有限差分法和有限体积法在计算流体中的应用

结构化网格有限体积法

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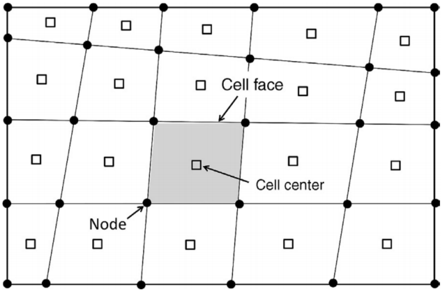
1. 引言
2. 有限体积法公式推导
3. 边界条件

引言

- 1965，Harlow and Welch 在美国 Los Alamos 国家实验室提出。

引言

- cells
- faces
- nodes
- $\bar{\phi}_o = \frac{1}{V_o} \int_{V_o} \phi dV$



一维扩散方程

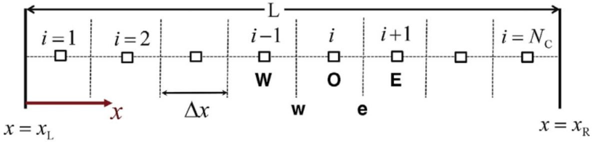
一维扩散方程为 (泊松方程):

$$\frac{d}{dx}\Gamma\left[\frac{d\phi}{dx}\right] = -S_\phi$$

通量:

$$J = -\Gamma\frac{d\phi}{dx}$$

一维 FVM 离散 stencil



$$\int_w^e \frac{d}{dx}\Gamma\left[\frac{d\phi}{dx}\right]dx = \int_w^e -S_\phi\, dx$$

$$\Downarrow$$

$$\Gamma\frac{d\phi}{dx}\bigg|_{e,i} - \Gamma\frac{d\phi}{dx}\bigg|_{w,i} = -S_i\Delta x_i$$

$$\Downarrow$$

$$J_{e,i} - J_{w,i} = S_i\Delta x_i$$

守恒性

局部守恒:

$$J_{e,i} - J_{w,i} = S_i \Delta x_i$$

整体守恒:

$$J_{e,N_C} - J_{w,1} = S_1 \Delta x_1 + S_2 \Delta x_2 + \cdots + S_{N_C} \Delta x_{N_C} = \int_{x_L}^{x_R} S dx$$

相对于有限差分法,有限体积法无论什么样的网格类型都可以保证守恒性与网格无关。

精度分析

泰勒公式：

$$\phi_{i+1} = \phi_e + \frac{\Delta x}{2}\frac{d\phi}{dx}\bigg|_e + \frac{1}{2}\left(\frac{\Delta x}{2}\right)^2 \frac{d^2\phi}{dx^2}\bigg|_e + \frac{1}{6}\left(\frac{\Delta x}{2}\right)^3 \frac{d^3\phi}{dx^3}\bigg|_e + \cdots$$

$$\phi_{i} = \phi_e - \frac{\Delta x}{2}\frac{d\phi}{dx}\bigg|_e + \frac{1}{2}\left(\frac{\Delta x}{2}\right)^2 \frac{d^2\phi}{dx^2}\bigg|_e - \frac{1}{6}\left(\frac{\Delta x}{2}\right)^3 \frac{d^3\phi}{dx^3}\bigg|_e + \cdots$$

$$\Downarrow$$

$$\frac{d\phi}{dx}\bigg|_e = \frac{\phi_{i+1}-\phi_i}{\Delta x} - \frac{\Delta x^2}{24}\frac{d^3\phi}{dx^3}\bigg|_e + \cdots$$

$$\Downarrow$$

$$\Gamma_{e,i}\frac{\phi_{i+1}-\phi_i}{\Delta x} - \Gamma_{w,i}\frac{\phi_i-\phi_{i-1}}{\Delta x} = -S_i\Delta x \quad \forall i = 2,\cdots,N_c - 1$$

均匀网格

扩散系数 =1：

$$\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{\Delta x^2}=-S_i \quad \forall i=2,\cdots,N_c-1$$

扩散系数变化:

$$\Gamma_{e,i}=\frac{\Gamma_i+\Gamma_{i+1}}{2},\Gamma_{w,i}=\frac{\Gamma_i+\Gamma_{i-1}}{2}$$

$$\left(\frac{\Gamma_i+\Gamma_{i+1}}{2\Delta x}+\frac{\Gamma_i+\Gamma_{i-1}}{2\Delta x}\right)\phi_i-\left(\frac{\Gamma_i+\Gamma_{i+1}}{2\Delta x}\right)\phi_{i+1}$$
$$-\left(\frac{\Gamma_i+\Gamma_{i-1}}{2\Delta x}\right)\phi_{i-1}=S_i\Delta x$$

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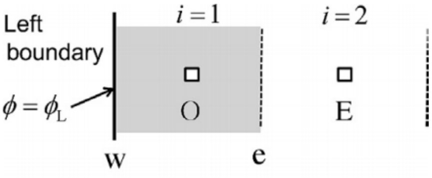
引言

- Dirichlet
- Neumann
- Robin



Dirichlet Condition

泰勒公式展开：

$$\phi_O = \phi_w + \frac{\Delta x}{2}\frac{d\phi}{dx}\bigg|_w + \frac{1}{2}\left(\frac{\Delta x}{2}\right)^2\frac{d^2\phi}{dx^2}\bigg|_w + \frac{1}{6}\left(\frac{\Delta x}{2}\right)^3\frac{d^3\phi}{dx^3}\bigg|_w + \cdots$$

整理可得：

$$\frac{d\phi}{dx}\bigg|_w = \frac{2}{\Delta x}(\phi_O - \phi_w) - \frac{\Delta x}{4}\frac{d^2\phi}{dx^2}\bigg|_w + \cdots$$

几阶格式？

二阶格式

泰勒公式展开:

$$\phi_E = \phi_w + \frac{3\Delta x}{2}\frac{d\phi}{dx}\bigg|_w + \frac{1}{2}\left(\frac{3\Delta x}{2}\right)^2\frac{d^2\phi}{dx^2}\bigg|_w + \frac{1}{6}\left(\frac{3\Delta x}{2}\right)^3\frac{d^3\phi}{dx^3}\bigg|_w + \cdots$$

$$\Downarrow$$

$$\frac{d\phi}{dx}\bigg|_w = \frac{9\phi_O - \phi_E - 8\phi_w}{3\Delta x} + \frac{1}{8}\Delta x^2\frac{d^3\phi}{dx^3}\bigg|_w + \cdots$$

$$= \frac{9\phi_O - \phi_E - 8\phi_L}{3\Delta x} + \frac{1}{8}\Delta x^2\frac{d^3\phi}{dx^3}\bigg|_w + \cdots$$

扩散系数

一阶格式：

$$\Gamma_O = \Gamma_w + \frac{\Delta x}{2}\frac{d\Gamma}{dx}\bigg|_w + \cdots$$

$$\Downarrow$$

$$\Gamma_w = \Gamma_O$$

二阶格式：

$$\Gamma_w = \frac{3\Gamma_O - \Gamma_E}{2}$$

左端

扩散系数变化:

$$\frac{\Gamma_1+\Gamma_2}{2}\left(\frac{\phi_2-\phi_1}{\Delta x}\right)-\frac{3\Gamma_1-\Gamma_2}{2}\left(\frac{9\phi_1-\phi_2-8\phi_L}{3\Delta x}\right)=-S_1\Delta x$$

$$\Downarrow$$

$$\begin{aligned}\frac{\Gamma_1+\Gamma_2}{2}\left(\frac{\phi_2-\phi_1}{\Delta x}\right)-\frac{3\Gamma_1-\Gamma_2}{2}\left(\frac{9\phi_1-\phi_2}{3\Delta x}\right)\\ =-S_1\Delta x-\frac{3\Gamma_1-\Gamma_2}{2}\left(\frac{8\phi_L}{3\Delta x}\right)\end{aligned}$$

右端

扩散系数变化:

$$\frac{3\Gamma_{N_c}+\Gamma_{N_c-1}}{2}\left(\frac{-9\phi_{N_c}-\phi_{N_c-1}}{3\Delta x}\right)-\frac{\Gamma_{N_c}+\Gamma_{N_c-1}}{2}\left(\frac{\phi_{N_c}-\phi_{N_c-1}}{\Delta x}\right)$$

$$=-S_{N_c}\Delta x-\frac{3\Gamma_{N_c}-\Gamma_{N_c-1}}{2}\left(\frac{8\phi_R}{3\Delta x}\right)$$

有限差分

$$
\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & & & & 0 \\
\frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} & 0 & \cdots & & & & 0 \\
 & \ddots & \ddots & \ddots & & & & & \\
\cdots & 0 & \frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} & 0 & \cdots & & 0 \\
 & & 0 & \frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} & 0 & & \\
 & & \cdots & 0 & \frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} & 0 & \cdots \\
 & & & & & \ddots & \ddots & \ddots & \\
0 & & & & \cdots & 0 & \frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} \\
0 & & & & & & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\ \phi_2 \\ \vdots \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \\ \vdots \\ \phi_{N-1} \\ \phi_N
\end{bmatrix}
=
\begin{bmatrix}
\phi_{\mathrm{L}} \\ -S_2 \\ \vdots \\ -S_{i-1} \\ -S_i \\ -S_{i+1} \\ \vdots \\ -S_{N-1} \\ \phi_{\mathrm{R}}
\end{bmatrix}.
\tag{6.25}
$$

有限体积

$$
\begin{bmatrix}
\frac{-4}{(\Delta x)^2} & \frac{4}{3(\Delta x)^2} & 0 & \cdots & \cdots & & & & 0 \\
\frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} & 0 & \cdots & & & & 0 \\
 & \ddots & \ddots & \ddots & & & & & \\
\cdots & 0 & \frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} & 0 & \cdots & & 0 \\
 & & 0 & \frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} & 0 & & \\
 & & \cdots & 0 & \frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} & 0 & \cdots \\
 & & & & & \ddots & \ddots & \ddots & \\
0 & & & & \cdots & 0 & \frac{1}{(\Delta x)^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} \\
0 & & & & & \cdots & 0 & \frac{4}{3(\Delta x)^2} & \frac{-4}{(\Delta x)^2}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\ \phi_2 \\ \vdots \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \\ \vdots \\ \phi_{N_C-1} \\ \phi_{N_C}
\end{bmatrix}
=
\begin{bmatrix}
-S_1 - \frac{8}{3(\Delta x)^2}\phi_{\mathrm{L}} \\ -S_2 \\ \vdots \\ -S_{i-1} \\ -S_i \\ -S_{i+1} \\ \vdots \\ -S_{N_C-1} \\ -S_{N_C} - \frac{8}{3(\Delta x)^2}\phi_{\mathrm{R}}
\end{bmatrix}.
\tag{6.26}
$$

Neumann Condition

第二类边界条件:

$$\left.\frac{d\phi}{dx}\right|_{x=x_L} = \left.\frac{d\phi}{dx}\right|_{w,l} = J_L$$

则方程变为:

$$\frac{\Gamma_1+\Gamma_2}{2}\left(\frac{\phi_2-\phi_1}{\Delta x}\right) - \frac{3\Gamma_1-\Gamma_2}{2}J_L = -S_1\Delta x$$

$$\Downarrow$$

$$\frac{\Gamma_1+\Gamma_2}{2}\left(\frac{\phi_2-\phi_1}{\Delta x}\right) = -S_1\Delta x + \frac{3\Gamma_1-\Gamma_2}{2}J_L$$

Neumman Condition

后处理时需要知道边界值：

$$\left.\frac{d\phi}{dx}\right|_w = \left(\frac{9\phi_1 - \phi_2 - 8\phi_L}{3\Delta x}\right) = J_L$$

$$\Downarrow$$

$$\phi_L = \frac{9\phi_1 - \phi_2 - 3J_L\Delta x}{8}$$

Robin Condition

第三类边界条件:

$$\alpha\phi_{w,1} + \beta \frac{d\phi}{dx}\bigg|_{w,1} = \gamma$$

泰勒公式展开:

$$\phi_O = \frac{\gamma}{\alpha} - \frac{\beta}{\alpha}\frac{d\phi}{dx}\bigg|_w + \frac{\Delta x}{2}\frac{d\phi}{dx}\bigg|_w + \frac{1}{2}\left(\frac{\Delta x}{2}\right)^2 \frac{d^2\phi}{dx^2}\bigg|_w + \frac{1}{6}\left(\frac{\Delta x}{2}\right)^3 \frac{d^3\phi}{dx^3}\bigg|_w + \cdots$$

$$\phi_E = \frac{\gamma}{\alpha} - \frac{\beta}{\alpha}\frac{d\phi}{dx}\bigg|_w + \frac{3\Delta x}{2}\frac{d\phi}{dx}\bigg|_w + \frac{1}{2}\left(\frac{3\Delta x}{2}\right)^2 \frac{d^2\phi}{dx^2}\bigg|_w + \frac{1}{6}\left(\frac{3\Delta x}{2}\right)^3 \frac{d^3\phi}{dx^3}\bigg|_w +$$

Robin Condition

推导可得：

$$\left.\frac{d\phi}{dx}\right|_w = \frac{9\phi_O - \phi_E - \frac{8\gamma}{\alpha}}{3\Delta x - \frac{8\beta}{\alpha}} + \frac{\frac{3}{8}\Delta x^3}{3\Delta x - \frac{8\beta}{\alpha}}\left.\frac{d^3\phi}{dx^3}\right|_w$$

边界上：

$$\frac{\Gamma_1+\Gamma_2}{2}\left(\frac{\phi_2-\phi_1}{\Delta x}\right) - \frac{3\Gamma_1-\Gamma_2}{2}\frac{9\phi_1-\phi_2-\frac{8\gamma}{\alpha}}{3\Delta x-\frac{8\beta}{\alpha}} = -S_1\Delta x$$

$$\Downarrow$$

$$\frac{\Gamma_1+\Gamma_2}{2}\left(\frac{\phi_2-\phi_1}{\Delta x}\right) - \frac{3\Gamma_1-\Gamma_2}{2}\frac{9\alpha\phi_1-\alpha\phi_2}{3\alpha\Delta x-8\beta}$$

$$= -S_1\Delta x - \frac{3\Gamma_1-\Gamma_2}{2}\frac{8\gamma}{3\alpha\Delta x-8\beta}$$

Robin Condition

后处理:

$$\alpha\phi_L + \beta\frac{9\phi_1 - \phi_2 - 8\phi_L}{3\Delta x} = \gamma$$

推出:

$$\phi_L = \frac{3\Delta x\gamma - 9\beta\phi_1 + \beta\phi_2}{3\alpha\Delta x - 8\beta}$$

Thanks!