\neg S_n).$$
But then w is inconsistent, contradicting what was just shown under 1.
3. w is witnessed. This is guaranteed by clause (ii) of the construction and the fact that

the $z_n \notin Var(S_n)$.

4. w is substitutionally witnessed. This is guaranteed by clause (iii) and the fact that the $z_n \notin Var(S_n)$. ■

LEMMA 7.9 (EXISTENCE LEMMA)
If w is a world in the canonical model for L, A a formula with $\Diamond A \in w$, and τ any transformation whose range excludes infinitely many variables of $\mathcal{L}$, then there is a world w' in the model such that $w \xrightarrow{\tau} w'$ and $A^\tau \in w'$.

PROOF FIXME:CHECK I first prove the lemma for logics L with substitution. Let $\Gamma = \{A^\tau\} \cup \{B^\tau : \Box B \in w\}$. Suppose Γ is not L-consistent. Then there are $B_1^\tau, \ldots, B_n^\tau$ with $\Box B_i \in w$ such that $\vdash_L B_1^\tau \wedge \ldots \wedge B_n^\tau \supset \neg A^\tau$. By definition **??**, this means that $\vdash_L (B_1 \wedge \ldots B_n \supset \neg A)^\tau$, and so $\vdash_L B_1 \wedge \ldots \wedge B_n \supset \neg A$ by (Sub$^\tau$). By (Nec) and (K), $\vdash_L \Box B_1 \wedge \ldots \wedge \Box B_n \supset \Box\neg A$. But then w contains both $\Diamond A$ and $\neg\Diamond A$, which is impossible because w is L-consistent. So Γ is L-consistent.

Since the range of τ excludes infinitely many variables, by the extensibility lemma, $\Gamma \subseteq H$ for some Henkin set H. Moreover, $w \xrightarrow{\tau} w'$ because $B^\tau \in H$ whenever for $\Box B \in w$.

Now for logics without substitution.

Let $S_1, S_2 \ldots$ enumerate all sentences in w of the form

$$x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B,$$

where $x_1, \ldots, x_n$ are zero or more distinct variables free in B. Let U be the "unused" $\mathcal{L}$-variables that are not in the range of τ. Let Z be an infinite subset of U such that $Z \backslash U$ is also infinite. For each $S_i = (x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B)$, let Z_{S_i} be a set of distinct variables $z_1, \ldots, z_n \in Z$ such that $Z_{S_i} \cap \bigcup_{j<i} Z_{S_j} = \emptyset$ (i.e. none of the z_i has been used for any earlier S_j). Abbreviate

$$\begin{aligned}
B_i &=_{df} [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau; \\
X_i &=_{df} x_1 = y_1 \wedge \ldots \wedge x_n = y_n; \\
Y_i &=_{df} y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau; \\
Z_i &=_{df} y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n.
\end{aligned}$$

(For $n = 0$, X_i, Y_i and Z_i are the tautology $\top$, and B_i is B^τ.)

Let $\Gamma^- = \{(Y_i \supset Z_i \wedge B_i) : S_i \in S_1, S_2, \ldots\}$, and let $\Gamma = \Gamma^- \cup \{A^\tau\}$.

Suppose for reductio that Γ is inconsistent. Then there are sentences $(Y_1 \supset Z_1 \wedge B_1), \ldots, (Y_m \supset Z_m \wedge B_m) \in \Gamma^-$ such that

$$\vdash_L \neg(A^\tau \wedge (Y_1 \supset Z_1 \wedge B_1) \wedge \ldots \wedge (Y_m \supset Z_m \wedge B_m)). \tag{1}$$

By (Nec),

$$\vdash_L \Box\neg(A^\tau \wedge (Y_1 \supset Z_1 \wedge B_1) \wedge \ldots \wedge (Y_m \supset Z_m \wedge B_m)). \tag{2}$$

Any member $(Y_i \supset Z_i \wedge B_i)$ of Γ^- has the form

$$y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \wedge [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau.$$

By $(\mathrm{CS_n})$,

$$\vdash_L x_1^\tau = y_1^\tau \wedge \ldots \wedge x_n^\tau = y_n^\tau \wedge \Box B^\tau \supset \\ \Box(y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \supset [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau). \quad (3)$$

Now w contains $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B$. So w^τ contains $x_1^\tau = y_1^\tau \wedge \ldots \wedge x_n^\tau = y_n^\tau \wedge \Box B^\tau$, which is the antecedent of (3). The consequent of (3) is $\Box(Z_i \supset B_i)$. Thus

$$w^\tau \vdash_L \Box(Z_1 \supset B_1) \wedge \ldots \wedge \Box(Z_m \supset B_m). \quad (4)$$

Let $\Delta = w^\tau \cup \{\Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m))\}$. So

$$\Delta \vdash_L \Box(Z_1 \supset B_1) \wedge \ldots \wedge \Box(Z_m \supset B_m); \quad (5)$$

$$\Delta \vdash_L \Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m)). \quad (6)$$

By (K) and (Nec), (5) and (6) yield

$$\Delta \vdash_L \Diamond(A^\tau \wedge (Y_1 \supset Z_1 \wedge B_1) \wedge \ldots \wedge (Y_m \supset Z_m \wedge B_m)). \quad (7)$$

By (2), it follows that Δ is inconsistent. This means that

$$w^\tau \vdash_L \neg\Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m)). \quad (8)$$

Now consider $Z_1 = (y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n)$. By $(\mathrm{LL_n^*})$ (or repeated application of $(\mathrm{LL^*})$),

$$\vdash_L y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \supset \Box\neg(A^\tau \wedge (y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n)) \\ \supset \Box\neg(A^\tau \wedge (y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau)), \quad (9)$$

because the z_i are not free in A^τ. In other words (and dropping the tautologous conjunct at the end),

$$\vdash_L Z_1 \supset \Box\neg(A^\tau \wedge (Y_1 \supset Z_1)) \supset \Box\neg A^\tau. \quad (10)$$

By the same reasoning,

$$\vdash_L Z_1 \wedge \ldots \wedge Z_m \supset \Box\neg(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m)) \supset \Box\neg A^\tau. \quad (11)$$

By (PC), (Nec) and (K), this means

$$\vdash_L Z_1 \wedge \ldots \wedge Z_m \supset \Diamond A^\tau \supset \Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m)). \quad (12)$$

Since $w^\tau \vdash_L \Diamond A^\tau$, (8) and (12) together entail

$$w^\tau \vdash_L \neg(Z_1 \wedge \ldots \wedge Z_m). \quad (13)$$

So there are $C_1, \ldots, C_k \in w$ such that

$$\vdash_L C_1^\tau \land \ldots \land C_k^\tau \supset \neg(Z_1 \land \ldots \land Z_m). \tag{14}$$

Each Z_i has the form $y_1^\tau = z_1 \land \ldots \land y_n^\tau = z_n$. All the z_i are pairwise distinct, and none of them occur in $C_1^\tau \land \ldots \land C_k^\tau$ (because the z_i are not in the range of τ) nor in any other Z_i. By (Sub*), we can therefore replace each z_i in (14) by the corresponding y_i^τ, turning Z_i into Y_i:

$$\vdash_L C_1^\tau \land \ldots \land C_k^\tau \supset \neg(Y_1 \land \ldots \land Y_m). \tag{15}$$

For any $Y_i = (y_1^\tau = y_1^\tau \land \ldots \land y_n^\tau = y_n^\tau)$, X_i is a sentence of the form $x_1 = y_1 \land \ldots \land x_n = y_n$. So X_i^τ is $x_1^\tau = y_1^\tau \land \ldots \land x_n^\tau = y_n^\tau$, and $\vdash_L X_i^\tau \supset Y_i$ by either (=R) or (Neg) and (∀=R). So (15) entails

$$\vdash_L C_1^\tau \land \ldots \land C_k^\tau \supset \neg(X_1^\tau \land \ldots \land X_m^\tau). \tag{16}$$

Thus by (Sub$^\tau$),

$$\vdash_L C_1 \land \ldots \land C_k \supset \neg(X_1 \land \ldots \land X_m). \tag{17}$$

Since $\{C_1, \ldots, C_k, X_1, \ldots, X_m\} \subseteq w$, it follows that w is inconsistent. Which it isn't. This completes the reductio.

So Γ is consistent. Since the infinitely many variables in $U \backslash Z$ do not occur in Γ, lemma 7.8 guarantees that $\Gamma \subseteq w'$ for some world w' in the canonical model for L. And of course, Γ was constructed so that w' satisfies the condition in definition 7.4 for $w \xrightarrow{\tau} w'$. This requires that for every formula B and variables $x_1 \ldots x_n$, $y_1, \ldots, y_n$ such that the $x_1 \ldots x_n$ are zero or more pairwise distinct members of $\mathit{Varf}(B)$, if $x_1 = y_1 \land \ldots \land x_n = y_n \land \Box B \in w$ and $y_1^\tau = y_1^\tau \land \ldots \land y_n^\tau = y_n^\tau \in w'$, then there are variables $z_1 \ldots z_n \notin \mathit{Var}(B^\tau)$ such that $z_1 = y_1^\tau \land \ldots \land z_n = y_n^\tau \land [z_1 \ldots z_n / x_1^\tau \ldots x_n^\tau] B^\tau \in w'$. By construction of Γ, whenever $x_1 = y_1 \land \ldots \land x_n = y_n \land \Box B \in w$, then there are suitable $z_1, \ldots, z_n$ such that $y_1^\tau = y_1^\tau \land \ldots \land y_n^\tau = y_n^\tau \supset y_1^\tau = z_1 \land \ldots \land y_n^\tau = z_n \land [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau \in w'$. So if $y_1^\tau = y_1^\tau \land \ldots \land y_n^\tau = y_n^\tau \in w'$, then $y_1^\tau = z_1 \land \ldots \land y_n^\tau = z_n \land [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau \in w'$. ■

Lemma 7.10 (Truth Lemma)
If $\mathcal{M} = \langle W, R, U, D, K, V \rangle$ is the canonical model for L, $w \in W$, and g is the canonical assignment on U_w, then for any $\mathcal{L}$-sentence A,

$$\mathcal{M} w, g \Vdash A \text{ iff } A \in w.$$

Proof FIXME by induction on A.

1. A is $Px_1 \ldots x_n$. $w, V \Vdash Px_1 \ldots x_n$ iff $\langle V_w(x_1), \ldots, V_w(x_n) \rangle \in V_w(P)$ by definition 2.8. By construction of V_w (definition 7.5), $V_w(x_i)$ is $[x_i]_w$ or undefined if $[x_i]_w = \emptyset$, and $V_w(P) = \{\langle [z_1]_w, \ldots, [z_n]_w \rangle : Pz_1 \ldots z_n \in w\}$. (For non-logical P, this is directly given by definition 7.5; for the identity predicate, $V_w(=)$ is $\{\langle d, d \rangle : d \in U_w\}$ by definition 2.8, which equals $\{\langle [z]_w, [z]_w \rangle : z = z \in w\} = \{\langle [z_1]_w, [z_2]_w \rangle : z_1 = z_2 \in w\}$ because the members of U_w are precisely the non-empty sets $[z]_w$.)

Now if $\langle V_w(x_1), \ldots, V_w(x_n)\rangle \in V_w(P)$, then $\langle [x_1]_w, \ldots, [x_n]_w\rangle \in \{\langle [z_1]_w, \ldots, [z_n]_w\rangle : Pz_1 \ldots z_n \in w\}$, where all the $[x_i]_w$ are non-empty (for $V_w(x_i)$ is defined). This means that there are variables $z_1, \ldots, z_n$ such that $\{x_1 = z_1, \ldots, x_n = z_n, Pz_1 \ldots z_n\} \subseteq w$. Then $Px_1 \ldots x_n \in w$ by (LL*).

In the other direction, if $Px_1 \ldots x_n \in w$, then $x_i = x_i \in w$ for all x_i in $x_1 \ldots x_n$ (see p. 67). Hence $\langle [x_1]_w, \ldots, [x_n]_w\rangle \in \{\langle [z_1]_w, \ldots, [z_n]_w\rangle : Pz_1 \ldots z_n \in w\}$, i.e. $\langle V_w(x_1), \ldots, V_w(x_n)\rangle \in V_w(P)$.

2. A is $\neg B$. $w, V \Vdash \neg B$ iff $w, V \nVdash B$ by definition 2.8, iff $B \notin w$ by induction hypothesis, iff $\neg B \in w$ by maximality of w.
3. A is $B \supset C$. $w, V \Vdash B \supset C$ iff $w, V \nVdash B$ or $w, V \Vdash C$ by definition 2.8, iff $B \notin w$ or $C \in w$ by induction hypothesis, iff $B \supset C \in w$ by maximality and consistency of w and the fact that $\vdash_L \neg B \supset (B \supset C)$ and $\vdash_L C \supset (B \supset C)$.
4. A is $\langle y : x\rangle B$. Assume first that $w, V \Vdash y \neq y$. So $V_w(y)$ is undefined, and it is not the case that $V_w(y)$ has multiple counterparts at any world. And then $w, V \Vdash \langle y : x\rangle B$ iff $w, V^{[y/x]} \Vdash B$ by definition 5.2, iff $w, V \Vdash [y/x]B$ by lemma 3.5, iff $[y/x]B \in w$ by induction hypothesis. Also by induction hypothesis, $y \neq y \in w$. By (SCN), $\vdash_L y \neq y \supset ([y/x]B \leftrightarrow \langle y : x\rangle B)$. So $[y/x]B \in w$ iff $\langle y : x\rangle B \in w$.

   Next, assume that $w, V \Vdash y = y$; so by induction hypothesis $y = y \in w$. Assume further that $\langle y : x\rangle B \notin w$. Then $\neg\langle y : x\rangle B \in w$ by maximality of w, and $\langle y : x\rangle \neg B \in w$ by (S¬). Since w is substitutionally witnessed and $y = y \in w$, there is a variable $z \notin \mathit{Var}(\langle y : x\rangle \neg B)$ such that $y = z \in w$ and $[z/x]\neg B \in w$. By induction hypothesis, $w, V \Vdash y = z$. Moreover, by definition **??**, $\neg[z/x]B \in w$, and so $[z/x]B \notin w$ by consistency of w. By induction hypothesis, $w, V \nVdash [z/x]B$. By definition 2.8, then $w, V \Vdash \neg[z/x]B$, i.e. $w, V \Vdash [z/x]\neg B$. Since z and x are modally separated in B, then $w, V^{[z/x]} \Vdash \neg B$ by lemma 3.5. But $V^{[z/x]}$ and $V^{[y/x]}$ agree on all variables at w, because $w, V \Vdash y = z$. So $w, V^{[y/x]} \Vdash \neg B$ by the locality lemma 2.10. So $w, V^{[y/x]} \nVdash B$ by definition 2.8, and $w, V \nVdash \langle y : x\rangle B$ by definition 5.2.

   In the other direction, assume $\langle y : x\rangle B \in w$. Since w is substitutionally witnessed and $y = y \in w$, there is a new variable z such that $y = z \in w$ and $[z/x]B \in w$. By induction hypothesis, $w, V \Vdash y = z$ and $w, V \Vdash [z/x]B$. Since z and x are modally separated in B, $w, V^{[z/x]} \Vdash B$ by lemma 3.5. As before $V^{[z/x]}$ and $V^{[y/x]}$ agree on all variables at w, because $w, V \Vdash y = z$; so $w, V^{[y/x]} \Vdash B$ by lemma 2.10 and $w, V \Vdash \langle y : x\rangle B$ by definition 5.2.
5. A is $\forall xB$. We first show that for any variable x, $w, V \Vdash Ex$ iff $Ex \in w$: $w, V \Vdash Ex$ iff $V_w(x) \in D_w$ by definition **??**, iff $[x]_w \in D_w$ by definition 7.5, iff $Ex \in w$ by definition 7.5.

   Now assume $\forall xB \in w$, and let y be any variable such that $Ey \in w$. As just shown, $w, V \Vdash Ey$. By (FUI**), $\exists x(x = y \wedge B) \in w$. By witnessing, there is a $z \notin \mathit{Var}(B)$ such that $z = y \wedge [z/x]B \in w$, and thus $z = y \in w$ and $[z/x]B \in w$. By induction hypothesis, $w, V \Vdash z = y$ and $w, V \Vdash [z/x]B$. By lemma 3.5, then $w, V^{[z/x]} \Vdash B$. And since $V_w(z) = V_w(y)$, it follows by lemma 2.10 that $w, V^{[y/x]} \Vdash B$. So if $\forall xB \in w$, then $w, V^{[y/x]} \Vdash B$ for all variables y with $Ey \in w$, i.e. with $V_w(y) \in D_w$. Since every member $[y]_w$ of D_w is denoted by some variable y under V_w, this means that $w, V' \Vdash B$ for all existential x-variants V' of V on w. So $w, V \Vdash \forall xB$.

Conversely, assume $\forall x B \notin w$. Then $\exists x \neg B \in w$; so by witnessing, $[y/x]\neg B \in w$ for some $y \notin Var(B)$ with $Ey \in w$. Then $\neg[y/x]B \in w$ and so $[y/x]B \notin w$. As shown above, $w, V \Vdash Ey$. Moreover, by induction hypothesis, $w, V \nVdash [y/x]B$. By lemma 3.5, then $w, V^{[y/x]} \nVdash B$. Let V' be the (existential) x-variant of V on w with $V'_w(x) = V^{[y/x]}_w(x)$. By the locality lemma, $w, V' \nVdash B$. So $w, V \nVdash \forall x B$.

6. A is $\Box B$. Assume $w, V \Vdash \Box B$. Then $w', V' \Vdash B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. We first show that if $w \xrightarrow{\tau} w'$, then $V_w \triangleright V^\tau_{w'}$. By definitions 2.7 and 7.5, $V_w \triangleright V^\tau_{w'}$ means that there is a transformation σ such that $w \xrightarrow{\sigma} w'$ and for every variable y, if there is a $z \in V_w(y)$ such that $[z^\sigma]_{w'} \in U_{w'}$ (i.e., if $V_w(y)$ has any σ-counterpart at w'), then there is a $z \in V_w(y)$ with $z^\sigma \in V^\tau_{w'}(y)$ (i.e., then $V^\tau_{w'}(y)$ is such a counterpart), otherwise $V^\tau_{w'}(y)$ is undefined. The relevant transformation σ will be τ. So what we'll show is this: for every variable y, if there is a $z \in V_w(y)$ such that $[z^\tau]_{w'} \in U_{w'}$, then there is a $z \in V_w(y)$ with $z^\tau \in V^\tau_{w'}(y)$, otherwise $V^\tau_{w'}(y)$ is undefined.
Let y be any variable. Assume first that there is a $z \in V_w(y)$ such that $[z^\tau]_{w'} \in U_{w'}$. Then $z = y \in w$ and $z^\tau = z^\tau \in w'$. By either (Neg) and (EI) or (=R), $\vdash_L z = y \supset y = y$; so $y = y \in w$. Moreover, by either (TE), (EI), (Nec) and (K) or (=R) and (Nec), $\vdash_L z = y \supset \Box(z = z \supset y = y)$; so $\Box(z = z \supset y = y) \in w$. By definition of $w \xrightarrow{\tau} w'$, then $z^\tau = z^\tau \supset y^\tau = y^\tau \in w'$. So $y^\tau = y^\tau \in w'$. Hence $y \in V_w(y)$ and $y^\tau \in [y^\tau]_{w'} = V_{w'}(y^\tau) = V^\tau_{w'}(y)$. Alternatively, assume there is no $z \in V_w(y)$ with $z^\tau = z^\tau \in w'$. Then either $V_w(y) = \emptyset$, in which case $y \neq y \in w$, and so $\Box(y \neq y) \in w$ by (NA), (EI), (Nec) and (K), and $y^\tau \neq y^\tau \in w'$ by definition of $w \xrightarrow{\tau} w'$, or else $V_w(y) \neq \emptyset$, but $z^\tau \neq z^\tau \in w'$ for all $z \in V_w(y)$, in which case, too, $y^\tau \neq y^\tau \in w'$ since $y \in V_w(y)$. Either way, $V_{w'}(y^\tau) = V^\tau_{w'}(y)$ is undefined.
We've shown that if $w, V \Vdash \Box B$, then for every w' and τ with $w \xrightarrow{\tau} w'$, $w', V^\tau \Vdash B$. By the transformation lemma, then $w', V \Vdash B^\tau$. By induction hypothesis, $B^\tau \in w'$. Now suppose $\Box B \notin w$. Then $\Diamond \neg B \in w$ by maximality of w. By the existence lemma, there is then a world w' and transformation τ with $w \xrightarrow{\tau} w'$ and $\neg B^\tau \in w'$. (Any transformation whose range excludes infinitely many variables will do.) But we've just seen that if $w \xrightarrow{\tau} w'$, then $B^\tau \in w'$. So if $w, V \Vdash \Box B$, then $\Box B \in w$.
For the other direction, assume $w, V \nVdash \Box B$. So $w', V' \nVdash B$ for some w', V' with wRw' and $V_w \triangleright V'_{w'}$. As before, $V_w \triangleright V'_{w'}$ means that there is a transformation τ with $w \xrightarrow{\tau} w'$ such that for every variable x, either there is a $y \in V_w(x)$ with $y^\tau \in V'_{w'}(x)$, or there is no $y \in V_w(x)$ with $y^\tau = y^\tau \in w'$, in which case $V'_{w'}(x)$ is undefined. Let τ be any transformation with $w \xrightarrow{\tau} w'$, and let $*$ be a substitution that maps each variable x in B to some member y of $V_w(x)$ with $y^\tau \in V'_{w'}(x)$, or to itself if there is no such y. Thus if $x \in Var(B)$ and $V'_{w'}(x)$ is defined, then $(*x)^\tau \in V'_{w'}(x)$, and so $V'_{w'}(x) = [(*x)^\tau]_{w'} = V^{\tau\cdot *}_{w'}(x)$. Alternatively, if $V'_{w'}(x)$ is undefined (so $*x = x$), then $V^{\tau\cdot *}_{w'}(x) = V^\tau_{w'}(x)$ is also undefined. The reason is that otherwise $V^\tau_{w'}(x) = [x^\tau]_{w'} \neq \emptyset$ and $x^\tau = x^\tau \in w'$; by definition of accessibility, then $\Box x \neq x \notin w$ and hence $x = x \in w$, as $\vdash_L x \neq x \supset \Box x \neq x$; so there is a $y \in V_w(x)$, namely x, such that $y^\tau = y^\tau \in w'$, in which case $V'_{w'}(x)$ cannot be undefined (by definition 2.7). So V' and $V^{\tau\cdot *}$ agree at w' on all variables in B. By lemma **??**, $w', V^{\tau\cdot *} \nVdash B$.
Now suppose for reductio that $\Box B \in w$. Let $x_1, \ldots, x_n$ be the variables x in $Var(B)$ with $(*x)^\tau \in V'_{w'}(x)$ (thus excluding empty variables as well as variables denot-

ing individuals without τ-counterparts at w'). For each such x_i, $*x_i \in V_w(x_i)$, and so $x_i = *x_i \in w$. If L is with substitution, then by (LL$_n$), $\langle *x_1, \ldots, *x_n : x_1, \ldots, x_n \rangle \Box B \in w$; so $\Box \langle *x_1, \ldots, *x_n : x_1, \ldots, x_n \rangle B \in w$ by (S$\Box$). By definition of $w \xrightarrow{\tau} w'$, then $\langle (*x_1)^\tau, \ldots, (*x_n)^\tau : x_1^\tau, \ldots, x_n^\tau \rangle B^\tau \in w'$. By substitutional witnessing, it follows that there are new variables $z_1, \ldots, z_n$ such that $z_i = (*x_i)^\tau \in w'$ and (hence) $[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau \in w'$. If L is without substitution, this fact – that there are new variables $z_1, \ldots, z_n$ such that $z_i = (*x_i)^\tau \in w'$ and $[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau \in w'$ – is guaranteed directly by definition of $w \xrightarrow{\tau} w'$ and the fact that $\Box B \in w$.
By induction hypothesis, $w', V \Vdash z_i = (*x_i)^\tau$ and $w', V \Vdash [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau$. Since the z_i are new, $w', V^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau]} \Vdash B^\tau$ by lemma 3.5. By the transformation lemma 3.9, then $w', V^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] \cdot \tau} \Vdash B$. However, for each x_i, $V_{w'}^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] \cdot \tau}(x_i) = V_{w'}^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau]}(x_i^\tau) = V_{w'}(z_i) = V_{w'}((*x_i)^\tau)$ (because $w', V \Vdash (*x_i)^\tau = z_i$) $= V_{w'}^\tau(*x_i) = V_{w'}^{\tau \cdot [*x_1, \ldots, *x_n / x_1, \ldots, x_n]}(x_i) = V_{w'}^{\tau \cdot *}(x_i)$. Similarly, if $x \in \mathit{Var}(B)$ is none of the $x_1, \ldots, x_n$, so $(*x)^\tau \notin V'_{w'}(x)$, then $*x$ is x by definition of $*$, and so $V_{w'}^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] \cdot \tau}(x) = V_{w'}^\tau(x) = V_{w'}^\tau(*x) = V_{w'}^{\tau \cdot *}(x)$. So $V^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] \cdot \tau}$ and $V_{w'}^{\tau \cdot *}$ agree at w' on all variables in B. By lemma 2.10, then $w', V^{\tau \cdot *} \Vdash B$ – contradiction. ■

8 Completeness

FIXME: This whole section

Recall that a logic L in some language of quantified modal logic is *(strongly) complete with respect to a class of models* $\mathbb{M}$ if every L-consistent set of formulas Γ is verified at some world in some model in $\mathbb{M}$. L is *characterised by* $\mathbb{M}$ if L is sound and complete with respect to $\mathbb{M}$.

The minimal positive and negative logics from sections **??** and **??** were designed to be complete with respect to the class of all positive and negative models, respectively. Let's confirm that this is the case.

> Theorem 8.1 (Completeness of P and P_s)
> The logics P and P_s are (strongly) complete with respect to the class of positive counterpart models.

Proof Let L range over P and P_s. We have to show that whenever a set of $\mathcal{L}$-formulas Γ is L-consistent, then there is some world in some positive counterpart model that verifies all members of Γ. By lemma 7.7, the canonical model $\mathcal{M}_L = \langle \mathcal{S}_L, V_L \rangle$ for L is a positive model. By the Extensibility Lemma, $\Gamma \subseteq w$ for some world w in $\mathcal{M}_L$, since none of the infinitely many variables Var^+ occur in Γ. By the truth lemma, then $w, V_L \Vdash_{\mathcal{S}_L} A$ for each $A \in \Gamma$. ■

Theorem 8.2 (Completeness of N and N_s)
The logics N and N_s are (strongly) complete with respect to the class of negative counterpart models.

Proof Let L range over N and N_s, and let Γ be an L-consistent set of $\mathcal{L}$-formulas. By lemma 7.7, the canonical model $\mathcal{M}_L = \langle \mathcal{S}_L, V_L \rangle$ for L is a negative model. By the Extensibility Lemma, $\Gamma \subseteq w$ for some world w in $\mathcal{M}_L$, since none of the infinitely many variables Var^+ occur in Γ. By the truth lemma, then $w, V_L \Vdash_{\mathcal{S}_L} A$ for each $A \in \Gamma$. ■

Together with the soundness theorems 4.3, 4.6, 5.7 and 5.8, it follows that P and P_s are characterized by the class of positive models, and N and N_s by the class of negative models.

In footnote 3 (on page 4) I mentioned that the introduction of multiple counterpart relations makes little difference to the base logic. Let's call a counterpart structure in which any two worlds are linked by at most one counterpart relation *unirelational*. As it turns out, P and P_s are also characterized by the class of unirelational positive models, and N and N_s by the class of unirelational negative models. The easiest way to see this is perhaps to note that all the lemmas in the previous section still go through if we define accessibility and counterparthood in canonical models by a fixed transformation τ whose range excludes infinitely many variables. The extensibility lemma 7.8 and existence lemma 7.9 are unaffected by this change; the only part that needs adjusting is the clause for $\Box B$ in the proof of the truth lemma 7.10, but the adjustments are straightforward.

Every quantified modal logic is strongly complete with respect to every class of models that contains its canonical model. However, on the traditional idea that logical truths should be true on any interpretation of the non-logical terms, an arguably more important kind of completeness is completeness with respect to all models with a certain type of *structure*.

Strictly speaking, we have two such notions, one for positive and one for negative logics.

Definition 8.3 (Positive completeness and characterisation)
A logic L in some language of quantified modal logic is *(strongly) positively complete with respect to a class of structures* $\mathbb{S}$ if every L-consistent set of formulas Γ is verified at some world in some positive model $\langle \mathcal{S}, V \rangle$ with $\mathcal{S} \in \mathbb{S}$. L is *positively characterised by* $\mathbb{S}$ if it is sound and positively complete with respect to $\mathbb{S}$.

Now we might try to show, first, that if a PML is canonical, then so is its quantified counterpart. Then we could try to show that the canonical model of the PML is in a class of frames F iff the opaque propositional guise of the canonical model of the quantified counterpart is in F. Then we'd have completeness transfer for all canonical logics.

> THEOREM 8.4 ((POSITIVE) COMPLETENESS TRANSFER)
> If L is a (unimodal) propositional modal logic that is complete with respect to the Kripke frames in some class F, then the PL is positively complete with respect to the total counterpart structures whose opaque propositional guise is in F.

[To be continued...]

[TODO: What is the minimal logic of functional structures? What kind of structures provides models for the minimal classical (non-free) QML? etc.]

[TODO: look at an example where we can prove completeness but classical Kripke semantics can't. S4M?]

[TODO: discuss need for multiple counterpart relations]