Generalising Kripke Semantics for Quantified Modal Logics

Wolfgang Schwarz

Unfinished draft, 23/01/2013

We turn now to what is arguably one of the least well behaved modal languages ever proposed: first-order modal logic.
[Blackburn and van Benthem 2007].

Contents

1 Introduction

Modal logic has outgrown its philosophical origins. What used to be the logic of possibility and necessity has become topic-neutral, with applications ranging from medical databases to the study of mathematical proofs. Part of this process has been a decline of interest in modal predicate logic. In [Hughes and Cresswell 1968] propositional modal logic appears as a stepping stone to modal predicate logic. In [Blackburn et al. 2001], modal predicate logic is hardly mentioned. There are two main reasons for this development. One is that modal predicate logics – on the standard semantics going back to Kripke, Kanger and Hintikka – have turned out to be complex and in many respects ill-behaved, as witnessed e.g. by the fact that completeness results often do not carry over from propositional to first-order logic. Alternative model theories have been proposed to overcome these problems, such as the meta-frame semantics of [Skvortsov and Shehtman 1993], but they generally lack the perspicuity of Kripke semantics. The other reason for the decline of modal predicate logic is that propositional modal logic itself has emerged as a fragment of first-order predicate logic, with the domain of "worlds" playing the role of "individuals". As emphasized in [Blackburn et al. 2001], the distinctive character of modal logic is not its subject matter, but its *perspective.* Statements of modal logic describe relational structures from the inside perspective of a particular node. Modal predicate logic therefore looks like a somewhat cumbersome hybrid, combining an internal perspective on one class of objects (the domain of the modal operators) with an external perspective on a possibly different class of objects (the domain of quantification).

Nevertheless, this hybrid perspective is useful and natural for many applications – not least the traditional philosophical topics of epistemic, deontic or temporal logic (or the logic of "metaphysical necessity", if there is such a thing). For instance, when reasoning about time, it is natural to take a perspective that is internal to the structure of times, so that what is true at one point may be false at another, but external to the structure of sticks and stones and people existing at the various times.

The main difficulties that arise for this approach concern *de re* statements like $\exists x \Diamond Fx$. If we assume that each world or time has an associated domain of individuals of things that exist at the relevant world or time, we can read $\exists x \Diamond Fx$ as true at w iff w's domain of individuals has a member that satisfies Fx at some point w' accessible from w. Accordingly, if some individual at w does not exist at w', $\exists x \Diamond \neg \exists y(x = y)$ is true (at w), despite the fact that $\neg \exists y(x = y)$ is unsatisfiable in standard predicate logic. This is why modal predicate logic often uses free logic as its predicate logic base (see e.g. [Garson 1984]). Another problem for the standard account is that it makes identity non-contingent. For example, if the domain of w contains two individuals x and y, then there can be no other point from the perspective of which these two individuals are identical: $\forall x \forall y(x \neq y \supset \Box x \neq y)$ is valid on the standard semantics. But arguably it is possible to believe of two individuals that they are one and the same, and then we may not want to count $\forall x \forall y(x \neq y \supset \Box x \neq y)$ as valid in epistemic modal logic. Moreover, $\forall x \forall y(x \neq y \supset \Box x \neq y)$ is not provable by the standard axioms and rules of first-order logic combined with those of the basic modal logic K. The standard account therefore cannot serve as model theory for one of the simplest modal predicate logics.

David Lewis [1968] once proposed an alternative to the standard account that (among other things) allows for contingent identity. On this account, the *de re* formula $\exists x \Diamond Fx$ is true at w if the domain of w contains an object whose *counterpart* at some accessible point w' satisfies Fx. If two objects at w have the same counterpart at w', $\exists x \exists y(x \neq y \wedge \Diamond(x = y))$ becomes true.

Unfortunately, Lewis combined his proposal with two further ideas, which have prevented its wider adoption in logic and philosophy. First, he held that ordinary individuals never exist at more than one possible world, so that an individual's counterparts at other worlds are never identical to the original individual. The analogous temporal position, defended in [Sider 2001], holds that ordinary individuals are temporarily unextended stages that have other stages as temporal counterparts. This is not how we intuitively think of sticks or stones or persons.

Second, Lewis effectively swapped the traditional, hybrid perspective of modal predicate logic for a thoroughly internal perspective, where statements are evaluated not relative to worlds, but relative to individuals at worlds; the counterpart relation then replaces the accessibility relation (see esp. [Lewis 1986: 230-235]). This has curious consequences for the logic determined by his semantics. For instance, the 'necessity of identity', $\Box\exists y(x=y)$, comes out valid, while basic distribution principles such as $\Box(A \wedge B) \supset \Box A$ become invalid (see [Hazen 1979], [Woollaston 1994]). These consequences can be explained, and the internalist approach has been developed into a powerful model-theoretic framework in the hands of Silvio Ghilardi, Giancarlo Meloni and Giovanna Corsi (see [Ghilardi and Meloni 1988], [Ghilardi and Meloni 1991], [Ghilardi 2001] [Corsi 2002], [Braüner and Ghilardi 2007: 591–616]). Nevertheless, the approach goes against the traditional conception of modal predicate logic.

In this essay, I will present and investigate a more conservative form of counterpart semantics. It retains the traditional, hybrid perspective of modal predicate logic and allows individuals to exist at several worlds or times. It deviates from standard Kripke semantics only in the assumption that *de re* formulas are evaluated by tracking individuals along the counterpart relation. Thus even if x itself exists at worlds accessible from w, what matters to the truth of $\Diamond Fx$ at w is whether x's counterparts at those worlds satisfy Fx. Kripke semantics is simply the special case where the counterpart relation is fixed as the identity relation. As we will see, lifting this restriction results in a fairly simple and intuitive account that overcomes some shortcomings of standard Kripke semantics.

2 Counterpart models

Like Kripke semantics, counterpart semantics comes in several flavours. A well-known choice in Kripke semantics is whether the domain of individuals that exist at a world is constant or variable. In counterpart semantics, strict identity across worlds is less important; the more relevant choice is whether every individual that exists at a world should have a counterpart at every other world.

If one allows for individuals to lack counterparts at some worlds, the next question is what can be said about things that don't exist. The alternatives are well-known from free logic. One option is that if x doesn't exist at w, then every atomic predication Fx is false at w. This is known as *negative*, or *single-domain* semantics. Alternatively, one may hold that non-existence is no bar to satisfying predicates, so Fx may be true at some worlds where x doesn't exist and false at others. The extension of F at a world must therefore be specified not only for things that exist at that world, but also for things that don't exist. This is known as *positive*, or *dual-domain* semantics. Both approaches are attractive for certain applications, so I will explore them in tandem.

In positive models, terms are never genuinely empty. Worlds are associated with an *inner domain* of individuals existing at that world, and an *outer domain* of individuals which, although they need not exist, may still fall in the extension of atomic predicates. Every individual at any world has at least one counterpart at every accessible world, if only in the outer domain. In negative models, we would like to do without the somewhat ghostly outer domains. Terms can go genuinely empty. However, in a modal setting, it is not enough to require that whenever x is empty, then atomic predications involving x are false. We must also require that modal predications of the form $\Diamond Fx$ are false for empty x. What doesn't exist can't have any counterparts. If $\Diamond Fx$ could be true and $\Diamond Fy$ false although x and y are both empty, we would have to introduce outer domains after all, so that the outer referent of x has an F-counterpart while that of y does not.

As a consequence, the logic of single-domain counterpart models validates some principles that are not derivable from the standard axioms and rules of negative free logic combined with those of the basic modal logic K, namely

(NA) $\neg Ex \supset \Box\neg Ex$,

(TE) $x = y \supset \Box(Ex \supset Ey)$.

Here Ex abbreviates $\exists y(x = y)$. (NA) reflects the fact that non-existent objects don't have any counterparts. (TE) says that if x is identical to y, and x has a counterpart at some accessible world, then y also has a counterpart at that world. This, too, is a consequence of shunning outer domains. If we had outer domains, an individual could have both existing and non-existing counterparts at a world, which would render (TE) false.

We can also offer counterpart models for negative modal predicate logics without (NA) and (TE). These are dual-domain models in which the extension of all predicates, including identity, is restricted to the inner domain. In this setting, (NA) requires that individuals that only figure in the outer domain of a world never have counterparts in the inner domain of another world. (What goes out of existence stays out of existence.) (TE) requires that if an individual in the inner domain of a world has a counterpart in the inner domain of another world, then all its counterparts at that world are in the inner domain. The two requirements are obviously independent, which shows the independence of (NA) and (TE).

As usual, a *model* combines an abstract *structure* with an *interpretation* of our language on that structure. The relevant structures are defined as follows. Note that the counterpart relation is a relation between pairs of an individual and a world, so that an individual x at w can for example be its own counterpart at w, but not at w' even though it also exists at w'. (This is a consequence of dropping the requirement of disjoint domains.)

Definition 2.1 (Counterpart structure)
A *counterpart structure* is a quintuple $\mathcal{S} = \langle W, R, U, D, C \rangle$, consisting of

1. a non-empty set W (of "points" or "worlds"),
2. a binary ("accessibility") relation R on W,
3. an ("outer domain") function U that assigns to each point $w \in W$ a set U_w,
4. an ("inner domain") function D that assigns to each point $w \in W$ a set $D_w \subseteq U_w$, and
5. a ("counterpart") relation C on $\{\langle d, w \rangle : w \in W, d \in U_w\}$

such that *either* $D = U$, *or* every individual at any world has at least one counterpart at any accessible world (i.e., if wRw' and $d \in U_w$, then there is a $d' \in U_{w'}$ with $\langle d, w \rangle C \langle d', w' \rangle$). In the first case, $\mathcal{S}$ is a *single-domain* structure, in the second case it is a *total* structure.

If every individual in a structure has a counterpart at every accessible world, and also $D = U$, then the structure is both single-domain and total.

Next, we define interpretations of the language of quantified modal logic on counterpart structures. To this end, we first have to say what that language is.

Definition 2.2 (Languages of QML)
Let *Const*, *Var*, *Pred* be disjoint sets of symbols such that *Const* has five distinct members $\neg, \supset, \forall, =$ and $\Box$, *Var* is countably infinite, and each member P of *Pred* is associated with a number n, called P's *arity*. The *standard language of quantified modal logic over Const, Var, Pred*, for short $\mathcal{L}$(*Const, Var, Pred*), is the set of formulas built from these ingredients by the usual construction rules

$$\mathit{wff} = Px_1 \ldots x_n \mid x = y \mid \neg A \mid (A \supset B) \mid \forall x A \mid \Box A,$$

where P is a member of *Pred* with arity n and $x_1, \ldots, x_n, x, y \in$ *Var*.

Nothing depends on the choice of *Const, Var* and *Pred*, and we could have fixed them once and for all in some arbitrary manner. I haven't done so not only to minimise arbitrariness, but also because we will later want to consider languages that result from others by adding further variables, and then we don't want to have defined interpretations (and logics) only for the original language.

Some notational conventions: I will often use '$\mathcal{L}$' as a metalinguistic variable ranging over languages $\mathcal{L}$(*Const, Var,Pred*) of quantified modal logic, and I will call the relevant members of *Var* and *Pred* *variables* and *predicates* (respectively) *of* $\mathcal{L}$. I use '$\neg$', '$\supset$', '$\forall$', '$=$' and '$\Box$' as metalinguistic variables for the corresponding ingredients of *Const*, 'x',

'y', 'z', 'v' (sometimes with indices or dashes) for members of *Var*, and 'F', 'G', 'P' for members of *Pred* with arity 1, 2 and n, respectively. Formulas involving '$\wedge$', '$\vee$', '$\leftrightarrow$', '$\exists$' and '$\Diamond$' are defined by the usual metalinguistic abbreviations. '$A_1 \wedge \ldots \wedge A_n$' stands for '$A_1$' if $n = 1$, or for '$(A_1 \wedge \ldots \wedge A_{n-1}) \wedge A_n$' if $n > 1$, or for an arbitrary tautology $\top$ (say, '$x{=}x \supset x{=}x$') if $n = 0$. For any expression or set of expressions A, *Var*(A) is the set of variables in (members of) A, and *Varf*(A) is the set of variables with free occurrences in (members of) A.

> Definition 2.3 (Interpretation)
> Let $\mathcal{S} = \langle W, R, U, D, C\rangle$ be a counterpart structure and $\mathcal{L}$ a language of quantified modal logic. An *interpretation function V for $\mathcal{L}$ on $\mathcal{S}$* is a function that assigns to each world $w \in W$ a function V_w such that
> 1. for every predicate P of $\mathcal{L}$, $V_w(P) \subseteq U_w^n$,
> 2. $V_w(=) = \{\langle d, d\rangle : d \in U_w\}$, and
> 3. for every variable x of $\mathcal{L}$, $V_w(x)$ is either undefined or in U_w.
>
> If $V_w(x)$ is undefined for some w and x, then V is called *partial*, otherwise it is *total*.

> Definition 2.4 (Counterpart model)
> A *counterpart model $\mathcal{M}$* for a language $\mathcal{L}$ consists of a counterpart structure $\mathcal{S}$ together with an interpretation function V for $\mathcal{L}$ on $\mathcal{S}$ such that either $\mathcal{S}$ is single-domain or both $\mathcal{S}$ and V are total. In the first case, $\mathcal{M}$ is a *negative* model; in the second case, it is a *positive* model.

Thus a counterpart model is a collection of free first-order models, with relations R and C that link models and their domains. Variables are non-rigid in the sense that their interpretation is world-relative. However, we will see at the end of this section that the truth-value of a formula A at a world w never depends on what V assigns to variables at worlds $w' \neq w$. For instance, when we evaluate $\Diamond Fx$ at w, we do not check whether Fx is true at some accessible world w', i.e. whether $V_{w'}(x) \in V_{w'}(F)$. Rather, we check whether some individuals at w' that are C-related to $V_w(x)$ are in $V_{w'}(F)$. $V_{w'}(x)$ only enters the picture when we evaluate formulas relative to w'. If we had a designated centre world w_{c} in each model, we could drop the world-relativity of V for individual variables.

Note that in a negative model, $D_w = U_w$ can be empty. In positive models, D_w may be empty, but U_w must have at least one member, since $V_w(x) \in U_w$.

Now let's specify how formulas of $\mathcal{L}$ are evaluated at worlds. For the semantics of quantifiers, we need the concept of an x-variant of V.

Definition 2.5 (Variant)
Let V and V' be interpretations on a structure $\mathcal{S}$. V' is an *x-variant of V on w* if V' differs from V at most in the value assigned to x at w. V' is an *existential* x-variant of V on w if in addition, $V'(x) \in D_w$.

$\forall xA$ will then be true at a world w under V iff A is true at w under all existential x-variants V' of V on w. This rule allows us to dispose with assignment functions and to use free variables as individual constants, which makes the semantics slightly simpler. You may have noticed that individual constants are not explicitly mentioned in definition 2.2. However, unlike e.g. in [Kripke 1963] and [Lewis 1968], the lack of individual constants plays no important role in the present account. Whenever you want to use an individual constant, simply use a variable that never gets bound. If you want, you may also add a clause to the syntax to the effect that a certain class of variables cannot be bound, and call these variables 'individual constants'. Nothing hangs on this way of handling quantifiers. If you prefer a more traditional treatment with assignment functions and a clear separation between constants and variables, it is trivial to translate between the two approaches (see [Bostock 1997: 81–90]).

At worlds considered as counterfactual, variables denote counterparts of the things they originally denoted. So we need an operation that shifts the value of terms to the counterparts of their original value.

Definition 2.6 (Image)
Let V and V' be interpretations on a structure $\mathcal{S}$. V' is a *w'-image of V at w* (for short, $V_w \triangleright V'_{w'}$) if

(i) for every world w in $\mathcal{S}$ and predicate P, $V_w(P) = V'_w(P)$, and
(ii) for every variable x, if $\langle V_w(x), w\rangle$ has a counterpart at w', then $V'_{w'}(x)$ is one of these counterparts, otherwise $V'_{w'}(x)$ is undefined.

If (i) holds, I will also say that V and V' *agree on all predicates*.

$V'_{w'}(x)$ can only be undefined in negative models. In positive models, this cannot happen because $V_w(x)$ is always defined and the counterpart relation is total.

Definition 2.7 (Truth)
The relation $w, V \Vdash_{\mathcal{S}} A$ ("A is *true at w in $\mathcal{S}$ under V*") between a world w in a structure $\mathcal{S}$, an interpretation function V on $\mathcal{S}$, and a sentence A is defined as follows.

$$w, V \Vdash_{\mathcal{S}} Px_1 \dots x_n \text{ iff } \langle V_w(x_1), \dots, V_w(x_n)\rangle \in V_w(P).$$

$w, V \Vdash_{\mathcal{S}} \neg A$	iff $w, V \nVdash_{\mathcal{S}} A$.
$w, V \Vdash_{\mathcal{S}} A \supset B$	iff $w, V \nVdash_{\mathcal{S}} A$ or $w, V \Vdash_{\mathcal{S}} B$.
$w, V \Vdash_{\mathcal{S}} \forall x A$	iff $w, V' \Vdash_{\mathcal{S}} A$ for all existential x-variants V' of V on w.
$w, V \Vdash_{\mathcal{S}} \Box A$	iff $w', V' \Vdash_{\mathcal{S}} A$ for all w', V' such that wRw' and $V_w \triangleright V'_{w'}$.

I will drop the subscript $\mathcal{S}$ when the structure is clear from context.

Now we can prove that the value that V assigns to variables at other worlds never matters when evaluating formulas at a given world. This follows from the following lemma.

Lemma 2.8 (Coincidence lemma)
Let A be a sentence in a language $\mathcal{L}$ of quantified modal logic, w a world in a structure $\mathcal{S}$, and V, V' interpretations for $\mathcal{L}$ on $\mathcal{S}$ such that V and V' agree on all predicates, and $V_w(x) = V'_w(x)$ for every variable x that is free in A. (In this case, I will say that V and V' *agree at w on the variables in A*.) Then

$$w, V \Vdash_{\mathcal{S}} A \text{ iff } w, V' \Vdash_{\mathcal{S}} A.$$

Proof by induction on A.

1. For atomic formulas, the claim is guaranteed directly by definition 2.7.
2. A is $\neg B$. $w, V \Vdash \neg B$ iff $w, V \nVdash B$ by definition 2.7, iff $w, V' \nVdash B$ by induction hypothesis, iff $w, V' \Vdash \neg B$ by definition 2.7.
3. A is $B \supset C$. $w, V \Vdash B \supset C$ iff $w, V \nVdash B$ or $w, V \Vdash C$ by definition 2.7, iff $w, V' \nVdash B$ or $w, V' \Vdash C$ by induction hypothesis, iff $w, V' \Vdash B \supset C$ by definition 2.7.
4. A is $\forall x B$. By definition 2.7, $w, V \Vdash \forall x B$ iff $w, V^* \Vdash B$ for all existential x-variants V^* of V on w. Each such x-variant V^* agrees at w with the x-variant V'^* of V' on w such that $V'^*(x) = V^*(x)$ on all variables in B. Conversely, each existential x-variant V'^* of V' on w agrees at w with the x-variant V^* of V on w with $V^*(x) = V'^*(x)$ on all variables in B. So by induction hypothesis, $w, V^* \Vdash B$ for all existential x-variants V^* of V on w iff $w, V'^* \Vdash B$ for all existential x-variants V'^* of V' on w, iff $w, V' \Vdash \forall x B$ by definition 2.7.
5. A is $\Box B$. By definition 2.7, $w, V \Vdash \Box B$ iff $w', V^* \Vdash B$ for all w', V^* such that wRw' and V^* is a w'-image of V at w; i.e. $V^*_{w'}$ assigns to each variable x some counterpart of $V_w(x)$ (or nothing if there is none). For all x free in B, the counterparts at w' of $V_w(x)$ at w are precisely the counterparts at w' of $V'_w(x)$ at w, since $V_w(x) = V'_w(x)$. So each w'-image of V at w agrees with some w'-image of V' on all variables in B and

vice versa. So by induction hypothesis, $w', V^* \Vdash B$ for all w', V^* such that wRw' and $V_w \triangleright V^*_{w'}$ iff $w', V'^* \Vdash B$ for all w', V'^* such that wRw' and $V_w \triangleright V'^*_{w'}$, iff $w, V' \Vdash \Box B$ by definition 2.7. ■

Corollary 2.9 (Locality lemma)
If two interpretations V and V' on a structure $\mathcal{S}$ agree on all predicates and if for all variables x, $V_w(x) = V'_w(x)$, then for any formula A, $w, V \Vdash_{\mathcal{S}} A$ iff $w, V' \Vdash_{\mathcal{S}} A$.

Proof Immediate from lemma 2.8. ■

Finally, we define the notions of semantic validity and consequence.

Definition 2.10 (Validity)
A sentence A of a language $\mathcal{L}$ is *valid in a model* $\mathcal{M} = \langle W, R, U, D, C, V \rangle$ iff A is true at all worlds in $\mathcal{M}$. A is *valid in a class of models* $\mathbb{M}$ (or in a structure $\mathcal{S}$) iff A is valid in all models that belong to $\mathbb{M}$ (or $\mathcal{S}$).

Definition 2.11 (Semantic consequence)
Let $\mathbb{M}$ be a set of models or structures. A formula A is a *(local) consequence* of a set of formulas Γ *in* $\mathbb{M}$ iff for all worlds w in all models in $\mathbb{M}$, whenever all members of Γ are true at w, then so is A. Two formulas A and B are *(locally) equivalent in* $\mathbb{M}$ iff they are consequences of one another in $\mathbb{M}$.

3 Substitution and disambiguation

Before we look at logical systems for our models, we need to talk a little bit about substitution.

We want to allow an individual x at w to have multiple counterparts at another world w'. Evaluated at w', the variable x becomes “ambiguous”: it denotes several things at once. Standard counterpart semantics supervaluates for the box and consequently subvaluates for the diamond: $\Box Fx$ is true iff Fx is true at all accessible worlds under all disambiguations of x.

A crucial question is whether the disambiguations are uniform or mixed: should $\Box Gxx$ be true iff at all accessible worlds, all x counterparts are G-related to *themselves* (uniform) or to *one another* (mixed)? On the mixed account, $\Box x = x$ becomes invalid, as does $\Box(Fx \vee \neg Fx)$, even if x exists at all worlds. (One can still have the corresponding unboxed principles if one restricts Necessitation.) Moreover, the semantics becomes more complicated because a mixed disambiguation cannot be represented by a standard interpretation function; so if we say that $\Box A$ is true relative to interpretation V iff A is true at all accessible worlds under all interpretation functions V' suitably related to V, we automatically get uniform disambiguations. Thus I have used uniform disambiguations in the previous section. (See section **??** for the alternative route.)

The present issue might remind you of the old observation that a sentence like 'Brutus killed himself' can be understood either as an application of a *monadic* predicate 'killing himself' to the subject Brutus, or as an application of the binary 'killing' to Brutus and Brutus. Peter Geach once suggested a syntactic mechanism for distinguishing these readings, by introducing an operator $\langle z : x, y\rangle$ that turns a binary expression into a unary expression: while Gxy is satisfied by pairs of individuals as values of x and y, $\langle z : x, y\rangle Gxy$ is satisfied by a single individual for z. The operator $\langle z : x, y\rangle$, which might be read 'z is an x and a y such that' acts as a quantifier that binds both x and y.

We might use a similar trick. On the uniform reading, $\Box x = x$ says that all counterparts of x are self-identical at all accessible worlds. To say that at all accessible worlds all x-counterparts are identical to all x-counterparts we could instead say $\langle x : y, z\rangle \Box y = z$. The effect of $\langle x : y, z\rangle$ is to introduce two variables y and z that co-refer with x. By using distinct but co-refering variables in a modal context, we can express relations between possibly distinct counterparts; by using the same variable, we make sure that the same counterpart must be assigned to every occurrence.

With $\langle x : y, z\rangle \Box y = z$, we actually end up with *three* co-referring variables: y and z are made to co-refer with x, but we also have x itself. Thus the job can also be done with $\langle x : y\rangle \Box x = y$ – read: 'x is a y such that ...'.

To see the use of this operator, consider the following two sentences, which look at first glance like simple applications of universal instantiation.

(1) $\forall x \Box Gxy \supset \Box Gyy$;

(2) $\forall x \Diamond Gxy \supset \Diamond Gyy$.

The first says that if all things x are such that all x-counterparts are G-related to all y-counterparts, then all y-counterparts are G-related to themselves. That must be true. (2), however, is not valid. If all things x are such that some x-counterpart is G-related to some y-counterpart, it only follows that some y-counterpart is G-related to some y-counterpart; it does not follow that some y-counterpart is G-related to itself.

With the two distinct variables x and y, the antecedent formula $\Diamond Gxy$ looks at arbitrary combinations of x-counterparts and y-counterparts, even if $x = y$. In the consequent, however, $\Diamond Gyy$ only looks at single y counterparts and checks whether one of them is

G-related to itself. To prevent this accidental "capturing" of y we can use the Geach quantifier and write

(2′) $\forall x \Diamond Gxy \supset \langle y : x \rangle \Diamond Gxy$

in place of (2).

The same issue arises with Leibniz' Law. In the pair

(3) $x = y \supset \Box Gxy \supset \Box Gyy;$

(4) $x = y \supset \Diamond Gxy \supset \Diamond Gyy,$

only the first sentence is valid. In (4), the substituted variable y again gets captured by the other occurrence of y in the scope of the diamond. To avoid this, we should write

(4′) $x = y \supset \Diamond Gxy \supset \langle y : x \rangle \Diamond Gxy$

in place of (4). The Geach quantifier $\langle y : x \rangle$ thus functions as an *object-language substitution operator*.

If we want to use this operator, we have to extend the language of quantified modal logic.

> Definition 3.1 (Languages of QML with substitution)
> A *language of quantified modal logic with substitution* is like a standard language of quantified modal logic (definition 2.2) except that the set *Const* has a further ingredient $\langle\ :\ \rangle$ with the construction rule that $\langle y : x \rangle A$ is a sentence of $\mathcal{L}$ whenever x, y are variables and A is a sentence of $\mathcal{L}$.

As for the semantics: just as $\forall x A$ is true relative to an interpretation V iff A is true relative to all x-variants of V (on the relevant domain), $\langle y : x \rangle A$ is true relative to V iff A is true relative to the x-variant of V that maps x to $V(y)$. In our modal framework:

> Definition 3.2 (Semantics for the substitution operator)
> $w, V \Vdash \langle y : x \rangle A$ iff $w, V' \Vdash A$, where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$.

Note that V' need not be an *existential* x-variant of V on w.

The coincidence lemma 2.8 is easily adjusted to languages with substitution, and corollary 2.9 follows as before. I won't go through the whole proof again. Here is the only new step in the induction:

A is $\langle y : x\rangle B$. $w, V \Vdash \langle y : x\rangle B$ iff $w, V^* \Vdash B$ where V^* is the x-variant of V on w with $V^*_w(x) = V_w(y)$. Let V'^* be the x-variant of V' on w with $V'^*_w(x) = V'_w(y)$. Then V^* and V'^* agree at w on all variables in B, so by induction hypothesis, $w, V^* \Vdash B$ iff $V'^*, w \Vdash B$. And this holds iff $w, V' \Vdash \langle y : x\rangle B$ by the semantics of $\langle y : x\rangle$. ■

[Here I might discuss the relationship between $\langle y : x\rangle A$ and $(\lambda x.A)y$.]

It is well-known that in first-order logic, careless substitution of variables can cause accidental capturing. For example,

(5) $\quad x = y \supset (\exists y(x \neq y) \supset \exists y(y \neq y))$

is not a valid instance of Leibniz's Law, because the variable y gets captured by the quantifier $\exists y$. There are two common ways to respond. One is to define substitution as a simple replacement of variables, and restrict principles like Leibniz's Law:

(LL$^-$) $\quad x = y \supset (A \supset [y/x]A)$ provided x is free in A and y is free for x in A,

where y is *free for x in A* if no occurrence of x in a lies in the scope of a y-quantifier. The other response is to use a more sophisticated definition of substitution on which $\exists y(y \neq y)$ does not count as a proper substitution instance of $\exists y(x \neq y)$.

Informally, $[y/x]A$ should say about y exactly what A says about x. More precisely, proper substitutions should satisfy the following condition, sometimes called the "substitution lemma":

$$w, V \Vdash [y/x]A \text{ iff } w, V^{[y/x]} \Vdash A,$$

where $V^{[y/x]}$ is like V except that it assigns to x (at any world) the value V assigns to y.

This goal can be achieved by applying the substitution to an alphabetic variant of the original formula in which the bound variables have been renamed so that capturing can't happen. Thus before x is replaced by y in $\exists y(x \neq y)$, the variable y is made free for x by renaming all bound occurrences of y by some new variable z. Instead of (5), a legitimate instance of Leibniz's Law then is

(5′) $\quad x = y \supset (\exists y(x \neq y) \supset \exists z(y \neq z))$.

The following definition uses this idea to prevent capturing of variables by quantifiers. (See [Bell and Machover 1977: 54-67], [Gabbay et al. 2009: 87–103] for alternatives.[1]

1 An unconventional aspect of the present definition is that $[y/x]A$ can replace bound occurrences of x. For example, $[y/x]\forall xFx$ is $\forall yFy$. This has the advantage of generalising nicely to polyadic substitutions and especially transformations, defined below.

Definition 3.3 (Classical Substitution)
For any variables x, y, z let $[y/x]z = y$ if $z = x$ (i.e., z is the same variable as x), otherwise $[y/x]z = z$. For formulas A, define $[y/x]A = A$ if $x = y$; otherwise

$$
\begin{aligned}
&[y/x]Px_1 \ldots x_n &&= P[y/x]x_1 \ldots [y/x]x_n \\
&[y/x]\neg A &&= \neg[y/x]A; \\
&[y/x](A \supset B) &&= [y/x]A \supset [y/x]B; \\
&[y/x]\forall zA &&= \begin{cases} \forall v[y/x][v/z]A & \text{if } z{=}y \text{ and } x \in \mathit{Varf}(A), \text{ or } z{=}x \text{ and } y \in \mathit{Varf}(A), \\ \forall[y/x]z[y/x]A & \text{otherwise,} \end{cases} \\
& &&\text{where } v \text{ is the alphabetically first variable not in } \mathit{Var}(A), x, y. \\
&[y/x]\langle y_2 : z\rangle A &&= \begin{cases} \langle [y/x]y_2 : v\rangle[y/x][v/z]A & \text{if } z{=}y \text{ and } x \in \mathit{Varf}(A) \text{ or } z{=}x \text{ and } y \in \mathit{Varf}(A), \\ \langle [y/x]y_2 : [y/x]z\rangle[y/x]A & \text{otherwise,} \end{cases} \\
& &&\text{where } v \text{ is the alphabetically first variable not in } \mathit{Var}(A), x, y, y_2. \\
&[y/x]\Box A &&= \Box[y/x]A.
\end{aligned}
$$

In standard predicate logic, the last two clauses are empty because there is neither a box nor a substitution quantifier; I've added them because we'll need them later. The clauses for the substitution quantifier are exactly parallel to those for the universal quantifier, and the underlying motivation is the same. For example, $[y/x]\langle y_2 : y\rangle x{\neq}y$ is $\langle y_2 : z\rangle y{\neq}z$ rather than $\langle y_2 : y\rangle y{\neq}y$.

The clause for the box treats substitution into modal contexts as unproblematic. However, as Lewis observed, in counterpart semantics modal operators effectively function as unselective binders that capture all variables in their scope. As a consequence, our definition of substitution does not satisfy the substitution lemma. For instance, $w, V^{[y/x]} \Vdash \Diamond Gxy$ does not imply $w, V \Vdash \Diamond Gyy$. Informally, $\Diamond Gxy$ says about the individual x that at some world, one of its counterparts is G-related to some y-counterpart; $\Diamond Gyy$ does not say the same thing about y, for it says that at some world, one of y's counterparts is G-related to *itself*.

People sometimes complain that counterpart semantics doesn't validate the necessity of identity

(NI) $\quad x{=}y \supset (\Box x{=}x \supset \Box x{=}y)$,

which is allegedly an instance of Leibniz's Law. But you can't just look at the shape of a formula to see whether it is a legitimate instance of Leibniz's Law. If $\Box$ is synonymous to $\forall y$, then (NI) is a tautological variant of (5) and clearly invalid. Similarly, there can be no argument about whether (NI) is a legitimate instance of Leibniz's Law in counterpart semantics. It definitely isn't.

The situation is then similar to the situation of naive substitution in first-order logic. We can either restrict principles like Leibniz's Law or adjust the definition of substitution so that it satisfies the substitution lemma.

A suitable restriction is that in a legitimate substitution $[y/x]A$, the variable y must be *modally free for* x in A, in the following sense.

> DEFINITION 3.4 (MODAL SEPARATION AND MODAL FREEDOM)
> Two variables x and y are *modally separated in* a formula A if no free occurrences of x and y in A lie in the scope of the same modal operator.
>
> y is *modally free for* x *in* A if either (i) $x = y$, or (ii) x and y are modally separated in A, or (iii) A has the form $\Box B$ and y is modally free for x in B.

For example, y is modally free for x in $\Box x\!=\!y$ or $\Box\Box\neg Gxy$ or $\Box\Diamond\neg\exists xGxy$, but not in $\Box\Diamond\neg Gxy$. Correspondingly,

$$x\!=\!y \supset (\Box x\!=\!y \supset \Box y\!=\!y) \text{ and}$$
$$x\!=\!y \supset (\Box\Box\neg Gxy \supset \Box\Box\neg Gyy) \text{ and}$$
$$x\!=\!y \supset (\Box\Box\neg\exists xGxy \supset \Box\Box\neg\exists zGyz)$$

are valid, while

$$x\!=\!y \supset (\Box\Diamond\neg Gxy \supset \Box\Diamond\neg Gyy)$$

is invalid.

If we don't want to restrict the substitution principles, can we redefine substitution so that it satisfies the substitution lemma? In standard languages of quantified modal logic, this is not easy, as we will see in a moment, after we've spelled out some special conditions under which classical substitution does its job even in counterpart models.

> DEFINITION 3.5 (INTERPRETATION UNDER SUBSTITUTION)
> For any interpretation V of a language $\mathcal{L}$ on a structure $\mathcal{S}$ and variables x, y of $\mathcal{L}$, $V^{[y/x]}$ is the interpretation that is like V except that for any world w in $\mathcal{S}$, $V^{[y/x]}_w(x) = V_w(y)$.

> LEMMA 3.6 (RESTRICTED SUBSTITUTION LEMMA, PRELIMINARY VERSION)
> Let A be a sentence in a language $\mathcal{L}$ (with or without substitution), $\mathcal{S}$ a counterpart structure for $\mathcal{L}$, w a world in $\mathcal{S}$, V an interpretation on $\mathcal{S}$. Then
>
> $$w, V^{[y/x]} \Vdash_{\mathcal{S}} A \text{ iff } w, V \Vdash_{\mathcal{S}} [y/x]A, \text{ provided } y \notin \mathit{Var}(A).$$

Proof If $y = x$, then $[y/x]A = A$ and $V^{[y/x]} = V$, so the result is trivial. Assume then that $y \neq x$. The proof is by induction on A.

1. A is $Px_1 \dots x_n$. $w, V^{[y/x]} \Vdash Px_1 \dots x_n$ iff $\langle V_w^{[y/x]}(x_1), \dots, V_w^{[y/x]}(x_n)\rangle \in V_w^{[y/x]}(P)$ by definition 2.7, iff $\langle V_w([y/x]x_1), \dots, V_w([y/x]x_n)\rangle \in V_w(P)$ by definition 3.5, iff $w, V \Vdash P[y/x]x_1 \dots [y/x]x_n$ by definition 2.7, iff $w, V \Vdash [y/x]Px_1 \dots x_n$ by definition 3.3.

2. A is $\neg B$. $w, V^{[y/x]} \Vdash \neg B$ iff $w, V^{[y/x]} \nVdash B$ by definition 2.7, iff $w, V \nVdash [y/x]B$ by induction hypothesis, iff $w, V \Vdash [y/x]\neg B$ by definitions 2.7 and 3.3.

3. A is $B \supset C$. $w, V^{[y/x]} \Vdash B \supset C$ iff $w, V^{[y/x]} \nVdash B$ or $w, V^{[y/x]} \Vdash C$ By definition 2.7, iff $w, V \nVdash [y/x]B$ or $w, V \Vdash [y/x]C$ by induction hypothesis, iff $w, V \Vdash [y/x](B \supset C)$ by definitions 2.7 and 3.3.

4. A is $\forall zB$. Since $y \notin Var(A)$, $[y/x]A = \forall[y/x]z[y/x]B$. Assume first that $z \neq x$. By definition 2.7, $w, V^{[y/x]} \Vdash \forall zB$ iff $w, V^{[y/x]\prime} \Vdash B$ for all existential z-variants $V^{[y/x]\prime}$ of $V^{[y/x]}$ on w. These $V^{[y/x]\prime}$ are precisely the functions $V'^{[y/x]}$ where V' is an existential z-variant of V on w. So, $w, V^{[y/x]} \Vdash \forall zB$ iff $w, V'^{[y/x]} \Vdash B$ for all existential z-variants V' of V on w. By induction hypothesis, $w, V'^{[y/x]} \Vdash B$ iff $w, V' \Vdash [y/x]B$. So $w, V^{[y/x]} \Vdash \forall zB$ iff $w, V' \Vdash \forall z[y/x]B$ by definition 2.7, iff $w, V \Vdash [y/x]\forall zB$ by definition 3.3.

   Alternatively, assume $z = x$. By definition 3.3, $[y/x]\forall zB$ is $\forall y[y/x]B$. Assume $w, V \nVdash \forall y[y/x]B$. By definition 2.7, then $w, V' \nVdash [y/x]B$ for some existential y-variant V' of V on w. By induction hypothesis, then $w, V'^{[y/x]} \nVdash B$. Let V^* be the (existential) x-variant of V on w with $V_w^*(x) = V_w'^{[y/x]}(x) = V_w'(y)$. V^* is a y-variant on w of $V'^{[y/x]}$, and y is not free in B, so by the coincidence lemma 2.8, $w, V^* \nVdash B$. But V^* is also an existential x-variant of $V^{[y/x]}$ on w. So $w, V^{[y/x]} \nVdash \forall xB$ by definition 2.7.

   Conversely, assume $w, V^{[y/x]} \nVdash \forall xB$. By definition 2.7, then $w, V^* \nVdash B$ for some existential x-variant of $V^{[y/x]}$ (and thus V) on w. Let V' be the (existential) y-variant of V on w with $V_w'(y) = V_w^*(x)$. Then $V'^{[y/x]}$ and V^* agree at w on all variables except y; in particular, $V_w'^{[y/x]}(x) = V_w'(y) = V_w^*(x)$. Since y is not free in B, by the coincidence lemma 2.8, $w, V'^{[y/x]} \nVdash B$. By induction hypothesis, $w, V' \nVdash [y/x]B$. And since V' is an existential y-variant of V on w, then $w, V \nVdash \forall y[y/x]B$ by definition 2.7.

5. A is $\langle y_2 : z\rangle B$. Since $y \notin Var(A)$, $[y/x]A = \langle [y/x]y_2 : [y/x]z\rangle[y/x]B$ by definition 3.3. By definition 3.2, $w, V \Vdash \langle [y/x]y_2 : [y/x]z\rangle[y/x]B$ iff $w, V' \Vdash [y/x]B$, where V' is the $[y/x]z$-variant of V on w with $V_w'([y/x]z) = V_w([y/x]y_2)$. By induction hypothesis, $w, V' \Vdash [y/x]B$ iff $w, V'^{[y/x]} \Vdash B$. Let $V^{[y/x]\prime}$ be the z-variant of $V^{[y/x]}$ on w with $V_w^{[y/x]\prime}(z) = V_w^{[y/x]}(y_2)$. Then $V^{[y/x]\prime}$ and $V'^{[y/x]}$ agree at w about all variables v in A: for $v = z$, $V_w^{[y/x]\prime}(z) = V_w([y/x]y_2) = V_w'([y/x]z) = V_w'^{[y/x]}(z)$; for $v = x \neq z$, $V_w^{[y/x]\prime}(x) = V_w^{[y/x]}(x) = V_w(y) = V_w'(y)$ (because $y \neq z$ and hence $[y/x]z \neq y$) $= V_w'([y/x]x) = V_w'^{[y/x]}(x)$; and for $v \notin \{x, y, z\}$, $V_w^{[y/x]\prime}(v) = V_w^{[y/x]}(v) = V_w(v) = V_w'(v) = V_w'^{[y/x]}(v)$. So by the coincidence lemma 2.8, $w, V'^{[y/x]} \Vdash B$ iff $w, V^{[y/x]\prime} \Vdash B$. And by definition 3.2, $w, V^{[y/x]\prime} \Vdash B$ iff $w, V^{[y/x]} \Vdash \langle y_2 : z\rangle B$.

6. A is $\Box B$. By definition 2.7, $w, V^{[y/x]} \Vdash \Box B$ iff $w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ with wRw' and $V_w^{[y/x]} \triangleright V_{w'}^{[y/x]\prime}$. On the other hand, $w, V \Vdash [y/x]\Box B$ iff $w, V \Vdash \Box[y/x]B$ (by definition 3.3), iff $w', V' \Vdash [y/x]B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. By induction hypothesis, $w', V'^{[y/x]} \Vdash B$ iff $w', V' \Vdash [y/x]B$. So we have to show that

   (1) $\quad w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ such that wRw' and $V_w^{[y/x]} \triangleright V_{w'}^{[y/x]\prime}$

   iff

   (2) $\quad w', V'^{[y/x]} \Vdash B$ for all w', V' such that wRw' and $V_w \triangleright V'_{w'}$.

   (1) implies (2) because every interpretation $V'^{[y/x]}$ with $V_w \triangleright V'_{w'}$ is also an interpretation $V^{[y/x]\prime}$ with $V_w^{[y/x]} \triangleright V_{w'}^{[y/x]\prime}$.

   Assume y is not free in $\Box B$, and that (2) holds. In order to derive (1), consider any w'-image $V^{[y/x]\prime}$ of $V^{[y/x]}$ at w. Let V^* be like $V^{[y/x]\prime}$ except that $V^*_{w'}(y) = V_{w'}^{[y/x]\prime}(x)$. Let V' be like V^* except that $V'_{w'}(x)$ is some counterpart of $V_w(x)$, or undefined if there is none. Then $V_w \triangleright V'_{w'}$. (In particular, $V'_{w'}(y) = V^*_{w'}(y) = V^{[y/x]\prime}(x)$ is some counterpart of $V_w^{[y/x]}(x) = V_w(y)$, or undefined if there is none.) So by (2), $w', V'^{[y/x]} \Vdash B$. But $V'^{[y/x]} = V^*$ (since $V^*_{w'}(x) = V^*_{w'}(y)$). So $w', V^* \Vdash B$. And since y is not free in B and V^* is a y-variant of $V^{[y/x]\prime}$ on w', by the coincidence lemma 2.8, $w', V^* \Vdash B$ iff $w', V^{[y/x]\prime} \Vdash B$. ■

A stronger version of this will be proved as lemma 3.9 below. We only need this version to verify that renaming bound variables (a.k.a. α-conversion) does not affect truth-values.

Definition 3.7 (Alphabetic variant)
A formula A' of a language of quantified modal logic (with or without substitution) is an *alphabetic variant of* a formula A if one of the following conditions is satisfied.

1. $A = A'$.
2. $A = \neg B$, $A' = \neg B'$, and B' is an alphabetic variant of B.
3. $A = B \supset C$, $A' = B' \supset C'$, and B', C' are alphabetic variants of B, C, respectively.
4. $A = \forall x B$, $A' = \forall z [z/x] B'$, B' is an alphabetic variant of B, and either $z = x$ or $z \notin \mathit{Var}(B')$.
5. $A = \langle y : x \rangle B$, $A' = \langle y : z \rangle [z/x] B'$, B' is an alphabetic variant of B, and either $z = x$ or $z \notin \mathit{Var}(B')$.
6. $A = \Box B$, $A' = \Box B'$, and B' is an alphabetic variant of A'.

Lemma 3.8 (alpha-conversion lemma)
If a formula A' is an alphabetic variant of a formula A, then for any world w in any structure $\mathcal{S}$ and any interpretation V on $\mathcal{S}$,

$$w, V \Vdash_{\mathcal{S}} A \text{ iff } w, V \Vdash_{\mathcal{S}} A'.$$

Proof by induction on A.

1. A is atomic. Then $A = A'$ and the claim is trivial.
2. A is $\neg B$. Then A' is $\neg B'$, where B' is an alphabetic variant of B. By induction hypothesis, $w, V \Vdash B$ iff $w, V \Vdash B'$. So $w, V \Vdash \neg B$ iff $w, V \Vdash \neg B'$ by definition 2.7.
3. A is $B \supset C$. Then A' is $B' \supset C'$, where B', C' are alphabetic variants of B, C, respectively. By induction hypothesis, $w, V \Vdash B$ iff $w, V \Vdash B'$ and $w, V \Vdash C$ iff $w, V \Vdash C'$. So $w, V \Vdash B \supset C$ iff $w, V \Vdash B' \supset C'$ by definition 2.7.
4. A is $\forall x B$. Then A' is either $\forall x B'$ or $\forall z[z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B')$. If B is $\forall x B'$, then by definition 2.7, $w, V \Vdash \forall x B$ iff $w, V' \Vdash B$ for all existential x-variants V' of V on w, which, by induction hypothesis, holds iff $w, V' \Vdash B'$ for all such V', i.e. (by definition 2.7 again) iff $w, V \Vdash \forall x B'$.

   Consider then the case where B is $\forall z[z/x]B'$, with $z \notin \mathit{Var}(B')$. This means that z is not free in B, because alphabetic variants never differ in their free variables. Now if $w, V \nVdash \forall z[z/x]B'$, then by definition 2.7 there is an existential z-variant V^* of V on w such that $w, V^* \nVdash [z/x]B'$. And then $w, (V^*)^{[z/x]} \nVdash B'$ by lemma 3.6. By induction hypothesis, then $w, (V^*)^{[z/x]} \nVdash B$. Let V' be the x-variant of V on w with $V'(x) = (V^*)^{[z/x]}(x) = V^*(z)$. Since z is not free in B, V' and $(V^*)^{[z/x]}$ agree at w about all free variables in B. So by the coincidence lemma 2.8, $w, V' \nVdash B$. And so $w, V \nVdash \forall x B$ by definition 2.7.

   The converse, that if $w, V \Vdash \forall z[z/x]B'$, then $w, V \Vdash \forall x B$, follows from the fact that $\forall x B$ is $\forall x[x/z][z/x]B'$ and thus an alphabetic variant of $\forall z[z/x]B'$.
5. A is $\langle y : x \rangle B$. Then A' is either $\langle y : x \rangle B'$ or $\langle y : z \rangle [z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B')$. Assume first that B is $\langle y : x \rangle B'$. By definition 3.2, $w, V \Vdash \langle y : x \rangle B$ iff $w, V' \Vdash B$ where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$. By induction hypothesis, this holds iff $w, V' \Vdash B'$, i.e. (by definition 2.7 again) iff $w, V \Vdash \langle y : x \rangle B'$.

   Consider then the case where B is $\langle y : z \rangle [z/x]B'$, with $z \notin \mathit{Var}(B')$. This means that z is not free in B, because alphabetic variants never differ in their free variables. By definition 2.7, $w, V \Vdash \langle y : x \rangle B$ iff $w, V' \Vdash B$, where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$. Let V^* be the z-variant of V on w with $V^*_w(z) = V'_w(x) = V_w(y)$. Since z is not free in B, V' and $(V^*)^{[z/x]}$ agree at w about all variables in B. So by the coincidence lemma 2.8, $w, V' \Vdash B$ iff $w, (V^*)^{[z/x]} \Vdash B$. By induction hypothesis, $w, (V^*)^{[z/x]} \Vdash B$ iff $w, (V^*)^{[z/x]} \Vdash B'$. By lemma 3.6, $w, (V^*)^{[z/x]} \Vdash B'$ iff $w, V^* \Vdash [z/x]B'$. And by definition 2.7, $w, V^* \Vdash [z/x]B'$ iff $w, V \Vdash \langle y : z \rangle [z/x]B'$.

6. A is $\Box B$. Then A' is $\Box B'$, where B' is an alphabetic variant of B. By definition 2.7, $w, V \Vdash \Box B$ iff $w', V' \Vdash B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$, and $w, V \Vdash \Box B'$ iff $w', V' \Vdash B'$ for all such w', V'. By induction hypothesis, $w', V' \Vdash B$ iff $w', V' \Vdash B'$. So $w, V \Vdash \Box B$ iff $w, V \Vdash \Box B'$ by definition 2.7. ∎

Now for the more general version of lemma 3.6.

Lemma 3.9 (Restricted substitution lemma)
Let A be a sentence in a language $\mathcal{L}$ of quantified modal logic (with or without substitution), $\mathcal{S}$ a counterpart structure for $\mathcal{L}$, w a world in $\mathcal{S}$, V an interpretation on $\mathcal{S}$. Then

(i) $w, V^{[y/x]} \Vdash_{\mathcal{S}} A$ iff $w, V \Vdash_{\mathcal{S}} [y/x]A$, provided that either
  (a) y and x are modally separated in A, or
  (b) it is not the case that $V_w(y)$ has multiple counterparts at any world.

(ii) if $w, V^{[y/x]} \Vdash_{\mathcal{S}} A$, then $w, V \Vdash_{\mathcal{S}} [y/x]A$, provided that y is modally free for x in A.

Proof If y and x are the same variable, then $V^{[y/x]}$ is V, and $[y/x]A$ is A; so trivially $w, V^{[y/x]} \Vdash A$ iff $w, V \Vdash [y/x]A$. Assume then that y and x are different variables. The proof is by induction on A.

For the base case, the provisos can be ignored: $w, V^{[y/x]} \Vdash Px_1 \ldots x_n$ iff $w, V \Vdash [y/x]Px_1 \ldots x_n$ by the same reasoning as in lemma 3.6. For complex A, the induction hypothesis is that (i) and (ii) hold for formulas of lower complexity, in particular for subformulas of A. Note that if one of the provisos of (i) and (ii) applies to A, then it also applies to subformulas for A. (For example, if y is modally free for x in A, then y is modally free for x in every subformula of A.) Moreover, if A is not of the form $\Box B$, then the proviso of (ii) entails proviso (a) of (i), because y is modally free for x in $A \neq \Box B$ only if y and x do not occur together in the scope of a modal operator in A.

1. A is $\neg B$. By definition 2.7, $w, V^{[y/x]} \Vdash \neg B$ iff $w, V^{[y/x]} \nVdash B$. Since a proviso of (i) or (ii) applies to A and therefore a proviso of (i) applies to B, by induction hypothesis, $w, V^{[y/x]} \nVdash B$ iff $w, V \nVdash [y/x]B$. And the latter holds iff $w, V \Vdash [y/x]\neg B$ by definitions 2.7 and 3.3.

2. A is $B \supset C$. By definition 2.7, $w, V^{[y/x]} \Vdash B \supset C$ iff $w, V^{[y/x]} \nVdash B$ or $w, V^{[y/x]} \Vdash C$. Since a proviso of (i) or (ii) applies to A and therefore a proviso of (i) applies to B and C, by induction hypothesis, $w, V^{[y/x]} \nVdash B$ iff $w, V \nVdash [y/x]B$, and $w, V^{[y/x]} \Vdash C$ iff $w, V \Vdash [y/x]C$. So $w, V^{[y/x]} \Vdash B \supset C$ iff $w, V \Vdash [y/x](B \supset C)$ by definitions 2.7 and 3.3.

3. A is $\forall zB$. Assume first that $[y/x]\forall zB$ is $\forall[y/x]z[y/x]B$, i.e. (by definition 3.3) neither $z = y$ and $x \in \mathit{Varf}(B)$ nor $z = x$ and $y \in \mathit{Varf}(B)$. By definition 2.7, $w, V \Vdash \forall[y/x]z[y/x]B$ iff $w, V' \Vdash [y/x]B$ for all existential $[y/x]z$-variants V' of V on w. Since a

proviso of (i) or (ii) applies to A and therefore a proviso of (i) applies to B, by induction hypothesis, $w, V' \Vdash [y/x]B$ iff $w, V'^{[y/x]} \Vdash B$.

Now assume $z \notin \{x, y\}$. Then $V'^{[y/x]}_w(x) = V'^{[y/x]}_w(y) = V'_w(y) = V_w(y)$ and $V'^{[y/x]}_w(z) = V'^{[y/x]}_w([y/x]z)$ is some arbitrary member of D_w. So the interpretations $V'^{[y/x]}$ coincide with the existential z-variants $V^{[y/x]\prime}$ of $V^{[y/x]}$ on w. Alternatively, if $z = x$, and thus $y \notin \mathit{Varf}(B)$, then $V'^{[y/x]}_w(x)$ is some arbitrary member of D_w, as is $V^{[y/x]\prime}_w(x)$. Similarly, if $z = y$ and thus $x \notin \mathit{fvar}(B)$, then $V'^{[y/x]}_w(y)$ is some arbitrary member of D_w, as is $V^{[y/x]\prime}_w(y)$. In either case, the interpretations $V'^{[y/x]}_w$ can be paired with the interpretations $V^{[y/x]\prime}$ such that the members of each pair agree at w about all free variables in B. So by the coincidence lemma 2.8, $w, V'^{[y/x]} \Vdash B$ for all existential $[y/x]z$-variants V' of V on w iff $w, V^{[y/x]\prime} \Vdash B$ for all existential z-variants $V^{[y/x]\prime}$ of $V^{[y/x]}$ on w, iff $w, V^{[y/x]} \Vdash \forall z B$ by definition 2.7.

Second, assume $[y/x]\forall z B$ is $\forall v[y/x][v/z]B$, for some new variable v. By the α-conversion lemma 3.8, $w, V^{[y/x]} \Vdash \forall z B$ iff $w, V^{[y/x]} \Vdash \forall v[v/z]B$. Since $v \notin \{x, y\}$, we can reason as above, with $[v/z]B$ in place of B, to show that $w, V \Vdash \forall v[y/x][v/z]B$ iff $w, V^{[y/x]} \Vdash \forall v[v/z]B$.

4. A is $\langle y_2 : z\rangle B$. This case is similar to the previous one. Assume first that $[y/x]\langle y_2 : z\rangle B$ is $\langle [y/x]y_2 : [y/x]z\rangle[y/x]B$, i.e. (by definition 3.3) neither $z = y$ and $x \in \mathit{Varf}(B)$ nor $z = x$ and $y \in \mathit{Varf}(B)$. By definition 2.7, $w, V \Vdash \langle [y/x]y_2 : [y/x]z\rangle[y/x]B$ iff $w, V' \Vdash [y/x]B$, where V' is the $[y/x]z$-variant of V on w with $V'_w([y/x]z) = V_w([y/x]y_2)$. Since a proviso of (i) or (ii) applies to A and therefore a proviso of (i) applies to B, by induction hypothesis, $w, V' \Vdash [y/x]B$ iff $w, V'^{[y/x]} \Vdash B$.

   Let V^* be the z-variant of $V^{[y/x]}$ on w with $V^*_w(z) = V_w([y/x]y_2)$. If $z \notin \{x, y\}$, then $V^*_w(x) = V^*_w(y) = V_w(y)$ and $V^*(z) = V_w([y/x]y_2)$. Moreover, $V'^{[y/x]}_w(x) = V'^{[y/x]}_w(y) = V_w(y)$ and $V'^{[y/x]}_w(z) = V'^{[y/x]}_w([y/x]z) = V_w([y/x]y_2)$. So $V'^{[y/x]}$ and V^* agree about all variables at w. Alternatively, if $z = x$, and thus $y \notin \mathit{Varf}(B)$, then $V'^{[y/x]}_w(x) = V_w([y/x]y_2) = V^*_w(x)$. Similarly, if $z = y$, and thus $x \notin \mathit{Varf}(B)$, then $V'^{[y/x]}_w(y) = V_w([y/x]y_2) = V^*_w(y)$. Either way, $V'^{[y/x]}$ and V^* agree at w about all free variables in B. By the coincidence lemma 2.8, $w, V'^{[y/x]} \Vdash B$ iff $w, V^* \Vdash B$, iff $w, V^{[y/x]} \Vdash \langle [y/x]y_2 : z\rangle B$ by definition 2.7.

5. A is $\Box B$. This is the interesting part. We have to go piecemeal.

   (i). By definition 2.7, $w, V^{[y/x]} \Vdash \Box B$ iff $w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ with wRw' and $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$. On the other hand, $w, V \Vdash [y/x]\Box B$ iff $w, V \Vdash \Box[y/x]B$ (by definition 3.3), iff $w', V' \Vdash [y/x]B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. Since the provisos of (i) carry over from $\Box B$ to B, by induction hypothesis, $w', V'^{[y/x]} \Vdash B$ iff $w', V' \Vdash [y/x]B$. So we have to show that

   (1) $\quad w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ such that wRw' and $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$

   iff

   (2) $\quad w', V'^{[y/x]} \Vdash B$ for all w', V' such that wRw' and $V_w \triangleright V'_{w'}$.

   (1) implies (2) because every interpretation $V'^{[y/x]}$ with $V_w \triangleright V'_{w'}$ is also an interpretation $V^{[y/x]\prime}$ with $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$. The converse, however, may fail: both $V'^{[y/x]}_{w'}$ and $V^{[y/x]\prime}_{w'}$

assign to x and y some counterpart of $V_w(y)$ (if there is any). But while $V'^{[y/x]}_{w'}$ assigns *the same* counterpart to x and y, $V^{[y/x]\prime}_{w'}$ may choose different counterparts for x and y.

If it is not the case that $V_w(y)$ has multiple counterparts at any accessible world w' from w, then this cannot happen. Thus under proviso (b), each $V^{[y/x]\prime}$ is also a $V'^{[y/x]}$, and so (2) implies (1).

For proviso (a), assume x and y do not both occur in the scope of a modal operator in $\Box B$. Then either x or y does not occur at all in $\Box B$. Assume first that x does not occur in $\Box B$. Then $[y/x]\Box B$ is $\Box B$ (by definition 3.3), and $w, V^{[y/x]} \Vdash \Box B$ iff $w, V \Vdash [y/x]\Box B$ by the coincidence lemma 2.8. Alternatively, assume that y does not occur in $\Box B$, and that (2) holds. In order to derive (1), consider any w'-image $V^{[y/x]\prime}$ of $V^{[y/x]}$ at w. Let V^* be like $V^{[y/x]\prime}$ except that $V^*_{w'}(y) = V^{[y/x]\prime}_{w'}(x)$. Let V' be like V^* except that $V'_{w'}(x)$ is some counterpart of $V_w(x)$, or undefined if there is none. Then $V_w \triangleright V'_{w'}$. (In particular, $V'_{w'}(y) = V^*_{w'}(y) = V^{[y/x]\prime}(x)$ is some counterpart of $V^{[y/x]}_w(x) = V_w(y)$, or undefined if there is none.) So by (2), $w', V'^{[y/x]} \Vdash B$. But $V'^{[y/x]} = V^*$ (since $V^*_{w'}(x) = V^*_{w'}(y)$). So $w', V^* \Vdash B$. And since $y \notin \mathit{Var}(B)$ and V^* is a y-variant of $V^{[y/x]\prime}$ on w', by the coincidence lemma 2.8, $w', V^* \Vdash B$ iff $w', V^{[y/x]\prime} \Vdash B$.

(ii). Assume $w, V^{[y/x]} \Vdash \Box B$. By definition 2.7, then $w', V^{[y/x]\prime} \Vdash B$ for all $w', V^{[y/x]\prime}$ with wRw' and $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$. As before, every interpretation $V'^{[y/x]}$ with $V_w \triangleright V'_{w'}$ is also an interpretation $V^{[y/x]\prime}$ with $V^{[y/x]}_w \triangleright V^{[y/x]\prime}_{w'}$. So $w', V'^{[y/x]} \Vdash B$ for all $w', V'^{[y/x]}$ with wRw' and $V_w \triangleright V'_{w'}$.

If y is modally free for x in $\Box B$, then y is modally free for x in B. Then by induction hypothesis, $w', V' \Vdash [y/x]B$ *if* $w', V'^{[y/x]} \Vdash B$. So $w', V' \Vdash [y/x]B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. By definition 2.7, this means that $w, V \Vdash \Box[y/x]B$, and so $w, V \Vdash [y/x]\Box B$ by definition 3.3. ■

The converse of (ii) is not true. E.g., $w, V \Vdash [y/x]\Box x\!=\!y$ does not imply $w, V^{[y/x]} \Vdash \Box x\!=\!y$. So the operation $[y/x]$, as defined in definition 3.3, does not always satisfy the "substitution lemma", not even when y is modally free for x.

Can we fix the definition? No – at least not if we allow for positive models. There is no operation Φ on sentences in standard languages of quantified modal logic such that in any (positive) model, $w, V^{[y/x]} \Vdash A$ iff $w, V \Vdash \Phi(A)$, and therefore no translation of $\langle y : x\rangle A$ into those languages. To prove this, we show that there are distinctions one can draw with $\langle y : x\rangle$ that cannot be drawn without it. In particular, the substitution quantifier allows us to say that an individual y has multiple counterparts at some accessible world: $\langle y : x\rangle \Diamond y \neq x$.

(In negative models, $\langle y : x\rangle A$ can be translated into $\exists x(x\!=\!y \wedge A) \vee (\neg Ey \wedge [y/x]A)$, which still has the downside of being very impractical, since $\Phi(A)$ can have much greater syntactic complexity than A.)

It is clear that $\Diamond y \neq y$ is not an adequate translation of $\langle y : x\rangle \Diamond x \neq y$. Before substituting y for x in $\Diamond x \neq y$, we would have to make x free for y by renaming the modally bound occurrence of y. However, the diamond, unlike the quantifier $\forall y$, binds

y in such a way that the domain over which it ranges (the counterparts of y's original referent) depends on the previous reference of y. So we can't just replace y by some other variable z, translating $\langle y : x\rangle\Diamond x\neq y$ as $\Diamond y\neq z$. This only works if z happens to corefer with y. Since we can't presuppose that there is always another name available for any given individual, we would somehow have to introduce a name z that corefers with y. For instance, we could translate $\Diamond x\neq y$ into $\exists z(y=z \wedge \Diamond x\neq z)$. Now x has become free for y in the scope of the diamond, so we can translate $\langle y : x\rangle\Diamond x\neq y$ as $\exists z(y=z \wedge \Diamond x\neq y)$. The problem with this is that the quantifier $\exists$ ranges only over existing objects, while $\langle y : x\rangle$ bears no such restriction. In positive models, $V_w(y)$ can have multiple counterparts even if it lies outside D_w, so that $\exists z(y = z \wedge \Diamond x \neq y)$ is false. (One would need an "outer quantifier" in place of $\exists$.)

Here is the full proof.

> **Theorem 3.10 (Undefinability of substitution)**
> There is no operation Φ on formulas A in a standard language $\mathcal{L}$ of quantified modal logic such that for all worlds w in all positive counterpart models $\langle\mathcal{S}, V\rangle$, $w, V \Vdash_{\mathcal{S}} \Phi(A)$ iff $w, V^{[y/x]} \Vdash_{\mathcal{S}} A$.

Proof Let $\mathcal{M}_1 = \langle\mathcal{S}_1, V\rangle$ be a positive counterpart model with $W = \{w\}$, $R = \{\langle w, w\rangle\}$, $U_w = \{x, y, y^*\}$, $D_w = \{x\}$, $C = \{\langle\langle w, d\rangle, \langle w, d\rangle\rangle : d \in U_w\}, V_w(y) = y$, $V_w(z) = x$ for every variable $z \neq y$, and $V_w(P) = \emptyset$ for all non-logical predicates P. Let $\mathcal{M}_2 = \langle\mathcal{S}_2, V\rangle$ be like $\mathcal{M}_1$ except that y^* is also a counterpart of y, i.e. $C = \{\langle\langle w, x\rangle, \langle w, x\rangle\rangle, \langle\langle w, y\rangle, \langle w, y\rangle\rangle, \langle\langle w, y^*\rangle, \langle w, y^*\rangle\rangle, \langle\langle w, y\rangle, \langle w, y^*\rangle\rangle\}$. Then $w, V^{[y/x]} \Vdash_{\mathcal{S}_2} \Diamond y\neq x$, but $w, V^{[y/x]} \nVdash_{\mathcal{S}_1} \Diamond y\neq x$.

On the other hand, every $\mathcal{L}$-sentence has the same truth-value at w under V in both models. We prove this by showing that for every $\mathcal{L}$-sentence A, $w, V \Vdash_{\mathcal{S}_1} A$ iff $w, V \Vdash_{\mathcal{S}_2} A$ iff $w, V^* \Vdash_{\mathcal{S}_2} A$, where V^* is the y-variant of V on w with $V^*_w(y) = V_w(y^*)$.

1. A is $Px_1 \ldots x_n$. It is clear that $w, V \Vdash_{\mathcal{S}_1} Px_1 \ldots x_n$ iff $w, V \Vdash_{\mathcal{S}_2} Px_1 \ldots x_n$ because the counterpart relation does not figure in the evaluation of atomic formulas. Moreover, for non-logical P, $w, V \nVdash_{\mathcal{S}_2} Px_1 \ldots x_n$ and $w, V^* \nVdash_{\mathcal{S}_2} Px_1 \ldots x_n$, because $V_w(P) = V^*_w(P) = \emptyset$. For the identity predicate, observe that $w, V \nVdash_{\mathcal{S}_2} u=v$ iff exactly one of u, v is y, since $V_w(z) = x$ for all terms $z \neq y$. For the same reason, $w, V^* \nVdash_{\mathcal{S}_2} u=v$ iff exactly one of u, v is y. So $w, V \Vdash_{\mathcal{S}_2} u=v$ iff $w, V^* \Vdash_{\mathcal{S}_2} u=v$.

2. A is $\neg B$. $w, V \Vdash_{\mathcal{S}_1} \neg B$ iff $w, V \nVdash_{\mathcal{S}_1} B$ by definition 2.7, iff $w, V \nVdash_{\mathcal{S}_2} B$ by induction hypothesis, iff $w, V \Vdash_{\mathcal{S}_2} \neg B$ by definition 2.7. Moreover, $w, V \nVdash_{\mathcal{S}_2} B$ iff $w, V^* \nVdash_{\mathcal{S}_2} B$ by induction hypothesis, iff $w, V^* \Vdash_{\mathcal{S}_2} \neg B$ by definition 2.7.

3. A is $B \supset C$. $w, V \Vdash_{\mathcal{S}_1} B \supset C$ iff $w, V \nVdash_{\mathcal{S}_1} B$ or $w, V \Vdash_{\mathcal{S}_1} C$ by definition 2.7, iff $w, V \nVdash_{\mathcal{S}_2} B$ or $w, V \Vdash_{\mathcal{S}_2} C$ by induction hypothesis, iff $w, V \Vdash_{\mathcal{S}_2} B \supset C$ by definition 2.7. Moreover, $w, V \nVdash_{\mathcal{S}_2} B$ or $w, V \Vdash_{\mathcal{S}_2} C$, iff $w, V^* \nVdash_{\mathcal{S}_2} B$ or $w, V^* \Vdash_{\mathcal{S}_2} C$ by induction hypothesis, iff $w, V^* \Vdash_{\mathcal{S}_2} B \supset C$ by definition 2.7.

4. A is $\forall zB$. Let v be a variable not in $Var(B) \cup \{y\}$. By lemma 3.8, $w, V \Vdash_{\mathcal{S}_1} \forall zB$ iff $w, V \Vdash_{\mathcal{S}_1} \forall v[v/z]B$. By definition 2.7, $w, V \Vdash_{\mathcal{S}_1} \forall v[v/z]B$ iff $w, V' \Vdash_{\mathcal{S}_1} [v/z]B$ for all existential v-variants V' of V on w. As $D_w = \{x\}$ and $V(v) = x$, the only such v-variant is V itself. So $w, V \Vdash_{\mathcal{S}_1} \forall zB$ iff $w, V \Vdash_{\mathcal{S}_1} [v/z]B$. By the same reasoning, $w, V \Vdash_{\mathcal{S}_2} \forall zB$ iff $w, V \Vdash_{\mathcal{S}_2} [v/z]B$. But by induction hypothesis, $w, V \Vdash_{\mathcal{S}_1} [v/z]B$ iff $w, V \Vdash_{\mathcal{S}_2} [v/z]B$. So $w, V \Vdash_{\mathcal{S}_1} \forall zB$ iff $w, V \Vdash_{\mathcal{S}_2} \forall zB$. Moreover, by induction hypothesis, $w, V \Vdash_{\mathcal{S}_2} [v/z]B$ iff $w, V^* \Vdash_{\mathcal{S}_2} [v/z]B$, iff $w, V^* \Vdash_{\mathcal{S}_2} \forall v[v/z]B$ because V^* is the only existential v-variant of V^* on w, iff $w, V^* \Vdash_{\mathcal{S}_2} \forall zB$ by lemma 3.8.
5. A is $\Box B$. In both structures, the only world accessible from w is w itself. Also in $\mathcal{S}_1$, V is the only w-image of V at w. So by definition 2.7, $w, V \Vdash_{\mathcal{S}_1} \Box B$ iff $w, V \Vdash_{\mathcal{S}_1} B$. In $\mathcal{S}_2$, there are two w-images of V at w: V and V^*. So $w, V \Vdash_{\mathcal{S}_2} \Box B$ iff both $w, V \Vdash_{\mathcal{S}_2} B$ and $w, V^* \Vdash_{\mathcal{S}_2} B$. By induction hypothesis, $w, V \Vdash_{\mathcal{S}_1} B$ iff both $w, V \Vdash_{\mathcal{S}_2} B$ and $w, V^* \Vdash_{\mathcal{S}_2} B$. So $w, V \Vdash_{\mathcal{S}_1} \Box B$ iff $w, V \Vdash_{\mathcal{S}_2} \Box B$. Moreover, in $\mathcal{S}_2$, V^* is the only w-image of V^* at w. So $w, V^* \Vdash_{\mathcal{S}_2} \Box B$ iff $w, V^* \Vdash_{\mathcal{S}_2} B$. By induction hypothesis, $w, V^* \Vdash_{\mathcal{S}_2} B$ iff $w, V \Vdash_{\mathcal{S}_2} B$. So $w, V^* \Vdash_{\mathcal{S}_2} \Box B$ iff both $w, V^* \Vdash_{\mathcal{S}_2} B$ and $w, V \Vdash_{\mathcal{S}_2} B$, which as we just saw holds iff $w, V \Vdash_{\mathcal{S}_2} \Box B$. ■

What we can do instead is introduce a new syntactic construction into the language that satisfies the substitution lemma by stipulation. This is what the substitution quantifier does. I have given its semantics in definition 3.2 by saying that $w, V \Vdash \langle y : x \rangle A$ iff $w, V' \Vdash A$, where V' is the x-variant of V on w with $V'_w(x) = V_w(y)$. By the locality lemma (corollary 2.9 of lemma 2.8), it immediately follows that

$$w, V \Vdash \langle y : x \rangle A \text{ iff } w, V^{[y/x]} \Vdash A.$$

In the following, I will consider both systems in extended languages that include the substitution quantifier $\langle y : x \rangle$ and systems in standard languages that exclude it. The advantage of having the substitution quantifier is that it not only adds welcome expressive resources to the language, but also makes the logic and model theory somewhat more streamlined, because the corresponding versions of principles like Leibniz's Law,

$$x = y \supset (A \supset \langle y : x \rangle A)$$

hold without restrictions.

It will be useful to have a notion of substitution that applies to several variables at once. To this end, let's generalise definition 3.3 (classical substitution).

> Definition 3.11 (Classical substitution, generalised)
> A **substitution** on a language $\mathcal{L}$ is a total function $\sigma : Var(\mathcal{L}) \to Var(\mathcal{L})$. If σ is injective, it is called a **transformation**. I write $[y_1, \ldots, y_n / x_1, \ldots, x_n]$ for the substitution that maps x_1 to y_1, …, x_n to y_n, and every other variable to itself.
>
> Application of a substitution σ to a formula A is defined as follows.

$$
\begin{aligned}
\sigma(Px_1 \dots x_n) &= P\sigma(x_1)\dots\sigma(x_n)\\
\sigma(\neg A) &= \neg\sigma(A);\\
\sigma(A \supset B) &= \sigma(A) \supset \sigma(B);\\
\sigma(\forall z A) &= \begin{cases} \forall v \sigma'([v/z]A) & \text{if there is an } x \text{ free in } \forall zA \text{ with } \sigma(x) = \sigma(z),\\ \forall\sigma(z)\sigma(A) & \text{otherwise,}\end{cases}\\
& \text{where } \sigma' \text{ is like } \sigma \text{ except that } \sigma'(v) = v, \text{ and } v \text{ is the alphabetically}\\
& \text{first variable not in } \sigma(A);\\
\sigma(\langle y_2 : z\rangle A) &= \begin{cases} \langle\sigma'(y_2) : v\rangle\sigma([v/z]A) & \text{if there is an } x \neq v \text{ in } \mathit{Varf}(A) \text{ with } \sigma(x) = \sigma(z),\\ \langle\sigma(y_2) : \sigma(z)\rangle\sigma(A) & \text{otherwise,}\end{cases}\\
& \text{where } \sigma' \text{ is like } \sigma \text{ except that } \sigma'(v) = v, \text{ and } v \text{ is the alphabetically}\\
& \text{first variable not in } \sigma(A);\\
\sigma(\Box A) &= \Box\sigma(A).
\end{aligned}
$$

I will also write σA or A^σ instead of $\sigma(A)$. If Γ is a set of formulas, I write $\sigma(\Gamma)$ or Γ^σ for $\{C^\tau : C \in \Gamma\}$.

Here is the corresponding generalisation of $V^{[y/x]}$.

Definition 3.12 (Interpretation under substitution, generalised)
For any interpretation V on a structure $\mathcal{S}$ and substitution σ, V^σ is the interpretation that is like V except that for any world w in $\mathcal{S}$ and variable x, $V^\sigma_w(x) = V_w(\sigma(x))$.

Substitutions can be composed. If σ and τ are substitutions, then $\tau\cdot\sigma$ is the substitution that maps each variable x to $\tau(\sigma(x))$. Observe that composition behaves differently in superscripts of formulas than in superscripts of interpretations: for formulas A,

$$(A^\sigma)^\tau = \tau(\sigma(A)) = A^{\tau\cdot\sigma},$$

but for interpretations V,

$$(V^\sigma)^\tau = V^{\sigma\cdot\tau}.$$

That's because $(V^\sigma)^\tau_w(x) = V^\sigma_w(\tau(x)) = V_w(\sigma(\tau(x)) = V_w(\sigma\cdot\tau(x)) = V^{\sigma\cdot\tau}_w(x)$.

Definition 3.11 draws attention to the class of injective substitutions, or transformations. A transformation never substitutes two distinct variables by the same variable. For instance, the identity substitution $[x/x]$ or the swapping operation $[x, y/y, x]$ are transformations. What's special about such substitutions is that they make capturing impossible: for the free variable y in $\forall x A(y)$ to be captured by the initial quantifier $\forall x$ after substitution, x and y have to be replaced by the same variable. Correspondingly, definition 3.11 entails that if σ is a transformation, then $\sigma(A)$ is simply A with all variables simultaneously replaced by their σ-value. Transformations satisfy the substitution lemma without any restrictions, even for modal formulas.

> LEMMA 3.13 (TRANSFORMATION LEMMA)
> For any world w in any structure $\mathcal{S}$, any interpretation V on $\mathcal{S}$, any formula A and transformation τ, $w, V^\tau \Vdash A$ iff $w, V \Vdash A^\tau$.

PROOF by induction on A.

1. $A = Px_1 \dots x_n$. $w, V^\tau \Vdash Px_1 \dots x_n$ iff $\langle V^\tau_w(x_1), \dots, V^\tau_w(x_n)\rangle \in V^\tau_w(P)$, iff $\langle V_w(x_1^\tau), \dots, V_w(x_n^\tau)\rangle \in V_w(P)$, iff $w, V \Vdash (Px_1 \dots x_n)^\tau$.
2. $A = \neg B$. $w, V^\tau \Vdash \neg B$ iff $w, V^\tau \nVdash B$, iff $w, V \nVdash B^\tau$ by induction hypothesis, iff $w, V \Vdash (\neg B)^\tau$.
3. $A = B \supset C$. $w, V^\tau \Vdash B \supset C$ iff $w, V^\tau \nVdash B$ or $w, V^\tau \Vdash C$, iff $w, V \nVdash B^\tau$ or $w, V \Vdash C^\tau$ by induction hypothesis, iff $w, V \Vdash (B \supset C)^\tau$.
4. $A = \langle y : x\rangle B$. By definition 3.2, $w, V^\tau \Vdash \langle y : x\rangle B$ iff $w, (V^\tau)^{[y/x]} \Vdash B$. Now $(V^\tau)^{[y/x]}_w(x) = V^\tau_w(y) = V_w(y^\tau) = V_w^{[y^\tau/x^\tau]}(x^\tau) = (V^{[y^\tau/x^\tau]})^\tau_w(x)$. And for any variable $z \neq x$, $(V^\tau)^{[y/x]}_w(z) = V^\tau_w(z) = V_w(z^\tau) = V_w^{[y^\tau/x^\tau]}(z^\tau)$ (because $z^\tau \neq x^\tau$, by injectivity of τ) $= (V^{[y^\tau/x^\tau]})^\tau_w(z)$. So $(V^\tau)^{[y/x]}$ coincides with $(V^{[y^\tau/x^\tau]})^\tau$ at w. By the locality lemma 2.9, $w, (V^\tau)^{[y/x]} \Vdash B$ iff $w, (V^{[y^\tau/x^\tau]})^\tau \Vdash B$. By induction hypothesis, the latter holds iff $w, V^{[y^\tau/x^\tau]} \Vdash B^\tau$, iff $w, V \Vdash \langle y^\tau : x^\tau\rangle B^\tau$ by definition 3.2, iff $w, V \Vdash (\langle y : x\rangle B)^\tau$ by definition 3.11.
5. $A = \forall x B$. Assume $w, V^\tau \nVdash \forall x B$. Then $w, V^* \nVdash B$ for some existential x-variant V^* of V^τ on w. Let V' be the (existential) x^τ-variant of V on w with $V'_w(x^\tau) = V^*_w(x)$. Then ${V'}^\tau_w(x) = V^*_w(x)$, and for any variable $z \neq x$, ${V'}^\tau_w(z) = V'_w(z^\tau) = V_w(z^\tau)$ (because $z^\tau \neq x^\tau$, by injectivity of τ) $= V^\tau_w(z) = V^*_w(z)$. So ${V'}^\tau$ coincides with V^* on w, and by locality (lemma 2.9), $w, {V'}^\tau \nVdash B$. By induction hypothesis, then $w, V' \nVdash B^\tau$. So there is an existential x^τ-variant V' of V on w such that $w, V' \nVdash B^\tau$. By definition 2.7, this means that $w, V \nVdash \forall x^\tau B^\tau$, and hence $w, V \nVdash (\forall x B)^\tau$ by definition 3.11.

   In the other direction, assume $w, V \nVdash (\forall x B)^\tau$, and thus $w, V \nVdash \forall x^\tau B^\tau$. Then $w, V' \nVdash B^\tau$ for some existential x^τ-variant V' of V on w, and by induction hypothesis $w, {V'}^\tau \nVdash B$. Let V^* be the (existential) x-variant of V^τ on w with $V^*_w(x) = V'_w(x^\tau)$. Then $V^*_w(x) = {V'}^\tau_w(x)$, and for any variable $z \neq x$, $V^*_w(z) = V^\tau_w(z) = V_w(z^\tau) = V'_w(z^\tau)$ (because $z^\tau \neq x^\tau$, by injectivity of τ) $= {V'}^\tau_w(z)$. So V^* coincides with ${V'}^\tau$ on w, and by locality (lemma 2.9), $w, V^* \nVdash B$. So there is an existential x-variant V^* of V^τ on w such that $w, V^* \nVdash B$. By definition 2.7, this means that $w, V^\tau \nVdash \forall x B$.

6. $A = \Box B$. Assume $w, V \nVdash \Box B^\tau$. Then $w', V' \nVdash B^\tau$ for some w', V' with wRw' and V' a w' image of V at w. This means that for all variables x, $V'_{w'}(x)$ is some counterpart at w' of $V_w(x)$ at w (if any, else undefined). By induction hypothesis, $w', V'^\tau \nVdash B$. Since for all x, $V'^\tau_{w'}(x) = V'_{w'}(x^\tau)$ and $V^\tau_w(x) = V_w(x^\tau)$, it follows that $V'^\tau_{w'}(x)$ is a counterpart at w' of $V^\tau_w(x)$ at w (if any, else undefined). So V'^τ is a w'-image of V^τ at w. Hence $w', V'^\tau \nVdash B$ for some w', V'^τ with wRw' and V'^τ a w'-image of V^τ at w. So $w, V^\tau \nVdash \Box B$.

   In the other direction, assume $w, V^\tau \nVdash \Box B$. Then $w', V^* \nVdash B$ for some w', V^* with wRw' and V^* a w' image of V^τ at w. This means that for all variables x, $V^*_{w'}(x)$ is some counterpart at w' of $V^\tau_w(x)$ at w (if any, else undefined). Let V' be like V except that for all variables x, $V'_{w'}(x^\tau) = V^*_{w'}(x)$, and for all $x \notin \mathrm{Ran}(\tau)$, $V'_{w'}(x)$ is an arbitrary counterpart at w' of $V_w(x)$ at w, or undefined if there is none. V' is a w' image of V at w. Moreover, V^* is V'^τ. By induction hypothesis, $w', V' \nVdash B^\tau$. So $w', V' \nVdash B^\tau$ for some w', V' with wRw' and V' a w' image of V at w. So $w, V \nVdash (\Box B)^\tau$. ■

For the substitution quantifier, we could introduce primitive polyadic quantifiers like $\langle y_1, y_2 : x_1, x_2\rangle$, which says '$y_1$ is an x_1 and y_2 an x_2 such that', and stipulate that

$$w, V \Vdash \langle y_1, y_2 : x_1, x_2\rangle A \text{ iff } w, V^{[y_1,y_2/x_1,x_2]} \Vdash A.$$

Geach's $\langle x : y, z\rangle$ is then equivalent to $\langle x, x : y, z\rangle$. But it turns out that $\langle y_1, y_2 : x_1, x_2\rangle$ is definable.

We can't simply say that $\langle y_1, y_2 : x_1, x_2\rangle A$ is $\langle y_1 : x_1\rangle\langle y_2 : x_2\rangle A$, since the bound variable x_1 might capture y_2, e.g. in the "swapping" operator $\langle x, y : y, x\rangle$. We must store the original value of y_2 in a temporary variable z: $\langle y_2 : z\rangle\langle y_1 : x_1\rangle\langle z : x_2\rangle$.

Definition 3.14 (Substitution sequences)
For any $n > 1$, sentence A and variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct, let $\langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A$ abbreviate $\langle y_n : z\rangle\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle\langle z : x_n\rangle A$, where z is the alphabetically first variable not in A or $x_1, \ldots, x_n$.

Lemma 3.15 (Substitution sequence semantics)
For any world w in any structure $\mathcal{S}$, any interpretation V on $\mathcal{S}$,

$$w, V \Vdash_{\mathcal{S}} \langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A \text{ iff } w, V^{[y_1,\ldots,y_n/x_1,\ldots,x_n]} \Vdash_{\mathcal{S}} A.$$

Proof By definition 3.14, $w, V \Vdash \langle y_1, \ldots, y_n : x_1, \ldots, x_n\rangle A$ iff $w, V \Vdash \langle y_n : z\rangle\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1}\rangle\langle z : x_n\rangle A$, for some z not in $x_1, \ldots, x_{n-1}, A$. By definition 3.2, $w, V \Vdash \langle y_n :$

$z\rangle\langle y_1,\ldots,y_{n-1} : x_1,\ldots,x_{n-1}\rangle\langle z : x_n\rangle A$ iff $w, V^{[y_n/z]} \Vdash \langle y_1,\ldots,y_{n-1} : x_1,\ldots,x_{n-1}\rangle\langle z : x_n\rangle A$, which by induction hypothesis holds iff $w, V^{[y_n/z]\cdot[y_1,\ldots,y_{n-1}/x_1,\ldots,x_{n-1}]} \Vdash \langle z : x_n\rangle A$. By definition 3.2 again, $w, V^{[y_n/z]\cdot[y_1,\ldots,y_{n-1}/x_1,\ldots,x_{n-1}]} \Vdash \langle z : x_n\rangle A$ iff $w, V^{[y_n/z]\cdot[y_1,\ldots,y_{n-1}/x_1,\ldots,x_{n-1}]\cdot[z/x_n]} \Vdash A$. Now $[y_n/z]\cdot[y_1,\ldots,y_{n-1}/x_1,\ldots,x_{n-1}]\cdot[z/x_n]$ is the function $\sigma : Var \to Var$ such that

$$\sigma(x) = [y_n/z]([y_1,\ldots,y_{n-1}/x_1,\ldots,x_{n-1}]([z/x_n](x))).$$

Since $z \notin x_1,\ldots,x_{n-1}$, this means that

$$\begin{aligned}
&\sigma(x_n) = y_n,\\
&\sigma(x_i) = y_i \text{ for } x_i \in \{x_1,\ldots,x_{n-1}\},\\
&\sigma(z) = y_n,
\end{aligned}$$

and $\sigma(x) = x$ for every other variable x. Since $z \notin Var(A)$, V^σ agrees at w with $V^{[y_1,\ldots,y_n/x_1,\ldots,x_n]}$ about all variables in A. So by the coincidence lemma 2.8, $w, V^\sigma \Vdash A$ iff $w, V^{[y_1,\ldots,y_n/x_1,\ldots,x_n]} \Vdash A$. ■

4 Logics

I now want to describe the minimal logics that are characterised by our semantics. Following tradition, a *logic* (or *system*) in this context is simply a set of formulas (relative to some language), and I will describe such sets by recursive clauses corresponding to the axioms and rules of a Hilbert-style calculus.

The following definition, standard in free logic, will keep formulas slightly shorter.

> DEFINITION 4.1 (EXISTENCE)
> For any variable variable x, let Ex abbreviate $\exists y(y{=}x)$, where y is the alphabetically first variable other than x.

Now recall that we have two kinds of models: positive models with two domains, and negative models with a single domain. The logic of positive models is essentially the combination of standard positive free logic with the minimal modal logic K. The only place to be careful is with substitution principles like Leibniz' Law, which either have to be expressed with object-language substitution or restricted as explained in the previous section.

Standard axiomatisations of free logics contain three principles that make use of substitution: Leibniz' Law,

(LL) $\vdash x{=}y \supset A \supset [y/x]A$,

Free Universal Instantiation,

(FUI) $\vdash \forall x A \supset (\exists x(x{=}y) \supset [y/x]A)$, and closure under first-order substitution,

(Sub) if $\vdash A$, then $\vdash [y/x]A$.

In languages without substitution, all three have to be restricted to the case where y is modally free for x in A. (If we add the unrestricted principles, we get logics for functional structures, as we'll see later.)

> Definition 4.2 (Positive quantified modal logics)
> The *positive (quantified modal) logic* is a function that maps each standard language $\mathcal{L}$ of quantified modal logic to a set L that contains
>
> (Taut) all propositional tautologies in $\mathcal{L}$,
>
> as well as all $\mathcal{L}$-instances of
>
> (UD) $\forall x A \supset (\forall x(A \supset B) \supset \forall x B)$,
>
> (VQ) $A \supset \forall x A$, provided x is not free in A,
>
> (FUI*) $\forall x A \supset (Ey \supset [y/x]A)$, provided y is modally free for x in A,
>
> ($\forall$Ex) $\forall x Ex$,
>
> (=R) $x = x$,
>
> (LL*) $x = y \supset A \supset [y/x]A$, provided y is modally free for x in A,
>
> (K) $\Box A \supset (\Box(A \supset B) \supset \Box B)$,
>
> and that is closed under modus ponens, universal generalisation, necessitation, and first-order substitution within $\mathcal{L}$:
>
> (MP) if $\vdash_L A$ and $\vdash_L A \supset B$, then $\vdash_L B$,
>
> (UG) if $\vdash_L A$, then $\vdash_L \forall x A$,
>
> (Nec) if $\vdash_L A$, then $\vdash_L \Box A$,
>
> (Sub*) if $\vdash_L A$, then $\vdash_L [y/x]A$, provided y is modally free for x in A.
>
> Here $\vdash_L A$ means $A \in L$. The logic that maps each language to the smallest set satisfying these constraints is the *minimal positive (quantified modal) logic* P.

As usual, I will often ignore the language-relativity of logics and e.g. write $\vdash_{\mathsf{P}} A$ instead of $\vdash_{\mathsf{P}(\mathcal{L})} A$. I will also

Standard positive free logic is axiomatised by (Taut), (UD), (VQ), (FUI), ($\forall$Ex), (=R), (LL), (MP), (UG), (Sub). P adds the modal principles (K) and (Nec) and restricts the principles for substitution.[2]

Standard negative free logic replaces (=R) and (∀Ex) by

(∀=R) $\forall x(x{=}x)$, and

(Neg) $Px_1 \ldots x_n \supset Ex_1 \wedge \ldots \wedge Ex_n$.

In our single-domain models, we need two further axioms:

(NA) $\neg Ex \supset \Box\neg Ex$, and

(TE) $x{=}y \supset \Box(Ex \supset Ey)$.

(NA) ("No Aliens") was already mentioned on p. 4. In effect, it says that if x doesn't exist, then x doesn't have any counterparts. (TE) says that if x is identical to y, and x has a counterpart at some world, then y also has a counterpart at that world.

(In negative dual-domain semantics, (NA) and (TE) are invalid while the other axioms of N are valid. Negative dual-domain models are like positive models except that all predicates, including identity, are restricted to the inner domain. Thus (NA) is false at w if some individual in the outer domain of w has a counterpart in the inner domain of some accessible world. (TE) is false at w if $V_w(x) \in D_w$ has both an existing and a non-existing counterpart at some accessible world w'; then some w'-image V' of V at w assigns the non-existing individual to y and the existing individual to x, rendering $Ex \supset Ey$ false at w' under V', wherefore $\Box(Ex \supset Ey)$ is false at w under V.)

As mentioned in section 2, (NA) should not be confused with the claim that no individual exists at any world that isn't a counterpart of something at the centre, which would require something like the Barcan Formula,

(BF) $\forall x \Box A \supset \Box \forall x A$.

But this isn't valid in the class of negative models. For example, if $W = \{w, w'\}$, wRw', $D_w = \emptyset$ and $D_{w'} = \{0\}$, then $w, V \Vdash \forall x \Box x{\neq}x$, but $w, V \nVdash \Box \forall x\, x{\neq}x$.

Definition 4.3 ((Strongly) negative logics)
A *(strongly) negative (quantified modal) logic* is a function that maps each standard language $\mathcal{L}$ to a set that contains all $\mathcal{L}$-instances of (Taut), (UD), (VQ), (FUI*), (LL*), (K), as well as

(Neg) $Px_1 \ldots x_n \supset Ex_1 \wedge \ldots \wedge Ex_n$,

2 [Fitting and Mendelsohn 1998] use $\forall x A \leftrightarrow A$ in place of (VQ), which precludes empty inner domains (as [Kutz 2000: 38] points out). The systems presented here do not validate the claim that something exists. To rule out empty inner domains in positive and negative models, $\exists x Ex$ would be needed as extra axiom.

(∀=R) $\forall x(x{=}x)$

(NA) $\neg Ex \supset \Box\neg Ex$,

(TE) $x{=}y \supset \Box(Ex \supset Ey)$,

and that is closed under (MP), (UG), (Nec) and (Sub*). Here $\vdash_L A$ means $A \in L$. The logic that maps each language to the smallest set satisfying these constraints is the *minimal (strongly) negative (quantified modal) logic* N.

Theorem 4.4 (Soundness of P)
Every member of P is valid in every positive counterpart model.

Proof We show that all P axioms are valid in every positive model, and that validity is closed under (MP), (UG), (Nec) and (Sub).

1. (Taut). Propositional tautologies are valid in every model by the standard satisfaction rules for the connectives.

2. (UD). Assume $w, V \Vdash \forall x(A \supset B)$ and $w, V \Vdash \forall xA$ in some model. By definition 2.7, then $w, V' \Vdash A \supset B$ and $w, V' \Vdash A$ for every existential x-variant V' of V on w, and so $w, V' \Vdash B$ for every such V'. Hence $w, V \Vdash \forall xB$.

3. (VQ). Suppose $w, V \nVdash A \supset \forall xA$ in some model. Then $w, V \Vdash A$ and $w, V \nVdash \forall xA$. If x is not free in A, then by the coincidence lemma 2.8, $w, V' \Vdash A$ for every x-variant V' of V on D_w; so $w, V \Vdash \forall xA$. Contradiction. So if x is not free in A, then $A \supset \forall xA$ is valid in every model.

4. (FUI*). Assume $w, V \Vdash \forall xA$ and $w, V \Vdash Ey$ in some model. By definition 2.7, then $w, V' \Vdash A$ for all existential x-variants V' of V on w. So in particular, $w, V^{[y:x]} \Vdash A$. If y is modally free for x in A, then by lemma 3.9, $w, V \Vdash [y/x]A$.

5. (∀Ex). By definition 2.7, $w, V \Vdash \forall xEx$ iff $w, V' \Vdash Ex$ for all existential x-variants V' of V on w, iff for all existential x-variants V' of V on w there is an existential y-variant V'' of V' on w such that $w, V'' \Vdash x{=}y$. But this is always the case: for any V', let V'' be $V'^{[x/y]}$.

6. (=R). By definition 2.3, $V_w(=) = \{\langle d, d\rangle : d \in U_w\}$, and by definition 2.3, $V_w(x) \in U_w$ in every positive model. So $w, V \Vdash x{=}x$ in every such model, by definition 2.7.

7. (LL*). Assume $w, V \Vdash x = y$, $w, V \Vdash A$, and y is modally free for x in A. Since $V_w(x) = V_w(y)$, V coincides with $V^{[y/x]}$ at w. So $w, V^{[y/x]} \Vdash A$ by the coincidence lemma 2.8. By lemma 3.9, $w, V^{[y/x]} \Vdash A$ only if $w, V \Vdash [y/x]A$. So $w, V \Vdash [y/x]A$.

8. (K). Assume $w, V \Vdash \Box(A \supset B)$ and $w, V \Vdash \Box A$. Then $w', V' \Vdash A \supset B$ and $w', V' \Vdash A$ for every w', V' such that wRw' and V' is a w'-image of V at w. Then $w', V' \Vdash B$ for any such w', V', and so $w, V \Vdash \Box B$.

9. (MP). Assume $w, V \Vdash A \supset B$ and $w, V \Vdash A$ in some model. By definition 2.7, then $w, V \Vdash B$ as well. So for any world w in any model, (MP) preserves truth at w.
10. (UG). Assume $w, V \nVdash \forall x A$ in some model $\mathcal{M}$. Then $w, V' \nVdash A$ for some existential x-variant V' of V on w. So A is invalid in a model like $\mathcal{M}$ but with V' as the interpretation function in place of V. Hence if A is valid in all positive models, then so is $\forall x A$.
11. (Nec). Assume $w, V \nVdash_{\mathcal{M}} \Box A$ in some model $\mathcal{M}$. Then $w', V' \nVdash A$ for some w' with wRw' and V' some w'-image of V at w. Let $\mathcal{M}^*$ be like $\mathcal{M}$ except with V' in place of V. $\mathcal{M}^*$ is a positive model. Since A is not valid in $\mathcal{M}^*$, it follows contrapositively that whenever A is valid in all positive models, then so is $\Box A$.
12. (Sub*). Assume $w, V \nVdash [y/x]A$ in some model $\langle \mathcal{S}, V \rangle$, and y is modally free for x in A. By lemma 3.9, then $w, V^{[y/x]} \nVdash A$. So A is invalid in the model $\langle \mathcal{S}, V^{[y/x]} \rangle$. Hence if A is valid in all positive models, then so is $[y/x]A$. ■

Theorem 4.5 (Soundness of N)
Every member of N is valid in every negative counterpart model.

Proof The cases for (Taut), (UD), (VQ), (FUI*), (LL*), (K), (MP), (UG), (Nec) and (Sub*) are essentially the same as in the previous proof. The remaining cases are

1. (Neg). Assume $w, V \Vdash Px_1 \ldots x_n$ in some model. By definition 2.3, $V_w(P) \subseteq U_w^n$, and by definition 2.4, $U_w = D_w$ in negative models. So $V_w(P) \subseteq D_w^n$. By definition 2.7, $w, V \Vdash Px_1 \ldots x_n$ therefore entails that $V_w(x_i) \in D_w$ for all $x_i \in x_1, \ldots, x_n$, and that $w, V \Vdash Ex_i$ for all such x_i.
2. ($\forall{=}$R). $w, V \Vdash \forall x(x{=}x)$ iff $w, V' \Vdash x{=}x$ for all existential x-variants V' of V on w. This is always the case, since by definition 2.3 and 2.3, $V_w(=) = \{\langle d, d \rangle : d \in D_w\}$ in negative models.
3. (NA). Assume $w, V \Vdash \neg Ex$. By definition 2.7, this means that $V_w(x) \notin D_w$, and therefore that $V_w(x)$ is undefined if the model is negative. But if $V_w(x)$ is undefined, then there is no world w' and individual d such that $\langle V_w(x), w \rangle C \langle d, w' \rangle$. By definitions 2.7 and 2.6, it follows that there is no world w' and interpretation V' with wRw' and $V_w \triangleright V'_{w'}$ such that $w', V' \Vdash Ex$. So then $w, V \Vdash \Box\neg Ex$ by definition 2.7. Thus $w, V \Vdash \neg Ex \supset \Box\neg Ex$.
4. (TE). Assume $w, V \Vdash x{=}y$. Then $V_w(x) = V_w(y)$ by definitions 2.3 and 2.7. Let w', V' be such that wRw' and $V_w \triangleright V'_{w'}$, and $w', V' \Vdash Ex$. By definitions 2.7 and 2.6, then $\langle V_w(x), w \rangle C \langle V'_{w'}(x), w' \rangle$. But then $w', V' \Vdash Ey$, because in a positive model $V'_{w'}(y) \notin D_{w'}$ only if $V_w(y)$ at w has no counterpart at w', and $V_w(y) = V_w(x)$. So if $w, V \Vdash x{=}y$, then $w, V \Vdash \Box(Ex \supset Ey)$, by definition 2.7, and so $w, V \Vdash x{=}y \supset \Box(Ex \supset Ey)$. ■

In the remainder of this section, I will prove a few properties derivable from the above axiomatisations. (Some of these will be needed later on in the completeness proof.) To this end, let $\mathcal{L}$ range over standard languages of quantified modal logic, and L over the corresponding sets $\mathsf{P}(\mathcal{L}), \mathsf{N}(\mathcal{L})$, and $\mathsf{C}(\mathcal{L})$.

LEMMA 4.6 (CLOSURE UNDER PROPOSITIONAL CONSEQUENCE)
For all $\mathcal{L}$-formulas $A_1, \ldots, A_n, B$,

(PC) if $\vdash_L A_1, \ldots, \vdash_L A_n$, and B is a propositional consequence of $A_1, \ldots, A_n$, then $\vdash_L B$.

PROOF If B is a propositional consequence of $A_1, \ldots, A_n$, then $A_1 \supset (\ldots \supset (A_n \supset B) \ldots)$ is a tautology. So by (Taut), $\vdash_L A_1 \supset (\ldots \supset (A_n \supset B) \ldots)$. If $\vdash_L A_1, \ldots, \vdash_L A_n$, then by n applications of (MP), $\vdash_L B$. ■

When giving proofs, I will often omit reference to (PC).

LEMMA 4.7 (REDUNDANT AXIOMS)
For any $\mathcal{L}$-formulas A and variables x,

(∀Ex) $\vdash_L \forall x Ex$,

(∀=R) $\vdash_L \forall x(x = x)$.

PROOF If L extends P, then (∀Ex) is an axiom. In N, we have $\vdash_L x = x \supset Ex$ by (Neg); so by (UG) and (UD), $\vdash_L \forall x x = x \supset \forall x Ex$. Since $\vdash_L \forall x(x = x)$ by (=R), $\vdash_L \forall x Ex$.

If L extends N, then (∀=R) is an axiom. In P, we have $\vdash_L x = x$ by (=R), and so (∀=R) by (UG). ■

LEMMA 4.8 (EXISTENCE AND SELF-IDENTITY)
If L is *negative*, then for any $\mathcal{L}$-variable x,

(NE) $\vdash_L Ex \leftrightarrow x = x$;

PROOF By (FUI*), $\vdash_L \forall x(x = x) \supset (Ex \supset x = x)$. By (∀=R), $\vdash_L \forall x(x = x)$. So $\vdash_L Ex \supset x = x$. Conversely, $x = x \supset Ex$ by (Neg). ■

Lemma 4.9 (Symmetry and transitivity of identity)
For any $\mathcal{L}$-variables x, y, z,

(=S) $\vdash_L x{=}y \supset y{=}x$;

(=T) $\vdash_L x{=}y \supset y{=}z \supset x{=}z$.

Proof For (= S), let v be some variable $\notin \{x, y\}$. Then

1.	$\vdash_L v{=}y \supset (v{=}x \supset y{=}x)$.	(LL*)
2.	$\vdash_L x{=}y \supset (x{=}x \supset y{=}x)$.	(1, (Sub*))
3.	$\vdash_L x{=}y \supset x{=}x$.	((=R), or (Neg) and (∀=R))
4.	$\vdash_L x{=}y \supset y{=}x$.	(2, 3)

For (= T),

1.	$\vdash_L x{=}y \supset y{=}x$.	(=S)
2.	$\vdash_L y{=}x \supset (y{=}z \supset x{=}z)$.	(LL*)
3.	$\vdash_L x{=}y \supset (y{=}z \supset x{=}z)$.	(1, 2)

■

Next we have proof-theoretic analogues of lemmas 3.8 and 3.13:

Lemma 4.10 (Syntactic alpha-conversion)
If A, A' are $\mathcal{L}$-formulas, and A' is an alphabetic variant of A, then

(AC) $\vdash_L A \leftrightarrow A'$.

Proof by induction on A.

1. A is atomic. Then $A = A'$ and $A \leftrightarrow A'$ is a propositional tautology.
2. A is $\neg B$. Then A' is $\neg B'$, where B' is an alphabetic variant of A'. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (PC), $\vdash_L \neg B \leftrightarrow \neg B'$.
3. A is $B \supset C$. Then A' is $B' \supset C'$, where B', C' are alphabetic variants of B, C, respectively. By induction hypothesis, $\vdash_L B \leftrightarrow B'$ and $\vdash_L C \leftrightarrow C'$. So by (PC), $\vdash_L (B \supset C) \leftrightarrow (B' \supset C')$.
4. A is $\forall xB$. Then A' is either $\forall xB'$ or $\forall z[z/x]B'$, where B' is an alphabetic variant of B and $z \notin Var(B')$. Assume first that A' is $\forall xB'$. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (UG) and (UD), $\vdash_L \forall xB \leftrightarrow \forall xB'$.

Alternatively, assume B is $\forall z[z/x]B'$ and $z \notin \mathit{Var}(B')$. Since B' differs from B at most in renaming bound variables, if z were free in B, then $z \in \mathit{Var}(B')$. So z is not free in B. Then

1.	$\vdash_L B \leftrightarrow B'$	(induction hypothesis)
2.	$\vdash_L [z/x]B \leftrightarrow [z/x]B'$	(1, (Sub*))
3.	$\vdash_L \forall xB \supset Ez \supset [z/x]B$	(FUI*)
4.	$\vdash_L \forall xB \supset Ez \supset [z/x]B'$	(2, 3)
5.	$\vdash_L \forall z\forall xB \supset \forall zEz \supset \forall z[z/x]B'$	(4, (UG), (UD))
6.	$\vdash_L \forall zEz$	($\forall$Ex)
7.	$\vdash_L \forall z\forall xB \supset \forall z[z/x]B'$	(5, 6)
8.	$\vdash_L \forall xB \supset \forall z\forall xB$	((VQ), z not free in B)
9.	$\vdash_L \forall xB \supset \forall z[z/x]B'$.	(7, 8)

Conversely,

10.	$\vdash_L \forall z[z/x]B' \supset Ex \supset [x/z][z/x]B'$	(FUI*)
11.	$\vdash_L \forall z[z/x]B' \supset Ex \supset B$	(1, 10, $z \notin \mathit{Var}(B')$)
12.	$\vdash_L \forall x\forall z[z/x]B' \supset \forall xB$	(11, (UG), (UD), ($\forall$Ex))
13.	$\vdash_L \forall z[z/x]B' \supset \forall x\forall z[z/x]B'$	(VQ)
14.	$\vdash_L \forall z[z/x]B' \supset \forall xB$	(12, 13)

5. A is $\Box B$. Then A' is $\Box B'$, where B' is an alphabetic variant of B. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. Then by (Nec), $\vdash_L \Box(B \leftrightarrow B')$, and by (K) and (PC), $\vdash_L \Box B \leftrightarrow \Box B'$. ■

Lemma 4.11 (Closure under transformations)
For any $\mathcal{L}$-formula A and transformation τ on $\mathcal{L}$,

(Sub$^\tau$) $\vdash_L A$ iff $\vdash_L A^\tau$.

Proof Assume $\vdash_L A$. Let $x_1, \dots, x_n$ be the variables in A. If $n = 0$, then $A = A^\tau$ and the result is trivial. If $n = 1$, then A^τ is $[x_1^\tau/x_1]A$, and x_1^τ is either x_1 itself or does not occur in A. In the first case, $[x_1^\tau/x_1]A = A$ and the result is again trivial. In the second case, x_1^τ is modally free for x_1 in A, and thus $\vdash_L [x_1^\tau/x_1]A$ by (Sub*).

Assume then that $n > 1$. Note first that $A^\tau = [x_n^\tau/v_n] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$, where $v_2, \dots, v_n$ are distinct variables not in A or A^τ. This is easily shown by induction on the subformulas B of A (ordered by complexity). To keep things short, let Σ abbreviate $[x_n^\tau/v_n] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]$.

1. If B is $Px_j \dots x_k$, then $x_j, \dots, x_k$ are variables from $x_1, \dots, x_n$, and $\Sigma B = Px_j^\tau \dots x_k^\tau = B^\tau$, by definitions 3.3 and 3.11.
2. If B is $\neg C$, then by induction hypothesis, $\Sigma C = C^\tau$, and hence $\neg\Sigma C = \neg C^\tau$. But $\Sigma\neg C$ is $\neg\Sigma C$ by definition 3.3, and $(\neg C)^\tau$ is $\neg C^\tau$ by definition 3.11.
3. The case for $C \supset D$ is analogous.
4. If B is $\forall zC$, then by induction hypothesis, $\Sigma C = C^\tau$. Since τ is injective, $\Sigma\forall zC$ is $\forall\Sigma z\Sigma C$ by definition 3.3, and $(\forall zC)^\tau$ is $\forall z^\tau C^\tau$ by definition 3.11. Moreover, since z is one of $x_1, \dots, x_n$, $\Sigma z = z^\tau$.
5. If B is $\Box C$, then by induction hypothesis, ΣC is C^τ, and hence $\Box\Sigma C$ is $\Box C^\tau$. But $\Sigma\Box C$ is $\Box\Sigma C$ by definition 3.3, and $(\Box C)^\tau$ is $\Box C^\tau$ by definition 3.11.

Now we show that L contains all "segments" of $[x_n^\tau/v_n] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$, beginning with the rightmost substitution, $[v_n/x_n]A$. Since v_n is modally free for x_n in A, by (Sub*), $\vdash_L [v_n/x_n]A$. Likewise, for each $1 < i < n$, v_i is modally free for x_i in $[v_{i+1}/x_{i+1}] \dots [v_n/x_n]A$. So $\vdash_L [v_2/x_2] \dots [v_n/x_n]A$.

With respect to $[x_1^\tau/x_1]$, we distinguish three cases. First, if $x_1 = x_1^\tau$, then $\vdash_L [x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$, because $[x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$ is $[v_2/x_2] \dots [v_n/x_n]A$. Second, if $x_1 \neq x_1^\tau$ and $x_1^\tau \notin \mathit{Var}(A)$, then $x_1^\tau \notin \mathit{Var}([v_2/x_2] \dots [v_n/x_n]A)$, since the $v_1, \dots, v_n$ are not in $\mathit{Var}(A)$ or $\mathit{Var}(A^\tau)$. So x_1^τ is modally free for x_1 in $[v_2/x_2] \dots [v_n/x_n]A$, and by (Sub*), $\vdash_L [x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$. Third, if $x_1 \neq x_1^\tau$ and $x_1^\tau \in \mathit{Var}(A)$, then x_1^τ must be one of $x_2, \dots, x_n$. Then again $x_1^\tau \notin \mathit{Var}([v_2/x_2] \dots [v_n/x_n]A)$, and so $\vdash_L [x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$ by (Sub*).

Next, x_2^τ is modally free for v_2 in $[x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$, because τ is injective and hence $x_2^\tau \neq x_1^\tau$, so x_2^τ does not occur in $[x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$. Hence $\vdash_L [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$. By the same reasoning, for each $2 < i \leq n$, x_i^τ is modally free for v_i in $[x_{i-1}^\tau/v_{i-1}] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$. So $\vdash_L [x_n^\tau/v_n] \dots [x_2^\tau/v_2][x_1^\tau/x_1][v_2/x_2] \dots [v_n/x_n]A$, i.e. $\vdash_L A^\tau$.

This proves the left-to-right direction of (Sub$^\tau$). The other direction immediately follows. Let $x_1^\tau, \dots, x_n^\tau$ be the variables in A^τ, and let σ be an arbitrary transformation that maps each x_i^τ back to x_i (i.e., to $(x_i^\tau)^{\tau^{-1}}$). By the left-to-right direction of (Sub$^\tau$), $\vdash_L A^\tau$ entails $\vdash_L (A^\tau)^\sigma$, and $(A^\tau)^\sigma$ is simply A. ■

Lemma 4.12 (Leibniz' Law with partial substitution)
Let A be a formula of $\mathcal{L}$, and x, y variables of $\mathcal{L}$. Let $[y//x]A$ be A with *one or more* occurrences of x replaced by y.

(LL$_\mathrm{p}^*$) $\vdash_L x = y \supset A \supset [y//x]A$, provided the following conditions are satisfied.

(i) $[y//x]A$ does not replace any occurrence of x in the scope of a quantifier binding x or y.

(ii) Either y is modally free for x in A, or $[y//x]A$ does not replace any occurrence of x in the scope of a modal operator in A that also contains y.

(iii) In the scope of any modal operator in A, $[y//x]A$ either replaces all or no occurrences of x by y.

PROOF Let $v \neq y$ be a variable not in $Var(A)$, and let $[v//x]A$ be like $[y//x]A$ except that all new occurrences of y are replaced by v: if $[y//x]A$ satisfies (i)–(iii), then so does $[y//x]A$ with all new occurrences of y replaced by v. Moreover, in the resulting formula $[v//x]A$ all occurrences of v are free and free for y, by clause (i); so $[y/v][v//x]A = [y//x]A$ by definition 3.3. By (LL*),

(1) $\vdash_L v{=}y \supset [v//x]A \supset [y/v][v//x]A$,

provided that y is modally free for v in $[v//x]A$, i.e. provided that either y is modally free for x in A, or $[v//x]A$ (and thus $[y//x]A$) does not replace any occurrence of x in the scope of a modal operator in A that also contains y. This is guaranteed by condition (ii). Since $[y/v][v//x]A$ is $[y//x]A$, (1) can be shortened to

(2) $\vdash_L v{=}y \supset [v//x]A \supset [y//x]A$.

By (Sub*), it follows that

(3) $\vdash_L [x/v](v{=}y \supset [v//x]A \supset [y//x]A)$,

provided that x is modally free for v in $v{=}y \supset [v//x]A \supset [y//x]A$. Since this isn't a formula of the form $\Box B$, x is modally free for v here iff no free occurrences of x and v lie in the scope of the same modal operator in $[v//x]A$. So whenever $[v//x]A$ (and thus $[y//x]A$) replaces some occurrences of x in the scope of a modal operator in A, then it must replace all occurrences of x in the scope of that operator. This is guaranteed by condition (iii). By definition 3.3, (3) can be simplified to

(4) $\vdash_L x{=}y \supset A \supset [y//x]A$.

■

I will never actually use (LL^*_p). I mention it only because Leibniz' Law is almost always stated for partial substitutions, and you may have wondered how the corresponding version looks in the present systems. Now you know. We could indeed have used (LL^*_p) as basic axiom instead of (LL*); (LL*) would then be derivable, because every formula A has an alphabetic variant A' such that $[y/x]A$ is an instance of $[y//x]A'$ that satisfies (i)–(iii) iff y is modally free for x in A, and because (LL*) is not used in the proof of lemma 4.10. I have chosen (LL*) as basic due to its much greater simplicity.[3]

The following facts will play an important role in the completeness proof.

3 Kutz's system uses the following version of (LL^*_p) ([Kutz 2000: 43]): [xxx check: surely he also requires freedom and freedom for]

(LL^K_p) $\vdash x{=}y \supset A \supset [y//x]A$, provided that

Lemma 4.13 (Leibniz' Law with sequences)
For any $\mathcal{L}$-formula A and variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct,

(LL$^*_n$) $\vdash_L x_1 = y_1 \land \ldots \land x_n = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A$, provided each y_i is modally free for x_i in $[y_1, \ldots, y_{i-1}/x_1, \ldots, x_{n-1}]A$.

For $i = 1$, the proviso is meant to say that y_1 is modally free for x_1 in A.

Proof By induction on n. For $n = 1$, (LL$^*_n$) is (LL*). Assume then that $n > 1$ and that each y_i in $y_1, \ldots, y_n$ is modally free for x_i in $[y_1, \ldots, y_{i-1}/x_1, \ldots, x_{n-1}]A$. Let z be some variable not in A, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$. So z is modally free for x_n in A. By (LL*),

(1) $\vdash_L x_n = z \supset A \supset [z/x_n]A$.

By induction hypothesis,

(2) $\vdash_L x_1 = y_1 \land \ldots \land x_{n-1} = y_{n-1} \supset [z/x_n]A \supset [y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$.

By assumption, y_n is modally free for x_n in $[y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}]A$. Then y_n is also modally free for z in $[y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$. So by (LL*),

(3) $\vdash_L z = y_n \supset [y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A \supset$
$[y_n/z][y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$.

But $[y_n/z][y_1, \ldots, y_{n-1}/x_1, \ldots, x_{n-1}][z/x_n]A$ is $[y_1, \ldots, y_n/x_1, \ldots, x_n]A$. Combining (1)–(3), we therefore have

(4) $\vdash_L x_1 = y_1 \land \ldots \land x_{n-1} = y_{n-1} \supset x_n = z \land z = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A$.

So by (Sub*),

(5) $\vdash_L x_1 = y_1 \land \ldots \land x_{n-1} = y_{n-1} \supset x_n = x_n \land x_n = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A$.

Since $\vdash_L x_n = y_n \supset x_n = x_n$ (by either (=R) or (Neg) and ($\forall$=R)), it follows that

(6) $\vdash_L x_1 = y_1 \land \ldots \land x_n = y_n \supset A \supset [y_1, \ldots, y_n/x_1, \ldots, x_n]A$.

■

(i) y is not free in the scope of a modal operator in A, and
(ii) in the scope of any modal operator in A, $[y//x]A$ either replaces all or no occurrences of x by y.

Evidently, this is a lot more restrictive than (LL$^*_p$). For example, (LL$^*_p$) validates

$\vdash x = y \supset \Box Gxy \supset \Box Gyy$ and
$\vdash x = y \supset (Fx \lor \Diamond Gxy) \supset (Fy \lor \Diamond Gxy)$,

which can't be derived in Kutz's system.

Lemma 4.14 (Cross-substitution)
For any $\mathcal{L}$-formula A and variables x, y,

(CS) $\vdash_L x = y \supset \Box A \supset \Box(y = z \supset [z/x]A)$, provided z is not free in A.

More generally, for any variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct,

(CS$_\mathrm{n}$) $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset \Box A \supset \Box(y_1 = z_1 \wedge \ldots \wedge y_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$, provided none of $z_1, \ldots, z_n$ is free in A.

Proof For (CS), assume z is not free in A. Then

1. $\vdash_L x = z \supset A \supset [z/x]A$. (LL*)
2. $\vdash_L A \supset (x = z \supset [z/x]A)$. (1)
3. $\vdash_L \Box A \supset \Box(x = z \supset [z/x]A)$. (2, (Nec), (K))
4. $\vdash_L x = y \supset \Box(x = z \supset [z/x]A) \supset \Box(y = z \supset [z/x]A)$. (LL*)
5. $\vdash_L x = y \supset \Box A \supset \Box(y = z \supset [z/x]A)$. (3, 4)

Step 4 is justified by the fact that x is not free in $[z/x]A$ and so x and y are modally separated in $x = z \supset [z/x]A$.

The proof for (CS$_\mathrm{n}$) is analogous. Assume none of $z_1, \ldots, z_n$ is free in A. Then

1. $\vdash_L x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset A \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A$. (LL$^*_\mathrm{n}$)
2. $\vdash_L A \supset (x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$. (1)
3. $\vdash_L \Box A \supset \Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$. (2, (Nec), (K))
4. $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset$
   $\Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A) \supset$
   $\Box(y_1 = z_1 \wedge \ldots \wedge y_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$. (LL$^*_\mathrm{n}$)
5. $\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset \Box A \supset$
   $\Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$. (3, 4)

Step 4 is justified by the fact that none of $x_1, \ldots, x_n$ is free in $[z_1, \ldots, z_n/x_1, \ldots, x_n]A$, and each y_i is modally free for x_i in $[y_1, \ldots, y_{i-1}/x_1, \ldots, x_{i-1}]\Box(x_1 = z_1 \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$, i.e. in $\Box(y_1 = z_1 \wedge \ldots \wedge y_{i-1} = z_{i-1} \wedge x_i = z_i \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A)$, because x_i and y_i are modally separated in $y_1 = z_1 \wedge \ldots \wedge y_{i-1} = z_{i-1} \wedge x_i = z_i \wedge \ldots \wedge x_n = z_n \supset [z_1, \ldots, z_n/x_1, \ldots, x_n]A$. ∎

Lemma 4.15 (Substitution-free Universal Instantiation)
For any $\mathcal{L}$-formula A and variables x, y,

$$\text{(FUI}^{**}\text{)} \vdash_L \forall xA \supset (Ey \supset \exists x(x{=}y \wedge A)).$$

Proof Let z be a variable not in $Var(A), x, y$.

1.	$\vdash_L z{=}y \supset Ey \supset Ez$	(LL*)
2.	$\vdash_L \forall xA \supset Ez \supset [z/x]A$	((FUI*), $z \notin Var(A)$)
3.	$\vdash_L \forall xA \wedge Ey \supset z{=}y \supset [z/x]A$	(1, 2)
4.	$\vdash_L \forall x(x{=}z \supset \neg A) \supset Ez \supset (z{=}z \supset [z/x]\neg A)$	((FUI*), $z \notin Var(A)$)
5.	$\vdash_L Ez \supset z{=}z$	((=R), or ($\forall$=R), (FUI*))
6.	$\vdash_L \forall x(x{=}z \supset \neg A) \supset Ez \supset [z/x]\neg A$	(4, 5)
7.	$\vdash_L Ez \supset [z/x]A \supset \exists x(x{=}z \wedge A)$	(6)
8.	$\vdash_L \forall xA \wedge Ey \supset z{=}y \supset \exists x(x{=}z \wedge A)$	(1, 3, 7)
9.	$\vdash_L z{=}y \supset \exists x(x{=}z \wedge A) \supset \exists x(x{=}y \wedge A)$	((LL*), $z \notin Var(A)$)
10.	$\vdash_L \forall xA \wedge Ey \supset z{=}y \supset \exists x(x{=}y \wedge A)$	(8, 9)
11.	$\vdash_L \forall z(\forall xA \wedge Ey) \supset \forall z(z{=}y \supset \exists x(x{=}y \wedge A))$	(10, (UG), (UD))
12.	$\vdash_L \forall xA \wedge Ey \supset \forall z(z{=}y \supset \exists x(x{=}y \wedge A))$	(11, (VQ))
13.	$\vdash_L \forall z(z{=}y \supset \exists x(x{=}y \wedge A)) \supset y{=}y \supset \exists x(x{=}y \wedge A)$	((FUI*), $z \notin Var(A)$)
14.	$\vdash_L Ey \supset y{=}y$	((=R), or ($\forall$=R), (FUI*))
15.	$\vdash_L \forall z(z{=}y \supset \exists x(x{=}y \wedge A)) \supset Ey \supset \exists x(x{=}y \wedge A)$	(13, 14)
16.	$\vdash_L \forall xA \supset Ey \supset \exists x(x{=}y \wedge A)$	(12, 15)

■

Incidentally, (FUI*) can also be derived from (FUI**), so we could just as well have used (FUI**) as basic axiom instead of (FUI*).

5 Logics with explicit substitution

Let's move on to languages with substitution. We first have to lay down some axioms governing the substitution operator. An obvious suggestion would be the lambda-conversion principle

$$\langle y : x \rangle A \leftrightarrow [y/x]A,$$

which would allow us to move back and forth between e.g. $\langle y : x \rangle Fx$ and Fy. But we've seen in lemma 3.9 that if things can have multiple counterparts, then these transitions are sound only under certain conditions: the move from $\langle y : x\rangle A$ to $[y/x]A$ requires that y is modally free for x in A, the other direction requires that y and x are modally separated in A. So we have the following somewhat more complex principles:

(SC1) $\langle y : x\rangle A \leftrightarrow [y/x]A$, provided y and x are modally separated in A.

(SC2) $\langle y : x\rangle A \supset [y/x]A$, provided y is modally free for x in A.

But now we need further principles telling us how $\langle y : x\rangle$ behaves when y is not modally free for x. For example, $\langle y : x\rangle\neg A$ should always entail $\neg\langle y : x\rangle A$, even if y is not modally free for x in A. More generally, the substitution operator commutes with every non-modal operator as long as there is no clash of bound variables:

(S$\neg$) $\langle y : x\rangle\neg A \leftrightarrow \neg\langle y : x\rangle A$,

(S$\supset$) $\langle y : x\rangle(A \supset B) \leftrightarrow (\langle y : x\rangle A \supset \langle y : x\rangle B)$,

(S$\forall$) $\langle y : x\rangle\forall zA \leftrightarrow \forall z\langle y : x\rangle A$, provided $z \notin \{x, y\}$,

(SS1) $\langle y : x\rangle\langle y_2 : z\rangle A \leftrightarrow \langle y_2 : z\rangle\langle y : x\rangle A$, provided $z \notin \{x, y\}$ and $y_2 \neq x$.

Substitution does not commute with the box. Roughly speaking, this is because $\langle y : x\rangle\Box A(x, y)$ says that at all accessible worlds, all counterparts x' and y' of y are $A(x', y')$, while $\Box\langle y : x\rangle A(x, y)$ says that at all accessible worlds, every counterpart $x' = y'$ of y is such that $A(x', y')$. In the first case, x' and y' may be different counterparts of y, while in the second case, they must be the same. Thus $\langle y : x\rangle\Box A$ entails $\Box\langle y : x\rangle A$, but the other direction holds only if either y does not have multiple counterparts at accessible worlds, or at most one of x and y occurs freely in A (including the special case where $x = y$).

(S$\Box$) $\langle y : x\rangle\Box A \supset \Box\langle y : x\rangle A$,

(S$\Diamond$) $\langle y : x\rangle\Diamond A \supset \Diamond\langle y : x\rangle A$, provided at most one of x, y is free in A.

These principles largely make (SC1) and (SC2) redundant. We only need to add the special case for substituting free variables in atomic formulas and in substitution operators, as well as a principle for vacuous substitutions:

(SAt) $\langle y : x\rangle Px_1 \dots x_n \leftrightarrow P[y/x]x_1 \dots [y/x]x_n$.

(SS2) $\langle y : x\rangle\langle x : z\rangle A \leftrightarrow \langle y : z\rangle\langle y : x\rangle A$.

(VS) $A \leftrightarrow \langle y : x\rangle A$, provided x is not free in A.

> Lemma 5.1 (Soundness of the substitution axioms)
> If $\mathcal{L}_s$ is a language of quantified modal logic with substitution, then every $\mathcal{L}_s$-instance of (S$\neg$), (S$\supset$), (S$\forall$), (SS1), (S$\Box$), (S$\Diamond$), (SAt), (SS2), and (VS) is valid in every (positive or negative) counterpart model.

Proof

1. (S$\neg$). $w, V \Vdash \langle y : x \rangle \neg A$ iff $w, V^{[y/x]} \Vdash \neg A$ by definition 3.2, iff $w, V^{[y/x]} \nVdash A$ by definition 2.7, iff $w, V \nVdash \langle y : x \rangle A$ by definition 3.2, iff $w, V \Vdash \neg \langle y : x \rangle A$ by definition 2.7.

2. (S$\supset$). $w, V \Vdash \langle y : x \rangle (A \supset B)$ iff $w, V^{[y/x]} \Vdash A \supset B$ by definition 3.2, iff $w, V^{[y/x]} \nVdash A$ or $w, V^{[y/x]} \Vdash B$ by definition 2.7, iff $w, V \nVdash \langle y : x \rangle A$ or $w, V \Vdash \langle y : x \rangle B$ by definition 3.2, iff $w, V \Vdash \langle y : x \rangle A \supset \langle y : x \rangle B$ by definition 2.7.

3. (S$\forall$). Assume $z \notin \{x, y\}$. Then the existential z-variants V' of $V^{[y/x]}$ on w coincide at w with the functions $(V^*)^{[y/x]}$ where V^* is an existential z-variant V^* of V on w. And so $w, V \Vdash \langle y : x \rangle \forall z A$ iff $w, V^{[y/x]} \Vdash \forall z A$ by definition 3.2, iff $w, V' \Vdash A$ for all existential z-variants V' of $V^{[y/x]}$ on w by definition 2.7, iff $w, (V^*)^{[y/x]} \Vdash A$ for all existential z-variants V^* of V on w, iff $w, V^* \Vdash \langle y : x \rangle A$ for all existential z-variants V^* of V on w by definition 3.2, iff $w, V \Vdash \forall z \langle y : x \rangle A$ by definition 2.7.

4. (SS1). Assume $z \notin \{x, y\}$ and $y_2 \neq x$. Then the function $[y/x] \cdot [y_2/z]$ is identical to the function $[y_2/z] \cdot [y/x]$. So $w, V \Vdash \langle y : x \rangle \langle y_2 : z \rangle A$ iff $w, V^{[y/x] \cdot [y_2/z]} \Vdash A$ by definition 3.2, iff $w, V^{[y_2/z] \cdot [y/x]} \Vdash A$, iff $w, V \Vdash \langle y_2 : z \rangle \langle y : x \rangle A$ by definition 3.2.

5. (S$\Box$). Assume $w, V \nVdash \Box \langle y : x \rangle A$. By definitions 2.7 and 3.2, this means that $w', V'^{[y/x]} \nVdash A$ for some w', V' such that wRw' and V' is a w'-image of V at w. Then $V'^{[y/x]}$ is also a w'-image of $V^{[y/x]}$ at w, since $V'^{[y/x]}_{w'}(x) = V'_{w'}(y)$ is some counterpart at w' of $V_w(y)$ at w (or undefined if there is none), and therefore some counterpart at w' of $V^{[y/x]}_w(x)$ at w (or undefined). So $w', V^* \nVdash A$ for some w', V^* such that wRw' and $V^{[y/x]}_w \triangleright V^*_{w'}$. So $w, V \Vdash \langle y : x \rangle \Box A$ by definitions 2.7 and 3.2.

6. (S$\Diamond$). Assume $w, V \Vdash \langle y : x \rangle \Diamond A$ and at most one of x, y is free in A. By definitions 2.7 and 3.2, $w', V^* \Vdash A$ for some w', V^* such that wRw' and V^* is a w'-image of $V^{[y/x]}$ at w. We have to show that there is a w'-image V' of V at w such that $w, V'^{[y/x]} \Vdash A$, since then $w, V \Vdash \Diamond \langle y : x \rangle A$.

   If $x = y$, then $V^*_{w'}(x) = V^*_{w'}(y)$ is a counterpart at w' of $V^{[y/x]}_w(x) = V^{[y/x]}_w(y) = V_w(x) = V_w(y)$ at w, so we can choose V^* itself as V'. We have $w, V'^{[y/x]} \Vdash A$ because $V'^{[y/x]} = V'$.

   Else if x is not free in A, let V' be some x-variant of V^* at w' such that $V^*_{w'}(x)$ is some counterpart at w' of $V_w(x)$ at w (or undefined if there is none). Since $V^*_{w'}(y)$ is a counterpart at w' of $V^{[y/x]}_w(y) = V_w(y)$ at w (or undefined if there is none), V' is a w'-image of V at w. Moreover, $V'^{[y/x]}$ and V^* agree at w' about all variables other than x; so by the coincidence lemma 2.8, $w', V'^{[y/x]} \Vdash A$.

   Else if y is not free in A, let V' be like V^* except that $V'_{w'}(y) = V^*_{w'}(x)$ and $V'_{w'}(x)$ is some counterpart at w' of $V_w(x)$ at w (or undefined if there is none). Since $V'_{w'}(y) = V^*_{w'}(x)$ is a counterpart at w' of $V^{[y/x]}_w(x) = V_w(y)$ at w (or undefined if there is none), V' is a w'-image of V at w. Moreover, $V'^{[y/x]}$ and V^* agree at w' about all variables other than y; in particular, $V'^{[y/x]}_{w'}(x) = V'_{w'}(y) = V^*_{w'}(x)$. So by the coincidence lemma 2.8, $w', V'^{[y/x]} \Vdash A$.

7. (SAt). $w, V \Vdash \langle y : x \rangle P x_1 \ldots x_n$ iff $w, V^{[y/x]} \Vdash P x_1 \ldots x_n$ by definition 3.2, iff $w, V \Vdash [y/x] P x_1 \ldots x_n$ by lemma 3.9.
8. (SS2). $w, V \Vdash \langle y : x \rangle \langle x : z \rangle A$ iff $w, V^{[y/x] \cdot [x/z]} \Vdash A$ by definition 3.2, iff $w, V^{[y/z] \cdot [y/x]} \Vdash A$ because $[y/x] \cdot [x/z] = [y/z] \cdot [y/x]$, iff $w, V \Vdash \langle y : z \rangle \langle y : x \rangle A$ by definition 3.2.
9. (VS). By definition 3.2, $w, V \Vdash \langle y : x \rangle A$ iff $w, V^{[y/x]} \Vdash A$. If x is not free in A, then $V^{[y/x]}$ agrees with V at w about all free variables in A. So by the coincidence lemma 2.8, $w, V^{[y/x]} \Vdash A$ iff $w, V \Vdash A$. So then $w, V \Vdash \langle y : x \rangle A$ iff $w, V \Vdash A$. ■

Definition 5.2 (Positive logics with substitution)
A *positive (quantified modal) logic with substitution* is a function that maps each language $\mathcal{L}_s$ of quantified modal logic with substitution to a set L that contains all $\mathcal{L}_s$-instances of the substitution axioms (S¬), (S⊃), (S∀), (SS1), (S□), (S◇), (SAt), (SS2), (VS), as well as all (Taut), (UD), (VQ), (∀Ex), (=R), (K),

(FUI$_\mathrm{s}$) $\forall x A \supset (Ey \supset \langle y : x \rangle A)$,

(LL$_\mathrm{s}$) $x = y \supset (A \supset \langle y : x \rangle A)$,

and that is closed under (MP), (UG), (Nec) and

(Sub$_\mathrm{s}$) if $\vdash_L A$, then $\vdash_L \langle y : x \rangle A$.

The smallest such logic is called P_s.

Definition 5.3 (Negative logics with substitution)
A *negative (quantified modal) logic with substitution* is a function that maps each language $\mathcal{L}_s$ to a set that contains all $\mathcal{L}_s$-instances of the substitution axioms (S¬), (S⊃), (S∀), (SS1), (S□), (S◇), (SAt), (SS2), (VS), as well as (Taut), (UD), (VQ), (Neg), (NA), (∀=R), (K), (FUI$_\mathrm{s}$), (LL$_\mathrm{s}$), and that is closed under (MP), (UG), (Nec) and (Sub$_\mathrm{s}$). The smallest such logic is called N_s.

Theorem 5.4 (Soundness of P_s)
Every member of P_s is valid in every positive counterpart model.

Proof We have to show that all P_s axioms are valid in every model, and that validity is closed under (MP), (UG), (Nec) and (Sub$_\mathrm{s}$). For (Taut), (UD), (VQ), (∀Ex), (=R), (K), (MP), (UG), (Nec), see the proof of theorem 4.4. For the substitution axioms, see lemma 5.1. The remaining cases are (FUI$_\mathrm{s}$), (LL$_\mathrm{s}$), and (Sub$_\mathrm{s}$).

1. (FUI$_s$). Assume $w, V \Vdash \forall x A$ and $w, V \Vdash Ey$ in some model. By definition 2.7, the latter means that $V_w(y) \in D_w$, and the former means that $w, V' \Vdash A$ for all existential x-variants V' of V on w. So in particular, $w, V' \Vdash A$, where V' is the x-variant of V on w with $V_w(x) = V_w(y)$. So $w, V \Vdash \langle y : x \rangle A$ by definition 3.2.
2. (LL$_s$). Assume $w, V \Vdash x = y$ and $w, V \Vdash A$. By definitions 2.7 and 2.3, then $V_w(x) = V_w(y)$. So $w, V \Vdash \langle y : x \rangle A$ by definition 3.2.
3. (Sub$_s$). Assume $w, V \nVdash \langle y : x \rangle A$ in some model $\mathcal{M} = \langle \mathcal{S}, V \rangle$. By definition 3.2, then $w, V' \nVdash A$, where V' is the x-variant of V on w with $V'(x) = V(y)$. So A is invalid in the model $\langle \mathcal{S}, V' \rangle$. Hence if A is valid in all positive models, then so is $\langle y : x \rangle A$. ■

Theorem 5.5 (Soundness of N_s)
Every member of N_s is valid in every negative counterpart model.

Proof All the cases needed here are covered in the proofs of theorem 4.5 and 5.4. ■

To derive some further properties of these systems, let $\mathcal{L}$ range over languages of quantified modal logic with substitution, and L over the corresponding sets $\mathsf{P}_s(\mathcal{L}), \mathsf{N}_s(\mathcal{L})$, and $\mathsf{C}_s(\mathcal{L})$.

Closure under propositional consequence and the validity of ($\forall$Ex) and ($\forall$=R) are proved just as for substitution-free logics (see lemmas 4.6 and 4.7). So we move on immediately to more interesting properties.

Lemma 5.6 (Substitution expansion)
If A is an $\mathcal{L}$-formula and x, y, z $\mathcal{L}$-variables, then

(SE1) $\vdash_L A \leftrightarrow \langle x : x \rangle A$;

(SE2) $\vdash_L \langle y : x \rangle A \leftrightarrow \langle y : z \rangle \langle z : x \rangle A$, provided z is not free in A.

Proof (SE1) is proved by induction on A.

1. A is atomic. Then $\vdash_L \langle x : x \rangle A \leftrightarrow [x/x]A$ by (SAt), and so $\vdash_L \langle x : x \rangle A \leftrightarrow A$ because $[x/x]A = A$.
2. A is $\neg B$. By induction hypothesis, $\vdash_L B \leftrightarrow \langle x : x \rangle B$. So by (PC), $\vdash_L \neg B \leftrightarrow \neg \langle x : x \rangle B$. And by $\langle S\neg \rangle$, $\vdash_L \langle x : x \rangle \neg B \leftrightarrow \neg \langle x : x \rangle B$.
3. A is $B \supset C$. By induction hypothesis, $\vdash_L B \leftrightarrow \langle x : x \rangle B$ and $\vdash_L C \leftrightarrow \langle x : x \rangle C$. So $\vdash_L (B \supset C) \leftrightarrow (\langle x : x \rangle B \supset \langle x : x \rangle C)$. And by $\langle S \supset \rangle$, $\vdash_L \langle x : x \rangle (B \supset C) \leftrightarrow (\langle x : x \rangle B \supset \langle x : x \rangle C)$.

4. A is $\forall zB$. If $z = x$, then $\vdash_L \forall xB \leftrightarrow \langle x : x\rangle\forall xB$ by (VS). If $z \neq x$, then by induction hypothesis, $\vdash_L B \leftrightarrow \langle x : x\rangle B$; by (UG) and (UD), $\vdash_L \forall zB \leftrightarrow \forall z\langle x : x\rangle B$; and $\vdash_L \langle x : x\rangle\forall zB \leftrightarrow \forall z\langle x : x\rangle B$ by (S$\forall$).

5. A is $\langle y : z\rangle B$. If $z = x$, then $\vdash_L \langle y : x\rangle B \leftrightarrow \langle x : x\rangle\langle y : x\rangle B$ by (VS). If $z \neq x$, then by induction hypothesis, $\vdash_L B \leftrightarrow \langle x : x\rangle B$; by (Sub$_s$) and (S$\supset$), $\vdash_L \langle y : z\rangle B \leftrightarrow \langle y : z\rangle\langle x : x\rangle B$; and $\vdash_L \langle x : x\rangle\langle y : z\rangle B \leftrightarrow \langle y : z\rangle\langle x : x\rangle B$ by (SS1) (if $y \neq x$) or (SS2) (if $y = x$).

6. A is $\Box B$. By (S$\Box$), $\vdash_L \langle x : x\rangle\Box B \supset \Box\langle x : x\rangle B$. Conversely, since at most one of x, x is free in $\neg B$, by (S$\Diamond$), $\vdash_L \langle x : x\rangle\Diamond\neg B \supset \Diamond\langle x : x\rangle\neg B$. Contraposing and unraveling the definition of the diamond, we have $\vdash_L \Box\neg\langle x : x\rangle\neg B \supset \neg\langle x : x\rangle\neg\Box\neg\neg B$. Since $\vdash_L \Box\neg\langle x : x\rangle\neg B \leftrightarrow \Box\langle x : x\rangle B$ and $\vdash_L \neg\langle x : x\rangle\neg\Box\neg\neg B \leftrightarrow \langle x : x\rangle B$ (by (S$\neg$), (Sub$_s$), (S$\supset$), (Nec) and (K)), this means that $\vdash_L \Box\langle x : x\rangle B \supset \langle x : x\rangle\Box B$.

As for (SE2): by (VQ), $\vdash_L \langle y : x\rangle A \leftrightarrow \langle y : z\rangle\langle y : x\rangle A$. And $\vdash_L \langle y : x\rangle\langle y : z\rangle A \leftrightarrow \langle y : z\rangle\langle y : x\rangle A$ by (SS1) (if $y \neq x$) or (SS2) (if $y = x$). Moreover, by (SS2), $\vdash_L \langle y : z\rangle\langle z : x\rangle A \leftrightarrow \langle y : x\rangle\langle y : z\rangle A$. So by (PC), $\vdash_L \langle y : x\rangle A \leftrightarrow \langle y : z\rangle\langle z : x\rangle A$. ■

Lemma 5.7 (Substituting bound variables)
For any $\mathcal{L}$-sentence A and variables x, y,

(SBV) $\forall xA \leftrightarrow \forall y\langle y : x\rangle A$, provided y is not free in A.

Proof

1.	$\vdash_L \forall y\langle y : x\rangle A \supset Ex \supset \langle x : y\rangle\langle y : x\rangle A$.	(FUI$_s$)
2.	$\vdash_L \langle x : y\rangle\langle y : x\rangle A \leftrightarrow A$.	((SE1), (SE2))
3.	$\vdash_L \forall x\forall y\langle y : x\rangle A \supset \forall xEx \supset \forall xA$.	(1, 2, (UG), (UD))
4.	$\vdash_L \forall x\forall y\langle y : x\rangle A \supset \forall xA$.	(3, ($\forall$Ex))
5.	$\vdash_L \forall y\langle y : x\rangle A \supset \forall x\forall y\langle y : x\rangle A$.	(VQ)
6.	$\vdash_L \forall y\langle y : x\rangle A \supset \forall xA$.	(4, 5)
7.	$\vdash_L \forall xA \supset Ey \supset \langle y : x\rangle A$.	(FUI$_s$)
8.	$\vdash_L \forall y\forall xA \supset \forall y\langle y : x\rangle A$.	(7, (UG), (UD), ($\forall$Ex))
9.	$\vdash_L \forall xA \supset \forall y\forall xA$.	((VQ), y not free in A)
10.	$\vdash_L \forall xA \supset \forall y\langle y : x\rangle A$.	(8, 9)
11.	$\vdash_L \forall xA \leftrightarrow \forall y\langle y : x\rangle A$.	(6, 10)

■

Lemma 5.8 (Substituting empty variables)
For any $\mathcal{L}$-sentence A and variables x, y,

(SEV) $\vdash_L x \neq x \land y \neq y \supset (A \leftrightarrow \langle y:x\rangle A)$.

Proof (SEV) is trivial if L is positive, in which case $\vdash_L x = x$. For negative L, it is proved by induction on A.

1. A is atomic. If $x \notin \mathit{Var}(A)$, then $\vdash_L A \leftrightarrow \langle y:x\rangle A$ by (VS), and so $\vdash_L x \neq x \land y \neq y \supset (A \leftrightarrow \langle y:x\rangle A)$ by (PC). If $x \in \mathit{Var}(A)$, then by (Neg)

   (1) $\quad \vdash_L x \neq x \land y \neq y \supset \neg A$.

   Also by (Neg), $\vdash_L x \neq x \land y \neq y \supset \neg[y/x]A$. By (SAt), $\vdash_L [y/x]A \leftrightarrow \langle y:x\rangle A$, and so $\vdash_L \neg[y/x]A \leftrightarrow \neg\langle y:x\rangle A$. So

   (2) $\quad \vdash_L x \neq x \land y \neq y \supset \neg\langle y:x\rangle A$.

   Combining (1) and (2) yields $\vdash_L x \neq x \land y \neq y \supset (A \leftrightarrow \langle y:x\rangle A)$.

2. A is $\neg B$. By induction hypothesis, $\vdash_L x \neq x \land y \neq y \supset (B \leftrightarrow \langle y:x\rangle B)$. So by (PC), $\vdash_L x \neq x \land y \neq y \supset (\neg B \leftrightarrow \neg\langle y:x\rangle B)$, and by (S$\neg$), $\vdash_L x \neq x \land y \neq y \supset (\neg B \leftrightarrow \langle y:x\rangle \neg B)$.

3. A is $B \supset C$. By induction hypothesis, $\vdash_L x \neq x \land y \neq y \supset (B \leftrightarrow \langle y:x\rangle B)$ and $\vdash_L x \neq x \land y \neq y \supset (C \leftrightarrow \langle y:x\rangle C)$. So by (PC), $\vdash_L x \neq x \land y \neq y \supset ((B \supset C) \leftrightarrow (\langle y:x\rangle B \supset \langle y:x\rangle C))$, and by (S$\supset$), $\vdash_L x \neq x \land y \neq y \supset ((B \supset C) \leftrightarrow \langle y:x\rangle(B \supset C))$.

4. A is $\forall z B$. We distinguish three cases.

   a) $z \notin \{x, y\}$. Then

1.	$\vdash_L x \neq x \land y \neq y \supset (B \leftrightarrow \langle y:x\rangle B)$	(ind. hyp.)
2.	$\vdash_L \forall z\, x \neq x \land \forall z\, y \neq y \supset (\forall z B \leftrightarrow \forall z\langle y:x\rangle B)$	(1, UG, UD)
3.	$\vdash_L x \neq x \land y \neq y \supset (\forall z B \leftrightarrow \forall z\langle y:x\rangle B)$	(2, VQ)
4.	$\vdash_L x \neq x \land y \neq y \supset (\forall z B \leftrightarrow \langle y:x\rangle \forall z B)$.	(3, (S$\forall$))

   b) $z = x$. Then A is $\forall x B$, and $\vdash_L \forall x B \leftrightarrow \langle y:x\rangle \forall x B$ by (VS). So $\vdash_L x \neq x \land y \neq y \supset (\forall x B \leftrightarrow \langle y:x\rangle \forall x B)$ by (PC).

   c) $z = y \neq x$. Then A is $\forall y B$. Let v be a variable not in $\mathit{Var}(A), x, y$.

1.	$\vdash_L x \neq x \land v \neq v \supset (B \leftrightarrow \langle v:x\rangle B)$.	(ind. hyp.)
2.	$\vdash_L \forall y x \neq x \land \forall y v \neq v \supset (\forall y B \leftrightarrow \forall y\langle v:x\rangle B)$.	(1, UG, UD)
3.	$\vdash_L x \neq x \land v \neq v \supset (\forall y B \leftrightarrow \forall y\langle v:x\rangle B)$.	(2, VQ)
4.	$\vdash_L x \neq x \land v \neq v \supset (\forall y B \leftrightarrow \langle v:x\rangle \forall y B)$.	(3, (S$\forall$))
5.	$\vdash_L \langle y:v\rangle x \neq x \land \langle y:v\rangle v \neq v \supset (\langle y:v\rangle \forall y B \leftrightarrow \langle y:v\rangle\langle v:x\rangle \forall y B)$.	(4, (Sub$_s$), (S$\supset$))
6.	$\vdash_L x \neq x \land y \neq y \supset (\langle y:v\rangle \forall y B \leftrightarrow \langle y:v\rangle\langle v:x\rangle \forall y B)$.	(5, (VS), (SAt))
7.	$\vdash_L x \neq x \land y \neq y \supset (\forall y B \leftrightarrow \langle y:v\rangle\langle v:x\rangle \forall y B)$.	(6, (VS))
8.	$\vdash_L x \neq x \land y \neq y \supset (\forall y B \leftrightarrow \langle y:x\rangle \forall y B)$.	(7, (SE2))

5. A is $\langle y_2 : z\rangle B$. We have four cases.

a) $z \notin \{x, y\}$ and $y_2 \neq x$. Then

1.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (B \leftrightarrow \langle y : x\rangle B)$	(ind. hyp.)
2.	$\vdash_L \langle y_2 : z\rangle x{\neq}x \wedge \langle y_2 : z\rangle y{\neq}y \supset (\langle y_2 : z\rangle B \leftrightarrow \langle y_2 : z\rangle\langle y : x\rangle B)$	(1, (Sub$_\mathrm{s}$), (S$\supset$))
3.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2 : z\rangle B \leftrightarrow \langle y_2 : z\rangle\langle y : x\rangle B)$	(2, (VS))
4.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2 : z\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : z\rangle B)$.	(3, (SS1))

b) $z \neq x$ and $y_2 = x$. Then A is $\langle x : z\rangle B$.

1.	$\vdash_L x{\neq}x \wedge z{\neq}z \supset (B \leftrightarrow \langle x : z\rangle B)$	(ind. hyp.)
2.	$\vdash_L \langle y : z\rangle x{\neq}x \wedge \langle y : z\rangle z{\neq}z \supset (\langle y : z\rangle B \leftrightarrow \langle y : z\rangle\langle x : z\rangle B)$	(1, (Sub$_\mathrm{s}$), (S$\supset$))
3.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y : z\rangle B \leftrightarrow \langle y : z\rangle\langle x : z\rangle B)$	(2, (SAt), $z \neq x$)
4.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y : z\rangle B \leftrightarrow \langle x : z\rangle B)$	(3, (VS), $z \neq x$)
5.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (B \leftrightarrow \langle y : x\rangle B)$	(ind. hyp.)
6.	$\vdash_L \langle y : z\rangle x{\neq}x \wedge \langle y : z\rangle y{\neq}y \supset (\langle y : z\rangle B \leftrightarrow \langle y : z\rangle\langle y : x\rangle B)$	(5, (Sub$_\mathrm{s}$),(S$\supset$))
7.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y : z\rangle B \leftrightarrow \langle y : z\rangle\langle y : x\rangle B)$	(6, (SAt), $z \neq x$)
8.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle x : z\rangle B \leftrightarrow \langle y : z\rangle\langle y : x\rangle B)$	(4, 7)
9.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle x : z\rangle B \leftrightarrow \langle y : x\rangle\langle x : z\rangle B)$.	(8, (SS2))

c) $z = x$. Then A is $\langle y_2 : x\rangle B$, and $\vdash_L \langle y_2 : x\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : x\rangle B$ by (VS). So $\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2 : x\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : x\rangle B)$ by (PC).

d) $z = y \neq x$ and $y_2 \neq x$. Then A is $\langle y_2 : y\rangle B$. Let v be a variable not in $\mathit{Var}(A), x, y, y_2$.

1.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset (B \leftrightarrow \langle v : x\rangle B)$.	(ind. hyp.)
2.	$\vdash_L \langle y_2 : y\rangle x{\neq}x \wedge \langle y_2 : y\rangle v{\neq}v \supset (\langle y_2 : y\rangle B \leftrightarrow \langle y_2 : y\rangle\langle v : x\rangle B)$.	(1, (Sub$_\mathrm{s}$), (S$\supset$))
3.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset (\langle y_2 : y\rangle B \leftrightarrow \langle y_2 : y\rangle\langle v : x\rangle B)$.	(2, (VS))
4.	$\vdash_L x{\neq}x \wedge v{\neq}v \supset (\langle y_2 : y\rangle B \leftrightarrow \langle v : x\rangle\langle y_2 : y\rangle B)$.	(3, (SS1), $y_2 \neq x$)
5.	$\vdash_L \langle y : v\rangle x{\neq}x \wedge \langle y : v\rangle v{\neq}v \supset (\langle y : v\rangle\langle y_2 : y\rangle B \leftrightarrow \langle y : v\rangle\langle v : x\rangle\langle y_2 : y\rangle B)$.	(4, (Sub$_\mathrm{s}$), (S$\supset$))
6.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y : v\rangle\langle y_2 : y\rangle B \leftrightarrow \langle y : v\rangle\langle v : x\rangle\langle y_2 : y\rangle B)$.	(5, (VS), (SAt))
7.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2 : y\rangle B \leftrightarrow \langle y : v\rangle\langle v : x\rangle\langle y_2 : y\rangle B)$.	(6, (VS))
8.	$\vdash_L x{\neq}x \wedge y{\neq}y \supset (\langle y_2 : y\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : y\rangle B)$.	(7, (SE2))

6. A is $\Box B$. Let v be a variable not in $\mathit{Var}(B)$.

1.	$\vdash_L x \neq x \wedge v \neq v \supset (B \leftrightarrow \langle v : x\rangle B)$.	(ind. hyp.)
2.	$\vdash_L \Box(x \neq x \wedge v \neq v) \supset (\Box B \leftrightarrow \Box\langle v : x\rangle B)$.	(1, (Nec), (K))
3.	$\vdash_L x \neq x \wedge v \neq v \supset \Box(x \neq x \wedge v \neq v)$	((NA), (NE), (Nec), (K))
4.	$\vdash_L x \neq x \wedge v \neq v \supset (\Box B \leftrightarrow \Box\langle v : x\rangle B)$.	(2, 3)
5.	$\vdash_L x \neq x \wedge v \neq v \supset (\Box B \leftrightarrow \langle v : x\rangle \Box B)$.	(4, (S$\Box$), (S$\Diamond$), $v \notin \mathit{Var}(B)$)
6.	$\vdash_L \langle y : v\rangle x \neq x \wedge \langle y : v\rangle v \neq v \supset (\langle y : v\rangle \Box B \leftrightarrow \langle y : v\rangle\langle v : x\rangle \Box B)$.	(5, (Sub$_s$), (S$\supset$))
7.	$\vdash_L x \neq x \wedge y \neq y \supset (\langle y : v\rangle \Box B \leftrightarrow \langle y : v\rangle\langle v : x\rangle \Box B)$.	(6, (SAt))
8.	$\vdash_L x \neq x \wedge y \neq y \supset (\Box B \leftrightarrow \langle y : v\rangle\langle v : x\rangle \Box B)$.	(7, (VS))
9.	$\vdash_L x \neq x \wedge y \neq y \supset (\Box B \leftrightarrow \langle y : x\rangle \Box B)$.	(8, (SE2))

Now we can prove (SC1) and (SC2). I will also prove that $\langle y : x\rangle A$ and $[y/x]A$ are provably equivalent conditional on $y \neq y$. Compare lemma 3.9 for a (slightly stronger) semantic version of this lemma.

> Lemma 5.9 (Substitution conversion)
> For any $\mathcal{L}$-formula A and variables x, y,
>
> (SC1) $\vdash_L \langle y : x\rangle A \leftrightarrow [y/x]A$, provided y and x are modally separated in A.
>
> (SC2) $\vdash_L \langle y : x\rangle A \supset [y/x]A$, provided y is modally free for x in A.
>
> (SCN) $\vdash_L y \neq y \supset (\langle y : x\rangle A \leftrightarrow [y/x]A)$.

Proof If x and y are the same variable, then by (SE1), $\vdash_L \langle x : x\rangle A \leftrightarrow [x/x]A$. Assume then that x and y are different variables. We first prove (SC1) and (SC2), by induction on A. Observe that if A is not a box formula $\Box B$, then by definition 3.4, y is modally free for x in A iff y and x are modally separated in A, in which case y and x are also modally separated in any subformula of A.

1. A is atomic. By (SAt), $\vdash_L \langle y : x\rangle A \leftrightarrow [y/x]A$ holds without any restrictions.

2. A is $\neg B$. If y and x are modally separated in A, then by induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$. So by (PC), $\vdash_L \neg\langle y : x\rangle B \leftrightarrow \neg[y/x]B$. By (S$\neg$) and definition 3.3, it follows that $\vdash_L \langle y : x\rangle\neg B \leftrightarrow [y/x]\neg B$.

3. A is $B \supset C$. If y and x are modally separated in A, then by induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$ and $\vdash_L \langle y : x\rangle C \leftrightarrow [y/x]C$. By (S$\supset$), $\vdash_L \langle y : x\rangle(B \supset C) \leftrightarrow (\langle y : x\rangle B \supset \langle y : x\rangle C)$. So $\vdash_L \langle y : x\rangle(B \supset C) \leftrightarrow ([y/x]B \supset [y/x]C)$, and so $\vdash_L \langle y : x\rangle(B \supset C) \leftrightarrow [y/x](B \supset C)$ by definition 3.3.

4. A is $\forall z B$. We have to distinguish four cases, assuming each time that y and x are modally separated in A.

a) $z \notin \{x, y\}$. By induction hypothesis, $\vdash_L \langle y : x \rangle B \leftrightarrow [y/x]B$. So by (UG) and (UD), $\vdash_L \forall z \langle y : x \rangle B \leftrightarrow \forall z [y/x]B$. Since $z \notin \{x, y\}$, $\vdash_L \langle y : x \rangle \forall z B \leftrightarrow \forall z \langle y : x \rangle B$ by (S$\forall$), and $\forall z [y/x]B$ is $[y/x]\forall z B$ by definition 3.3; so $\vdash_L \langle y : x \rangle \forall z B \leftrightarrow [y/x]\forall z B$.

b) $z = y$ and $x \notin \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\forall z B$ is $\forall y [y/x]B$.

1.	$\vdash_L \langle y : x \rangle B \leftrightarrow [y/x]B$.	(induction hypothesis)
2.	$\vdash_L \forall y \langle y : x \rangle B \leftrightarrow \forall y [y/x]B$.	(1, (UG), (UD))
3.	$\vdash_L B \leftrightarrow \langle y : x \rangle B$.	((VS), $x \notin \mathit{Varf}(B)$)
4.	$\vdash_L \forall y B \leftrightarrow \forall y \langle y : x \rangle B$.	(3, (UG), (UD))
5.	$\vdash_L \forall y B \leftrightarrow \langle y : x \rangle \forall y B$.	((VS), $x \notin \mathit{Varf}(B)$)
6.	$\vdash_L \langle y : x \rangle \forall y B \leftrightarrow \forall y [y/x]B$.	(2, 4, 5)

c) $z = x$ and $y \notin \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\forall z B$ is $\forall y [y/x]B$.

1.	$\vdash_L \langle y : x \rangle B \leftrightarrow [y/x]B$.	(induction hypothesis)
2.	$\vdash_L \forall y \langle y : x \rangle B \leftrightarrow \forall y [y/x]B$.	(1, (UG), (UD))
3.	$\vdash_L \forall x B \leftrightarrow \forall y \langle y : x \rangle B$.	((SBV), $y \notin \mathit{Varf}(B)$)
4.	$\vdash_L \forall x B \leftrightarrow \langle y : x \rangle \forall x B$.	(VS)
5.	$\vdash_L \langle y : x \rangle \forall x B \leftrightarrow \forall y [y/x]B$.	(2, 3, 4)

d) $z = x$ and $y \in \mathit{Varf}(B)$, or $z = y$ and $x \in \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\forall z B$ is $\forall v [y/x][v/z]B$ for some variable $v \notin \mathit{Var}(B) \cup \{x, y\}$. Since v and z are modally separated in B, by induction hypothesis $\vdash_L \langle v : z \rangle B \leftrightarrow [v/z]B$. So by (UG) and (UD), $\vdash_L \forall v \langle v : z \rangle B \leftrightarrow \forall v [v/z]B$. By (SBV), $\vdash_L \forall z B \leftrightarrow \forall v \langle v : z \rangle B$. So $\vdash_L \forall z B \leftrightarrow \forall v [v/z]B$. Moreover, as $z \in \{x, y\}$, y and x are modally separated in $[v/z]B$. So by induction hypothesis, $\vdash_L \langle y : x \rangle [v/z]B \leftrightarrow [y/x][v/z]B$. Then

1.	$\vdash_L \forall z B \leftrightarrow \forall v [v/z]B$	(as just shown)
2.	$\vdash_L \langle y : x \rangle \forall z B \leftrightarrow \langle y : x \rangle \forall v [v/z]B$	(1, (Sub$^{\mathrm{s}}$), (S$\neg$), (S$\supset$))
3.	$\vdash_L \langle y : x \rangle \forall v [v/z]B \leftrightarrow \forall v \langle y : x \rangle [v/z]B$.	(S$\forall$)
4.	$\vdash_L \langle y : x \rangle \forall z B \leftrightarrow \forall v \langle y : x \rangle [v/z]B$.	(2, 3)
5.	$\vdash_L \langle y : x \rangle [v/z]B \leftrightarrow [y/x][v/z]B$.	(induction hypothesis)
6.	$\vdash_L \forall v \langle y : x \rangle [v/z]B \leftrightarrow \forall v [y/x][v/z]B$.	(5, (UG), (UD))
7.	$\vdash_L \langle y : x \rangle \forall z B \leftrightarrow \forall v [y/x][v/z]B$.	(4, 6)

5. A is $\langle y_2 : z \rangle B$. Again we have four cases, assuming x and y are modally separated in A.

a) $z \notin \{x, y\}$. By definition 3.3, then $[y/x]\langle y_2 : z\rangle B$ is $\langle [y/x]y_2 : z\rangle [y/x]B$.

1.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle\langle y : x\rangle B$	((SS1) or (SS2))
2.	$\vdash \langle y : x\rangle B \leftrightarrow [y/x]B$	(induction hypothesis)
3.	$\vdash \langle [y/x]y_2 : z\rangle(\langle y : x\rangle B \leftrightarrow [y/x]B)$	(2, ($\mathrm{Sub_s}$))
4.	$\vdash \langle [y/x]y_2 : z\rangle\langle y : x\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle [y/x]B$	(3, (S$\supset$), (S$\neg$))
5.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle [y/x]B.$	(1, 4)

b) $z = y$ and $x \notin \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\langle y_2 : z\rangle B$ is $\langle [y/x]y_2 : y\rangle [y/x]B$. By induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$. So by ($\mathrm{Sub_s}$) and (S$\supset$), $\vdash_L \langle [y/x]y_2 : y\rangle\langle y : x\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle [y/x]B$. If $y_2 = x$, then $\vdash_L \langle y : x\rangle\langle y_2 : y\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle\langle y : x\rangle B$ by (SS2). If $y_2 \neq x$, then

1.	$\vdash_L \langle y_2 : y\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : y\rangle B$	((VS), $x \notin \mathit{Varf}(\langle y_2 : y\rangle B)$)
2.	$\vdash_L B \leftrightarrow \langle y : x\rangle B$	((VS), $x \notin \mathit{Varf}(B)$)
3.	$\vdash_L \langle y_2 : y\rangle B \leftrightarrow \langle y_2 : y\rangle\langle y : x\rangle B$	(1, ($\mathrm{Sub_s}$), (S$\supset$))
4.	$\vdash_L \langle y : x\rangle\langle y_2 : y\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle\langle y : x\rangle B$	(1, 3)

So either way $\vdash_L \langle y : x\rangle\langle y_2 : y\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle\langle y : x\rangle B$. So $\vdash_L \langle y : x\rangle\langle y_2 : y\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle [y/x]B$.

c) $z = x$ and $y \notin \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\langle y_2 : z\rangle B$ is ([y/x]$\mathrm{y_2}$: y)$[y/x]B$. By induction hypothesis, $\vdash_L \langle y : x\rangle B \leftrightarrow [y/x]B$. So by ($\mathrm{Sub_s}$) and (S$\supset$), $\vdash_L \langle [y/x]y_2 : y\rangle\langle y : x\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle [y/x]B$. Since $y \notin \mathit{Varf}(B)$, by (SE2), $\vdash_L \langle [y/x]y_2 : y\rangle\langle y : x\rangle B \leftrightarrow \langle [y/x]y_2 : x\rangle B$. Moreover, $\vdash_L \langle [y/x]y_2 : x\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : x\rangle B$ by either (VS) (if $x \neq y_2$) or by (SE1), ($\mathrm{Sub_s}$) and (S$\supset$) (if $x = y_2$). So $\vdash_L \langle y : x\rangle\langle y_2 : x\rangle B \leftrightarrow \langle [y/x]y_2 : y\rangle\langle y : x\rangle B$.

d) $z = x$ and $y \in \mathit{Varf}(B)$, or $z = y$ and $x \in \mathit{Varf}(B)$. By definition 3.3, then $[y/x]\langle y_2 : z\rangle B$ is $\langle [y/x]y_2 : v\rangle [y/x][v/z]B$, where $v \notin \mathit{Var}(B) \cup \{x, y, y_2\}$.

1.	$\vdash \langle v : z\rangle B \leftrightarrow [v/z]B$	(induction hypothesis)
2.	$\vdash \langle y_2 : v\rangle\langle v : z\rangle B \leftrightarrow \langle y_2 : v\rangle [v/z]B$	(1, ($\mathrm{Sub_s}$), (S$\supset$), (S$\neg$))
3.	$\vdash \langle y_2 : z\rangle B \leftrightarrow \langle y_2 : v\rangle\langle v : z\rangle B$	(SE2)
4.	$\vdash \langle y_2 : z\rangle B \leftrightarrow \langle y_2 : v\rangle [v/z]B$	(2, 3)

Since $z \in \{x, y\}$, x and y are modally separated in $[v/z]B$. So:

5.	$\vdash \langle y : x\rangle [v/z]B \leftrightarrow [y/x][v/z]B$	(ind. hyp.)
6.	$\vdash \langle [y/x]y_2 : v\rangle\langle y : x\rangle [v/z]B \leftrightarrow \langle [y/x]y_2 : v\rangle [y/x][v/z]B$	(5, ($\mathrm{Sub_s}$), (S$\supset$))
7.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle y : x\rangle\langle y_2 : v\rangle [v/z]B$	(4, ($\mathrm{Sub_s}$), (S$\supset$))
8.	$\vdash \langle y : x\rangle\langle y_2 : v\rangle [v/z]B \leftrightarrow \langle [y/x]y_2 : v\rangle\langle y : x\rangle [v/z]B$	((SS1) or (SS2))
9.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : v\rangle\langle y : x\rangle [v/z]B$	(7, 8)
10.	$\vdash \langle y : x\rangle\langle y_2 : z\rangle B \leftrightarrow \langle [y/x]y_2 : v\rangle [y/x][v/z]B$	(6, 9)

6. A is $\Box B$. For (SC1), assume x and y are modally separated in A. Then they are also modally separated in B, so by induction hypothesis, $\vdash_L \langle y:x\rangle B \leftrightarrow [y/x]B$. By (Nec) and (K), then $\vdash_L \Box\langle y:x\rangle B \leftrightarrow \Box[y/x]B$. By (S$\Box$), $\vdash_L \langle y:x\rangle\Box B \supset \Box\langle y:x\rangle B$. Since at most one of x, y is free in B, by (S$\Diamond$), $\vdash_L \langle y:x\rangle\Diamond\neg B \supset \Diamond\langle y:x\rangle\neg B$; so $\vdash_L \Box\langle y:x\rangle B \supset \langle y:x\rangle\Box B$ (by (S$\neg$), (Sub$_s$), (S$\supset$), (Nec), (K)). So $\vdash_L \langle y:x\rangle\Box B \leftrightarrow \Box[y/x]B$. Since $\Box[y/x]B$ is $[y/x]\Box B$ by definition 3.3, this means that $\vdash_L \langle y:x\rangle\Box B \leftrightarrow [y/x]\Box B$.

   For (SC2), assume y is modally free for x in $\Box B$. Then y is modally free for x in B, so by induction hypothesis, $\vdash \langle y:x\rangle B \supset [y/x]B$. By (Nec) and (K), then $\vdash \Box\langle y:x\rangle B \supset \Box[y/x]B$. By (S$\Box$), $\vdash \langle y:x\rangle\Box B \supset \Box\langle y:x\rangle B$. So $\vdash \langle y:x\rangle\Box B \supset \Box[y/x]B$.

Here is the proof for (SCN). The first three clauses are very similar.

1. A is atomic. Then $\vdash_L \langle y:x\rangle A \leftrightarrow [y/x]A$ as we've seen above, and so $\vdash_L y\neq y \supset (\langle y:x\rangle A \leftrightarrow [y/x]A)$ by (PC).

2. A is $\neg B$. By induction hypothesis, $\vdash_L y\neq y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$. So by (PC), $\vdash_L y\neq y \supset (\neg\langle y:x\rangle B \leftrightarrow \neg[y/x]B)$. By (S$\neg$) and definition 3.3, it follows that $\vdash_L y\neq y \supset (\langle y:x\rangle\neg B \leftrightarrow [y/x]\neg B)$.

3. A is $B \supset C$. By induction hypothesis, $\vdash_L y\neq y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$ and $\vdash_L y\neq y \supset (\langle y:x\rangle C \leftrightarrow [y/x]C)$. By (S$\supset$), $\vdash_L y\neq y \supset (\langle y:x\rangle(B\supset C) \leftrightarrow (\langle y:x\rangle B \supset \langle y:x\rangle C))$. So $\vdash_L y\neq y \supset (\langle y:x\rangle(B\supset C) \leftrightarrow ([y/x]B \supset [y/x]C))$, and so $\vdash_L y\neq y \supset (\langle y:x\rangle(B\supset C) \leftrightarrow [y/x](B\supset C))$ by definition 3.3.

4. A is $\forall zB$. If $z \notin \{x,y\}$, then by induction hypothesis, $\vdash_L y\neq y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$. So by (UG) and (UD), $\vdash_L \forall z\, y\neq y \supset (\forall z\langle y:x\rangle B \leftrightarrow \forall z[y/x]B)$. Since $z \notin \{x,y\}$, $\vdash_L \langle y:x\rangle\forall zB \leftrightarrow \forall z\langle y:x\rangle B$ by (S$\forall$), and $\vdash_L y\neq y \supset \forall z\, y\neq y$ by (VQ), and $\forall z[y/x]B$ is $[y/x]\forall zB$ by definition 3.3; so $\vdash_L y\neq y \supset (\langle y:x\rangle\forall zB \leftrightarrow [y/x]\forall zB)$.

   Alternatively, if $z \in \{x,y\}$, then either x or y is not free in A, and thus x and y are modally separated in A. By (SC2), then $\vdash_L \langle y:x\rangle\forall zB \leftrightarrow [y/x]\forall zB$, and so by (PC), $\vdash_L y\neq y \supset (\langle y:x\rangle\forall zB \leftrightarrow [y/x]\forall zB)$.

5. A is $\langle y_2:z\rangle B$. If $z \notin \{x,y\}$, then by induction hypothesis, $\vdash_L y\neq y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$. So by (Sub$_s$) and (S$\supset$), $\vdash_L \langle [y/x]y_2 : z\rangle y\neq y \supset (\langle [y/x]y_2 : z\rangle\langle y:x\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle[y/x]B)$. By (VS), $\langle [y/x]y_2 : z\rangle y\neq y \leftrightarrow y\neq y$. And by (SS1) or (SS2), $\langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle\langle y:x\rangle B$. So $\vdash_L y\neq y \supset (\langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow \langle [y/x]y_2 : z\rangle[y/x]B)$. But by definition 3.3, $[y/x]\langle y_2:z\rangle B$ is $\langle [y/x]y_2 : y\rangle[y/x]B$.

   Alternatively, if $z \in \{x,y\}$, then either x or y is not free in A, and thus x and y are modally separated in A. By (SC2), then $\vdash_L \langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow [y/x]\langle y_2:z\rangle B$, and so by (PC), $\vdash_L y\neq y \supset (\langle y:x\rangle\langle y_2:z\rangle B \leftrightarrow [y/x]\langle y_2:z\rangle B)$.

6. A is $\Box B$. Then

1.	$\vdash_L y{\neq}y \supset (\langle y:x\rangle B \leftrightarrow [y/x]B)$.	(ind. hyp.)
2.	$\vdash_L \Box y{\neq}y \supset (\Box\langle y:x\rangle B \leftrightarrow \Box[y/x]B)$.	(1, (Nec), (K))
3.	$\vdash_L y{\neq}y \supset \Box y{\neq}y$.	((=R) or (NA), (NE) and (Nec
4.	$\vdash_L y{\neq}y \supset (\Box\langle y:x\rangle B \leftrightarrow \Box[y/x]B)$.	(2, 3)
5.	$\vdash_L y{\neq}y \supset \langle y:x\rangle(x{\neq}x \wedge y{\neq}y)$	((SAt), (S$\supset$), (S$\neg$))
6.	$\vdash_L (x{\neq}x \wedge y{\neq}y) \supset \Box(x{\neq}x \wedge y{\neq}y)$.	((=R) or (NA), (NE), (Nec) an
7.	$\vdash_L \Box(x{\neq}x \wedge y{\neq}y) \supset (\Box B \leftrightarrow \Box\langle y:x\rangle B)$.	((SEV), (Nec), (K))
8.	$\vdash_L (x{\neq}x \wedge y{\neq}y) \supset (\Box B \leftrightarrow \Box\langle y:x\rangle B)$.	(6, 7)
9.	$\vdash_L \langle y:x\rangle(x{\neq}x \wedge y{\neq}y) \supset (\langle y:x\rangle\Box B \leftrightarrow \langle y:x\rangle\Box\langle y:x\rangle B)$.	(8, (Sub$_s$), (S$\supset$))
10.	$\vdash_L \langle y:x\rangle(x{\neq}x \wedge y{\neq}y) \supset (\langle y:x\rangle\Box B \leftrightarrow \Box\langle y:x\rangle B)$.	(9, (VS))
11.	$\vdash_L y{\neq}y \supset (\langle y:x\rangle\Box B \leftrightarrow \Box\langle y:x\rangle B)$.	(7, 10)
12.	$\vdash_L y{\neq}y \supset (\langle y:x\rangle\Box B \leftrightarrow [y/x]\Box B)$.	(4, 13, def. 3.3)

■

Lemma 5.10 (Syntactic alpha-conversion)
If A, A' are $\mathcal{L}$-formulas, and A' is an alphabetic variant of A, then

(AC) $\vdash_L A \leftrightarrow A'$.

Proof Induction on A.

1. A is atomic. Then $A = A'$ and $\vdash_L A \leftrightarrow A'$ by (Taut).
2. A is $\neg B$. Then A' is $\neg B'$ with B' an alphabetic variant of B. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. By (PC), $\vdash_L \neg B \leftrightarrow \neg B'$.
3. A is $B \supset C$. Then A' is $B' \supset C'$ with B', C' alphabetic variants of B, C, respectively. By induction hypothesis, $\vdash_L B \leftrightarrow B'$ and $\vdash_{sC} C \leftrightarrow C'$. By (PC), then $\vdash_L (B \supset C) \leftrightarrow (B' \supset C')$.
4. A is $\forall x B$. Then A' is either $\forall x B'$ or $\forall z[z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B')$. Assume first that A' is $\forall x B'$. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (UG) and (UD), $\vdash_L \forall x B \leftrightarrow \forall x B'$.

   Alternatively, assume A' is $\forall z[z/x]B'$ and $z \notin \mathit{Var}(B')$. Since B' differs from B at most in renaming bound variables, if z were free in B, then $z \in \mathit{Var}(B')$. So z is not free in

B. Then

1.	$\vdash_L B \leftrightarrow B'$.	induction hypothesis
2.	$\vdash_L \langle z : x\rangle B \leftrightarrow \langle z : x\rangle B'$.	(1, (Sub$_s$), (S$\neg$))
3.	$\vdash_L \langle z : x\rangle B' \leftrightarrow [z/x]B'$.	((SC1), $z \notin \mathit{Var}(B')$)
4.	$\vdash_L \langle z : x\rangle B \leftrightarrow [z/x]B'$.	(2, 3)
5.	$\vdash_L \forall z\langle z : x\rangle B \leftrightarrow \forall z[z/x]B'$.	(4, (UG), (UD))
6.	$\vdash_L \forall xB \leftrightarrow \forall z\langle z : x\rangle B$.	((SBV), z not free in B)
7.	$\vdash_L \forall xB \leftrightarrow \forall z[z/x]B'$.	(5, 6)

5. A is $\langle y : x\rangle B$. Then A' is either $\langle y : x\rangle B'$ or $\langle y : z\rangle[z/x]B'$, where B' is an alphabetic variant of B and $z \notin \mathit{Var}(B)$. Assume first that A' is $\langle y : x\rangle B'$. By induction hypothesis, $\vdash_L B \leftrightarrow B'$. So by (Sub$_s$) and (S$\supset$), $\vdash_L \langle y : x\rangle B \leftrightarrow \langle y : x\rangle B'$.

   Alternatively, assume A' is $\langle y : z\rangle[z/x]B'$ and $z \notin \mathit{Var}(B')$. Again, it follows that z is not free in B. So

1.	$\vdash_L B \leftrightarrow B'$.	induction hypothesis
2.	$\vdash_L \langle z : x\rangle B \leftrightarrow \langle z : x\rangle B'$.	(1, (Sub$_s$), (S$\supset$))
3.	$\vdash_L \langle z : x\rangle B' \leftrightarrow [z/x]B'$.	((SC1), $z \notin \mathit{Var}(B')$)
4.	$\vdash_L \langle z : x\rangle B \leftrightarrow [z/x]B'$.	(2, 3)
5.	$\vdash_L \langle y : z\rangle\langle z : x\rangle B \leftrightarrow \langle y : z\rangle[z/x]B'$.	(4, (Sub$_s$), (S$\supset$))
6.	$\vdash_L \langle y : z\rangle\langle z : x\rangle B \leftrightarrow \langle y : x\rangle B$.	((SE2), z not free in B)
7.	$\vdash_L \langle y : x\rangle B \leftrightarrow \langle y : z\rangle[z/x]B'$.	(5, 6)

6. A is $\Box A'$. Then B is $\Box B'$ with B' an alphabetic variant of A'. By induction hypothesis, $\vdash_L A' \leftrightarrow B'$. Then by (Nec), $\vdash_L \Box(A' \leftrightarrow B')$, and by (K), $\vdash_L \Box A' \leftrightarrow \Box B'$. ■

Theorem 5.11 (Substitution and non-substitution logics)
For any $\mathcal{L}$-formula A and variables x, y,

(FUI*) $\vdash_L \forall xA \supset (Ey \supset [y/x]A)$, provided y is modally free for x in A,

(LL*) $\vdash_L x = y \supset A \supset [y/x]A$, provided y is modally free for x in A,

(Sub*) if $\vdash_L A$, then $\vdash_L [y/x]A$, provided y is modally free for x in A.

It follows that $\mathsf{P}(\mathcal{L}) \subseteq \mathsf{P}_s(\mathcal{L})$ and $\mathsf{N}(\mathcal{L}) \subseteq \mathsf{N}_s(\mathcal{L})$.

Proof Assume y is modally free for x in A. Then by (SC2), $\vdash_L \langle y : x\rangle A \supset [y/x]A$. By (FUI$_s$), $\vdash_L \forall xA \supset (Ey \supset \langle y : x\rangle A)$, so by (PC), $\vdash_L \forall xA \supset (Ey \supset [y/x]A)$. Similarly, by (LL$_s$), $\vdash_L x = y \supset A \supset \langle y : x\rangle A$, so by (PC), $\vdash_L x = y \supset A \supset [y/x]A$. Finally, by (Sub$_s$), if $\vdash_L A$, then $\vdash_L \langle y : x\rangle A$, so then $\vdash_L [y/x]A$ by (PC). ■

Lemma 5.12 (Symmetry and transitivity of identity)
For any $\mathcal{L}$-variables x, y, z,

(=S) $\vdash_L x = y \supset y = x$;

(=T) $\vdash_L x = y \supset y = z \supset x = z$.

Proof Immediate from lemma 5.11 and lemma 4.9. ∎

Lemma 5.13 (Variations on Leibniz' Law)
If A is an $\mathcal{L}$-formula and x, y, y' are $\mathcal{L}$-variables, then

(LV1) $\vdash_L \langle y : x \rangle A \wedge x = y \supset A$.

(LV2) $\vdash_L \langle y : x \rangle A \wedge y = y' \supset [y'/x]A$, provided y' is modally free for x in A.

Proof (LV1). Let z be an $\mathcal{L}$-variable not in $Var(A)$. Then

1.	$\vdash_L x = z \supset \langle z : x \rangle A \supset \langle x : z \rangle \langle z : x \rangle A$.	(LL$_{\mathrm{s}}$)
2.	$\vdash_L x = z \supset \langle z : x \rangle A \supset \langle x : x \rangle A$.	(1, (SE2), $z \notin Var(A)$)
3.	$\vdash_L x = z \supset \langle z : x \rangle A \supset A$.	(2, (SE1))
4.	$\vdash_L \langle y : z \rangle x = z \supset \langle y : z \rangle \langle z : x \rangle A \supset \langle y : z \rangle A$.	(3, (VS), (S$\supset$))
5.	$\vdash_L x = z \supset \langle y : z \rangle \langle z : x \rangle A \supset \langle y : z \rangle A$.	(4, (SAt))
6.	$\vdash_L x = z \supset \langle y : x \rangle A \supset \langle y : z \rangle A$.	(5, (SE2), $z \notin Var(A)$)
7.	$\vdash_L x = z \supset \langle y : x \rangle A \supset A$.	(6, (VS), $z \notin Var(A)$).

(LV2).

1.	$\vdash_L x = y \wedge y = y' \supset x = y'$.	(=T)
2.	$\vdash_L A \wedge x = y' \supset [y'/x]A$.	((LL*), y' m.f. in A)
3.	$\vdash_L A \wedge x = y \wedge y = y' \supset [y'/x]A$.	(1, 2)
4.	$\vdash_L \langle y : x \rangle A \wedge \langle y : x \rangle x = y \wedge \langle y : x \rangle y = y' \supset \langle y : x \rangle [y'/x]A$.	(3, (Sub$_{\mathrm{s}}$), (S$\neg$), (S$\supset$))
5.	$\vdash_L y = y \supset \langle y : x \rangle x = y$.	(SAt)
6.	$\vdash_L y = y' \supset y = y$.	((LL*), (=S))
7.	$\vdash_L y = y' \supset \langle y : x \rangle y = y'$.	(VS)
8.	$\vdash_L \langle y : x \rangle A \wedge y = y' \supset \langle y : x \rangle [y'/x]A$.	(4, 5, 6, 7)
9.	$\vdash_L \langle y : x \rangle [y'/x]A \supset [y'/x]A$.	(VS)
10.	$\vdash_L \langle y : x \rangle A \wedge y = y' \supset [y'/x]A$.	(8, 9)

∎

Lemma 5.14 (Leibniz' Law with sequences)
For any $\mathcal{L}$-formula A and variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the $x_1, \ldots, x_n$ are pairwise distinct,

$$(\mathrm{LL_n}) \vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset \langle y_1, \ldots, y_n : x_1, \ldots, x_n \rangle A.$$

Proof For $n = 1$, $(\mathrm{LL_n})$ is $(\mathrm{LL_s})$. Assume then that $n > 1$. To keep formulas in the following proof at a managable length, let $\underline{\phi(i)}$ abbreviate the sequence $\phi(1), \ldots, \phi(n-1)$. For example, $\langle \underline{y_i} : \underline{x_i} \rangle$ is $\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1} \rangle$. Let z be the alphabetically first variable not in A or $x_1, \ldots, x_n$. Now

1.	$\vdash_L x_n = y_n \supset \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A.$	$(\mathrm{LL_s})$
2.	$\vdash_L \langle y_n : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : z \rangle \langle z : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A.$	(SE1)
3.	$\vdash_L \langle z : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	((SS1) or (SS2))
4.	$\vdash_L \langle y_n : z \rangle \langle z : x_n \rangle \langle \underline{y_i} : \underline{x_i} \rangle A$ $\supset \langle y_n : z \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	(3, $(\mathrm{Sub_s})$, $(\mathrm{S}\supset)$)
5.	$\vdash_L x_n = y_n \supset \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : z \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	(1, 2, 4)
6.	$\vdash_L x_n = z \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$ $\supset \langle z : x_n \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	$(\mathrm{LL_s})$
7.	$\vdash_L x_n = z \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$ $\supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A.$	(6, (SS1))
8.	$\vdash_L z = x_n \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A$ $\supset \langle x_n : z \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A.$	$(\mathrm{LL_s})$
9.	$\vdash_L z = x_n \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle \langle z : x_n \rangle A$ $\supset \langle \underline{y_i} : \underline{x_i} \rangle \langle x_n : z \rangle \langle z : x_n \rangle \langle z : x_n \rangle A.$	(8, (SS1), (SS2))
10.	$\vdash_L \langle x_n : z \rangle \langle z : x_n \rangle \langle z : x_n \rangle A \leftrightarrow \langle z : x_n \rangle A$	((SE1), (SE2))
11.	$\vdash_L \langle \underline{y_i} : \underline{x_i} \rangle \langle x_n : z \rangle \langle z : x_n \rangle \langle z : x_n \rangle A \supset \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$	(10, $(\mathrm{Sub_s})$, $(\mathrm{S}\supset)$)
12.	$\vdash_L z = x_n \supset x_n = z$	(=S)
13.	$\vdash_L z = x_n \supset \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A \supset \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	(7, 9, 11, 12)
14.	$\vdash_L x_n = y_n \supset \langle y_n : z \rangle z = x_n$	((=S), (SAt))
15.	$\vdash_L x_n = y_n \supset \langle y_n : z \rangle \langle \underline{[z/x_n] y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A$ $\supset \langle y_n : z \rangle \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	13, 14, $(\mathrm{Sub_s})$, $(\mathrm{S}\supset)$
16.	$\vdash_L x_n = y_n \supset \langle \underline{y_i} : \underline{x_i} \rangle A \supset \langle y_n : z \rangle \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	5, 15
17.	$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_{n-1} = y_{n-1} \supset A \supset \langle \underline{y_i} : \underline{x_i} \rangle A.$	(induction hypothesis)
18.	$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset \langle y_n : z \rangle \langle \underline{y_i} : \underline{x_i} \rangle \langle z : x_n \rangle A.$	(16, 17)
19.	$\vdash_L x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset A \supset \langle y_1, \ldots, y_n : x_1, \ldots, x_n \rangle A.$	(18, def. 3.14)

■

LEMMA 5.15 (CLOSURE UNDER TRANSFORMATIONS)
For any $\mathcal{L}$-formula A and transformation τ on $\mathcal{L}$,

(Sub$^\tau$) $\vdash_L A$ iff $\vdash_L A^\tau$.

PROOF The proof is exactly as in lemma 4.11. ■

6 Canonical models

DEFINITION 6.1 (SYNTACTIC CONSEQUENCE AND CONSISTENCY)
Let A be a formula of some language $\mathcal{L}$, Γ a set of $\mathcal{L}$-formulas, and L a logic. A is a *syntactic consequence* of Γ in $L(\mathcal{L})$, for short: $\Gamma \vdash_{L(\mathcal{L})} A$, iff there are 0 or more sentences $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_{L(\mathcal{L})} B_1 \wedge \ldots \wedge B_n \supset A$. (For $n = 0$, $B_1 \wedge \ldots \wedge B_n \supset A$ is A.)

Γ is $L(\mathcal{L})$-*consistent* iff there are no members $A_1, \ldots, A_n$ of Γ such that $\vdash_L \neg(A_1 \wedge \ldots \wedge A_n)$.

Let's hold fixed a particular language $\mathcal{L}$, with or without substitution. A logic L is *weakly complete* with respect to a class of models $\mathbb{M}$ iff L contains every formula valid in $\mathbb{M}$: whenever A is valid in $\mathbb{M}$, then $\vdash_L A$. Equivalently, every formula $A \notin L$ is false at some world in some model in $\mathbb{M}$. L is *strongly complete* with respect to $\mathbb{M}$ iff whenever A is a semantic consequence of a set of formulas Γ in $\mathbb{M}$, then $\Gamma \vdash_L A$. Since $\Gamma \nvdash_L A$ iff $\Gamma \cup \{\neg A\}$ is L-consistent, and A is a semantic consequence of Γ in $\mathbb{M}$ iff no world in any model in $\mathbb{M}$ verifies all members of $\Gamma \cup \{\neg A\}$ (see definition 2.11), this means that L is strongly complete with respect to $\mathbb{M}$ iff for every L-consistent set of formulas Γ there is a world in some model in $\mathbb{M}$ at which all members of Γ are true.

I will use the canonical model technique for proving strong completeness. Let me briefly review the basic idea. We associate with each logic L a *canonical model* $\mathcal{M}_L$. The worlds of $\mathcal{M}_L$ are construed as maximal L-consistent sets of formulas, and it is shown that a formula A is true at a world in $\mathcal{M}_L$ iff A is a member of that world. Since every L-consistent set of formulas can be extended to a maximal L-consistent set, it follows that every L-consistent set of formulas is verified at some world in $\mathcal{M}_L$, and therefore that L is strongly complete with respect to every model class that contains $\mathcal{M}_L$.

To ensure that what's true at a world are precisely the formulas it contains, the interpretation V in a canonical model assigns to each variable x at each world w some individual $[x]_w$ and to each predicate P at w the set of n-tuples $\langle [x_1]_w, \ldots, [x_n] \rangle$ such that $Px_1 \ldots x_n \in w$. The customary way to make this work is to identify $[x]_w$ with the class of variables z such that $x = z \in w$. The domains therefore consist of equivalence classes of variables.

A well-known problem now arises from the fact that first-order logic does not require every individual to have a name. This means that there are consistent sets Γ that contain $\exists x Fx$ as well as $\neg Fx_i$ for every variable x_i. If we extend Γ to a maximal consistent set w and apply the construction just outlined, then $V_w(F)$ would be the empty set. So we would have $w, V \Vdash \neg\exists x Fx$, although $\exists x Fx \in w$. To avoid this, one requires that the worlds in a canonical model are all *witnessed* so that whenever an existential formula $\exists x Fx$ is in w, then some witnessing instance Fy is in w as well. But we still want the set Γ to be verified at some world. So the worlds are construed in a larger language $\mathcal{L}^*$ that adds infinitely many new variables to the original language $\mathcal{L}$. The new variables may then serve as witnesses. (In the new language, there are again consistent sets of sentences that are not included in any world, but not so in the old language.)

In modal logic, this problem reappears in another form. Assume Γ contains $\Diamond\exists x Fx$ but also $\Box\neg Fx_i$ for every $\mathcal{L}^*$-variable x_i. Using Kripke semantic, we then need a world w' accessible from the Γ-world that verifies all instances of $\neg Fx_i$, as well as $\exists x Fx$. But then w' isn't witnessed!

A standard way out is to stipulate that worlds in $\mathcal{M}_L$ must be *modally witnessed* in the sense that whenever $\Diamond\exists x A \in w$, then $\Diamond[y/x]A \in w$ for some (possibly new) variable y. Metaphysically speaking, this means that whenever it is possible that something is so-and-so, then we can point at some object at the actual world which is possibly so-and-so. In single-domain models, this has the unfortunate consequence of rendering the Barcan Formula valid. In dual-domain semantics, the "modal witness" can come from the outer domain, so that $\forall x \Box Fx$ does not entail $\Box\forall x Fx$. However, if the relevant logic is classical rather than free, the Barcan Formula still comes out valid, despite the fact that it is not entailed by the principles of classical first-order logic and K.

Counterpart semantics provides a less drastic way out that does not introduce undesired validities and thereby reduce the applicability of the canonical model technique. In counterpart semantics, the truth of $\Box\neg Fx_i$ at some world w only requires that $\neg Fx_i$ is true at w' under all *w'-images* V' of V at w – i.e. under interpretations V' such that $V'_{w'}(x_i)$ is some counterpart of $V_w(x_i) = [x_1]_w$. Suppose, for example, that $[x_1]_w = \{x_1\}$ and each individual $[x_i]_w$ at w has $[x_{i+1}]_{w'}$ as their unique counterpart at w'. Then the truth of $\Box\neg Fx_1, \Box\neg Fx_2$, etc. at w only requires that $\neg Fx_1, \neg Fx_2$, etc. are true at w' under an assignment of $[x_2]_{w'}$ to x_1, $[x_3]_{w'}$ to x_2, etc. So $Fx_2, Fx_3, \ldots \in w'$, but the variable x_1 becomes available to serve as a witness for $\Diamond\exists x Fx$.

Here we exploit the fact that in counterpart semantics truth at a world "considered as actual" can come apart from truth at a world "considered as counterfactual". In the canonical model, membership in a world only coincides with truth at the world "as actual": $A \in w$ iff $w, V \Vdash A$. So if w' contains Fx_1, then $w', V \Vdash Fx_1$. On the other hand, when we look at w' ("as counterfactual") from the perspective of w, we evaluate formulas not by the original interpretation function V, but by an image V' of V. Given that $V'_{w'}(x_1) = [x_2]_{w'}$ and $Fx_2 \notin w'$, $w', V' \nVdash Fx_1$.

However, this creates a complication. Suppose we want to show that for every formula A and world w in $\mathcal{M}_L$,

(6) $\quad \Box A \in w$ iff $w, V \Vdash \Box A$,

where V is the interpretation function of $\mathcal{M}_L$. Proceeding by induction on complexity of A, we can assume that for all w,

(7) $\quad A \in w$ iff $w, V \Vdash A$.

In standard Kripke semantics, we now only need to stipulate that w' is accessible from w iff w' contains all A for which w contains $\Box A$. This means that $\Box A \in w$ iff $A \in w'$ for all w-accessible w'; by (7), the latter holds iff $w', V \Vdash A$ for all such w', i.e. iff $w, V \Vdash \Box A$ by the semantics of the box. In counterpart semantics, this line of thought no longer goes through, since $w, V \Vdash \Box A$ only means that A is true at all accessible worlds w' *considered as counterfactual*: $w', V' \Vdash A$. By contrast, (7) only considers worlds *as actual*; it does not tell us that $A \in w$ iff $w, V' \Vdash A$.

In the following construction, the counterpart relation in canonical models will be specified by means of a variable transformation τ, so that $[x]_w$ at w always has $[x^\tau]_{w'}$ at w' as counterpart (unless $[x^\tau]_{w'}$ is empty – see below). So if $[x]_w = \{x, y\}$, then the counterparts at w' of $[x]_w$ are $[x^\tau]_{w'}$ and $[y^\tau]_{w'}$. Since $[x^\tau]_{w'} = V_{w'}(x^\tau)$ and $V_{w'}(x^\tau) = V^\tau_{w'}(x)$, it follows that V^τ is always a w'-image of V at w.

If V^τ is in fact the only w'-image of V at w, then our problem is solved. Define accessibility so that wRw' iff w' contains A^τ whenever w contains $\Box A$. So $\Box A \in w$ iff $A^\tau \in w'$ for every w' accessible from w. The induction hypothesis (7) tells us that $A^\tau \in w'$ iff $w', V \Vdash A^\tau$. By the transformation lemma (lemma 3.13), $w', V^\tau \Vdash A$ iff $w', V \Vdash A^\tau$: A is true at w' as counterfactual iff A^τ is true at w' as actual. So $\Box A \in w$ iff $w', V^\tau \Vdash A$ for all w' accessible from w. If V^τ is the only w'-image, it follows that $\Box A \in w$ iff $w, V \Vdash \Box A$.

The remaining problem is that V^τ may not be the only w'-image of V at w. For example, assume w contains $x = y$ but not $\Box x = y$. Some accessible w' then contains $x^\tau \neq y^\tau$. I.e., the individual $[x]_w = [y]_w = \{x, y, \ldots\}$ at w has two counterparts at w', $[x^\tau]_{w'}$ and $[y^\tau]_{w'}$, which V^τ assigns (at w') to x and y, respectively. But then there will also be another w'-image of V at w which assigns, for example, $[y^\tau]_{w'}$ to both x and y.

Sometimes this is harmless. Assume w also contains $\Box Fx$. We want to show that $w, V \Vdash \Box Fx$ and thus that $w', V' \Vdash Fx$ for all accessible w' and w'-images V' of V. In particular, at the above world w', both $[x^\tau]_{w'}$ and $[y^\tau]_{w'}$ must fall in $V_{w'}(F)$. Since w contains $\Box Fx$, we know that w' contains Fx^τ, so $[x^\tau]_{w'} \in V_{w'}(F)$ by construction of the canonical interpretation V. What about $[y^\tau]_{w'}$? Well, since w contains $\Box Fx$ and $x = y$, then it also contains $\Box Fy$, by (LL*). So w' contains Fy^τ and $[y^\tau]_{w'} \in V_{w'}(F)$. The upshot is that the truth-value of Fx at w' considered as counterfactual does not vary between V^τ and other w'-images of V.

Unfortunately, this is not always the case. Assume again that w contains $x = y$ but not $\Box x = y$, so that some accessible world w' contains $x^\tau \neq y^\tau$. Assume further that w contains $\Box\Diamond x \neq y$, and that we are working with a positive logic without explicit substitution. By construction of accessibility, w' contains $\Diamond x^\tau \neq y^\tau$. To verify $\Box\Diamond x \neq y$ at w, we need to ensure that $w', V' \Vdash \Diamond x \neq y$ for all w'-images V' of V at w, not just for V^τ. Consider the image V' that assigns $[y^\tau]_{w'}$ to both x and y. In order to satisfy $w', V' \Vdash \Diamond x \neq y$, $[y^\tau]_{w'}$ must have two counterparts at some world w'' accessible from w'. But so far, we have no guarantee that this is the case. There has to be a variable z other than y^τ such that w' contains $z = y^\tau$ as well as $\Diamond z \neq y^\tau$. The latter ensures that $z^\tau \neq y^{\tau\tau} \in w''$ for some accessible w''; $[z^\tau]_{w''}$ and $[y^{\tau\tau}]_{w''}$ are then both counterparts at w'' of $[y^\tau]_{w'}$. Hence we stipulate that if w' does not contain $z = y^\tau$ and $\Diamond z \neq y^\tau$ for some suitable z, then w' is not accessible from w. In general, if w contains $\Box A$ as well as $x = y$, and x is free in A, then for w' to be accessible from w, we require that it must contain not only A^τ, but also $z = y^\tau$ and $[z/x^\tau]A^\tau$, for some z not free in A^τ.

This requirement might be easier to understand if we consider the same situation in a language with substitution. Here $\Box\Diamond x \neq y$ and $x = y$ entail $\Box\langle y : x\rangle\Diamond x \neq y$ (by (LL_s) and (S$\Box$)). By the original, simple definition of accessibility, each world w' accessible from w must contain $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$, which says that $[y^\tau]_{w'}$ has multiple counterparts at some accessible world w''. Before we worry about images other than V^τ, we ought to make sure that $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$ is true at w' under V^τ. Again, this requires there to be a variable z other than y^τ such that w' contains $z = y^\tau$ and $\Diamond z \neq y^\tau$. In effect, z is a kind of witness for the substitution formula $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$. Just as an existential formula $\exists x A$ must be witnessed by an instance $[z/x]A$, a substitution formula $\langle y : x\rangle A$ must be witnessed by $[z/x]A$ together with $z = y$. Loosely speaking, $\langle y : x\rangle A(x)$ says that y is identical to some x such that $A(x)$. In a canonical model, we want a concrete witness z so that y is identical to z and $A(z)$. y itself may not serve that purpose, because $\langle y : x\rangle A(x)$ does not guarantee $A(y)$.

This requirement of substitutional witnessing entails that if w contains $\Box A$, then any accessible w' contains not only A^τ, but also $z = y^\tau$ and $[z/x^\tau]A^\tau$ (for some suitable z). So we don't need to complicate the accessibility relation. In our example, since w' contains A^τ whenever w contains $\Box A$, w' contains $\langle y^\tau : x^\tau\rangle\Diamond x^\tau \neq y^\tau$, which settles that $[y^\tau]_{w'}$ has two counterparts at some accessible world. Without substitution, $\langle y^\tau : x^\tau \rangle \Diamond x^\tau \neq y^\tau$ is inexpressible (see lemma 3.10). So we have to limit the accessible worlds by requiring membership of the relevant witnessing formulas in addition to A^τ.

On to the details. Let $\mathcal{L} = \langle \mathcal{C}, \mathcal{V}, \mathcal{P} \rangle$ be some language with or without substitution and L a positive or strongly negative quantified modal logic applied to $\mathcal{L}$. Define the extended language $\mathcal{L}^* = \langle \mathcal{C}, \mathcal{V}^*, \mathcal{P} \rangle$, where $\mathcal{V}^*$ is $\mathcal{V}$ with the addition of infinitely many new variables $\mathcal{V}^+$.

Definition 6.2 (Henkin set)
A *Henkin set* for L is a set H of $\mathcal{L}^*$-formulas that is

1. *L-consistent*: there are no $A_1, \ldots, A_n \in H$ with $\vdash_{L(\mathcal{L}^*)} \neg(A_1 \wedge \ldots \wedge A_n)$,
2. *maximal*: for every $\mathcal{L}^*$-formula A, H contains either A or $\neg A$,
3. *witnessed*: whenever H contains an existential formula $\exists x A$, then there is a variable $y \notin Var(A)$ such that H contains $[y/x]A$ as well as Ey, and
4. *substitutionally witnessed*: whenever H contains a substitution formula $\langle y : x \rangle A$ as well as $y = y$, then there is a variable $z \notin Var(\langle y : x \rangle A)$ such that H contains $y = z$.

I write $\mathcal{H}_L$ for the class of Henkin sets for L in $\mathcal{L}^*$.

If L is without substitution, the fourth clause is trivial. Above I said that witnessing a substitution formula $\langle y : x \rangle A$ requires $y = z$ as well as $[z/x]A$, but in fact $y = z$ is enough, since $[z/x]A$ follows from $\langle y : x \rangle A$ and $y = z$ by (LV2) (lemma 5.13). I have also added the condition that H contains $y = y$. In negative logics, a Henkin set may contain $y \neq y$ as well as $\langle y : x \rangle A$; adding $y = z$ would render the set inconsistent.

The requirement of substitutional witnessing generalises to substitution sequences: if H contains a substitution formula $\langle y_1, \ldots, y_n : x_1, \ldots, x_n \rangle A$ as well as $y_i = y_i$ for all y_i in $y_1, \ldots, y_n$, then there are (distinct) new variables $z_1, \ldots, z_n$ such that H contains $y_1 = z_1, \ldots, y_n = z_n$ as well as $[z_1, \ldots, z_n / x_1, \ldots, x_n]A$. This is easily proved by induction on n. Suppose H contains $\langle y_1, \ldots, y_n : x_1, \ldots, x_n \rangle A$. By definition 3.14, this is $\langle y_n : v \rangle \langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1} \rangle \langle v : x_n \rangle A$, where v is new. Witnessing requires $y_n = z_n \in H$ and (hence) $[z_n/v]\langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1} \rangle \langle v : x_n \rangle A = \langle y_1, \ldots, y_{n-1} : x_1, \ldots, x_{n-1} \rangle \langle z_n : x_n \rangle A \in H$ for some new z_n. By induction hypothesis, the latter means that there are (distinct) $z_1, \ldots, z_{n-1} \notin Var(\langle z_n : x_n \rangle A)$ such that H contains $y_1 = z_1, \ldots, y_{n-1} = z_{n-1}$ as well as $[z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]\langle z_n : x_n \rangle A$. Since all the x_i and z_i are pairwise distinct, $[z_1, \ldots, z_{n-1}/x_1, \ldots, x_{n-1}]\langle z_n : x_n \rangle A$ is $\langle z_n :$

$x_n\rangle[z_1,\ldots,z_{n-1}/x_1,\ldots,x_{n-1}]A$. By (SC1), it follows that $[z_n/x_n][z_1,\ldots,z_{n-1}/x_1,\ldots,x_{n-1}]A = [z_1,\ldots,z_n/x_1,\ldots,x_n]A \in H$.

Definition 6.3 (Variable classes)
For any Henkin set H, define $\sim_H$ to be the relation on $Var(\mathcal{L}^*)$ such that $x \sim_H y$ iff $x\!=\!y \in H$. For any variable x, let $[x]_H$ be $\{y : x \sim_H y\}$.

Lemma 6.4 ($\sim$-Lemma)
$\sim_H$ is transitive and symmetrical.

Proof Immediate from lemmas 4.9 and 5.12. ■

Now let τ be some arbitrary, but fixed transformation on $\mathcal{L}^*$ that maps each $\mathcal{L}^*$-variable x to a variable $x^\tau \in \mathcal{V}^+$. Recall that transformations are injective substitutions. So each $\mathcal{L}^*$-variable x has a unique "successor" $x^\tau \in \mathcal{V}^*$, which in turn has a unique successor $x^{\tau\tau} \in \mathcal{V}^*$, etc.

Definition 6.5 (Canonical model)
The *canonical model* $\langle W, R, D, C, V\rangle$ for L is defined as follow.

1. The *worlds* W are the Henkin sets $\mathcal{H}_L$.
2. For each $w \in W$, the *outer domain* U_w comprises the non-empty sets $[x]_w$, where x is a $\mathcal{L}^*$-variable.
3. For each $w \in W$, the *inner domain* D_w comprises the sets $[x]_w$ for which $Ex \in w$.
4. The *interpretation* V is such that for all $\mathcal{L}^*$-variables x, $V_w(x)$ is either $[x]_w$ or undefined if $[x]_w = \emptyset$, and for all non-logical predicates P, $V_w(P) = \{\langle [x_1]_w,\ldots,[x_n]_w\rangle : Px_1\ldots x_n \in w\}$.
5. If L is with substitution, then the *accessibility relation* R holds between world w and world w' iff for every formula A, if $\Box A \in w$, then $A^\tau \in w'$.

   If L is without substitution, then the *accessibility relation* R holds between world w and world w' iff for every formula A and variables $x_1\ldots x_n, y_1,\ldots,y_n$ ($n \geq 0$) such that the $x_1\ldots x_n$ are pairwise distinct members of *Varf*(A), if $x_1 = y_1 \land \ldots \land x_n = y_n \land \Box A \in w$ and $y_1^\tau = y_1^\tau \land \ldots \land y_n^\tau = y_n^\tau \in w'$, then there are variables $z_1\ldots z_n \notin Var(A^\tau)$ such that $z_1 = y_1^\tau \land \ldots \land z_n = y_n^\tau \land [z_1\ldots z_n/x_1^\tau\ldots x_n^\tau]A^\tau \in w'$.

6. $\langle d, w \rangle$ has $\langle d', w' \rangle$ as a *counterpart* iff $d \in U_w, d' \in U_{w'}$, and there is an $x \in d$ such that $x^\tau \in d'$.

A few comments on this definition. Clause 4 takes into account the fact that in negative logics, $\neg Ex$ entails $x \neq y$ for every variable y. So if $\neg Ex \in w$, then $[x]_w$ is the empty set. However, we don't want to say that empty terms denote the empty set (so that $\emptyset \in D_w$, and $x = x$ would have to be true). Instead, the canonical interpretation assigns to each variable x at w the set $[x]_w$, *unless that set is empty*, in which case $V_w(x)$ remains undefined. Similarly, clause 6 ensures that $[x]_w$ at w has $[x^\tau]_{w'}$ as counterpart at w' only if $[x^\tau]_{w'} \neq \emptyset$.

The term '$\{\langle [x_1]_w, \ldots, [x_n]_w \rangle : Px_1 \ldots x_n \in w\}$' in clause 4 is meant to denote the set of n-tuples $\langle d_1, \ldots, d_n \rangle$ for which there are variables $x_1, \ldots, x_n$ such that $d_1 = [x_1]_w$ and $\ldots$ and $d_n = [x_n]_w$ and $Px_1 \ldots x_n \in w$. These d_i are guaranteed to be non-empty because $x_i = x_i \in w$ whenever $Px_1 \ldots x_n \in w$: if L is positive, then $\vdash_L z_i = z_i$ by (=R); if L is negative, then $\vdash_L Pz_1 \ldots z_n \supset Ez_i$ by (Neg) and hence $\vdash_L Pz_1 \ldots z_n \supset z_i = z_i$ by ($\forall$=R) and (FUI*).

Clause 5 adds the witnessing requirements on accessible world as explained above, but generalised to multiple variables and negative logics. (In this case, the generalised version for n variable pairs is not entailed by the requirement for a single pair, unlike in the case of substitutional witnessing.) Note that the $x_1, \ldots, x_n$ need not be *all* the free variables in A. Also recall from p.5 that a conjunction of zero sentences is the tautology $\top$; so for $n = 0$, the accessibility requirement says that if $\top \wedge \Box A \in w$ and $\top \in w'$, then $\top \wedge A^\tau \in w'$ – equivalently: if $\Box A \in w$, then $A^\tau \in w'$.

Lemma 6.6 (Charge of canonical models)
If L is positive, then the canonical model for L is positive. If L is strongly negative, then the canonical model for L is negative.

Proof If L is positive, then for all $\mathcal{L}^*$-variables x, every Henkin set for L contains $x = x$ (by (=R)). So $[x]_w$ is never empty. Nor is $[x^\tau]_{w'}$, for any world w'. So everything at any world has a counterpart at every other world. So the canonical model for a positive logic is positive.

If L is strongly negative, then every Henkin set for L contains $x = x \supset Ex$, for all $\mathcal{L}^*$-variables x (by (Neg)). So $[x]_w \neq \emptyset$ iff $Ex \in w$, which means that $D_w = U_w$ for all worlds w in the model. So the canonical model for a strongly negative logic is negative. ■

Lemma 6.7 (Extensibility Lemma)
If Γ is an L-consistent set of $\mathcal{L}^*$-sentences in which infinitely many $\mathcal{L}^*$-variables do not occur, then there is a Henkin set $H \in \mathcal{H}_L$ such that $\Gamma \subseteq H$.

Proof Let $S_1, S_2, \dots$ be an enumeration of all $\mathcal{L}^*$-sentences, and $z_1, z_2, \dots$ an enumeration of the unused $\mathcal{L}^*$-variables in such a way that $z_i \notin \mathit{Var}(S_1 \wedge \dots \wedge S_i)$. Let $\Gamma_0 = \Gamma$, and define Γ_n for $n \geq 1$ as follows.

(i) If $\Gamma_{n-1} \cup \{S_n\}$ is not L-consistent, then $\Gamma_n = \Gamma_{n-1}$;

(ii) else if S_n is an existential formula $\exists x$, then $\Gamma_n = \Gamma_{n-1} \cup \{\exists x A, [z_n/x]A, Ez_n\}$;

(iii) else if S_n is a substitution formula $\langle y : x \rangle A$, then $\Gamma_n = \Gamma_{n-1} \cup \{\langle y : x \rangle A, y{=}y \supset y{=}z_n\}$;

(iv) else $\Gamma_n = \Gamma_{n-1} \cup \{S_n\}$.

Define w as the union of all Γ_n. We show that w is a Henkin set for L.

1. w is L-consistent. This is shown by proving that Γ_0 is L-consistent and that whenever Γ_{n-1} is L-consistent, then so is Γ_n. It follows that no finite subset of w is L-inconsistent, and hence that w itself is L-consistent. The base step, that Γ_0 is L-consistent is given by assumption. Now assume (for $n > 0$) that Γ_{n-1} is L-consistent. Then Γ_n is constructed by applying one of (i)–(iv).

   a) If case (i) in the construction applies, then $\Gamma_n = \Gamma_{n-1}$, and so Γ_n is also L-consistent.

   b) Assume case (ii) in the construction applies, and suppose that $\Gamma_n = \Gamma_{n-1} \cup \{\exists x A, [z_n/x]A, Ez\}$ is L-inconsistent. Then there is a finite subset $\{C_1, \dots, C_m\} \subseteq \Gamma_{n-1}$ such that

      1. $\vdash_L \neg(C_1 \wedge \dots \wedge C_m \wedge \exists x A \wedge [z_n/x]A \wedge Ez_n)$.

      Let $\underline{C}$ abbreviate $C_1 \wedge \dots \wedge C_m$. Then

2.	$\vdash_L \underline{C} \wedge \exists x A \supset (Ez_n \supset \neg[z_n/x]A)$	(1)
3.	$\vdash_L \forall z_n(\underline{C} \wedge \exists x A) \supset \forall z_n Ez_n \supset \forall z_n \neg[z_n/x]A$	(2, (UG), (UD))
4.	$\vdash_L \underline{C} \wedge \exists x A \supset \forall z_n(\underline{C} \wedge \exists x A)$	((VQ), z_n not in Γ_{n-1})
5.	$\vdash_L \underline{C} \wedge \exists x A \supset \forall z_n Ez_n \supset \forall z_n \neg[z_n/x]A.$	(3, 4)
6.	$\vdash_L \underline{C} \wedge \exists x A \supset \forall z_n \neg[z_n/x]A.$	(5, (∀Ex))
7.	$\vdash_L \forall z_n \neg[z_n/x]A \leftrightarrow \forall x \neg A$	((AC), $z_n \notin \mathit{Var}(A)$)
8.	$\vdash_L \underline{C} \wedge \exists x A \supset \neg\exists x A.$	(6, 7)

      So $\{C_1, \dots C_m, \exists x A\}$ is not L-consistent, contradicting the assumption that clause (ii*) applies.

   c) Assume case (iii) in the construction applies (hence L is with substitution), and suppose that $\Gamma_n = \Gamma_{n-1} \cup \{\langle y : x \rangle A, y{=}y \supset y{=}z_n\}$ is L-inconsistent. Then there is a finite subset $\{C_1, \dots, C_m\} \subseteq \Gamma_{n-1}$ such that

      1. $\vdash_L \neg(\underline{C} \wedge \langle y : x \rangle A \wedge (y{=}y \supset y{\neq}z))$.

(As before, $\underline{C}$ is $C_1 \wedge \ldots \wedge C_m$.) But then

2.	$\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset y{=}y \wedge y{\neq}z_n$	(1)
3.	$\vdash_L \langle y : z_n\rangle(\underline{C} \wedge \langle y : x\rangle A \supset y{=}y \wedge y{\neq}z_n)$	(2, (Sub$_s$))
4.	$\vdash_L \langle y : z_n\rangle(\underline{C} \wedge \langle y : x\rangle A) \supset \langle y : z_n\rangle y{=}y \wedge \langle y : z_n\rangle y{\neq}z_n$	(3, (S$\supset$), (S$\neg$))
5.	$\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset \langle y : z_n\rangle(\underline{C} \wedge \langle y : x\rangle A)$	((VS), z_n not in Γ_{n-1}, S_n)
6.	$\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset \langle y : z_n\rangle y{=}y \wedge \langle y : z_n\rangle y{\neq}z_n$	(4, 5)
7.	$\vdash_L \langle y : z_n\rangle y{\neq}z_n \leftrightarrow y{\neq}y$	(SAt)
8.	$\vdash_L \langle y : z_n\rangle y{\neq}y \leftrightarrow y{\neq}y$	(SAt)
9.	$\vdash_L \underline{C} \wedge \langle y : x\rangle A \supset (y{=}y \wedge y{\neq}y)$.	(6, 7, 8)

So $\{C_1, \ldots, C_m, \langle y : x\rangle A\}$ is L-inconsistent, contradicting the assumption that clause (iii) applies.

d) Assume case (iv) in the construction applies. Then $\Gamma_n = \Gamma_{n-1} \cup \{S_n\}$ is L-consistent, since otherwise case (i) would have applied.

2. w is maximal. Assume some formula S_n is not in w. Then case (i) applied to S_n, so $\Gamma_{n-1} \cup \{S_n\}$ is not L-consistent. So there are $C_1, \ldots, C_m \in \Gamma_{n-1}$ such that $\vdash_L C_1 \wedge \ldots C_m \supset \neg S_n$. Similarly, if $S_k = \neg S_n$ is not in w, then there are $D_1, \ldots, D_l \in \Gamma_{k-1}$ such that $\vdash_L D_1 \wedge \ldots D_l \supset \neg S_k$. By (PC), it follows that there are $C_1, \ldots, C_m, D_1, \ldots D_l \in w$ such that

$$\vdash_L C_1 \wedge \ldots \wedge C_m \wedge D_1 \wedge \ldots \wedge D_l \supset (\neg S_n \wedge \neg\neg S_n).$$

But then w is inconsistent, contradicting what was just shown under 1.

3. w is witnessed. This is guaranteed by clause (ii) of the construction, respectively.

4. w is substitutionally witnessed. This is guaranteed by clause (iii) and the fact that the $z_n \notin \mathit{Var}(S_n)$. ■

Lemma 6.8 (Existence Lemma)
If w is a world in the canonical model for L, and A is a formula with $\Diamond A \in w$, then there is a world w' in the model with wRw' and $A^\tau \in w'$.

Proof I first prove the lemma for logics L with substitution. Let $\Gamma = \{A^\tau\} \cup \{B^\tau : \Box B \in w\}$. Suppose Γ is not L-consistent. Then there are $B_1^\tau, \ldots, B_n^\tau$ with $\Box B_i \in w$ such that $\vdash_L B_1^\tau \wedge \ldots \wedge B_n^\tau \supset \neg A^\tau$. By definition 3.3, this means that $\vdash_L (B_1 \wedge \ldots B_n \supset \neg A)^\tau$, and so $\vdash_L B_1 \wedge \ldots \wedge B_n \supset \neg A$ by (Sub$^\tau$). By (Nec) and (K), $\vdash_L \Box B_1 \wedge \ldots \wedge \Box B_n \supset \Box\neg A$. But then w contains both $\Diamond A$ and $\neg\Diamond A$, which is impossible because w is L-consistent. So Γ is L-consistent.

The variables in Γ all have the form x^τ with x a variable of $\mathcal{L}_s^*$. Hence the infinitely many variables of $\mathcal{L}_s$ do not occur in Γ. By the extensibility lemma, $\Gamma \subseteq H$ for some Henkin set H. Moreover, H is accessible from w because it contains B^τ for all B for which $\Box B \in w$.

Now for logics without substitution.

Let $S_1, S_2 \ldots$ enumerate all sentences in w of the form

$$x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B,$$

where $x_1, \ldots, x_n$ are zero or more distinct variables free in B. Let Z be an infinite set of $\mathcal{L}$-variables such that $Var(\mathcal{L}) \backslash U$ is also infinite. For each $S_i = (x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B)$, let Z_{S_i} be a set of distinct variables $z_1, \ldots, z_n \in Z$ such that $Z_{S_i} \cap \bigcup_{j<i} Z_{S_j} = \emptyset$ (i.e. none of the z_i has been used for any earlier S_j). Abbreviate

$$\begin{aligned}
B_i &=_{df} [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau; \\
X_i &=_{df} x_1 = y_1 \wedge \ldots \wedge x_n = y_n; \\
Y_i &=_{df} y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau; \\
Z_i &=_{df} y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n.
\end{aligned}$$

(For $n = 0$, X_i, Y_i and Z_i are the tautology $\top$, and B_i is B^τ.)

Let $\Gamma^- = \{(Y_i \supset Z_i \wedge B_i) : S_i \in S_1, S_2, \ldots\}$, and let $\Gamma = \Gamma^- \cup \{A^\tau\}$.

Suppose for reductio that Γ is inconsistent. Then there are sentences $(Y_1 \supset Z_1 \wedge B_1), \ldots, (Y_m \supset Z_m \wedge B_m) \in \Gamma^-$ such that

(1) $\quad \vdash_L \neg(A^\tau \wedge (Y_1 \supset Z_1 \wedge B_1) \wedge \ldots \wedge (Y_m \supset Z_m \wedge B_m))$.

By (Nec),

(2) $\quad \vdash_L \Box\neg(A^\tau \wedge (Y_1 \supset Z_1 \wedge B_1) \wedge \ldots \wedge (Y_m \supset Z_m \wedge B_m))$.

Any member $(Y_i \supset Z_i \wedge B_i)$ of Γ^- has the form

$$y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \wedge [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau.$$

By ($\mathrm{CS_n}$),

(3) $\quad \vdash_L x_1^\tau = y_1^\tau \wedge \ldots \wedge x_n^\tau = y_n^\tau \wedge \Box B^\tau \supset$
$\Box(y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \supset [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau)$.

Now w contains $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B$. So w^τ contains $x_1^\tau = y_1^\tau \wedge \ldots \wedge x_n^\tau = y_n^\tau \wedge \Box B^\tau$, which is the antecedent of (3). The consequent of (3) is $\Box(Z_i \supset B_i)$. Thus

(4) $\quad w^\tau \vdash_L \Box(Z_1 \supset B_1) \wedge \ldots \wedge \Box(Z_m \supset B_m)$.

Let $\Delta = w^\tau \cup \{\Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m))\}$. So

(5) $\quad \Delta \vdash_L \Box(Z_1 \supset B_1) \wedge \ldots \wedge \Box(Z_m \supset B_m)$;

(6) $\quad \Delta \vdash_L \Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m))$.

By (K) and (Nec), (5) and (6) yield

(7) $\quad \Delta \vdash_L \Diamond(A^\tau \wedge (Y_1 \supset Z_1 \wedge B_1) \wedge \ldots \wedge (Y_m \supset Z_m \wedge B_m))$.

By (2), it follows that Δ is inconsistent. This means that

(8) $\quad w^\tau \vdash_L \neg\Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m))$.

Now consider $Z_1 = (y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n)$. By ($\mathrm{LL}_\mathrm{n}^*$) (or repeated application of (LL^*)),

(9) $$\vdash_L y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \supset \Box\neg(A^\tau \wedge (y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n)) \supset \Box\neg(A^\tau \wedge (y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau)),$$

because the z_i are not free in A^τ. In other words (and dropping the tautologous conjunct at the end),

(10) $\quad \vdash_L Z_1 \supset \Box\neg(A^\tau \wedge (Y_1 \supset Z_1)) \supset \Box\neg A^\tau$.

By the same reasoning,

(11) $\quad \vdash_L Z_1 \wedge \ldots \wedge Z_m \supset \Box\neg(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m)) \supset \Box\neg A^\tau$.

By (PC), (Nec) and (K), this means

(12) $\quad \vdash_L Z_1 \wedge \ldots \wedge Z_m \supset \Diamond A^\tau \supset \Diamond(A^\tau \wedge (Y_1 \supset Z_1) \wedge \ldots \wedge (Y_m \supset Z_m))$.

Since $w^\tau \vdash_L \Diamond A^\tau$, (8) and (12) together entail

(13) $\quad w^\tau \vdash_L \neg(Z_1 \wedge \ldots \wedge Z_m)$.

So there are $C_1, \ldots, C_k \in w$ such that

(14) $\quad \vdash_L C_1^\tau \wedge \ldots \wedge C_k^\tau \supset \neg(Z_1 \wedge \ldots \wedge Z_m)$.

Each Z_i has the form $y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n$. All the z_i are pairwise distinct, and none of them occur in $C_1^\tau \wedge \ldots \wedge C_k^\tau$ (because the z_i are not in the range of τ) nor in any other Z_i. By (Sub^*), we can therefore replace each z_i in (14) by the corresponding y_i^τ, turning Z_i into Y_i:

(15) $\quad \vdash_L C_1^\tau \wedge \ldots \wedge C_k^\tau \supset \neg(Y_1 \wedge \ldots \wedge Y_m)$.

For any $Y_i = (y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau)$, X_i is a sentence of the form $x_1 = y_1 \wedge \ldots \wedge x_n = y_n$. So X_i^τ is $x_1^\tau = y_1^\tau \wedge \ldots \wedge x_n^\tau = y_n^\tau$, and $\vdash_L X_i^\tau \supset Y_i$ by either (=R) or (Neg) and ($\forall$=R). So (15) entails

(16) $\quad \vdash_L C_1^\tau \wedge \ldots \wedge C_k^\tau \supset \neg(X_1^\tau \wedge \ldots \wedge X_m^\tau)$.

Thus by (Sub^τ),

(17) $\quad \vdash_L C_1 \wedge \ldots \wedge C_k \supset \neg(X_1 \wedge \ldots \wedge X_m)$.

Since $\{C_1, \ldots, C_k, X_1, \ldots, X_m\} \subseteq w$, it follows that w is inconsistent. Which it isn't. This completes the reductio.

So Γ is consistent. Since the infinitely many variables of $\mathcal{L}$ outside Z do not occur in Γ, lemma 6.7 guarantees that $\Gamma \subseteq w'$ for some world w' in the canonical model for L. And of course, Γ was constructed so that w' satisfies the condition in definition 6.5 for accessibility from w. This requires that for every formula B and variables $x_1 \ldots x_n$, $y_1, \ldots, y_n$ such that the $x_1 \ldots x_n$ are zero or more pairwise distinct members of $\mathit{Varf}(B)$, if $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B \in w$ and $y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \in w'$, then there are variables $z_1 \ldots z_n \notin \mathit{Var}(B^\tau)$ such that $z_1 = y_1^\tau \wedge \ldots \wedge z_n = y_n^\tau \wedge [z_1 \ldots z_n / x_1^\tau \ldots x_n^\tau]B^\tau \in w'$. By construction of Γ, whenever $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge \Box B \in w$, then there are suitable $z_1, \ldots, z_n$ such that $y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \supset y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \wedge [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau]B^\tau \in w'$. So if $y_1^\tau = y_1^\tau \wedge \ldots \wedge y_n^\tau = y_n^\tau \in w'$, then $y_1^\tau = z_1 \wedge \ldots \wedge y_n^\tau = z_n \wedge [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau]B^\tau \in w'$. ■

Lemma 6.9 (Truth Lemma)
For any sentence A and world w in the canonical model $\mathcal{M}_L = \langle W, R, D, C, V \rangle$ for L,

$$w, V \Vdash A \text{ iff } A \in w.$$

Proof by induction on A.

1. A is $Px_1 \ldots x_n$. $w, V \Vdash Px_1 \ldots x_n$ iff $\langle V_w(x_1), \ldots, V_w(x_n) \rangle \in V_w(P)$ by definition 2.7. By construction of V_w (definition 6.5), $V_w(x_i)$ is $[x_i]_w$ or undefined if $[x_i]_w = \emptyset$, and $V_w(P) = \{\langle [z_1]_w, \ldots, [z_n]_w \rangle : Pz_1 \ldots z_n \in w\}$. (For non-logical P, this is directly given by definition 6.5; for the identity predicate, $V_w(=)$ is $\{\langle d, d \rangle : d \in U_w\}$ by definition 2.7, which equals $\{\langle [z]_w, [z]_w \rangle : z = z \in w\} = \{\langle [z_1]_w, [z_2]_w \rangle : z_1 = z_2 \in w\}$ because the members of U_w are precisely the non-empty sets $[z]_w$.)

   Now if $\langle V_w(x_1), \ldots, V_w(x_n) \rangle \in V_w(P)$, then $\langle [x_1]_w, \ldots, [x_n]_w \rangle \in \{\langle [z_1]_w, \ldots, [z_n]_w \rangle : Pz_1 \ldots z_n \in w\}$, where all the $[x_i]_w$ are non-empty (for $V_w(x_i)$ is defined). This means that there are variables $z_1, \ldots, z_n$ such that $\{x_1 = z_1, \ldots, x_n = z_n, Pz_1 \ldots z_n\} \subseteq w$. Then $Px_1 \ldots x_n \in w$ by (LL*).

   In the other direction, if $Px_1 \ldots x_n \in w$, then $x_i = x_i \in w$ for all x_i in $x_1 \ldots x_n$ (see p. 60). Hence $\langle [x_1]_w, \ldots, [x_n]_w \rangle \in \{\langle [z_1]_w, \ldots, [z_n]_w \rangle : Pz_1 \ldots z_n \in w\}$, i.e. $\langle V_w(x_1), \ldots, V_w(x_n) \rangle \in V_w(P)$.

2. A is $\neg B$. $w, V \Vdash \neg B$ iff $w, V \nVdash B$ by definition 2.7, iff $B \notin w$ by induction hypothesis, iff $\neg B \in w$ by maximality of w.

3. A is $B \supset C$. $w, V \Vdash B \supset C$ iff $w, V \nVdash B$ or $w, V \Vdash C$ by definition 2.7, iff $B \notin w$ or $C \in w$ by induction hypothesis, iff $B \supset C \in w$ by maximality and consistency of w and the fact that $\vdash_L \neg B \supset (B \supset C)$ and $\vdash_L C \supset (B \supset C)$.

4. A is $\langle y : x\rangle B$. Assume first that $w, V \Vdash y \neq y$. So $V_w(y)$ is undefined, and it is not the case that $V_w(y)$ has multiple counterparts at any world. And then $w, V \Vdash \langle y : x\rangle B$ iff $w, V^{[y/x]} \Vdash B$ by definition 3.2, iff $w, V \Vdash [y/x]B$ by lemma 3.9, iff $[y/x]B \in w$ by induction hypothesis. Also by induction hypothesis, $y \neq y \in w$. By (SCN), $\vdash_L y \neq y \supset ([y/x]B \leftrightarrow \langle y : x\rangle B)$. So $[y/x]B \in w$ iff $\langle y : x\rangle B \in w$.

   Next, assume that $w, V \Vdash y = y$; so by induction hypothesis $y = y \in w$. Assume further that $\langle y : x\rangle B \notin w$. Then $\neg\langle y : x\rangle B \in w$ by maximality of w, and $\langle y : x\rangle \neg B \in w$ by (S$\neg$). Since w is substitutionally witnessed and $y = y \in w$, there is a variable $z \notin \mathit{Var}(\langle y : x\rangle \neg B)$ such that $y = z \in w$ and $[z/x]\neg B \in w$. By induction hypothesis, $w, V \Vdash y = z$. Moreover, by definition 3.3, $\neg[z/x]B \in w$, and so $[z/x]B \notin w$ by consistency of w. By induction hypothesis, $w, V \nVdash [z/x]B$. By definition 2.7, then $w, V \Vdash \neg[z/x]B$, i.e. $w, V \Vdash [z/x]\neg B$. Since z and x are modally separated in B, then $w, V^{[z/x]} \Vdash \neg B$ by lemma 3.9. But $V^{[z/x]}$ and $V^{[y/x]}$ agree on all variables at w, because $w, V \Vdash y = z$. So $w, V^{[y/x]} \Vdash \neg B$ by the locality lemma 2.9. So $w, V^{[y/x]} \nVdash B$ by definition 2.7, and $w, V \nVdash \langle y : x\rangle B$ by definition 3.2.

   In the other direction, assume $\langle y : x\rangle B \in w$. Since w is substitutionally witnessed and $y = y \in w$, there is a new variable z such that $y = z \in w$ and $[z/x]B \in w$. By induction hypothesis, $w, V \Vdash y = z$ and $w, V \Vdash [z/x]B$. Since z and x are modally separated in B, $w, V^{[z/x]} \Vdash B$ by lemma 3.9. As before $V^{[z/x]}$ and $V^{[y/x]}$ agree on all variables at w, because $w, V \Vdash y = z$; so $w, V^{[y/x]} \Vdash B$ by lemma 2.9 and $w, V \Vdash \langle y : x\rangle B$ by definition 3.2.

5. A is $\forall x B$. We first show that for any variable x, $w, V \Vdash Ex$ iff $Ex \in w$: $w, V \Vdash Ex$ iff $V_w(x) \in D_w$ by definitions 4.1 and 3.3, iff $[x]_w \in D_w$ by definition 6.5, iff $Ex \in w$ by definition 6.5.

   Now assume $\forall x B \in w$, and let y be any variable such that $Ey \in w$. As just shown, $w, V \Vdash Ey$. By (FUI**), $\exists x(x = y \wedge B) \in w$. By witnessing, there is a $z \notin \mathit{Var}(B)$ such that $z = y \wedge [z/x]B \in w$, and thus $z = y \in w$ and $[z/x]B \in w$. By induction hypothesis, $w, V \Vdash z = y$ and $w, V \Vdash [z/x]B$. By lemma 3.9, then $w, V^{[z/x]} \Vdash B$. And since $V_w(z) = V_w(y)$, it follows by lemma 2.9 that $w, V^{[y/x]} \Vdash B$. So if $\forall x B \in w$, then $w, V^{[y/x]} \Vdash B$ for all variables y with $Ey \in w$, i.e. with $V_w(y) \in D_w$. Since every member $[y]_w$ of D_w is denoted by some variable y under V_w, this means that $w, V' \Vdash B$ for all existential x-variants V' of V on w. So $w, V \Vdash \forall x B$.

   Conversely, assume $\forall x B \notin w$. Then $\exists x \neg B \in w$; so by witnessing, $[y/x]\neg B \in w$ for some $y \notin \mathit{Var}(B)$ with $Ey \in w$. Then $\neg[y/x]B \in w$ and so $[y/x]B \notin w$. As shown above, $w, V \Vdash Ey$. Moreover, by induction hypothesis, $w, V \nVdash [y/x]B$. By lemma 3.9, then $w, V^{[y/x]} \nVdash B$. Let V' be the (existential) x-variant of V on w with $V'_w(x) = V^{[y/x]}_w(x)$. By the locality lemma, $w, V' \nVdash B$. So $w, V \nVdash \forall x B$.

6. A is $\Box B$. Assume $w, V \Vdash \Box B$. Then $w', V' \Vdash B$ for all w', V' with wRw' and $V_w \triangleright V'_{w'}$. We first show that if wRw', then $V_w \triangleright V^{\tau}_{w'}$. By definitions 2.6 and 6.5, this means that for every variable y, if there is a $z \in V_w(y)$ such that $[z^{\tau}]_{w'} \in U_{w'}$ (i.e., if $\langle V_w(y), w\rangle$ has any counterpart at w'), then there is a $z \in V_w(y)$ with $z^{\tau} \in V^{\tau}_{w'}(y)$ (i.e., then $\langle V^{\tau}_{w'}(y), w'\rangle$ is such a counterpart), otherwise $V^{\tau}_{w'}(y)$ is undefined. So let y be any variable. Assume first that there is a $z \in V_w(y)$ such that $[z^{\tau}]_{w'} \in U_{w'}$. Then $z = y \in w$ and $z^{\tau} = z^{\tau} \in w'$.

By either (Neg) and (NE) or (=R), $\vdash_L z=y \supset y=y$; so $y=y \in w$. Moreover, by either (TE), (NE), (Nec) and (K) or (=R) and (Nec), $\vdash_L z=y \supset \Box(z=z \supset y=y)$; so $\Box(z=z \supset y=y) \in w$. By definition of accessibility, then $z^\tau=z^\tau \supset y^\tau=y^\tau \in w'$. So $y^\tau=y^\tau \in w'$. Hence $y \in V_w(y)$ and $y^\tau \in [y^\tau]_{w'} = V_{w'}(y^\tau) = V^\tau_{w'}(y)$. Alternatively, assume there is no $z \in V_w(y)$ with $z^\tau=z^\tau \in w'$. Then either $V_w(y)=\emptyset$, in which case $y \neq y \in w$, and so $\Box(y \neq y) \in w$ by (NA), (NE), (Nec) and (K), and $y^\tau \neq y^\tau \in w'$ by definition of R, or else $V_w(y) \neq \emptyset$, but $z^\tau \neq z^\tau \in w'$ for all $z \in V_w(y)$, in which case, too, $y^\tau \neq y^\tau \in w'$ since $y \in V_w(y)$. Either way, $V_{w'}(y^\tau) = V^\tau_{w'}(y)$ is undefined.

We've shown that if $w, V \Vdash \Box B$, then for every w' with wRw', $w', V^\tau \Vdash B$. By the transformation lemma, then $w', V \Vdash B^\tau$. By induction hypothesis, $B^\tau \in w'$. Now suppose $\Box B \notin w$. Then $\Diamond\neg B \in w$ by maximality of w. By the existence lemma, there is then a world w' with wRw' and $\neg B^\tau \in w'$. But we've just seen that $B^\tau \in w'$. So if $w, V \Vdash \Box B$, then $\Box B \in w$.

For the other direction, assume $w, V \nVdash \Box B$. So $w', V' \nVdash B$ for some w', V' with wRw' and $V_w \triangleright V'_{w'}$. As before, $V_w \triangleright V'_{w'}$ means that for every variable x, either there is a $y \in V_w(x)$ with $y^\tau \in V'_{w'}(x)$, or there is no $y \in V_w(x)$ with $y^\tau=y^\tau \in w'$, in which case $V'_{w'}(x)$ is undefined. Let $*$ be a substitution that maps each variable x in B to some member y of $V_w(x)$ with $y^\tau \in V'_{w'}(x)$, or to itself if there is no such y. Thus if $x \in Var(B)$ and $V'_{w'}(x)$ is defined, then $(*x)^\tau \in V'_{w'}(x)$, and so $V'_{w'}(x) = [(*x)^\tau]_{w'} = V^{\tau \cdot *}_{w'}(x)$. Alternatively, if $V'_{w'}(x)$ is undefined (so $*x = x$), then $V^{\tau \cdot *}_{w'}(x) = V^\tau_{w'}(x)$ is also undefined: otherwise $V^\tau_{w'}(x) = [x^\tau]_{w'} \neq \emptyset$ and $x^\tau=x^\tau \in w'$; by definition of accessibility, then $\Box x=x \notin w$ and hence $x=x \in w$, as $\vdash_L x \neq x \supset \Box x \neq x$; so there is a $y \in V_w(x)$, namely x, such that $y^\tau=y^\tau \in w'$, in which case $V'_{w'}$ cannot be undefined (by definition 2.6). So V' and $V^{\tau \cdot *}$ agree at w' on all variables in B. By lemma 2.8, $w', V^{\tau \cdot *} \nVdash B$.

Now suppose for reductio that $\Box B \in w$. Let $x_1, \ldots, x_n$ be the variables x in $Var(B)$ with $(*x)^\tau \in V'_{w'}(x)$ (thus excluding empty variables as well as variables denoting individuals without counterparts at w'.) For each such x_i, $*x_i \in V_w(x_i)$, and so $x_i = *x_i \in w$. If L is with substitution, then by (LL$_\mathrm{n}$), $\langle *x_1, \ldots, *x_n : x_1, \ldots, x_n \rangle \Box B \in w$; so $\Box\langle *x_1, \ldots, *x_n : x_1, \ldots, x_n \rangle B \in w$ by (S$\Box$). By construction of R, then $\langle (*x_1)^\tau, \ldots, (*x_n)^\tau : x_1^\tau, \ldots, x_n^\tau \rangle B^\tau \in w'$. By substitutional witnessing, it follows that there are new variables $z_1, \ldots, z_n$ such that $z_i = (*x_i)^\tau \in w'$ and (hence) $[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau \in w'$. If L is without substitution, this fact – that there are new variables $z_1, \ldots, z_n$ such that $z_i = (*x_i)^\tau \in w'$ and $[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau \in w'$ – is guaranteed directly by definition of R and the fact that $\Box B \in w$.

By induction hypothesis, $w', V \Vdash z_i = (*x_i)^\tau$ and $w', V \Vdash [z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] B^\tau$. Since the z_i are new, $w', V^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau]} \Vdash B^\tau$ by lemma 3.9. By the transformation lemma 3.13, then $w', V^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] \cdot \tau} \Vdash B$. However, for each x_i, $V^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] \cdot \tau}_{w'}(x_i) = V^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau]}_{w'}(x_i^\tau) = V_{w'}(z_i) = V_{w'}((*x_i)^\tau)$ (because $w', V \Vdash (*x_i)^\tau = z_i$) $= V^\tau_{w'}(*x_i) = V^{\tau \cdot [*x_1, \ldots, *x_n / x_1, \ldots, x_n]}_{w'}(x_i) = V^{\tau \cdot *}_{w'}(x_i)$. Similarly, if $x \in Var(B)$ is none of the $x_1, \ldots, x_n$, so $(*x)^\tau \notin V'_{w'}(x)$, then $*x$ is x by definition of $*$, and so $V^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] \cdot \tau}_{w'}(x) = V^\tau_{w'}(x) = V^\tau_{w'}(*x) = V^{\tau \cdot *}_{w'}(x)$. So $V^{[z_1, \ldots, z_n / x_1^\tau, \ldots, x_n^\tau] \cdot \tau}$ and $V^{\tau \cdot *}_{w'}$ agree at w' on all variables in B. By lemma 2.9, then $w', V^{\tau \cdot *} \Vdash B$ – contradiction. ∎

7 Completeness results

Definition 7.1 (Completeness and characterisation)
Let L be a logic in some language of quantified modal logic, and $\mathbb{M}$ a class of structures or models for that language.

L is *(strongly) complete with respect to* $\mathbb{M}$ if for every L-consistent set of formulas Γ there is a world in some model in $\mathbb{M}$ at which all members of Γ are true. L is *characterised by* $\mathbb{M}$ if L is sound and complete with respect to $\mathbb{M}$.

L is *model-complete* if it is characterised by some class of models. L is *structure-complete* if it is characterised by some class of structures.

An immediate consequence of the canonical model construction is that every quantified modal logic is strongly complete with respect to some class of models, namely its canonical model. Of course, this isn't a particularly interesting class of models.

Theorem 7.2 (Completeness of P and P_s)
The logics P and P_s are strongly complete with respect to the class of positive counterpart models.

Proof Let L range over P and P_s. We have to show that whenever a set of formulas Γ is L-consistent, then there is some world in some positive counterpart model that verifies all members of Γ. By lemma 6.6, the canonical model $\mathcal{M}_L = \langle \mathcal{S}_L, V_L \rangle$ for L is a positive model. By the Extensibility Lemma, $\Gamma \subseteq w$ for some world w in $\mathcal{M}_L$. By the truth lemma, then $w, V_L \Vdash_{\mathcal{S}_L} A$ for each $A \in \Gamma$. ∎

Theorem 7.3 (Completeness of N and N_s)
The logics N and N_s are strongly complete with respect to the class of negative counterpart models.

Proof Let L range over N and N_s, and let Γ be an L-consistent set of formulas. By lemma 6.6, the canonical model $\mathcal{M}_L = \langle \mathcal{S}_L, V_L \rangle$ for L is a negative model. By the Extensibility Lemma, $\Gamma \subseteq w$ for some world w in $\mathcal{M}_L$. By the truth lemma, then $w, V_L \Vdash_{\mathcal{S}_L} A$ for each $A \in \Gamma$. ∎

So $\mathsf{P}, \mathsf{P}_s, \mathsf{N}, \mathsf{N}_s$ are structure-complete.

[To be continued...]

References

J. L. Bell and M. Machover [1977]: *A Course in Mathematical Logic*. Noth-Holland Publishing Company

Patrick Blackburn, Maarten de Rijke and Yde Venema [2001]: *Modal Logic*. Cambridge: Cambridge University Press

Patrick Blackburn and Johan van Benthem [2007]: "Modal Logic: A Semantic Perspective". In [Blackburn et al. 2007], 1–85

Patrick Blackburn, Johan van Benthem and Frank Wolter (Eds.) [2007]: *Handbook of Modal Logic*. Dordrecht: Elsevier

David Bostock [1997]: *Intermediate Logic*. Oxford: Clarendon Press

Torben Braüner and Silvio Ghilardi [2007]: "First-order Modal Logic". In [Blackburn et al. 2007], 549–620

Giovanna Corsi [2002]: "Counterpart Semantics. A Foundational Study on Quantified Modal Logics". Manuscript

Melvin Fitting and Richard L. Mendelsohn [1998]: *First-Order Modal Logic*. Dordrecht: Kluwer

Dov Gabbay, Valentin Shehtman and Dmitrij Skvortsov [2009]: *Quantification in Non-classical Logic*, vol Vol. 1. Dordrecht: Elsevier

James W. Garson [1984]: "Quantification in Modal Logic". In D. Gabbay and F. Guenthner (Eds.) *Handbook of Philosophical Logic,* vol 2. Dordrecht: Reidel, 249–307

Silvio Ghilardi [2001]: "Substitution, quantifiers and identity in modal logic". In E. Morscher and A. Hieke (Eds.) *New essays in free logic,* Dordrecht: Kluwer, 87–115

Silvio Ghilardi and Giancarlo Meloni [1988]: "Modal and tense predicate logic: Models in presheaves and categorical conceptualization". In *Categorical algebra and its applications,* 130–142

— [1991]: "Philosophical and mathematical investigations in first-order modal logic". In *Problemi fondazionali in teoria del significato. Atti del convegno di Pontignano,* Firenze: Olsckhi, 77–107

Allen P. Hazen [1979]: "Counterpart-Theoretic Semantics for Modal Logic". *Journal of Philosophy*, 76: 319–338

G. E. Hughes and Max J. Cresswell [1968]: *An Introduction to Modal Logic*. London: Methuen

— [1996]: *A New Introduction to Modal Logic*. London and New York: Routledge

Markus Kracht and Oliver Kutz [2002]: "The semantics of modal predicate logic I. Counterpart frames". In Frank Wolter, Heinrich Wansing, Maarten de Rijke and Michael Zakharayaschev (Eds.) *Advances in Modal Logic,* vol 3. World Scientific Publishing Company

— [2005]: "The semantics of modal predicate logic II. Modal individuals revisited". In Reinhard Kahle (Ed.) *Intensionality,* Los Angeles: A. K. Peters

Saul Kripke [1963]: "Semantical considerations on modal logics". *Acta Philosophica Fennica*, 16: 83–94

Oliver Kutz [2000]: "Kripke-Typ-Semantiken für die modale Prädikatenlogik". Diplomarbeit, Humboldt University Berlin

David Lewis [1968]: "Counterpart Theory and Quantified Modal Logic". *Journal of Philosophy*, 65: 113–126

— [1986]: *On the Plurality of Worlds*. Malden (Mass.): Blackwell

Theodore Sider [2001]: *Four-Dimensionalism*. Oxford: Clarendon Press

D. P. Skvortsov and V. B. Shehtman [1993]: "Maximal Kripke–Type Semantics for Modal and Superintuitionistic Predicate Logics". *Annals of Pure and Applied Logic*, 63: 69–101

Lin Woollaston [1994]: "Counterpart Theory as a Semantics for Modal Logic". *Logique et Analyse*, 147–148: 255–263