有限差分法和有限体积法在计算流体中的应用

时间项离散

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1. 引言
2. 稳态和时间推进
3. Explicit, Forward Euler
4. Implict, Back Euler

1 引言

2 稳态和时间推进

3 Explicit, Forward Euler

4 Implict, Back Euler

引言

我猜中了前头，没有猜中结果，大话西游(白晶晶)

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引言

- $\frac{\partial T}{\partial t} = \alpha \nabla^2 T + Q_{gen}$
- 瞬态和稳态，定常非定常

两个角度

- 时间推进
- 忽略时间

pros and cons

	优点	缺点
非定常	不要考虑解的存在性	因此需要大量时间步迭代计算
	矩阵性质通常优于定常矩阵	
		有时间项截断误差
定常	可以直接求解	解的唯一性存疑
	没有时间项截断误差	

四个计算方法

- 显式 (前向欧拉)
- 隐式 (后项欧拉)
- Crank-Nicolson
- ADI

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Explicit, Forward Euler

$$\left.\frac{\partial \phi}{\partial t}\right|_{i,j,n} = \frac{\phi_{i,j,n+1} - \phi_{i,j,n}}{\Delta t} - \frac{\Delta t}{2}\left.\frac{\partial^2 \phi}{\partial t^2}\right|_{i,j,n} + \dots$$

$$\Downarrow$$

$$\frac{\phi_{i,j,n+1} - \phi_{i,j,n}}{\Delta t} = \alpha \left[\frac{\phi_{i+1,j,n} - 2\phi_{i,j,n} + \phi_{i-1,j,n}}{\Delta x^2} + \frac{\phi_{i,j+1,n} - 2\phi_{i,j,n} + \phi_{i,j11,n}}{\Delta y^2} \right]$$

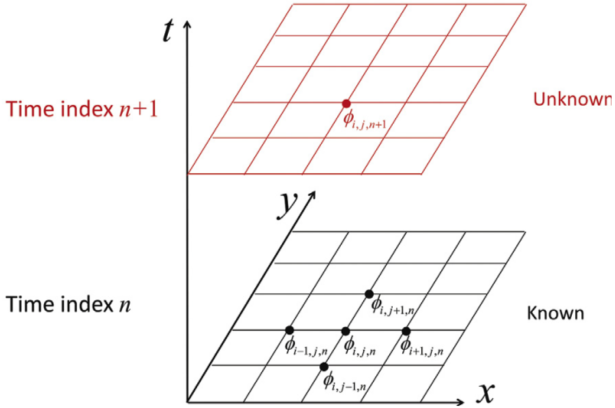
$$+\epsilon(\Delta t, \Delta x^2, \Delta y^2)$$

$$\Downarrow$$

$$\phi_{i,j,n+1} = \phi_{i,j,n} + \alpha \Delta t \left[\frac{\phi_{i+1,j,n} - 2\phi_{i,j,n} + \phi_{i-1,j,n}}{\Delta x^2} + \frac{\phi_{i,j+1,n} - 2\phi_{i,j,n} + \phi_{i,j-1,n}}{\Delta y^2} \right]$$

$$+S_{i,j}\Delta t$$

Explicit, Forward Euler



One dimension

$$\frac{\phi_{j,n+1}-\phi_{j,n}}{\Delta t}=\alpha\left[\frac{\phi_{j+1,n}-2\phi_{j,n}+\phi_{j-1,n}}{\Delta x^2}\right]+S_{i,j}$$

$$\frac{\epsilon_{j,n+1}-\epsilon_{j,n}}{\Delta t}=\alpha\left[\frac{\epsilon_{j+1,n}-2\epsilon_{j,n}+\epsilon_{j-1,n}}{\Delta x^2}\right]$$

$$\epsilon_{j,n}=\sum_{m=0}^{M-1}C_{m,n}\exp\left(im\pi\frac{j}{M-1}\right)=\sum_{m=0}^{M-1}C_{m,n}\exp\left(ij\theta_m\right)$$

One dimension

$$C_{m,n} = \exp\left(\beta_m n\Delta t\right)$$

$$C_{m,n} = \exp\left(\beta_m n\Delta t\right) = \exp([\beta_{m,r} + i\beta_{m,i}]n\Delta t)$$

$$= \underbrace{\exp(\beta_{m,r} n\Delta t)}_{applitude} \cdot \underbrace{\exp(i\beta_{m,i} n\Delta t)}_{phase\ \ angle}$$

设 $\xi_m = \exp\left(\beta_m n\Delta t\right)$ 必须满足:

$$\|\xi_m\| \leq 1$$

error analysis

$$\sum_{m=0}^{M-1} \xi_m^{n+1} \exp(ij\theta_m) - \sum_{m=0}^{M-1} \xi_m^{n} \exp(ij\theta_m) =$$

$$\frac{\alpha\Delta t}{\Delta x^2}\left[\sum_{m=0}^{M-1} \xi_m^{n} \exp(i[j+1]\theta_m) - 2\sum_{m=0}^{M-1} \xi_m^{n} \exp(ij\theta_m) + \sum_{m=0}^{M-1} \xi_m^{n} \exp(i[j-1]\theta_m)\right]$$

error analysis

$$\xi_m \exp(ij\theta_m) - \exp(ij\theta_m) =$$

$$\frac{\alpha\Delta t}{\Delta x^2}\left[\exp(i[j+1]\theta_m) - 2\exp(ij\theta_m) + \exp(i[j-1]\theta_m)\right]$$

$$\Downarrow$$

$$\xi_m - 1 = \frac{\alpha\Delta t}{\Delta x^2}[\exp(i\theta_m) - 2 + \exp(-i\theta_m)] = \frac{\alpha\Delta t}{\Delta x^2}[2\cos(\theta_m) - 2]$$

$$\Downarrow$$

$$\xi_m = 1 - \frac{\alpha\Delta t}{\Delta x^2}\left[4\sin^2\left(\frac{\theta_m}{2}\right)\right]$$

stability criterion

对于一维瞬态方程，稳定性标准为：

$$\frac{\alpha\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

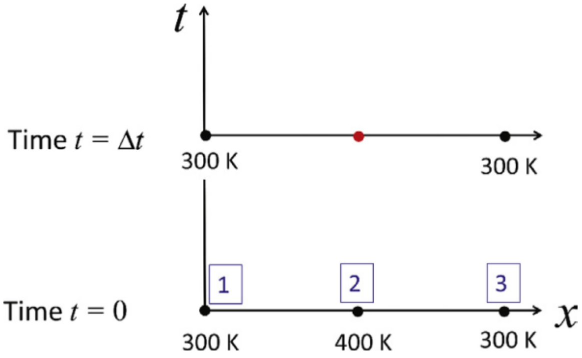
此标准成为:网格傅里叶数

二维情况为：

$$\alpha\Delta t[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}] \leq \frac{1}{2}$$

例题 1

$$\phi_{2,1} = \phi_{2,0} + \frac{\alpha \Delta t}{\Delta x^2}[\phi_{3,0} - 2\phi_{2,0} + \phi_{1,0}]$$



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Implicit, Back Euler

$$\frac{\partial \phi}{\partial t}\Big|_{i,j,n} = \frac{\phi_{i,j,n+1} - \phi_{i,j,n}}{\Delta t} + \frac{\Delta t}{2}\frac{\partial^2 \phi}{\partial t^2}\Big|_{i,j,n} + \dots$$

$$\Downarrow$$

$$\frac{\phi_{i,j,n+1} - \phi_{i,j,n}}{\Delta t} = \alpha[\frac{\phi_{i+1,j,n+1} - 2\phi_{i,j,n+1} + \phi_{i-1,j,n+1}}{\Delta x^2}$$

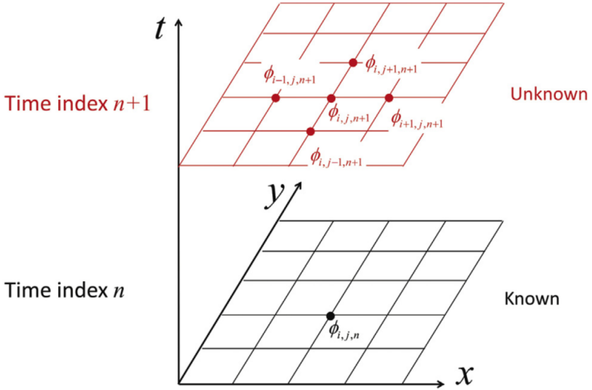
$$+\frac{\phi_{i,j+1,n+1} - 2\phi_{i,j,n+1} + \phi_{i,j+1,n+1}}{\Delta y^2}] + \epsilon(\Delta t, \Delta x^2, \Delta y^2)$$

$$\Downarrow$$

$$\left(\frac{1}{\Delta t} + \frac{2\alpha}{\Delta x^2} + \frac{2\alpha}{\Delta y^2}\right)\phi_{i,j,n+1} - \frac{\alpha}{\Delta x^2}\phi_{i+1,j,n+1} - \frac{\alpha}{\Delta x^2}\phi_{i-1,j,n+1}$$

$$-\frac{\alpha}{\Delta y^2}\phi_{i,j+1,n+1} - \frac{\alpha}{\Delta y^2}\phi_{i,j-1,n+1} = \frac{1}{\Delta t}\phi_{i,j,n} + S_{i,j}\Delta t$$

Implicit, Back Euler



One dimension

$$\frac{\epsilon_{j,n+1} - \epsilon_{j,n}}{\Delta t} = \alpha \left[\frac{\epsilon_{j+1,n+1} - 2\epsilon_{j,n+1} + \epsilon_{j-1,n+1}}{\Delta x^2} \right]$$

error analysis

$$
\sum_{m=0}^{M-1} \xi_m^{n+1} \exp(ij\theta_m) - \sum_{m=0}^{M-1} \xi_m^{n} \exp(ij\theta_m) =
$$

$$
\frac{\alpha \Delta t}{\Delta x^2} \left[\sum_{m=0}^{M-1} \xi_m^{n+1} \exp(i[j+1]\theta_m) - 2 \sum_{m=0}^{M-1} \xi_m^{n+1} \exp(ij\theta_m) \right.
$$

$$
\left. + \sum_{m=0}^{M-1} \xi_m^{n+1} \exp(i[j-1]\theta_m) \right]
$$

error analysis

$$\xi_m \exp(ij\theta_m) - \exp(ij\theta_m) =$$

$$\frac{\alpha\Delta t}{\Delta x^2}\xi_m\left[\exp(i[j+1]\theta_m) - 2\exp(ij\theta_m) + \exp(i[j-1]\theta_m)\right]$$

$$\Downarrow$$

$$\xi_m - 1 = \frac{\alpha\Delta t}{\Delta x^2}\xi_m[\exp(i\theta_m) - 2 + \exp(-i\theta_m)] = \frac{\alpha\Delta t}{\Delta x^2}\xi_m[2\cos(\theta_m) - 2]$$

$$\Downarrow$$

$$\xi_m - 1 = -\frac{\alpha\Delta t}{\Delta x^2}\xi_m\left[4\sin^2(\frac{\theta_m}{2})\right]$$

stability criterion

$$\xi_m - 1 = -\frac{\alpha \Delta t}{\Delta x^2} \xi_m \left[4 \sin^2 \left(\frac{\theta_m}{2} \right) \right]$$

$$\Downarrow$$

$$\xi_m = \frac{1}{1 + \frac{4\alpha \Delta t}{\Delta x^2} sin^2 \left(\frac{\theta_m}{2} \right)}$$

此格式意味:无条件稳定, unconditionally stable

- backward
- CrankNicolson <0, 0.9, 1>
- Euler
- localEuler
- steadyState

Thanks!