有限差分法和有限体积法在计算流体中的应用

结构化网格有限体积法-对流项计算格式

汪洋

武汉理工大学交通学院

2021 年 2 月

1. 对流扩散通量格式
2. Upwind Scheme

引言

$advection - diffusion\ quation$，对流扩散方程

$$J = J_A + J_D = \rho u \phi - \Gamma \frac{d\phi}{dx}$$

ρu 是单位面积的质量流动率

ϕ 是输运的标量

$$\frac{d}{dx}\left[\rho u \phi - \Gamma \frac{d\phi}{dx}\right] = S_\phi$$

一维有限体积法

$$\int_{w,i}^{e,i} \frac{d}{dx}\left[\rho u\phi - \Gamma\frac{d\phi}{dx}\right] = \int_{w,i}^{e,i} S_\phi dx = S_\phi dx$$

$$\Downarrow$$

$$\left[\rho u\phi - \Gamma\frac{d\phi}{dx}\right]_{e,i} - \left[\rho u\phi - \Gamma\frac{d\phi}{dx}\right]_{w,i} = J_{e,i} - J_{w,i} = S_i\Delta x_i$$

有限体积法处理 ρu

这里假设，$\rho u_{e,i}$ 等量已知。两种处理方式

- 交错网格 (staggered mesh)
- 同位网格 (collocated mesh)

一维

$$\rho u|_{e,i}\left(\frac{\phi_{i+1}+\phi_i}{2}\right)-\Gamma_{e,i}\left(\frac{\phi_{i+1}-\phi_i}{\Delta x}\right)-\rho u|_{w,i}\left(\frac{\phi_{i-1}+\phi_i}{2}\right)$$
$$+\Gamma_{w,i}\left(\frac{\phi_i-\phi_{i-1}}{\Delta x}\right)=S_i\Delta x$$
$$\Downarrow$$
$$\left(\frac{\Gamma_{e,i}}{\Delta x}+\frac{\Gamma_{w,i}}{\Delta x}+\frac{1}{2}(\rho u|_{e,i}-\rho u|_{w,i})\right)\phi_i$$
$$-\left(\frac{\Gamma_{e,i}}{\Delta x}-\frac{1}{2}\rho u|_{e,i}\right)\phi_{i+1}-\left(\frac{\Gamma_{w,i}}{\Delta x}+\frac{1}{2}\rho u|_{w,i}\right)\phi_{i-1}=S_i\Delta x$$

一维简化情况

$$\left(\frac{2\Gamma}{\Delta x}\phi_i - \left(\frac{\Gamma}{\Delta x} - \frac{1}{2}\rho u\right)\phi_{i+1} - \left(\frac{\Gamma}{\Delta x} + \frac{1}{2}\rho u\right)\phi_{i-1} = S_i \Delta x\right.$$

$$\Downarrow$$

$$2\phi_i - \left(1 - \frac{Pe_\Delta}{2}\right)\phi_{i+1} - \left(1 + \frac{Pe_\Delta}{2}\right)\phi_{i-1} = S_i \frac{\Delta x^2}{\Gamma}$$

其中:$Pe_\Delta = \rho u \Delta x / \Gamma$ 是无量纲数，称为网格 Peclet 数。

Peclet 数讨论

- 当 $Pe \gg 1$，对流主导
- 当 $Pe \ll 1$，扩散主导
- 思考如果 $|Pe| \geq 2$ 时什么情况。

所以思考中央差分格式。

1. 对流扩散通量格式
2. Upwind Scheme

Upwind Scheme

$$\phi_i = \begin{cases} \phi_i \;\; if \;\; \rho u|_{e,i} > 0 \\ \phi_{i+1} \;\; if \;\; \rho u|_{e,i} < 0 \\ \phi_{i-1} \;\; if \;\; \rho u|_{w,i} > 0 \\ \phi_i \;\; if \;\; \rho u|_{w,i} < 0 \end{cases}$$

Upwind Scheme $\rho u > 0$

$$\rho u \phi_i - \Gamma_{e,i}\left(\frac{\phi_{i+1}-\phi_i}{\Delta x}\right) - \rho u \phi_{i-1} + \Gamma_{w,i}\left(\frac{\phi_i - \phi_{i-1}}{\Delta x}\right) = S_i \Delta x$$

$$\Downarrow$$

$$(2+Pe)\phi_i - \phi_{i+1} - (1+Pe)\phi_{i-1} = S_i \frac{\Delta x^2}{\Gamma}$$

一阶格式，精度不高。但是很好用。

比较两个公式

$$\left[\rho u\phi_i - (\Gamma - \frac{\Delta x}{2})\frac{d\phi}{dx}\Big|_{e,i}\right] - \left[\rho u\phi_{i-1} - (\Gamma - \frac{\Delta x}{2})\frac{d\phi}{dx}\Big|_{w,i}\right] = S_i\Delta x_i$$

$$\left[\rho u\phi_i - (\Gamma)\frac{d\phi}{dx}\Big|_{e,i}\right] - \left[\rho u\phi_{i-1} - (\Gamma)\frac{d\phi}{dx}\Big|_{w,i}\right] = S_i\Delta x_i$$

一种重要单词：smearing

Seconde Order Upwind Scheme

有一点点不同:

$$\phi_{i-1} = \phi_e - \frac{3\Delta x}{2}\frac{d\phi}{dx}\bigg|_{e,i} + \frac{1}{2}\frac{d^2\phi}{dx^2}\bigg|_{e,i} - \frac{1}{6}\frac{d^3\phi}{dx^3}\bigg|_{e,i} + \cdots$$

$$\Downarrow$$

$$\phi_e = \frac{3\phi_i - \phi_{i-1}}{2} + \frac{3}{8}(\Delta x^2)\frac{d^2\phi}{dx^2}\bigg|_{e,i} + \cdots$$

$$\Downarrow$$

$$\rho u\left(\frac{3\phi_i - \phi_{i-1}}{2}\right) - \Gamma_{e,i}\left(\frac{\phi_{i+1} - \phi_i}{\Delta x}\right) - \rho u\left(\frac{3\phi_{i-1} - \phi_{i-2}}{2}\right)$$

$$+\Gamma_{w,i}\left(\frac{\phi_i - \phi_{i-1}}{\Delta x}\right) = S_i\Delta x$$

Seconde Order Upwind Scheme

$$\phi_{e,i} = \begin{cases} \dfrac{3\phi_i - \phi_{i-1}}{2} & if \quad \rho u|_{e,i} > 0 \\ \dfrac{3\phi_{i+1} - \phi_{i+2}}{2} & if \quad \rho u|_{e,i} < 0 \end{cases}$$

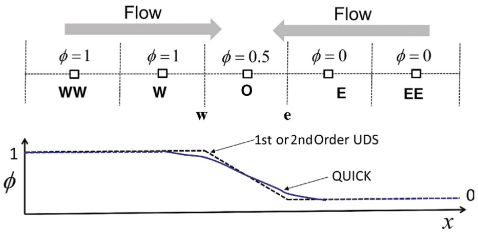
$$\phi_{w,i} = \begin{cases} \dfrac{3\phi_{i-1} - \phi_{i-2}}{2} & if \quad \rho u|_{w,i} > 0 \\ \dfrac{3\phi_i - \phi_{i+1}}{2} & if \quad \rho u|_{w,i} < 0 \end{cases}$$

QUICK

$$\phi_{e,i} = \begin{cases} \frac{3\phi_{i+1}}{8} + \frac{3\phi_i}{4} - \frac{\phi_{i-1}}{8} if & \rho u|_{e,i} > 0 \\ \frac{3\phi_i}{8} + \frac{3\phi_{i+1}}{4} - \frac{\phi_{i+2}}{8} if & \rho u|_{e,i} < 0 \end{cases}$$

$$\phi_{w,i} = \begin{cases} \frac{3\phi_i}{8} + \frac{3\phi_{i-1}}{4} - \frac{\phi_{i-2}}{8} if & \rho u|_{e,i} > 0 \\ \frac{3\phi_{i-1}}{8} + \frac{3\phi_i}{4} - \frac{\phi_{i+1}}{8} if & \rho u|_{e,i} < 0 \end{cases}$$

一个讨论


Flow
Flow
$\phi=1$
$\phi=1$
$\phi=0.5$
$\phi=0$
$\phi=0$
WW
W
O
E
EE
w
e
1
ϕ
1st or 2nd Order UDS
QUICK
0
x

边界条件

Dirichlet condition:

$$\phi_L = \phi_{e,1} - \Delta x \left.\frac{d\phi}{dx}\right|_{e,1} + \frac{1}{2}\Delta x^2 \left.\frac{d^2\phi}{dx^2}\right|_{e,1} - \frac{1}{6}\Delta x^3 \left.\frac{d^3\phi}{dx^3}\right|_{e,1} + \cdots$$

$$\Downarrow$$

对于二阶迎风:

$$\phi_{e,1} = 2\phi_1 - \phi_L + \frac{1}{4}\Delta x^2 \left.\frac{d^2\phi}{dx^2}\right|_{e,1} + \cdots$$

对于 QUICK:

$$\phi_{e,1} = \phi_1 - \frac{1}{3}\phi_L + \frac{1}{3}\phi_2 - \frac{1}{24}\Delta x^3 \left.\frac{d^3\phi}{dx^3}\right|_{e,1} + \cdots$$

讨论

- 数值格式应该保证在两个单元或者节点间单调增 (减)。物理真实。
- 当时用高阶格式时，会有局部极值点，导致越界。
- 一阶迎风格式不会出现这种问题。
- 最近几十年发明了许多保证单调性的格式。例如：essentially nonoscillatory(ENO)，total variation diminishing(TVD)

一个单词，overshoot,undershoot

OpenFOAM