有限差分法和有限体积法在计算流体中的应用

非结构化网格

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1. 法向和切向通量
2. 边界条件
3. 系数矩阵组装和求解

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法向和切向通量分析

对于 ϕ 的在面 f 上的通量计算如下:

$$(\nabla\phi)_f = \left.\frac{\partial\phi}{\partial x}\right|_f \widehat{\mathbf{i}} + \left.\frac{\partial\phi}{\partial y}\right|_f \widehat{\mathbf{j}} = \left[(\nabla\phi)_f \cdot \widehat{\mathbf{i}}\right]\widehat{\mathbf{i}} + \left[(\nabla\phi)_f \cdot \widehat{\mathbf{j}}\right]\widehat{\mathbf{j}}$$

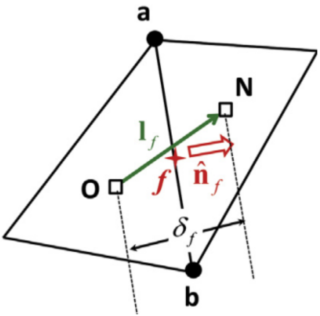
$$\Downarrow$$

$$(\nabla\phi)_f = \left[(\nabla\phi)_f \cdot \widehat{\mathbf{n}}_f\right]\widehat{\mathbf{n}}_f + \left[(\nabla\phi)_f \cdot \widehat{\mathbf{t}}_f\right]\widehat{\mathbf{t}}_\mathbf{f}$$

$$\Downarrow$$

$$(\nabla\phi)_f \cdot \mathbf{l}_f = \left[(\nabla\phi)_f \cdot \widehat{\mathbf{n}}_f\right]\widehat{\mathbf{n}}_f \cdot \mathbf{l}_f + \left[(\nabla\phi)_f \cdot \widehat{\mathbf{t}}_f\right]\widehat{\mathbf{t}}_\mathbf{f} \cdot \mathbf{l}_f$$

$$(\nabla\phi)_f\cdot\widehat{\mathbf{n}}_f=\frac{(\nabla\phi_f\cdot\widehat{\mathbf{l}}_f}{\delta_f}-\frac{\left[(\nabla\phi)_f\cdot\widehat{\mathbf{t}}_f\right]\widehat{\mathbf{t}}_f\cdot\widehat{\mathbf{l}}_f}{\delta_f}$$



泰勒公式展开

$$\phi_N = \phi_f + \left.\frac{\partial \phi}{\partial x}\right|_f (x_n - x_f) + \left.\frac{\partial \phi}{\partial y}\right|_f (y_n - y_f)$$

$$\frac{1}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_f (x_n - x_f)^2 + \frac{1}{2}\left.\frac{\partial^2 \phi}{\partial y^2}\right|_f (y_n - y_f)^2 + \frac{1}{2}\left.\frac{\partial^2 \phi}{\partial x \partial y}\right|_f (x_n - x_f)(y_n - y_f)$$

$$\phi_O = \phi_f + \left.\frac{\partial \phi}{\partial x}\right|_f (x_o - x_f) + \left.\frac{\partial \phi}{\partial y}\right|_f (y_o - y_f)$$

$$\frac{1}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_f (x_o - x_f)^2 + \frac{1}{2}\left.\frac{\partial^2 \phi}{\partial y^2}\right|_f (y_o - y_f)^2 + \frac{1}{2}\left.\frac{\partial^2 \phi}{\partial x \partial y}\right|_f (x_o - x_f)(y_o - y_f)$$

泰勒公式展开

$$
\phi_n - \phi_o = \left.\frac{\partial \phi}{\partial x}\right|_f (x_n - x_o) + \left.\frac{\partial \phi}{\partial y}\right|_f (y_n - y_o)
$$

$$
\frac{1}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_f \left[(x_n - x_f)^2 - (x_o - x_f)^2\right] + \frac{1}{2}\left.\frac{\partial^2 \phi}{\partial y^2}\right|_f \left[(y_n - y_f)^2 - (y_o - y_f)^2\right]
$$

$$
+\frac{1}{2}\left.\frac{\partial^2 \phi}{\partial x \partial y}\right|_f \left[(x_n - x_f)(y_n - y_f) - (x_n - x_f)(y_n - y_f)\right]
$$

$$
\Downarrow
$$

$$
\phi_n - \phi_o \approx (\nabla \phi)_f \cdot \mathbf{l}_f
$$

切向量通量

$$\epsilon = \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}\bigg|_f \left[(x_n - x_f)^2 - (x_o - x_f)^2\right] + \frac{1}{2}\frac{\partial^2\phi}{\partial y^2}\bigg|_f \left[(y_n - y_f)^2 - (y_o - y_f)^2\right]$$

$$+\frac{1}{2}\frac{\partial^2\phi}{\partial x\partial y}\bigg|_f \left[(x_n - x_f)(y_n - y_f) - (x_n - x_f)(y_n - y_f)\right]$$

$$\Downarrow$$

$$(\nabla\phi)_f \cdot \widehat{\mathbf{n}}_f = \frac{\phi_n - \phi_o}{\delta_f} - \frac{\left[(\nabla\phi)_f \cdot \widehat{\mathbf{t}}_f\right]\widehat{\mathbf{t}}_f \cdot \widehat{\mathbf{l}}_f}{\delta_f} = \frac{\phi_n - \phi_o}{\delta_f} - \frac{\mathbf{J}_{T,f}}{\delta_f}$$

只有当 $\mathbf{l}_f$ 与 $\mathbf{n}_f$ 平行时，切向通量才等于零

法向量通量

$$(\nabla\phi)_f \cdot \widehat{\mathbf{t}}_f \approx \frac{\phi_a - \phi_b}{|\mathbf{t}_f|}$$

$$J_{T,f} = \left[(\nabla\phi)_f \cdot \widehat{\mathbf{t}}_f\right]\widehat{\mathbf{t}}_f \cdot \mathbf{l}_f$$

$$\left[\frac{\phi_a - \phi_b}{A_f}\right]\frac{(x_a - x_b)(x_n - x_o) + (y_a - y_b)(y_n - y_o)}{A_f}$$

通项公式

$$\sum_{f=1}^{N_{f,o}} \Gamma_f \left(\frac{\phi_{N(f)} - \phi_o}{\delta_f} - \left[\frac{\phi_{a(f)} - \phi_{b(f)}}{\delta_f A_f} \right] \hat{\mathbf{t}}_f \cdot \mathbf{l}_f \right) A_f = S_{\phi,O} V_O$$

思考下，这个公式，怎么处理。

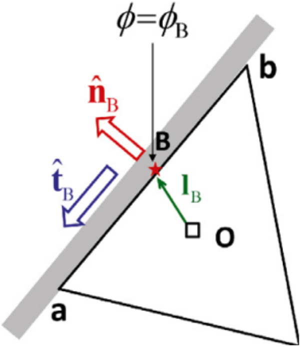
通项公式

$$\sum_{f=1}^{N_{f,o}} \Gamma_f \left(\frac{\phi_{N(f)} - \phi_o}{\delta_f} \right) A_f = S_{\phi,O} V_O + \sum_{f=1}^{N_{f,o}} \Gamma_f \left(\left[\frac{\phi^*_{a(f)} - \phi^*_{b(f)}}{\delta_f A_f} \right] \widehat{\mathbf{t}}_f \cdot \mathbf{l}_f \right) A_f$$

讨论显隐式

- 显示格式，不易收敛，对网格质量要求高，但是计算工作量小。
- 隐式格式，易收敛，对网格要求低，但是计算工作量大，需要更多内存。
- skewness，就是 $\mathbf{l}_f$ 和 $\mathbf{n}_f$ 是否平行。角度越大，越不好
- 一般希望不要大于 60，极限不要大于 75。这是一般经验，不是强调。
- 学会使用网格分析软件分析网格奇异性。

Dirichlet 边界条件



Dirichlet 边界条件

对于 ϕ 的在边界面上的通量计算如下:

$$\sum_{\substack{f=1,\\ f\neq b}}^{N_{f,o}} \Gamma_f \left[(\nabla\phi)_f \cdot \widehat{\mathbf{n}}_f \right] A_f + \Gamma_B \left[(\nabla\phi)_B \cdot \widehat{\mathbf{n}}_B \right] A_B = S_{\phi,o} V_o$$

$$\Downarrow$$

$$(\nabla\phi)_B = \left[(\nabla\phi)_B \cdot \widehat{\mathbf{n}}_B \right] \widehat{\mathbf{n}}_B + \left[(\nabla\phi)_B \cdot \widehat{\mathbf{t}}_B \right] \widehat{\mathbf{t}}_{\mathbf{B}}$$

$$\Downarrow$$

$$(\nabla\phi)_B \cdot \mathbf{l}_B = \left[(\nabla\phi)_B \cdot \widehat{\mathbf{n}}_B \right] \widehat{\mathbf{n}}_B \cdot \mathbf{l}_B + \left[(\nabla\phi)_B \cdot \widehat{\mathbf{t}}_B \right] \widehat{\mathbf{t}}_{\mathbf{B}} \cdot \mathbf{l}_B$$

推导

$$(\nabla\phi)_B \cdot \widehat{\mathbf{n}}_B = \frac{\phi_B - \phi_o}{\delta_B} - \frac{\left[(\nabla\phi)_B \cdot \widehat{\mathbf{t}}_B\right]\widehat{\mathbf{t}}_B \cdot \mathbf{l}_B}{\delta_B} = \frac{\phi_B - \phi_o}{\delta_B} - \frac{\mathbf{J}_{T,B}}{\delta_B}$$

$$(\nabla\phi)_B \cdot \widehat{\mathbf{t}}_B \approx \frac{\phi_a - \phi_b}{|\mathbf{t}_B|}$$

$$J_{T,B} = \left[(\nabla\phi)_B \cdot \widehat{\mathbf{t}}_B\right]\widehat{\mathbf{t}}_B \cdot \mathbf{l}_B$$

$$\left[\frac{\phi_a - \phi_b}{A_B}\right]\frac{(x_a - x_b)(x_n - x_o) + (y_a - y_b)(y_n - y_o)}{A_B}$$

通项公式

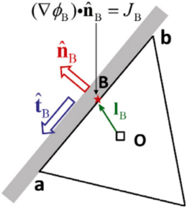
$$\sum_{\substack{f=1,\\ f\neq b}}^{N_{f,o}} \Gamma_f \left(\frac{\phi_{N(f)} - \phi_o}{\delta_f} - \left[\frac{\phi_{a(f)} - \phi_{b(f)}}{\delta_f A_f} \right] \widehat{\mathbf{t}}_f \cdot \mathbf{l}_f \right) A_f$$

$$+ \Gamma_B \left(\frac{\phi_B - \phi_o}{\delta_B} - \left[\frac{\phi_{a(B)} - \phi_{b(B)}}{\delta_B A_B} \right] \widehat{\mathbf{t}}_B \cdot \mathbf{l}_B \right) A_B = S_{\phi,O} V_O$$

讨论

- 如果边界的值相等，是什么情况
- 几阶精度呢？
- 能否构造高阶？

Neumann 边界条件



$$(\nabla\phi)_B \cdot \widehat{\mathbf{n}}_B = J_B$$

$$\sum_{\substack{f=1,\\ f\neq b}}^{N_{f,o}} \Gamma_f \left(\frac{\phi_{N(f)} - \phi_o}{\delta_f} - \left[\frac{\phi_{a(f)} - \phi_{b(f)}}{\delta_f A_f} \right] \widehat{\mathbf{t}}_f \cdot \mathbf{l}_f \right) A_f + \Gamma_B J_B A_B = S_{\phi,o} V_o$$

思考下，后处理，ϕ_B 的值如何处理

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系数矩阵公式:

$$A_O\phi_O + \sum_{j=1}^{N_{f,O}} A_j\phi_j = -S_O V_O + S_{skew,O} = Q_O$$

$$A_O = \sum_{f=1}^{N_{f,O}} \frac{\Gamma_f A_f}{\delta_f}$$

$$A_j = -\frac{\Gamma_j A_j}{\delta j}$$

$$S_{skew,O} = -\sum_{f=1}^{N_{f,O}} \Gamma_f \left(\frac{\phi^*_{a(f)} - \phi^*_{b(f)}}{\delta_f}\right)\widehat{\mathbf{t}}_f \cdot \mathbf{l}_f$$

伪码

```
for icell = 1: ncells
    Ao(icell) = 0
    Anb(icell, :) = 0
    for ifc = 1 : nface(icell)
        gface = link_cell_to_face(icell, ifc)
        if(bface(gface) == 1) skip
        ic1 = link_face_to_cell(gface, 1)
        ic2 = link_face_to_cell(grace, 2)
        gamma(gface) = wf(gface)*gamma(ic1) + (1 - wf)*
            gamma(ic2)
        Ao(icell) = Ao(icell) + gammaf(gface)*areaf(gface)
            /delta(gface)
        Anb(icell, ifc) = - gammaf(gface)*areaf(gface)/
            delta(gface)
    end
end
```

伪码

```
for icell = 1: ncells
    skew(icell) = 0
    for ifc = 1 : nface(icell)
        gface = link_cell_to_face(icell, ifc)
        vb = link_face_to_vertex(gface, 1)
        va = link_face_to_vertex(gface, 2)
        ic1 = link_face_to_cell(gface, 1)
        ic2 = link_face_to_cell(gface, 2)
        Lf(1) = xc(ic2) - xc(ic1)
        Lf(2) = yc(ic2) - yc(ic1)
        tf(1) = xv(va) - xv(vb)
        tf(2) = yv(va) - yv(vb)
        utf(:) = tf(:) / area(ifc)
        utf_dot_Lf = utf(1)*Lf(1) + utf(2)*Lf(2)

        skew(icell) = skew(icell) - gammaf(ifc)*utf_dot_Lf*(phiv(va) - phiv(vb))/
             deltaf(gface)
    end
    Q(icell) = skew(icell) - source(icell)*vol(icell)
    end
end
```

伪码

```
for bface = 1: nbfaces
    gface = link_bfaces_to_face(bface)
    ic = link_face_to_cell(gface, 1)
    Q(ic) = Q(ic) + gammaf(gface)*areaf(gface)*phib(bface)
        /delta(gface)
end
```

OpenFOAM