1 Language $\mathcal{L}$

In this section we define an imperative language $\mathcal{L}$ with memory allocation and deallocation primitives, and for simplification we only use pointers as values.

The syntax of the language $\mathcal{L}$ is as follows.

$$
\begin{array}{rcl}
x, y, z, \ldots \text{ (variables)} & \in & \mathbf{Var} \\
s \text{ (statements)} & ::= & \mathbf{skip} \mid s_1; s_2 \mid *x \leftarrow y \mid \mathbf{free}(x) \\
& \mid & \mathbf{let}\ x = \mathbf{malloc}()\ \mathbf{in}\ s \mid \mathbf{let}\ x = \mathbf{null}\ \mathbf{in}\ s \\
& \mid & \mathbf{let}\ x = y\ \mathbf{in}\ s \mid \mathbf{let}\ x = *y\ \mathbf{in}\ s \\
& \mid & \mathbf{ifnull}\ (*x)\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2 \mid f(\vec{x}) \\
& \mid & \mathbf{const}(*x)s \mid \mathbf{endconst}(*x) \\
d \text{ (proc. defs.)} & ::= & \{f \mapsto (x_1, \ldots, x_n)s\} \\
D \text{ (definitions)} & ::= & \langle d_1 \cup \cdots \cup d_n \rangle \\
P \text{ (programs)} & ::= & \langle D, s \rangle
\end{array}
$$

Notation $\vec{x}$ is for a finite sequence $\{x_1, ..., x_n\}$, where we assume that each element is distinct; $[\vec{x'}/\vec{x}]s$ is for a term obtained by replacing each free occurrence of $\vec{x}$ in s with variables $\vec{x'}$.

The **Var** is a countably infinite set of *variables* and each variable is a pointer. The statement **skip** means "does nothing". The statement $s_1; s_2$ is a sequential execution of s_1 and s_2. The statement $*x \leftarrow y$ updates the content of cell which is pointed to by x with the value y. The statement $\mathbf{free}(x)$ deallocates a memory cell which is pointed to by pointer x. The statement $\mathbf{let}\ x = e\ \mathbf{in}\ s$ evaluates the expression e, binds x to the result, and executes s. The expression $\mathbf{malloc}()$ allocates a new memory cell. The expression **null** evaluates to the null pointer. The expression $*y$ means dereferencing a memory cell pointed to by y. The statement $\mathbf{ifnull}\ (*x)\mathbf{then}\ s_1\mathbf{else}\ s_2$ executes s_1 if $*x$ is **null** and executes s_2 otherwise. The statement $f(\vec{x})$ expresses a procedure f with arguments $\vec{x}$. The statement $\mathbf{const}(*x)s$ means $(*x)$ is a constant in statement s; the statement $\mathbf{endconst}(*x)$ means from this point $(*x)$ maybe not constant.

The d represents a procedure definition which maps a procedure name f to its procedure body $(\vec{x})s$; The D represents a set of procedure definitions $\langle d_1 \cup \ldots d_n \rangle$, and each definition is distinct; The pair $\langle D, s \rangle$ represents a program, where D is a set of definitions and s is a main statement; the E represents evaluation context.

1.1 Operational semantics

In this section we introduce operational semantics of language $\mathcal{L}$. We assume there is a countable infinite set $\mathcal{H}$ of *heap addresses* ranged over by l.

We use a configuration $\langle H, R, s, n, C \rangle$ to express a run-time state. Each elements in the configuration is as follows.

- H, a *heap*, is a finite mapping from $\mathcal{H}$ to $\mathcal{H} \cup \{\mathbf{null}\}$;

- R, an *environment*, is a finite mapping from **Var** to $\mathcal{H} \cup \{\mathbf{null}\}$;
- s is the statement that is being executed;
- n is a natural number that represents the number of memory cells available for allocation.
- C is a set of actions, which contains $\mathbf{const}(*x)$, $\mathbf{null}(*x)$ and $\neg\mathbf{null}(*x)$.

The operational semantics of the language $\mathcal{L}$ is given by a labeled transition relation $\langle H, R, s, n, C\rangle \xrightarrow{\rho}_D \langle H', R', s', n', C'\rangle$. The label ρ is as follows.

$$\rho \text{ (label)} \quad ::= \quad \mathbf{malloc} \mid \mathbf{free} \mid \mathbf{null}(*x) \mid \neg\mathbf{null}(*x) \mid \tau$$

The ρ, an *action*, is **malloc**, **free**, or τ. The action **malloc** expresses an allocation of a memory cell; **free** expresses a deallocation of a memory cell; τ expresses the other actions. We often omit τ in $\xrightarrow{\tau}_D$. We use a metavariable σ for a finite sequence of actions $\rho_1 \dots \rho_n$. We write $\xrightarrow{\rho_1 \cdots \rho_n}_D$ for $\xrightarrow{\rho_1}_D \xrightarrow{\rho_2}_D \dots \xrightarrow{\rho_n}_D$. We write $\xRightarrow{\rho}_D$ for $\longrightarrow^*_D \xrightarrow{\rho}_D \longrightarrow^*_D$. We write $\xRightarrow{\rho_1 \cdots \rho_n}_D$ for $\xRightarrow{\rho_1}_D \dots \xRightarrow{\rho_n}_D$.

Notation the $\mathbf{Dom}(f)$ is a mapping from function name f to its domain; for a map f, the $f\{x \mapsto v\}$ and $f \backslash x$ are defined as follows:

$$\begin{aligned} f\{x \mapsto v\}(w) &= \begin{cases} v & \text{if } x = w \\ f(w) & \text{otherwise.} \end{cases} \\ (f \backslash x)(w) &= \begin{cases} \text{undefined} & \text{if } x = w \\ f(w) & \text{otherwise.} \end{cases} \end{aligned}$$

and $filter(C, *x)$ is defined by a pseudcode as follows:

$$\begin{aligned} filter(C, *x) \quad = \quad & let\ C' = C - \mathbf{const}(*x)\ in \\ & if\ \mathbf{const}(*x) \in\ C'\ then\ return\ C' \\ & else\ return\ C' \backslash \{\mathbf{null}(*x), \neg\mathbf{null}(*x)\} \end{aligned}$$

Figure 1 depicts the relation $\xrightarrow{\rho}_D$. Several important rules are listed as follows.

- Sem-ConstSkip: That a memory cell pointed to by $x$is no longer a constant is expressed by doing nothing.
- Sem-ConstSeq: That a memory cell pointed to by x should be a constant in a stamtement s is expressed by adding a statement $\mathbf{endconst}(*x)$ at the end of statement s.
- Sem-Free: Deallocation of a memory cell pointed to by x is expressed by deleting the entry for $R(x)$ from the heap. This action increments the number of available cells (i.e., n) by one (i.e., $n + 1$).
- Sem-Malloc and Sem-OutOfMem: Allocation of a memory cell is expressed by adding a fresh entry to the heap. This action is allowed only if the number of available cells is positive; if the number is zero, then the configuration leads to an error state **OutOfMemory**.

- SEM-ASSIGNEXN,SEM-FREEEXN,SEM-DEREFEXN and SEM-FREEEXN : These rules express an illegal access to memory. If such action is performed, then the configuration leads to exceptional state **MemEx**. This state **MemEx** is not seen as an erroneous state in the current paper, hence a well-typed program may lead to these states. The command **free**(x) , if x is a null pointer, leads to **MemEx** in the current semantics, although it is equivalent to **skip** in the C language.
- SEM-CONSTEXN: expresses that if a constant $*x$ is changed in s it will raise **ConstEx** exception.

Our goal is to guarantee *total* memory-leak freedom and reject memory leaks. By our language $\mathcal{L}$, they are formally defined as follows:

Definition 1 (total memory-leak freedom)**.** *A program* $\langle D, s\rangle$ *is* totally memory-leak free *if there is a natural number* n *such that it does not require more than* n *cells.*

Definition 2 (Memory leak)**.** *A configuration* $\langle H, R, s, n, C\rangle$ goes overflow *if there is* σ *such that* $\langle H, R, s, n, C\rangle \overset{\sigma}{\Longrightarrow}$ **OutOfMemory**. *A program* $\langle D, s\rangle$ consumes at least n cells *if* $\langle\emptyset, \emptyset, s, n, \emptyset\rangle$ *goes overflow.*

2 Type system

2.1 Types

The syntax of the types is as follows.

$$
\begin{array}{lll}
P \quad \text{(behavioral types)} & ::= & \mathbf{0} \mid P_1; P_2 \mid \mathbf{malloc} \mid \mathbf{free} \mid \alpha \mid \mu\alpha.P \\
& & \mid (x)P \mid (*x)(P_1, P_2) \mid \mathbf{const}(*x)P \mid \mathbf{endconst}(*x) \\
\Gamma \quad \text{(variable type environment)} & ::= & \{x_1, x_2, \ldots, x_n\} \\
\Psi \quad \text{(dependent function type)} & ::= & (\vec{x})P \\
\Theta \quad \text{(function type environment)} & ::= & \{f_1 : \Psi_1, \ldots, f_n : \Psi_n\}
\end{array}
$$

Behavioral types ranged over by P express the abstaction of behaviors of a program. The type $\mathbf{0}$ represents the do-nothing behavior; the type $P_1; P_2$ represents the sequential execution of P_1 and P_2; The type **malloc** represents an allocation of a memory cell exactly once; the type **free** represents a deallocation; the type $\mu\alpha.P$ represents the behavior of α defined by the recursive equation $\alpha = P$; the type $(*x)(P_1, P_2)$ represents that P_1 or P_2 is obtained dependent on $*x$; the type $P_1 + P_2$ represents the choice between P_1 and P_2; the α is a type variable; the type $\mathbf{const}(*x)P$ represents that $*x$ is a constant in behavioral type P ; the type $\mathbf{endconst}(*x)$ represents $*x$ no longer be a constant from this point.

A type environments for variables ranged over by Γ is a set of variables. Since our interest is the behavior of a program, not the types of values, a variable type environment does not carry information on the types of variables.

Dependent function types ranged over by Ψ represents the behavior of a function; $\vec{x}$ is the formal arguments of the function.

Function types ranged over by Θ is a mapping from function names to dependent function types.

$$\frac{C' = filter(C, *x)}{\langle H, R, \textbf{endconst}(*x), n, C\rangle \rightarrow_D \langle H, R, \textbf{skip}, n, C'\rangle} \quad (\textsc{Sem-ConstSkip})$$

$$\langle H, R, \textbf{const}(*x)s, n, C\rangle \rightarrow_D \langle H, R, s; \textbf{endconst}(*x), n, C \cup \{\textbf{const}(*x)\}\rangle \quad (\textsc{Sem-ConstSeq})$$

$$\langle H, R, \textbf{skip}; s, n, C\rangle \longrightarrow_D \langle H, R, s, n, C\rangle \quad (\textsc{Sem-Skip})$$

$$\frac{\langle H, R, s_1, n, C\rangle \xrightarrow{\rho}_D \langle H', R', s_1', n', C'\rangle}{\langle H, R, s_1; s_2, n, C\rangle \xrightarrow{\rho}_D \langle H', R', s_1'; s_2, n', C'\rangle} \quad (\textsc{Sem-Seq})$$

$$\frac{x' \notin \textbf{Dom}(R)}{\langle H,\ R,\ \textbf{let}\ x = \textbf{null}\ \textbf{in}\ s, n, C\rangle \longrightarrow_D \langle H,\ R\{x' \mapsto \textbf{null}\},\ [x'/x]\, s, n, C\rangle} \quad (\textsc{Sem-LetNull})$$

$$\frac{x' \notin \textbf{Dom}(R)}{\langle H,\ R,\ \textbf{let}\ x = y\ \textbf{in}\ s, n, C\rangle \longrightarrow_D \langle H,\ R\{x' \mapsto R(y)\},\ [x'/x]\, s, n, C\rangle} \quad (\textsc{Sem-LetEq})$$

$$\frac{H(R(x)) = \textbf{null}, \textbf{const}(*x) \notin C}{\langle H,\ R,\ \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ s_2,\ n, C\rangle \xrightarrow{\textbf{null}(*x)}_D \langle H,\ R,\ s_1,\ n, C\rangle} \quad (\textsc{Sem-IfNullT})$$

$$\frac{H(R(x)) \neq \textbf{null}, \textbf{const}(*x) \notin C}{\langle H,\ R,\ \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ s_2,\ n, C\rangle \xrightarrow{\neg\textbf{null}(*x)}_D \langle H,\ R,\ s_2,\ n, C\rangle} \quad (\textsc{Sem-IfNullF})$$

$$\frac{H(R(x)) = \textbf{null}, \textbf{const}(*x) \in C}{\langle H,\ R,\ \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ s_2,\ n, C\rangle \xrightarrow{\textbf{null}(*x)}_D \langle H,\ R,\ s_1,\ n, C \cup \{\textbf{null}(*x)\}\rangle} \quad (\textsc{Sem-IfConstNullT})$$

$$\frac{H(R(x)) \neq \textbf{null}, \textbf{const}(*x) \in C}{\langle H,\ R,\ \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ s_2,\ n, C\rangle \xrightarrow{\neg\textbf{null}(*x)}_D \langle H,\ R,\ s_2,\ n, C \cup \{\neg\textbf{null}(*x)\}\rangle} \quad (\textsc{Sem-IfConstNullF})$$

$$\frac{\forall z. R(x) = R(z) \Rightarrow \textbf{const}(*x) \notin C}{\langle H\{R(x) \mapsto v\}, R, *x \leftarrow y, n, C\rangle \longrightarrow_D \langle H\{R(x) \mapsto R(y)\}, R, \textbf{skip}, n, C\rangle} \quad (\textsc{Sem-Assign})$$

$$\frac{x' \notin \textbf{Dom}(R) \qquad R(y) \in \textbf{Dom}(H)}{\langle H,\ R,\ \textbf{let}\ x = *y\ \textbf{in}\ s, n, C\rangle \longrightarrow_D \langle H,\ R\{x' \mapsto H(R(y))\},\ [x'/x]\, s, n, C\rangle} \quad (\textsc{Sem-LetDeref})$$

$$\frac{R(x) \neq \textbf{null}\ \text{and}\ R(x) \in \textbf{Dom}(H)}{\langle H\{R(x) \mapsto v\},\ R,\ \textbf{free}(x), n, C\rangle \xrightarrow{\textbf{free}}_D \langle H \backslash R(x),\ R,\ \textbf{skip}, n+1, C\rangle} \quad (\textsc{Sem-Free})$$

$$\frac{l \notin \textbf{Dom}(H) \qquad n > 0 \qquad x' \notin \textbf{Dom}(H) \cup \textbf{Dom}(R) \cup fv(C)}{\langle H,\ R,\ \textbf{let}\ x = \textbf{malloc}()\ \textbf{in}\ s, n, C\rangle \xrightarrow{\textbf{malloc}}_D \langle H\{l \mapsto v\},\ R\{x' \mapsto l\},\ [x'/x]\, s, n-1, C\rangle} \quad (\textsc{Sem-Malloc})$$

$$\frac{D(f) = (\vec{y})s}{\langle H,\ R,\ f(\vec{x}), n, C\rangle \longrightarrow_D \langle H,\ R,\ [\vec{x}/\vec{y}]\, s, n, C\rangle} \quad (\textsc{Sem-Call})$$

$$\frac{R(x) = \textbf{null}\ \text{or}\ R(x) \notin \textbf{Dom}(H)}{\langle H,\ R,\ \textbf{free}(x), n, C\rangle \xrightarrow{\textbf{free}}_D \textbf{MemEx}} \quad (\textsc{Sem-FreeExn})$$

$$\frac{R(x) = \textbf{null}\ \text{or}\ R(x) \notin \textbf{Dom}(H)}{\langle H,\ R,\ *x \leftarrow y, n, C\rangle \longrightarrow_D \textbf{MemEx}} \quad (\textsc{Sem-AssignExn})$$

$$\frac{R(y) = \textbf{null}\ \text{or}\ R(y) \notin \textbf{Dom}(H)}{\langle H,\ R,\ \textbf{let}\ x = *y\ \textbf{in}\ s, n, C\rangle \longrightarrow_D \textbf{MemEx}} \quad (\textsc{Sem-DerefExn})$$

$$\frac{\exists z. \textbf{const}(*z) \in C\ \text{and}\ R(x) = R(z)}{\langle H\{R(x) \mapsto v\}, R, *x \leftarrow y, n, C\rangle \longrightarrow_D \textbf{ConstEx}} \quad (\textsc{Sem-AssignConstExn})$$

$$\langle H,\ R,\ \textbf{let}\ x = \textbf{malloc}()\ \textbf{in}\ s, 0, C\rangle \xrightarrow{\textbf{malloc}}_D \textbf{OutOfMemory} \quad (\textsc{Sem-OutOfMem})$$

Figure 1: Operational semantics of $\mathcal{L}$.

Figure 2 depicts semantics of behavioral types with dependent types, and they are given by the labeled transition system. The relation $\langle P, C\rangle \xrightarrow{\rho} \langle P', C'\rangle$ means that P can make an action ρ, and P turns into P' after it makes action ρ; C and C' record constant value environment before and after making action ρ respectively.

2.2 Typing rules

The type judgment for statements is of the form $\Theta; \Gamma \vdash s : P$, which represents that under the function type environment Θ and the variable type environment Γ, the abstracted behavioral type of statement s is P.

Before showing typing rules for statements in Figure 3, we need explain several important definitions. The first one is $OK_n(P, C)$, a predicate, where P represents the behavior of a program which consumes at most n memory cells under constant value environment C.

Definition 3 ($\sharp_\rho(\sigma)$)**.** *$\sharp_\rho(\sigma)$ is the number of ρ in the sequence σ.*

Definition 4. *$OK_n(P, C)$ holds if $\forall P'$ and σ. if $\langle P, C\rangle \xrightarrow{\sigma} \langle P', C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$*

Intuitively, $OK_n(P, C)$ represents at very running steps, the number of memory cells a program consumed will not exceed the number of memory cells the program requires.

Definition 5 (Subtyping)**.** *$C \vdash P_1 \leq P_2$ is the largest relation such that, for any P_1', C' and ρ, if $\langle P_1, C\rangle \xrightarrow{\rho} \langle P_1', C'\rangle$, then there exists P_2' such that $\langle P_2, C\rangle \xRightarrow{\rho} \langle P_2',' \rangle$ and $C' \vdash P_1' \leq P_2'$. We write $P_1 \leq P_2$ if $C \vdash P_1 \leq P_2$ for any C.*

Figure 3 shows the typing rules. For example, the rule T-IfNull represents the behavior of **ifnull** $(*x)$ **then** s_1 **else** s_2 is abstracted as $(*x)(P_1, P_2)$ where P_1 and P_2 are the behavior of s_1 and s_2 respectively; this conditional statement means that executing s_1 if $(*x)$ is a null pointer, otherwise s_2. The typing rule T-Program represents a program requires at most n memory cells during running under the predication $OK_n(P, C)$, where P is behavioral type of statement s.

2.3 Type soundness

Theorem 2.1. *If $\vdash \langle D, s\rangle : n$ for some n, then $\langle D, s\rangle$ is totally memory-leak free.*

The proof is based on the following lemmas: preservation and lack of immediate overflow.

Definition 6. *consistency(H, R, C): for all x. (1) if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ (3) if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.*

Definition 7. *we write $\Theta \vdash \langle H, R, s, n, C\rangle : \langle P, C\rangle$, if there exists Γ such that $\Theta; \Gamma \vdash s : P$, $OK_n(P, C)$, consistency(H, R, C) and $\Gamma \subseteq \mathbf{Dom}(R)$.*

Lemma 2.2 (Preservation)**.** *suppose that $\Theta \vdash \langle H, R, s, n, C\rangle : \langle P, C\rangle$, if $\langle H, R, s, n, C\rangle \xrightarrow{\rho} \langle H', R', s', n', C'\rangle$ then $\exists P'$ and C' s.t. (1) $\Theta \vdash \langle H', R', s', n', C'\rangle : \langle P', C'\rangle$ and (2) $\langle P, C\rangle \xRightarrow{\rho} \langle P', C'\rangle$.*

Lemma 2.3 (Lack of immediate overflow)**.** *If $\Theta \vdash \langle H, R, s, n, C\rangle : \langle P, C\rangle$, then $\langle H, R, s, n, C\rangle \not\xrightarrow{\mathbf{malloc}}$* **OutOfMemory**.

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$$\langle \mathbf{0}; P, C\rangle \to \langle P, C\rangle \qquad \text{(Tr-Skip)}$$

$$\langle \mathbf{free}, C\rangle \xrightarrow{\mathbf{free}} \langle \mathbf{0}, C\rangle \quad \text{(Tr-Free)} \qquad \langle \mu\alpha.P, C\rangle \to \langle [\mu\alpha.P/\alpha]P, C\rangle \ \text{(Tr-Rec)}$$

$$\langle P_1 + P_2, C\rangle \to \langle P_1, C\rangle \ \text{(Tr-ChoiceL)} \qquad \langle P_1 + P_2, C\rangle \to \langle P_2, C\rangle \ \text{(Tr-ChoiceR)}$$

$$\frac{\langle P_1, C\rangle \xrightarrow{\rho} \langle P_1', C'\rangle}{\langle P_1; P_2, C\rangle \xrightarrow{\rho} \langle P_1'; P_2, C'\rangle} \qquad \text{(Tr-Seq)}$$

$$\langle \mathbf{malloc}, C\rangle \xrightarrow{\mathbf{malloc}} \langle 0, C\rangle \qquad \text{(Tr-Malloc)}$$

$$\frac{x' is fresh}{\langle (x)P, C\rangle \to \langle [x'/x]P, C\rangle} \qquad \text{(Tr-Bind)}$$

$$\langle \mathbf{const}(*x)P, C\rangle \to \langle P; \mathbf{endconst}(*x), C \cup \{\mathbf{const}(*x)\}\rangle \qquad \text{(Tr-Const)}$$

$$\frac{C' = filter(C, *x)}{\langle \mathbf{endconst}(*x), C\rangle \to \langle \mathbf{0}, C'\rangle} \qquad \text{(Tr-Endconst)}$$

$$\frac{\mathbf{const}(*x) \notin C}{\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1, C\rangle} \ \text{(Tr-NotConst1)} \qquad \frac{\mathbf{const}(*x) \notin C}{\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\neg\mathbf{null}(*x)} \langle P_2, C\rangle} \ \text{(Tr-NotConst2)}$$

$$\frac{\mathbf{null}(*x) \in C \qquad \mathbf{const}(*x) \in C}{\langle (*x)(P_1, P_2), C\rangle \to \langle P_1, C\rangle} \ \text{(Tr-NullIn)} \qquad \frac{\neg\mathbf{null}(*x) \in C \qquad \mathbf{const}(*x) \in C}{\langle (*x)(P_1, P_2), C\rangle \to \langle P_2, C\rangle} \ \text{(Tr-NNullIn)}$$

$$\frac{\mathbf{null}(*x), \neg\mathbf{null}(*x) \notin C \qquad \mathbf{const}(*x) \in C}{\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1, C \cup \mathbf{null}(*x)\rangle} \qquad \text{(Tr-NNullNotIn1)}$$

$$\frac{\mathbf{null}(*x), \neg\mathbf{null}(*x) \notin C \qquad \mathbf{const}(*x) \in C}{\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\neg\mathbf{null}(*x)} \langle P_2, C \cup \neg\mathbf{null}(*x)\rangle} \qquad \text{(Tr-NNullNotIn2)}$$

Figure 2: semantics of behavioral types with dependent types.

$$\Theta;\Gamma \vdash \mathbf{skip} : \mathbf{0} \quad \text{(T-Skip)}$$

$$\frac{\Theta;\Gamma \vdash s_1 : P_1 \quad \Theta;\Gamma \vdash s_2 : P_2}{\Theta;\Gamma \vdash s_1; s_2 : P_1; P_2} \quad \text{(T-Seq)}$$

$$\Theta;\Gamma, x, y \vdash *x \leftarrow y : \mathbf{0} \quad \text{(T-Assign)}$$

$$\Theta;\Gamma, x \vdash \mathbf{free}(x) : \mathbf{free} \quad \text{(T-Free)}$$

$$\frac{\Theta;\Gamma, x \vdash s : P}{\Theta;\Gamma \vdash \mathbf{let}\ x = \mathbf{malloc}()\ \ \mathbf{in}\ s : \mathbf{malloc}; (x)P} \quad \text{(T-Malloc)}$$

$$\frac{\Theta;\Gamma, x, y \vdash s : P}{\Theta;\Gamma, y \vdash \mathbf{let}\ x = y\ \mathbf{in}\ s : [y/x]P} \quad \text{(T-LetEq)}$$

$$\frac{\Theta;\Gamma, x, y \vdash s : P}{\Theta;\Gamma, y \vdash \mathbf{let}\ x = *y\ \mathbf{in}\ s : (x)P} \quad \text{(T-LetDeref)}$$

$$\frac{\Theta;\Gamma, x \vdash s : P}{\Theta;\Gamma \vdash \mathbf{let}\ x = \mathbf{null}\ \mathbf{in}\ s : (x)P} \quad \text{(T-LetNull)}$$

$$\Theta;\Gamma, x \vdash \mathbf{endconst}(*x) : \mathbf{endconst}(*x) \quad \text{(T-Endconst)}$$

$$\frac{\Theta;\Gamma, x \vdash s : P}{\Theta;\Gamma, x \vdash \mathbf{const}(*x)s : \mathbf{const}(*x)P} \quad \text{(T-Const)}$$

$$\frac{\Theta;\Gamma, x \vdash s_1 : P_1 \qquad \Theta;\Gamma, x \vdash s_2 : P_2}{\Theta;\Gamma, x \vdash \mathbf{ifnull}\ (*x)\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2 : (*x)(P_1, P_2)} \quad \text{(T-IfNull)}$$

$$\Theta, f : (\vec{y})P; \Gamma, \vec{x} \vdash f(\vec{x}) : P[\vec{x}/\vec{y}] \quad \text{(T-Call)}$$

$$\frac{\Theta;\Gamma \vdash s : P_1 \qquad P_1 \leq P_2}{\Theta;\Gamma \vdash s : P_2} \quad \text{(T-Sub)}$$

$$\frac{\Theta(f) = (\vec{x})P \qquad \mathbf{Dom}(D) = \mathbf{Dom}(\Theta) \qquad \Theta; x_1, \ldots, x_n \vdash s : P \text{ for each } f \mapsto (x_1, \ldots, x_n)s \in D}{\vdash D : \Theta} \quad \text{(T-Def)}$$

$$\frac{\vdash D : \Theta \qquad \Theta;\emptyset \vdash s : P \quad OK_n(P, C)}{\vdash \langle D, s \rangle : n} \quad \text{(T-Program)}$$

Figure 3: typing rules

3 Proof of Lemmas

Lemma 3.1. *If* $OK_n(P,C)$ *and* $\langle P,C\rangle \xrightarrow{\rho} \langle P',C'\rangle$*, then*

- $OK_{n-1}(P',C')$ *if* $\rho = \mathbf{malloc}$*,*
- $OK_{n+1}(P',C')$ *if* $\rho = \mathbf{free}$*,*
- $OK_n(P',C')$ *if* $\rho = Otherwise$

Proof. By induction on $\langle P,C\rangle \xrightarrow{\rho} \langle P',C'\rangle$.

- Case $P = \mathbf{0}; P'$ and $\langle \mathbf{0}; P', C\rangle \rightarrow \langle P', C\rangle$

  We need to prove $OK_n(P',C)$. Assume that $OK_n(P',C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P',C\rangle \xrightarrow{\sigma} \langle Q,C'\rangle$, $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n(\mathbf{0};P',C)$ holds, we have if $\langle \mathbf{0};P',C\rangle \rightarrow \langle P',C\rangle \xrightarrow{\sigma} \langle Q,C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which are in contradiction to the assumption $\sharp_m(\sigma) - \sharp_f(\sigma) > n$. Therefore, $OK_n(P',C)$ holds.

- Case $P = \mathbf{malloc}$ and $\langle \mathbf{malloc}, C\rangle \xrightarrow{\mathbf{malloc}} \langle 0, C\rangle$

  we need to prove $OK_{n-1}(0,C)$, which means we need to prove that for all σ and Q, if $\langle 0,C\rangle \xrightarrow{\sigma} \langle Q,C'\rangle$ then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n-1$. There is no σ and Q such that $\langle 0,C\rangle \xrightarrow{\sigma} \langle Q,C'\rangle$. Therefore, $OK_{n-1}(0,C)$ holds.

- Case $P = \mathbf{free}$ and $\langle \mathbf{free}, C\rangle \xrightarrow{\mathbf{free}} \langle \mathbf{0}, C\rangle$

  We need to prove $OK_{n+1}(\mathbf{0},C)$, which means we need to prove $\forall\sigma$ and Q if $\langle \mathbf{0},C\rangle \xrightarrow{\sigma} \langle Q,C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n+1$. There is no Q and σ s.t. $\langle \mathbf{0},C\rangle \xrightarrow{\sigma} \langle Q,C\rangle$, so (1) holds. Therefore, $OK(\mathbf{0},C)$ holds.

- Case $P = \mathbf{endconst}(*x)$ and $\frac{C' = filter(C,*x)}{\langle \mathbf{endconst}(*x),C\rangle \rightarrow \langle \mathbf{0},C'\rangle}$

  We need to prove $OK_n(\mathbf{0},C')$, which means we need to prove $\forall\sigma$ and Q if $\langle \mathbf{0},C\rangle \xrightarrow{\sigma} \langle Q,C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$ and (2) $OK(C')$ holds. There is no Q and σ s.t. $\langle \mathbf{0},C\rangle \xrightarrow{\sigma} \langle Q,C\rangle$. So $OK_n(\mathbf{0},C')$ holds.

- Case $P = (x)P'$ and $\frac{x' is fresh}{\langle (x)P',C\rangle \rightarrow \langle [x'/x]P',C\rangle}$

  We need to prove $OK_n([x'/x]P',C)$. Assuming that $OK_n([x'/x]P',C)$ does not hold. Then we have $\exists\sigma$ and Q s.t. $\langle [x'/x]P',C\rangle \xrightarrow{\sigma} \langle Q,C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of $OK_n((x)P',C)$, we have if $\langle (x)P',C\rangle \rightarrow \langle [x'/x]P',C\rangle \xrightarrow{\sigma} \langle Q,C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$. Therefore we get the contradiction.

  Therefore $OK_n([x'/x]P',C)$ holds.

- Case $P = (*x)(P_1,P_2)$ and $\frac{\mathbf{const}(*x) \notin C}{\langle (*x)(P_1,P_2),C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1,C\rangle}$

  We need to prove $OK_n(P_1,C)$. Assume that $OK_n(P_1,C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P_1,C\rangle \xrightarrow{\sigma} \langle Q,C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption $\sharp_m(\sigma) - \sharp_f(\sigma) > n$. Therefore, $OK_n(P_1, C)$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\frac{\mathbf{const}(*x)\notin C}{\langle (*x)(P_1,P_2),C\rangle \to \langle P_2, C\rangle}$

  We need to prove $OK_n(P_2, C)$. Assume that $OK_n(P_2, C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P_2, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\neg\mathbf{null}(*x)} \langle P_2, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption. Therefore, $OK_n(P_2, C)$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\frac{\mathbf{null}(*x)\in C \qquad \mathbf{const}(*x)\in C}{\langle (*x)(P_1,P_2),C\rangle \to \langle P_1, C\rangle}$

  We need to prove $OK_n(P_1, C)$. Assume that $OK_n(P_1, C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P_1, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \to \langle P_1, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption. Therefore, $OK_n(P_1, C)$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\frac{\neg\mathbf{null}(*x)\in C \qquad \mathbf{const}(*x)\in C}{\langle (*x)(P_1,P_2),C\rangle \to \langle P_2, C\rangle}$

  We need to prove $OK_n(P_2, C)$. Assume that $OK_n(P_2, C)$ does not hold. Then we have $\exists\sigma$ and Q s.t. $\langle P_2, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \to \langle P_2, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption. Therefore, $OK_n(P_2, C)$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\frac{\mathbf{null}(*x), \neg\mathbf{null}(*x)\notin C \qquad \mathbf{const}(*x)\in C}{\langle (*x)(P_1,P_2),C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1, C\cup\{\mathbf{null}(*x)\}\rangle}$

  We need to prove $OK_n(P_1, C \cup \{\mathbf{null}(*x)\})$. Assume that $OK_n(P_1, C \cup \{\mathbf{null}(*x)\})$ does not hold. Then we have $\exists\sigma$ and Q s.t. $\langle P_1, C \cup \{\mathbf{null}(*x)\}\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1, C \cup \{\mathbf{null}(*x)\}\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$. Therefore, we get the contradiction and $OK_n(P_1, C \cup \{\mathbf{null}(*x)\})$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\frac{\mathbf{null}(*x), \neg\mathbf{null}(*x)\notin C \qquad \mathbf{const}(*x)\in C}{\langle (*x)(P_1,P_2),C\rangle \xrightarrow{\neg\mathbf{null}(*x)} \langle P_2, C\cup\{\neg\mathbf{null}(*x)\}\rangle}$

  Similar to the above.

- Case $P = \mathbf{const}(*x)P'$ and $\langle \mathbf{const}(*x)P', C\rangle \to \langle P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x)\rangle$

  We need to prove $OK_n(P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x))$. Assume that $OK_n(P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x))$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x)\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n(\mathbf{const}(*x)P', C)$ holds, we have if $\langle \mathbf{const}(*x)P', C\rangle \to \langle P; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x)\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption. Therefore, $OK_n(P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x))$ holds.

- Case $P = \mu\alpha.P'$ and $\langle \mu\alpha.P', C\rangle \to \langle [\mu\alpha.P'/\alpha]P', C\rangle$

  We need to prove $OK_n([\mu\alpha.P'/\alpha]P', C)$. Assume that $OK_n([\mu\alpha.P'/\alpha]P', C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle [\mu\alpha.P'/\alpha]P', C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n(\mu\alpha.P', C)$ holds, we have if $\langle \mu\alpha.P', C\rangle \to \langle [\mu\alpha.P'/\alpha]P', C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \le n$, which is a contradiction. Therefore, $OK([\mu\alpha.P'/\alpha]P', C)$ holds.

- Case $P = P_1; P_2$ and $\dfrac{\langle P_1, C\rangle \xRightarrow{\rho} \langle P_1', C'\rangle}{\langle P_1; P_2, C\rangle \xRightarrow{\rho} \langle P_1'; P_2, C'\rangle}$

  We need to prove $OK_{n'}(P_1'; P_2, C)$, where n' is determined by

$$n' = \begin{cases} n+1 & \rho = \mathbf{free} \\ n-1 & \rho = \mathbf{malloc} \\ n & \text{Otherwise.} \end{cases}$$

  Assume that $OK_{n'}(P_1'; P_2, C')$ does not hold. Then, we have $\exists\sigma$, Q and C'' s.t. $\langle P_1'; P_2, C\rangle \xrightarrow{\sigma} \langle Q, C''\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n'$.

  From the definition of that $OK_n(P_1; P_2, C)$ holds, we have if $\langle P_1; P_2, C\rangle \xRightarrow{\rho} \langle P_1'; P_2, C'\rangle \xrightarrow{\sigma} \langle Q, C''\rangle$, then $\sharp_m(\rho\sigma) - \sharp_f(\rho\sigma) \le n$.

  Then we get $n' + \sharp_m(\rho) - \sharp_f(\rho) < \sharp_m(\rho) + \sharp_m(\sigma) - \sharp_f(\rho) - \sharp_f(\sigma) \le n$. For any ρ, the $n' + \sharp_m(\rho) - \sharp_f(\rho) = n$, therefore we get a contradiction. Therefore, $OK_{n'}(P_1; P_2, F')$ holds.

$\square$

Lemma 3.2. *If* $consistency(H, R, C)$ *and* $\langle H, R, s, n, C\rangle \xrightarrow{\rho} \langle H', R', s', n', C'\rangle$*, then* $consistency(H', R', C')$*.*

Proof. By induction on $\langle H, R, s, n, C\rangle \xrightarrow{\rho} \langle H', R', s', n', C'\rangle$

- Case: $\langle H, R, \mathbf{const}(*y)s, n, C\rangle \to \langle H, R, s; \mathbf{endconst}(*y), n', C \cup \mathbf{const}(*y)\rangle$.

  We need to prove $consistency(H, R, C \cup \mathbf{const}(*y))$. Form assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H, R, C \cup \mathbf{const}(*y))$, we chose z arbitrarily. (3) Assuming $\mathbf{null}(*z) \in C \cup \mathbf{const}(*y)$. This implies $\mathbf{null}(*z) \in C$. By using (1), we have $\mathbf{const}(*z) \in C$, and this implies $\mathbf{const}(*z) \in C \cup \mathbf{const}(*y)$. Assuming $H(R(z))$ is defined, then $H(R(z)) = null$ from (1). (4) Assuming $\neg\mathbf{null}(*z) \in C \cup \mathbf{const}(*y)$. This implies $\neg\mathbf{null}(*z) \in C$. By using (2), we have $\mathbf{const}(*z) \in C$. This implies $\mathbf{const}(*z) \in C \cup \mathbf{const}(*y)$. Assuming $H(R(z))$ is defined, then $H(R(z)) \neq null$ from (2).

  Therefore, $consistency(H, R, C \cup \mathbf{const}(*y))$ holds.

- Case: $\langle H, R, \mathbf{endconst}(*y), n, C\rangle \to \langle H, R, skip, n, C'\rangle$ where $C' = filter(C, *y)$.

  We need to prove $consistency(H, R, C')$ where $C' = filter(C, *y)$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then

$H(R(x)) = null$ and (2) $\forall x$.if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

To prove $consistency(H, R, C')$, we chose z arbitrarily. From the definition of function $filter(C, *y)$, we know (3) if $\mathbf{const}(*y) \in (C - \mathbf{const}(*y))$ then we have $C' = C - \mathbf{const}(*y)$, otherwise we have $C' = (C - \mathbf{const}(*y)) \backslash \{\mathbf{null}(*x), \neg\mathbf{null}(*x)\}$. From (3) we have $C' \in C$. (4) Assuming $\mathbf{null}(*z) \in C'$, this implies $\mathbf{null}(*z) \in C$. From (1) we have $\mathbf{const}(*z) \in C$. Now we want to get $\mathbf{const}(*z) \in C'$. We should consider two cases: $z \neq y$ and $z = y$. If $z \neq y$ then we have $\mathbf{const}(*z) \in C'$ from (3); if $z = y$, and because $\mathbf{null}(*z) \in C'$, then we have $\mathbf{const}(*z) \in C'$. Assuming $H(R(z))$ is defined, then we have $H(R(z)) = null$ from (1). (5) Assuming $\neg\mathbf{null}(*z) \in C'$. The similar to (4).

Therefore, $consistency(H, R, C')$ holds.

- Case: $\langle H, R, \mathbf{free}(y), n, C\rangle \xrightarrow{\mathbf{free}} \langle H\backslash R(y), R, skip, n+1, C\rangle$.

  We need to prove $consistency(H\backslash R(y), R, C$. From assumption $consistency(H, R, C)$, we have (1) $\forall x$.if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x$.if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H\backslash R(y), R, C$, we chose z arbitrarily. (3) Assuming $\mathbf{null}(*z) \in C$. By using (1), we have $\mathbf{const}(*z) \in C$. Assuming $H(R(z))$ is defined, we have $H(R(z)) = null$ from (1). We know that y has been deallocated from H, so if $z = y$ then $H\backslash R(y)(R(z))$ is not defined, otherwise $H\backslash R(y)(R(z))$ is defined and $H\backslash R(y)(R(z)) = H(R(z)) = null$. (4) Assuming $\mathbf{null}(*z) \in C$, By using (2), we have $\mathbf{const}(*z) \in C$. Assuming $H(R(z))$ is defined, we have $H(R(z)) \neq null$ from (2). We know that y has been deallocated from H, so if $z = y$ then $H\backslash R(y)(R(z))$ is not defined, otherwise $H\backslash R(y)(R(z))$ is defined and $H\backslash R(y)(R(z)) = H(R(z)) \neq null$.

  Therefore, $consistency(H\backslash R(y), R, C$ holds.

- Case: $\langle H, R, \mathbf{let}\ y = \mathbf{malloc\ in}\ s, n, C\rangle \xrightarrow{\mathbf{malloc}} \langle H\{l \mapsto v\}, R\{x' \mapsto l\}, [x'/y]s, n', C\rangle$ where $x' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$

  We need to prove $consistency(H\{l \mapsto v\}, R\{x' \mapsto l\}, C)$. From assumption $consistency(H, R, C)$, we have (1) $\forall x$.if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x$.if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H\{l \mapsto v\}, R\{x' \mapsto l\}, C)$, we chose z arbitrarily. (3) Assuming $\mathbf{null}(*z) \in C$. From (1) we have $\mathbf{const}(*z) \in C$. Assuming $H(R(z))$ is defined, we have $H(R(z)) = null$ from (1). We have $x' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$, so $z \neq x'$. Therefore we get $H\{l \mapsto v\}(R\{x' \mapsto l\}z) = H(R(z)) = null$. (4) Assuming $\neg\mathbf{null}(*z) \in C$. similar to (3).

  Therefore, $consistency(H\{l \mapsto v\}, R\{x' \mapsto l\}, C)$ holds.

- Case: $\langle H, R, skip; s, n, C\rangle \to \langle H, R, s, n', C\rangle$.

  Obviously, $consistency(H, R, C)$ holds form assumption.

- Case: $\langle H\{R(w) \mapsto v\}, R, *w \leftarrow y, n, C\rangle \rightarrow \langle H\{R(w) \mapsto R(y)\}, R, skip, n, C\rangle$ where $\forall z.R(w) = R(z) \Rightarrow \mathbf{const}(*z) \notin C$

  We need to prove $consistency(H\{R(w) \mapsto R(y)\}, R, C)$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H\{R(w) \mapsto v\}(R(x))$ is defined then $H\{R(w) \mapsto v\}(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H\{R(w) \mapsto v\}(R(x))$ is defined then $H\{R(w) \mapsto v\}(R(x)) = null$.

  To prove $consistency(H\{R(w) \mapsto R(y)\}, R, C)$, we chose m arbitrarily. (3) Assuming $\mathbf{null}(*m) \in C$. By using (1), we have $\mathbf{const}(*m) \in C$. Then assuming $H\{R(w) \mapsto v\}(R(m))$ is defined, we have $H\{R(w) \mapsto v\}(R(m)) = null$ from (1). Because we know $\forall z.R(w) = R(z) \Rightarrow \mathbf{const}(*z) \notin C$ and $\mathbf{const}(*m) \in C$, we have $m \neq w$, then we have $H\{R(w) \mapsto R(y)\}(R(m)) = H\{R(w) \mapsto v\}(R(m)) = null$. (4) Assuming $\neg\mathbf{null}(*m) \in C$. Similar to (3).

  Therefore, $consistency(H\{R(w) \mapsto R(y)\}, R, C)$ holds.

- Case: $\langle H, R, \mathbf{let}\ z = y\ \mathbf{in}\ s, n, C\rangle \rightarrow \langle H, R\{z' \mapsto R(y)\}, [z'/z]s, n, C\rangle$ where $z' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$

  We need to prove $consistency(H, R\{z' \mapsto R(y)\}, C$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H, R\{z' \mapsto R(y)\}, C$, we chose m arbitrarily. (3) Assuming $\mathbf{null}(*m) \in C$. By using (1), we have $\mathbf{const}(*m) \in C$. Then assuming $H(R(m))$ is defined, we have $H(R(m)) = null$ from (1). Because we have $z' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$, we have $m \neq z'$, then we have $H(R\{z' \mapsto R(y)\}(m)) = H(R(m)) = null$. (4) Assuming $\neg\mathbf{null}(*m) \in C$. By using (2), we have $\mathbf{const}(*m) \in C$. Then assuming $H(R(m))$ is defined, we have $H(R(m)) \neq null$ from (2). Because we have $z' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$, we have $m \neq z'$, then we have $H(R\{z' \mapsto R(y)\}(m)) = H(R(m)) \neq null$.

  Therefore, $consistency(H, R\{z' \mapsto R(y)\}, C)$ holds.

- Case: $\langle H, R, \mathbf{ifnull}\ (*y)\ \mathbf{then}\ s_1\ \mathbf{else}\ \ s_2, n, C\rangle \xrightarrow{\mathbf{null}(*y)} \langle H, R, s_1, n, C\rangle$ where $H(R(y)) = null$ and $\mathbf{const}(*y) \notin C$

  Obviously, $consistency(H, R, C)$ holds from assumption.

- Case: $\langle H, R, \mathbf{ifnull}\ (*y)\ \mathbf{then}\ s_1\ \mathbf{else}\ \ s_2, n, C\rangle \xrightarrow{\neg\mathbf{null}(*y)} \langle H, R, s_2, n, C\rangle$ where $H(R(y)) \neq null$ and $\mathbf{const}(*y) \notin C$

  Obviously, $consistency(H, R, C)$ holds from assumption.

- Case: $\langle H, R, \mathbf{ifnull}\ (*y)\ \mathbf{then}\ s_1\ \mathbf{else}\ \ s_2, n, C\rangle \xrightarrow{\mathbf{null}(*y)} \langle H, R, s_1, n, C \cup \mathbf{null}(*y)\rangle$ where $H(R(y)) = null$ and $\mathbf{const}(*y) \in C$

  We need to prove $consistency(H, R, C \cup \mathbf{null}(*y)$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H, R, C \cup \mathbf{null}(*y)$, we chose z arbitrarily. (3) Assuming $\mathbf{null}(*z) \in C \cup \mathbf{null}(*y)$. This implies $\mathbf{null}(*z) \in C$. By using (1), we have $\mathbf{const}(*z) \in C$. This implies

$\mathbf{const}(*z) \in C \cup \mathbf{null}(*y)$. Assuming $H(R(z))$ is defined, then we have $H(R(z)) = \mathit{null}$ from (1). (4) Assuming $\neg\mathbf{null}(*z) \in C \cup \mathbf{null}(*y)$. This implies $\neg\mathbf{null}(*z) \in C$. By using (2), we have $\mathbf{const}(*z) \in C$. This implies $\mathbf{const}(*z) \in C \cup \mathbf{null}(*y)$. Assuming $H(R(z))$ is defined, then we have $H(R(z)) \neq \mathit{null}$ from (2).

Therefore, $\mathit{consistency}(H, R, C \cup \mathbf{null}(*y))$ holds.

□

Proof of Lemma 2.2: By induction on the derivation of $\langle H, R, s, n, C\rangle \xrightarrow{\rho} \langle H', R', s', n', C'\rangle$.

- Case: $\langle H, R, \mathbf{const}(*x)s, n, C\rangle \to \langle H, R, s; \mathbf{endconst}(*x), n, C \cup \{\mathbf{const}(*x)\}\rangle$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, \mathbf{const}(*x)s, n, C\rangle : \langle P, C\rangle$, we have $\Theta; \Gamma \vdash \mathbf{const}(*x)s : P$, $OK_n(P, C)$ and $\mathit{consistency}(H, R, C)$. From the inversion of typing rules, we get $\Theta; \Gamma \vdash s : P''$ and $\mathbf{const}(*x)P'' \leq P$ for some P''. By subtyping, we have $P''; \mathbf{endconst}(*x) \leq Q$ and $\langle P, C\rangle \Longrightarrow \langle Q, C \cup \{\mathbf{const}(*x)\}\rangle$ for some Q.

  we need to find P' and C' s.t. $\Theta; \Gamma \vdash s; \mathbf{endconst}(*x) : P'$, $OK_n(P', C')$, $\langle P, C'\rangle \Longrightarrow \langle P', C'\rangle$ and $\mathit{consistency}(H, R, C')$. Taking Q as P' and $C \cup \{\mathbf{const}(*x)\}$ as C'. Therefore $\langle P, C\rangle \to \langle P', C'\rangle$ holds, and then $OK_n(P', C')$ and $\mathit{consistency}(H, R, C')$ hold from Lemma 4.1 and Lemma 4.2. From $\Theta; \Gamma \vdash s; \mathbf{endconst}(*x) : P''; \mathbf{endconst}(*x)$, $P''; \mathbf{endconst}(*x) \leq Q$ and T-Sub, $\Theta; \Gamma \vdash s; \mathbf{endconst}(*x) : P'$ holds.

- Case: $\langle H, R, \mathbf{endconst}(*x), n, C\rangle \to \langle H, R, \mathbf{skip}, n, C'\rangle$ where $C' = \mathit{filter}(C, *x)$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, \mathbf{endconst}(*x), n, C\rangle : \langle P, C\rangle$, we have $\Theta; \Gamma \vdash \mathbf{endconst}(*x) : P$, $OK_n(P, C)$ and $\mathit{consistency}(H, R, C)$. From the inversion of typing rules, we get $\Theta; \Gamma \vdash \mathbf{endconst}(*x) : \mathbf{endconst}(*x)$ and $\mathbf{endconst}(*x) \leq P$. By subtyping , we get $0 \leq Q$ and $\langle P, C\rangle \to \langle Q, C\rangle$ for some Q.

  we need to find P' and C' s.t. $\Theta; \Gamma \vdash \mathbf{skip} : P'$, $OK_n(P', C')$, $\langle P, C\rangle \Longrightarrow P', C'\rangle$ and $\mathit{consistency}(H, R, C')$. Taking Q as P' and C as C', then $\langle P, C\rangle \to \langle P', C'\rangle$ holds, and then $OK_n(P', C')$ and $\mathit{consistency}(H, R, C')$ hold from Lemma 4.1 and Lemma 4.2 . From T-Skip, T-Sub and $0 \leq Q$, then $\Theta; \Gamma \vdash \mathbf{skip} : P'$ holds.

- Case: $\langle H, R, \mathbf{free}(x)\ , n, C\rangle \xrightarrow{\mathbf{free}} \langle H', R, \mathbf{skip}, n+1, C\rangle$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, \mathbf{free}(x)\ , n, C\rangle : \langle P, C\rangle$, we have $OK_n(P, C)$, $\mathit{consistency}(H, R, C)$ and $\Theta; \Gamma \vdash \mathbf{free}(x) : P$. From inversion of the typing rules, we have $\Theta; \Gamma \vdash \mathbf{free}(x) : \mathbf{free}$ and $\mathbf{free} \leq P$. By the subtyping, we have $\langle P, F\rangle \xRightarrow{\mathbf{free}} \langle Q, C\rangle$ and $\mathbf{0} \leq Q$ for some Q.

  We need to find P' and C' such that $\langle P, C\rangle \xRightarrow{\mathbf{free}} \langle P', C'\rangle$, $\Theta; \Gamma \vdash \mathbf{skip} : P'$, and $OK_{n+1}(P', C')$. Take Q as P' and C as C'. Then, $\langle P, C\rangle \xRightarrow{\mathbf{free}} \langle P', C'\rangle$ holds, and $OK_{n+1}(P', C')$ holds from Lemma 4.1. We also have $\Theta; \Gamma \vdash \mathbf{skip} : P'$ from T-Skip, $\mathbf{0} \leq Q$ and T-Sub.

- Case: $\langle H, R, \mathbf{let}\ x = \mathbf{malloc}()\ \mathbf{in}\ s, n, C\rangle \xrightarrow{\mathbf{malloc}} \langle H', R', [x'/x]s, n-1, C\rangle$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, \mathbf{let}\ x = \mathbf{malloc}()\ \mathbf{in}\ s, n, C\rangle : \langle P, C\rangle$, we have $\Theta; \Gamma \vdash \mathbf{let}\ x = \mathbf{malloc}()\ \mathbf{in}\ s : P$, $OK_n(P, C)$ and $\mathit{consistency}(H, R, C)$. By the inversion of typing rules, we have $\Theta; \Gamma, x \vdash s : P''$ and $\mathbf{malloc}; (x)P'' \leq P$ for some P''. By subtyping, we get $\langle P, C\rangle \xRightarrow{\mathbf{malloc}} \langle Q, F\rangle$ and $[x'/x]P'' \leq Q$ for some Q.

We need to find P' and C' such that $\Theta;\Gamma,x' \vdash [x'/x]s : P'$, $\langle P,C\rangle \xrightarrow{\textbf{malloc}} \langle P',C'\rangle$, $consistency(H',R',C')$ and $OK_{n-1}(P',C')$. Take Q as P' and C as C'. Then $\langle P,C\rangle \xrightarrow{\textbf{malloc}} \langle P',C'\rangle$ holds, and then $OK_{n-1}(P',C')$ and $consistency(H',R',C')$ hold by Lemma 4.1 and Lemma 4.2. From $\Theta;\Gamma,x \vdash s : P''$ and $\textbf{malloc};(x)P'' \leq P$, we have $\Theta;\Gamma,x'' \vdash [x''/x]s : [x''/x]P''$ and $\textbf{malloc};(x)P'' \leq P$, and then by the definition of subtyping we have $[x''/x]P'' \leq Q'$ for some Q'. Therefore, we get $\Theta;\Gamma,x'' \vdash [x''/x]s : Q'$. Take x'' as x' and Q' as P', then $\Theta;\Gamma,x' \vdash [x'/x]s : P'$ holds.

- Case: $\langle H,R,\textbf{skip};s,n,C\rangle \to \langle H,R,s,n,C\rangle$

  From the assumption $\Theta;\Gamma \vdash \langle H,R,\textbf{skip};s,n,C\rangle : \langle P,C\rangle$, we have $\Theta;\Gamma \vdash \textbf{skip};s : P$, $OK_n(P,C)$ and $consistency(H,R,C)$. From the inversion of the typing rules, we get $\Theta;\Gamma \vdash s : P''$ and $0;P'' \leq P$. From the definition of subtyping, we have $\langle P,C\rangle \Longrightarrow \langle Q,C\rangle$ and $P'' \leq Q$ for some Q.

  We need to find P' and C' such that $\Theta;\Gamma \vdash s : P'$ and $\langle P,C\rangle \to \langle P',C'\rangle$ and $OK_n(P',C')$. Take Q as P' and C as C'. Then $\langle P,C\rangle \Longrightarrow \langle P',C'\rangle$ holds, and then $OK_n(P',C')$ and $consistency(H,R,C')$ hold. We also have $\Theta;\Gamma \vdash s : P'$ from T-SUB, $\Gamma \vdash s : P''$ and $P'' \leq Q$.

- Case: $\langle H,R,*x \leftarrow y,n,C\rangle \to \langle H',R,\textbf{skip},n,C\rangle$

  From the assumption $\Theta;\Gamma \vdash \langle H,R,*x \leftarrow y,n,C\rangle : \langle P,C\rangle$, we have $\Theta;\Gamma \vdash *x \leftarrow y : P$, $OK_n(P,C)$ and $consistency(H,R,C)$. From the inversion of typing rules, we have $0 \leq P$.

  We need to find P' and C' such that $\Theta;\Gamma \vdash \textbf{skip} : P'$, $\langle P,C\rangle \Longrightarrow \langle P',C'\rangle$ and $OK_n(P',C')$. Take P as P' and C as C'. Then $\langle P,C\rangle \Longrightarrow \langle P',C'\rangle$ holds, and then $OK_n(P',C')$ and $consistency(H',R,C')$ hold from Lemma 4.1 and Lemma 4.2. We also have $\Theta;\Gamma \vdash \textbf{skip} : P'$ from T-SKIP, $0 \leq P$ and T-SUB.

- Case: $\langle H,R,\textbf{let } x = y \textbf{ in } s,n,C\rangle \to \langle H,R',[x'/x]s,n,C\rangle$

  From the assumption $\Theta;\Gamma \vdash \langle H,R,\textbf{let } x = y \textbf{ in } s,n,C\rangle : \langle P,C\rangle$, we have $\Theta;\Gamma,y \vdash \textbf{let } x = y \textbf{ in } s : P$, $OK_n(P,C)$ and $consistency(H,R,C)$. From the inversion of typing rules, we have $\Theta;\Gamma,x,y \vdash s : P''$ and $\textbf{let } x = y \textbf{ in } P'' \leq P$ for some P''. By subtying, we have $\langle P,C\rangle \to \langle Q,C\rangle$ and $[x'/x]P'' \leq Q$ for some Q.

  We need to find P' and C' such that $\Theta;\Gamma,x',y \vdash [x'/x]s : P'$, $\langle P,C\rangle \to \langle P',C'\rangle$, $OK_n(P',C')$ and $consistency(H,R',C')$. Take Q as P' and C as C'. Then $\langle P,C\rangle \Longrightarrow \langle P',C'\rangle$ and $OK_n(P',C')$ hold. From $\Theta;\Gamma,x,y \vdash s : P''$ and $\textbf{let } x = y \textbf{ in } P'' \leq P$, we have $\Theta;\Gamma,x'',y \vdash [x''/x]s : [x''/x]P''$ and $\textbf{let } x'' = y \textbf{ in } [x''/x]P'' \leq P$, and then by subtying we have $[x''/x]P'' \leq Q'$ for some Q'. Therefore, we have $\Theta;\Gamma,x'',y \vdash [x''/x]s : Q'$. Take x'' as x' and Q' as P', then $\Theta;\Gamma,x',y \vdash [x'/x]s : P'$ holds.

- Case: $\langle H,R,\textbf{let } x = \textbf{null in } s,n\rangle \to \langle H,R',[x'/x]s,n\rangle$

  Similar to the above.

- Case: $\langle H,R,\textbf{let } x = *y \textbf{ in } s,n\rangle \to \langle H,R',[x'/x]s,n\rangle$

  Similar to the above.

- Case: $\langle H,R,\textbf{ifnull } (*x) \textbf{ then } s_1 \textbf{ else } s_2,n,C\rangle \xrightarrow{\textbf{null}(*x)} \langle H,R,s_1,n,C\rangle$ if $H(R(x)) = \textbf{null}$ and $\textbf{const}(*x) \notin C$

From assumption $\Theta;\Gamma \vdash \langle H, R, \textbf{ifnull } (*x) \textbf{ then } s_1 \textbf{ else } s_2, n, C\rangle : \langle P, C\rangle$, we have $\Theta;\Gamma \vdash \textbf{ifnull } (*x) \textbf{ then } s_1 \textbf{ else } s_2 : P$, $OK_n(P, C)$ and $consistency(H, R, C)$. From the inversion of typing rules, we have $\Theta;\Gamma \vdash s_1 : P_1$, $\Theta;\Gamma \vdash s_2 : P_2$ and $(*x)(P_1, P_2) \leq P$. According to the rule Tr-NotConst1 and $\textbf{const}(*x) \notin C$, we have $\langle (*x)(P_1, P_2)\rangle \xrightarrow{\textbf{null}(*x)} \langle P_1, C\rangle$, and then by definition of subtyping, we get $\langle P, C\rangle \xRightarrow{\textbf{null}(*x)} \langle Q, C\rangle$ and $P_1 \leq Q$ for some Q.

We need to find P' and C' such that $\Theta;\Gamma \vdash s_1 : P'$, $\langle P, C\rangle \xRightarrow{\textbf{null}(*x)} \langle P', C'\rangle$ and $OK_n(P', C')$. Take Q as P' and C as C'. Then $\langle P, C\rangle \xRightarrow{\textbf{null}(*x)} \langle P', C'\rangle$ and $OK_n(P', C')$ hold. We also have $\Theta;\Gamma \vdash s_1 : P'$ from T-Sub, $\Theta;\Gamma \vdash s_1 : P_1$ and $P_1 \leq Q$.

- Case: $\langle H, R, \textbf{ifnull } (*x) \textbf{ then } s_1 \textbf{ else } s_2, n, C\rangle \xrightarrow{\neg\textbf{null}(*x)} \langle H, R, s_1, n, C\rangle$ if $H(R(x)) \neq \textbf{null}$ and $\textbf{const}(*x) \notin C$

  Similar to the above.

- Case: $\langle H, R, \textbf{ifnull } (*x) \textbf{ then } s_1 \textbf{ else } s_2, n, C\rangle \xrightarrow{\textbf{null}(*x)} \langle H, R, s_1, n, C'\rangle$ if $H(R(x)) = \textbf{null}$, $\textbf{const}(*x) \in C$ and $C' = C \cup \{\textbf{null}(*x)\}$

  From assumption $\Theta;\Gamma \vdash \langle H, R, \textbf{ifnull } (*x) \textbf{ then } s_1 \textbf{ else } s_2, n, C\rangle : \langle P, C\rangle$, we have $\Theta;\Gamma \vdash \textbf{ifnull } (*x) \textbf{ then } s_1 \textbf{ else } s_2 : P$, $OK_n(P, C)$ and $consistency(H, R, C)$. From the inversion of typing rules, we have $\Theta;\Gamma \vdash s_1 : P_1$, $\Theta;\Gamma \vdash s_2 : P_2$ and $(*x)(P_1, P_2) \leq P$. According to rule Tr-NNullNotIn1, $\textbf{const}(*x) \in C$ and $C' = C \cup \textbf{null}(*x)$, we have $\langle (*x)(P_1, P_2)\rangle \xrightarrow{\textbf{null}(*x)} \langle P_1, C \cup \textbf{null}(*x)\rangle$, and then by the definition of subtyping, we get $\langle P, C\rangle \xRightarrow{\textbf{null}(*x)} \langle Q, C \cup \{\textbf{null}(*x)\}\rangle$ and $P_1 \leq Q$ for some Q.

  We need to find P' and C' such that $\Theta;\Gamma \vdash s_1 : P'$, $\langle P, C\rangle \xRightarrow{\textbf{null}(*x)} \langle P', C'\rangle$, $OK_n(P', C')$ and $consistency(H, R, C')$. Take Q as P' and $C \cup \{\textbf{null}(*x)\}$ as C'. Then $\langle P, C\rangle \xrightarrow{\textbf{null}(*x)} \langle P', c'\rangle$ holds, and then $OK_n(P', C')$ and $consistency(H, R, C')$hold by Lemma 4.1 and Lemma 4.2. We also have $\Theta;\Gamma \vdash s_1 : P'$ from T-Sub, $\Theta;\Gamma \vdash s_1 : P_1$ and $P_1 \leq Q$.

- Case: $\langle H, R, \textbf{ifnull } (*x) \textbf{ then } s_1 \textbf{ else } s_2, n, C\rangle \xrightarrow{\neg\textbf{null}(*x)} \langle H, R, s_2, n, C'\rangle$ if $H(R(x)) \neq \textbf{null}$, $\textbf{const}(*x) \in C$ and $C' = C \cup \{\neg\textbf{null}(*x)\}$

  Similar to the above proof.

- Case: $\langle H, R, s_1; s_2, n, C\rangle \rightarrow \langle H', R', s_1'; s_2, n', C'\rangle$

  From the assumption $\Theta;\Gamma \vdash \langle H, R, s_1; s_2, n, C\rangle : \langle P, C\rangle$, we have $\Theta;\Gamma \vdash s_1; s_2 : P$, $OK_n(P, C)$ and $consistency(H, R, C)$. By inversion of typing rules, we have $\Theta;\Gamma \vdash s_1 : P_1$, $\Theta;\Gamma \vdash s_2 : P_2$ and $P_1; P_2 \leq P$ for some P_1 and P_2.

  By IH on $\langle H, R, s_1, n, C\rangle$ with derivation $\langle H, R, s_1, n, C\rangle \xrightarrow{\rho} \langle H', R', s_1', n', C'\rangle$, we have $\exists P_1', C_1'$ s.t. $\Theta;\Gamma \vdash \langle H', R', s_1', n', C'\rangle : \langle P_1', C_1'\rangle$ and $\langle P_1, C\rangle \xrightarrow{\rho} \langle P_1', C_1'\rangle$.

  By subtyping we have $\langle P, C\rangle \xrightarrow{\rho} \langle Q, C_1'\rangle$ and $P_1'; P_2 \leq Q$ for some Q.

  We need to find P' and C' s.t. $\langle P, C\rangle \xrightarrow{\rho} \langle P', C'\rangle$, $OK_n(P', C')$ and $\Theta;\Gamma \vdash s_1'; s_2 : P'\rangle$. Take Q as P' and C_1' as C', $\langle P, C\rangle \xrightarrow{\rho} \langle P', C'\rangle$ and $OK_n(P', C')$ hold. By T-Sub, $\Theta;\Gamma \vdash s_1'; s_2 : P_1'; P_2$ and $P_1'; P_2 \leq Q$, we have $\Theta;\Gamma \vdash s_1'; s_2 : P'$ holds.

□

We write $\langle H, R, s, n, C\rangle \xrightarrow{\rho}$ if there is a transition $\xrightarrow{\rho}$ from $\langle H, R, s, n, C\rangle$.

Lemma 3.3. *If* $\Theta \vdash \langle H, R, s, n, C\rangle : \langle P, C\rangle$ *and* $\langle H, R, s, n, C\rangle \stackrel{\rho}{\Longrightarrow}$ *and* $\rho \in \{\mathbf{malloc}, \mathbf{free}, \mathbf{null}(*x), \neg\mathbf{null}(*x)\}$*, then there exists* P' *and* C' *such that* $\langle P, C\rangle \stackrel{\rho}{\Longrightarrow} \langle P', C'\rangle$.

Proof. Induction on the derivation of $\Theta; \Gamma \vdash \langle H, R, s, n, C\rangle : \langle P, C\rangle$. □

Proof of Lemma 2.3:

By contradiction. Assume $\langle H, R, s, n, C\rangle \xrightarrow{\rho}$ **OutOfMemory**. Then, n is 0 and $\rho = \mathbf{malloc}$ from SEM-OUTOFMEM. From the assumption we have $\Theta; \Gamma \vdash s : P$ and $OK_0(P, C)$. From Lemma 4.3, there exists P' and C' such that $\langle P, C\rangle \stackrel{\mathbf{malloc}}{\Longrightarrow} \langle P', C'\rangle$. However, this contradicts $OK_0(P, C)$.

□

Proof of Theorem 2.1:

We have $\Theta; \emptyset \vdash s : P, \vdash D : \Theta$, $OK_n(P, C)$ and *consistency*(H, R, C).

Suppose that there exists σ such that $\langle \emptyset, \emptyset, s, n, C\rangle \xrightarrow{\sigma} \langle H', R', s', n', C'\rangle \xrightarrow{\rho}$ **OutOfMemory**. Then, $n' = 0$ and $\rho = \mathbf{malloc}$. From Lemma 2.2, there exists P' and C' such that $\Theta; \Gamma' \vdash s' : P'$, $\langle P, C\rangle \stackrel{\sigma}{\Longrightarrow} \langle P', C'\rangle$, and $OK_0(P', C')$; hence $\langle H', R', s', 0\rangle \xrightarrow{\mathbf{malloc}}$. However, this contradicts Lemma 2.3.

□

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4 Proof of Lemmas

Lemma 4.1. *If* $OK_n(P, C)$ *and* $\langle P, C\rangle \xrightarrow{\rho} \langle P', C'\rangle$*, then*

- $OK_{n-1}(P', C')$ *if* $\rho = \mathbf{malloc}$*,*
- $OK_{n+1}(P', C')$ *if* $\rho = \mathbf{free}$*,*
- $OK_n(P', C')$ *if* $\rho =$ *Otherwise*

Proof. By induction on $\langle P, C\rangle \xrightarrow{\rho} \langle P', C'\rangle$.

- Case $P = \mathbf{0}; P'$ and $\langle \mathbf{0}; P', C\rangle \rightarrow \langle P', C\rangle$

  We need to prove $OK_n(P', C)$. Assume that $OK_n(P', C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P', C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n(\mathbf{0}; P', C)$ holds, we have if $\langle \mathbf{0}; P', C\rangle \rightarrow \langle P', C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which are in contradiction to the assumption $\sharp_m(\sigma) - \sharp_f(\sigma) > n$. Therefore, $OK_n(P', C)$ holds.

- Case $P = \mathbf{malloc}$ and $\langle \mathbf{malloc}, C\rangle \xrightarrow{\mathbf{malloc}} \langle 0, C\rangle$

  we need to prove $OK_{n-1}(0, C)$, which means we need to prove that for all σ and Q, if $\langle 0, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n - 1$. There is no σ and Q such that $\langle 0, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$. Therefore, $OK_{n-1}(0, C)$ holds.

- Case $P = \mathbf{free}$ and $\langle \mathbf{free}, C\rangle \xrightarrow{\mathbf{free}} \langle \mathbf{0}, C\rangle$

  We need to prove $OK_{n+1}(\mathbf{0}, C)$, which means we need to prove $\forall\sigma$ and Q if $\langle \mathbf{0}, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n+1$. There is no Q and σ s.t. $\langle \mathbf{0}, C\rangle \xrightarrow{\sigma} \langle Q, C\rangle$, so (1) holds. Therefore, $OK(\mathbf{0}, C)$ holds.

- Case $P = \mathbf{endconst}(*x)$ and $\frac{C' = filter(C, *x)}{\langle \mathbf{endconst}(*x), C\rangle \rightarrow \langle \mathbf{0}, C'\rangle}$

  We need to prove $OK_n(\mathbf{0}, C')$, which means we need to prove $\forall\sigma$ and Q if $\langle \mathbf{0}, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$ and (2) $OK(C')$ holds. There is no Q and σ s.t. $\langle \mathbf{0}, C\rangle \xrightarrow{\sigma} \langle Q, C\rangle$. So $OK_n(\mathbf{0}, C')$ holds.

- Case $P = (x)P'$ and $\frac{x' is fresh}{\langle (x)P', C\rangle \rightarrow \langle [x'/x]P', C\rangle}$

  We need to prove $OK_n([x'/x]P', C)$. Assuming that $OK_n([x'/x]P', C)$ does not hold. Then we have $\exists\sigma$ and Q s.t. $\langle [x'/x]P', C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of $OK_n((x)P', C)$, we have if $\langle (x)P', C\rangle \rightarrow \langle [x'/x]P', C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$. Therefore we get the contradiction.

  Therefore $OK_n([x'/x]P', C)$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\frac{\mathbf{const}(*x) \notin C}{\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1, C\rangle}$

  We need to prove $OK_n(P_1, C)$. Assume that $OK_n(P_1, C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P_1, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption $\sharp_m(\sigma) - \sharp_f(\sigma) > n$. Therefore, $OK_n(P_1, C)$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\frac{\mathbf{const}(*x) \notin C}{\langle (*x)(P_1, P_2), C\rangle \rightarrow \langle P_2, C\rangle}$

  We need to prove $OK_n(P_2, C)$. Assume that $OK_n(P_2, C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P_2, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\neg\mathbf{null}(*x)} \langle P_2, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption. Therefore, $OK_n(P_2, C)$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\frac{\mathbf{null}(*x) \in C \qquad \mathbf{const}(*x) \in C}{\langle (*x)(P_1, P_2), C\rangle \rightarrow \langle P_1, C\rangle}$

  We need to prove $OK_n(P_1, C)$. Assume that $OK_n(P_1, C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P_1, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \rightarrow \langle P_1, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption. Therefore, $OK_n(P_1, C)$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\frac{\neg\mathbf{null}(*x) \in C \qquad \mathbf{const}(*x) \in C}{\langle (*x)(P_1, P_2), C\rangle \rightarrow \langle P_2, C\rangle}$

We need to prove $OK_n(P_2, C)$. Assume that $OK_n(P_2, C)$ does not hold. Then we have $\exists\sigma$ and Q s.t. $\langle P_2, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \rightarrow \langle P_2, C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption. Therefore, $OK_n(P_2, C)$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\dfrac{\mathbf{null}(*x), \neg\mathbf{null}(*x) \notin C \qquad \mathbf{const}(*x) \in C}{\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1, C \cup \{\mathbf{null}(*x)\}\rangle}$

  We need to prove $OK_n(P_1, C \cup \{\mathbf{null}(*x)\})$. Assume that $OK_n(P_1, C \cup \{\mathbf{null}(*x)\})$ does not hold. Then we have $\exists\sigma$ and Q s.t. $\langle P_1, C \cup \{\mathbf{null}(*x)\}\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n((*x)(P_1, P_2), C)$ holds, we have if $\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P_1, C \cup \{\mathbf{null}(*x)\}\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$. Therefore, we get the contradiction and $OK_n(P_1, C \cup \{\mathbf{null}(*x)\})$ holds.

- Case $P = (*x)(P_1, P_2)$ and $\dfrac{\mathbf{null}(*x), \neg\mathbf{null}(*x) \notin C \qquad \mathbf{const}(*x) \in C}{\langle (*x)(P_1, P_2), C\rangle \xrightarrow{\neg\mathbf{null}(*x)} \langle P_2, C \cup \{\neg\mathbf{null}(*x)\}\rangle}$

  Similar to the above.

- Case $P = \mathbf{const}(*x)P'$ and $\langle \mathbf{const}(*x)P', C\rangle \rightarrow \langle P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x)\rangle$

  We need to prove $OK_n(P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x))$. Assume that $OK_n(P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x))$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x)\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n(\mathbf{const}(*x)P', C)$ holds, we have if $\langle \mathbf{const}(*x)P', C\rangle \rightarrow \langle P; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x)\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is in contradiction to the assumption. Therefore, $OK_n(P'; \mathbf{endconst}(*x), C \cup \mathbf{const}(*x))$ holds.

- Case $P = \mu\alpha.P'$ and $\langle \mu\alpha.P', C\rangle \rightarrow \langle [\mu\alpha.P'/\alpha]P', C\rangle$

  We need to prove $OK_n([\mu\alpha.P'/\alpha]P', C)$. Assume that $OK_n([\mu\alpha.P'/\alpha]P', C)$ does not hold. Then, we have $\exists\sigma$ and Q s.t. $\langle [\mu\alpha.P'/\alpha]P', C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n$.

  From the definition of that $OK_n(\mu\alpha.P', C)$ holds, we have if $\langle \mu\alpha.P', C\rangle \rightarrow \langle [\mu\alpha.P'/\alpha]P', C\rangle \xrightarrow{\sigma} \langle Q, C'\rangle$, then $\sharp_m(\sigma) - \sharp_f(\sigma) \leq n$, which is a contradiction. Therefore, $OK([\mu\alpha.P'/\alpha]P', C)$ holds.

- Case $P = P_1; P_2$ and $\dfrac{\langle P_1, C\rangle \xRightarrow{\rho} \langle P_1', C'\rangle}{\langle P_1; P_2, C\rangle \xRightarrow{\rho} \langle P_1'; P_2, C'\rangle}$

  We need to prove $OK_{n'}(P_1'; P_2, C)$, where n' is determined by

  $$n' = \begin{cases} n+1 & \rho = \mathbf{free} \\ n-1 & \rho = \mathbf{malloc} \\ n & \text{Otherwise.} \end{cases}$$

  Assume that $OK_{n'}(P_1'; P_2, C')$ does not hold. Then, we have $\exists\sigma$, Q and C'' s.t. $\langle P_1'; P_2, C\rangle \xrightarrow{\sigma} \langle Q, C''\rangle$ and $\sharp_m(\sigma) - \sharp_f(\sigma) > n'$.

  From the definition of that $OK_n(P_1; P_2, C)$ holds, we have if $\langle P_1; P_2, C\rangle \xRightarrow{\rho} \langle P_1'; P_2, C'\rangle \xrightarrow{\sigma} \langle Q, C''\rangle$, then $\sharp_m(\rho\sigma) - \sharp_f(\rho\sigma) \leq n$.

Then we get $n' + \sharp_m(\rho) - \sharp_f(\rho) < \sharp_m(\rho) + \sharp_m(\sigma) - \sharp_f(\rho) - \sharp_f(\sigma) \le n$. For any ρ, the $n' + \sharp_m(\rho) - \sharp_f(\rho) = n$, therefore we get a contradiction. Therefore, $OK_{n'}(P_1; P_2, F')$ holds.

$\square$

Lemma 4.2. *If* $consistency(H, R, C)$ *and* $\langle H, R, s, n, C\rangle \xrightarrow{\rho} \langle H', R', s', n', C'\rangle$*, then* $consistency(H', R', C')$*.*

Proof. By induction on $\langle H, R, s, n, C\rangle \xrightarrow{\rho} \langle H', R', s', n', C'\rangle$

- Case: $\langle H, R, \mathbf{const}(*y)s, n, C\rangle \to \langle H, R, s; \mathbf{endconst}(*y), n', C \cup \mathbf{const}(*y)\rangle$.

  We need to prove $consistency(H, R, C \cup \mathbf{const}(*y))$. Form assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H, R, C \cup \mathbf{const}(*y))$, we chose z arbitrarily. (3) Assuming $\mathbf{null}(*z) \in C \cup \mathbf{const}(*y)$. This implies $\mathbf{null}(*z) \in C$. By using (1), we have $\mathbf{const}(*z) \in C$, and this implies $\mathbf{const}(*z) \in C \cup \mathbf{const}(*y)$. Assuming $H(R(z))$ is defined, then $H(R(z)) = null$ from (1). (4) Assuming $\neg\mathbf{null}(*z) \in C \cup \mathbf{const}(*y)$. This implies $\neg\mathbf{null}(*z) \in C$. By using (2), we have $\mathbf{const}(*z) \in C$. This implies $\mathbf{const}(*z) \in C \cup \mathbf{const}(*y)$. Assuming $H(R(z))$ is defined, then $H(R(z)) \neq null$ from (2).

  Therefore, $consistency(H, R, C \cup \mathbf{const}(*y))$ holds.

- Case: $\langle H, R, \mathbf{endconst}(*y), n, C\rangle \to \langle H, R, skip, n, C'\rangle$ where $C' = filter(C, *y)$.

  We need to prove $consistency(H, R, C')$ where $C' = filter(C, *y)$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H, R, C')$, we chose z arbitrarily. From the definition of function $filter(C, *y)$, we know (3) if $\mathbf{const}(*y) \in (C - \mathbf{const}(*y))$ then we have $C' = C - \mathbf{const}(*y)$, otherwise we have $C' = (C - \mathbf{const}(*y)) \backslash \{\mathbf{null}(*x), \neg\mathbf{null}(*x)\}$. From (3) we have $C' \in C$. (4) Assuming $\mathbf{null}(*z) \in C'$, this implies $\mathbf{null}(*z) \in C$. From (1) we have $\mathbf{const}(*z) \in C$. Now we want to get $\mathbf{const}(*z) \in C'$. We should consider two cases: $z \neq y$ and $z = y$. If $z \neq y$ then we have $\mathbf{const}(*z) \in C'$ from (3); if $z = y$, and because $\mathbf{null}(*z) \in C'$, then we have $\mathbf{const}(*z) \in C'$. Assuming $H(R(z))$ is defined, then we have $H(R(z)) = null$ from (1). (5) Assuming $\neg\mathbf{null}(*z) \in C'$. The similar to (4).

  Therefore, $consistency(H, R, C')$ holds.

- Case: $\langle H, R, \mathbf{free}(y), n, C\rangle \xrightarrow{\mathbf{free}} \langle H \backslash R(y), R, skip, n+1, C\rangle$.

  We need to prove $consistency(H \backslash R(y), R, C$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H \backslash R(y), R, C$, we chose z arbitrarily. (3) Assuming $\mathbf{null}(*z) \in C$. By using (1), we have $\mathbf{const}(*z) \in C$. Assuming $H(R(z))$ is defined, we have $H(R(z)) = null$ from (1). We know that y has been deallocated from H, so if $z = y$ then $H \backslash R(y)(R(z))$

is not defined, otherwise $H\backslash R(y)(R(z))$ is defined and $H\backslash R(y)(R(z)) = H(R(z)) = null$. (4) Assuming $\mathbf{null}(*z) \in C$, By using (2), we have $\mathbf{const}(*z) \in C$. Assuming $H(R(z))$ is defined, we have $H(R(z)) \neq null$ from (2). We know that y has been deallocated from H, so if $z = y$ then $H\backslash R(y)(R(z))$ is not defined, otherwise $H\backslash R(y)(R(z))$ is defined and $H\backslash R(y)(R(z)) = H(R(z)) \neq null$.

Therefore, $consistency(H\backslash R(y), R, C$ holds.

- Case: $\langle H, R, \mathbf{let}\ y = \mathbf{malloc\ in}\ s, n, C\rangle \xrightarrow{\mathbf{malloc}} \langle H\{l \mapsto v\}, R\{x' \mapsto l\}, [x'/y]s, n', C\rangle$ where $x' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$

  We need to prove $consistency(H\{l \mapsto v\}, R\{x' \mapsto l\}, C)$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H\{l \mapsto v\}, R\{x' \mapsto l\}, C)$, we chose z arbitrarily. (3) Assuming $\mathbf{null}(*z) \in C$. From (1) we have $\mathbf{const}(*z) \in C$. Assuming $H(R(z))$ is defined, we have $H(R(z)) = null$ from (1). We have $x' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$, so $z \neq x'$. Therefore we get $H\{l \mapsto v\}(R\{x' \mapsto l\}z) = H(R(z)) = null$. (4) Assuming $\neg\mathbf{null}(*z) \in C$. similar to (3).

  Therefore, $consistency(H\{l \mapsto v\}, R\{x' \mapsto l\}, C)$ holds.

- Case: $\langle H, R, skip; s, n, C\rangle \rightarrow \langle H, R, s, n', C\rangle$.

  Obviously, $consistency(H, R, C)$ holds form assumption.

- Case: $\langle H\{R(w) \mapsto v\}, R, *w \leftarrow y, n, C\rangle \rightarrow \langle H\{R(w) \mapsto R(y)\}, R, skip, n, C\rangle$ where $\forall z.R(w) = R(z) \Rightarrow \mathbf{const}(*z) \notin C$

  We need to prove $consistency(H\{R(w) \mapsto R(y)\}, R, C)$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H\{R(w) \mapsto v\}(R(x))$ is defined then $H\{R(w) \mapsto v\}(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H\{R(w) \mapsto v\}(R(x))$ is defined then $H\{R(w) \mapsto v\}(R(x)) = null$.

  To prove $consistency(H\{R(w) \mapsto R(y)\}, R, C)$, we chose m arbitrarily. (3) Assuming $\mathbf{null}(*m) \in C$. By using (1), we have $\mathbf{const}(*m) \in C$. Then assuming $H\{R(w) \mapsto v\}(R(m))$ is defined, we have $H\{R(w) \mapsto v\}(R(m)) = null$ from (1). Because we know $\forall z.R(w) = R(z) \Rightarrow \mathbf{const}(*z) \notin C$ and $\mathbf{const}(*m) \in C$, we have $m \neq w$, then we have $H\{R(w) \mapsto R(y)\}(R(m)) = H\{R(w) \mapsto v\}(R(m)) = null$. (4) Assuming $\neg\mathbf{null}(*m) \in C$. Similar to (3).

  Therefore, $consistency(H\{R(w) \mapsto R(y)\}, R, C)$ holds.

- Case: $\langle H, R, \mathbf{let}\ z = y\ \mathbf{in}\ s, n, C\rangle \rightarrow \langle H, R\{z' \mapsto R(y)\}, [z'/z]s, n, C\rangle$ where $z' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$

  We need to prove $consistency(H, R\{z' \mapsto R(y)\}, C$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H, R\{z' \mapsto R(y)\}, C$, we chose m arbitrarily. (3) Assuming $\mathbf{null}(*m) \in C$. By using (1), we have $\mathbf{const}(*m) \in C$. Then assuming $H(R(m))$ is defined, we have

$H(R(m)) = null$ from (1). Because we have $z' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$, we have $m \neq z'$, then we have $H(R\{z' \mapsto R(y)\}(m)) = H(R(m)) = null$. (4) Assuming $\neg\mathbf{null}(*m) \in C$. By using (2), we have $\mathbf{const}(*m) \in C$. Then assuming $H(R(m))$ is defined, we have $H(R(m)) \neq null$ from (2). Because we have $z' \notin \mathbf{Dom}(H) \cup \mathbf{Dom}(R) \cup fv(C)$, we have $m \neq z'$, then we have $H(R\{z' \mapsto R(y)\}(m)) = H(R(m)) \neq null$.

Therefore, $consistency(H, R\{z' \mapsto R(y)\}, C)$ holds.

- Case: $\langle H, R, \mathbf{ifnull}\ (*y)\ \mathbf{then}\ s_1\ \mathbf{else}\ \ s_2, n, C\rangle \xrightarrow{\mathbf{null}(*y)} \langle H, R, s_1, n, C\rangle$ where $H(R(y)) = null$ and $\mathbf{const}(*y) \notin C$

  Obviously, $consistency(H, R, C)$ holds from assumption.

- Case: $\langle H, R, \mathbf{ifnull}\ (*y)\ \mathbf{then}\ s_1\ \mathbf{else}\ \ s_2, n, C\rangle \xrightarrow{\neg\mathbf{null}(*y)} \langle H, R, s_2, n, C\rangle$ where $H(R(y)) \neq null$ and $\mathbf{const}(*y) \notin C$

  Obviously, $consistency(H, R, C)$ holds from assumption.

- Case: $\langle H, R, \mathbf{ifnull}\ (*y)\ \mathbf{then}\ s_1\ \mathbf{else}\ \ s_2, n, C\rangle \xrightarrow{\mathbf{null}(*y)} \langle H, R, s_1, n, C \cup \mathbf{null}(*y)\rangle$ where $H(R(y)) = null$ and $\mathbf{const}(*y) \in C$

  We need to prove $consistency(H, R, C \cup \mathbf{null}(*y)$. From assumption $consistency(H, R, C)$, we have (1) $\forall x.$if $\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) = null$ and (2) $\forall x.$if $\neg\mathbf{null}(*x) \in C$, then $\mathbf{const}(*x) \in C$ and if $H(R(x))$ is defined then $H(R(x)) \neq null$.

  To prove $consistency(H, R, C \cup \mathbf{null}(*y)$, we chose z arbitrarily. (3) Assuming $\mathbf{null}(*z) \in C \cup \mathbf{null}(*y)$. This implies $\mathbf{null}(*z) \in C$. By using (1), we have $\mathbf{const}(*z) \in C$. This implies $\mathbf{const}(*z) \in C \cup \mathbf{null}(*y)$. Assuming $H(R(z))$ is defined, then we have $H(R(z)) = null$ from (1). (4) Assuming $\neg\mathbf{null}(*z) \in C \cup \mathbf{null}(*y)$. This implies $\neg\mathbf{null}(*z) \in C$. By using (2), we have $\mathbf{const}(*z) \in C$. This implies $\mathbf{const}(*z) \in C \cup \mathbf{null}(*y)$. Assuming $H(R(z))$ is defined, then we have $H(R(z)) \neq null$ from (2).

  Therefore, $consistency(H, R, C \cup \mathbf{null}(*y))$ holds.

$\square$

Proof of Lemma 2.2: By induction on the derivation of $\langle H, R, s, n, C\rangle \xrightarrow{\rho} \langle H', R', s', n', C'\rangle$.

- Case: $\langle H, R, \mathbf{const}(*x)s, n, C\rangle \rightarrow \langle H, R, s; \mathbf{endconst}(*x), n, C \cup \{\mathbf{const}(*x)\}\rangle$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, \mathbf{const}(*x)s, n, C\rangle : \langle P, C\rangle$, we have $\Theta; \Gamma \vdash \mathbf{const}(*x)s : P$, $OK_n(P, C)$ and $consistency(H, R, C)$. From the inversion of typing rules, we get $\Theta; \Gamma \vdash s : P''$ and $\mathbf{const}(*x)P'' \leq P$ for some P''. By subtyping, we have $P''; \mathbf{endconst}(*x) \leq Q$ and $\langle P, C\rangle \Longrightarrow \langle Q, C \cup \{\mathbf{const}(*x)\}\rangle$ for some Q.

  we need to find P' and C' s.t. $\Theta; \Gamma \vdash s; \mathbf{endconst}(*x) : P'$, $OK_n(P', C')$, $\langle P, C'\rangle \Longrightarrow \langle P', C'\rangle$ and $consistency(H, R, C')$. Taking Q as P' and $C \cup \{\mathbf{const}(*x)\}$ as C'. Therefore $\langle P, C\rangle \rightarrow \langle P', C'\rangle$ holds, and then $OK_n(P', C')$ and $consistency(H, R, C')$ hold from Lemma 4.1 and Lemma 4.2. From $\Theta; \Gamma \vdash s; \mathbf{endconst}(*x) : P''; \mathbf{endconst}(*x)$, $P''; \mathbf{endconst}(*x) \leq Q$ and T-Sub, $\Theta; \Gamma \vdash s; \mathbf{endconst}(*x) : P'$ holds.

- Case: $\langle H, R, \mathbf{endconst}(*x), n, C\rangle \to \langle H, R, \mathbf{skip}, n, C'\rangle$ where $C' = \mathit{filter}(C, *x)$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, \mathbf{endconst}(*x), n, C\rangle : \langle P, C\rangle$, we have $\Theta; \Gamma \vdash \mathbf{endconst}(*x):$ P, $OK_n(P, C)$ and $\mathit{consistency}(H, R, C)$. From the inversion of typing rules, we get $\Theta; \Gamma \vdash$ $\mathbf{endconst}(*x) : \mathbf{endconst}(*x)$ and $\mathbf{endconst}(*x) \leq P$. By subtyping , we get $0 \leq Q$ and $\langle P, C\rangle \to \langle Q, C\rangle$ for some Q.

  we need to find P' and C' s.t. $\Theta; \Gamma \vdash \mathbf{skip} : P'$, $OK_n(P', C')$, $\langle P, C\rangle \implies P', C'\rangle$ and $\mathit{consistency}(H, R, C')$. Taking Q as P' and C as C', then $\langle P, C\rangle \to \langle P', C'\rangle$ holds, and then $OK_n(P', C')$ and $\mathit{consistency}(H, R, C')$ hold from Lemma 4.1 and Lemma 4.2 . From T-Skip, T-Sub and $0 \leq Q$, then $\Theta; \Gamma \vdash \mathbf{skip} : P'$ holds.

- Case: $\langle H, R, \mathbf{free}(x)\ , n, C\rangle \xrightarrow{\mathbf{free}} \langle H', R, \mathbf{skip}, n+1, C\rangle$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, \mathbf{free}(x)\ , n, C\rangle : \langle P, C\rangle$, we have $OK_n(P, C)$, $\mathit{consistency}(H, R, C)$ and $\Theta; \Gamma \vdash \mathbf{free}(x) : P$. From inversion of the typing rules, we have $\Theta; \Gamma \vdash \mathbf{free}(x) : \mathbf{free}$ and $\mathbf{free} \leq P$. By the subtyping, we have $\langle P, F\rangle \xRightarrow{\mathbf{free}} \langle Q, C\rangle$ and $\mathbf{0} \leq Q$ for some Q.

  We need to find P' and C' such that $\langle P, C\rangle \xRightarrow{\mathbf{free}} \langle P', C'\rangle$, $\Theta; \Gamma \vdash \mathbf{skip} : P'$, and $OK_{n+1}(P', C')$. Take Q as P' and C as C'. Then, $\langle P, C\rangle \xRightarrow{\mathbf{free}} \langle P', C'\rangle$ holds, and $OK_{n+1}(P', C')$ holds from Lemma 4.1. We also have $\Theta; \Gamma \vdash \mathbf{skip} : P'$ from T-Skip, $\mathbf{0} \leq Q$ and T-Sub.

- Case: $\langle H, R, \mathbf{let}\ x = \mathbf{malloc}()\ \mathbf{in}\ s, n, C\rangle \xrightarrow{\mathbf{malloc}} \langle H', R', [x'/x]s, n-1, C\rangle$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, \mathbf{let}\ x = \mathbf{malloc}()\ \mathbf{in}\ s, n, C\rangle : \langle P, C\rangle$, we have $\Theta; \Gamma \vdash$ $\mathbf{let}\ x = \mathbf{malloc}()\ \mathbf{in}\ s : P$, $OK_n(P, C)$ and $\mathit{consistency}(H, R, C)$. By the inversion of typing rules, we have $\Theta; \Gamma, x \vdash s : P''$ and $\mathbf{malloc}; (x)P'' \leq P$ for some P''. By subtyping, we get $\langle P, C\rangle \xRightarrow{\mathbf{malloc}} \langle Q, F\rangle$ and $[x'/x]P'' \leq Q$ for some Q.

  We need to find P' and C' such that $\Theta; \Gamma, x' \vdash [x'/x]s : P'$, $\langle P, C\rangle \xRightarrow{\mathbf{malloc}} \langle P', C'\rangle$, $\mathit{consistency}(H', R', C')$ and $OK_{n-1}(P', C')$. Take Q as P' and C as C'. Then $\langle P, C\rangle \xRightarrow{\mathbf{malloc}} \langle P', C'\rangle$ holds, and then $OK_{n-1}(P', C')$ and $\mathit{consistency}(H', R', C')$ hold by Lemma 4.1 and Lemma 4.2. From $\Theta; \Gamma, x \vdash s : P''$ and $\mathbf{malloc}; (x)P'' \leq P$, we have $\Theta; \Gamma, x'' \vdash [x''/x]s : [x''/x]P''$ and $\mathbf{malloc}; (x)P'' \leq P$, and then by the definition of subtyping we have $[x''/x]P'' \leq Q'$ for some Q'. Therefore, we get $\Theta; \Gamma, x'' \vdash [x''/x]s : Q'$. Take x'' as x' and Q' as P', then $\Theta; \Gamma, x' \vdash [x'/x]s : P'$ holds.

- Case: $\langle H, R, \mathbf{skip}; s, n, C\rangle \to \langle H, R, s, n, C\rangle$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, \mathbf{skip}; s, n, C\rangle : \langle P, C\rangle$, we have $\Theta; \Gamma \vdash \mathbf{skip}; s : P$, $OK_n(P, C)$ and $\mathit{consistency}(H, R, C)$. From the inversion of the typing rules, we get $\Theta; \Gamma \vdash$ $s : P''$ and $0; P'' \leq P$. From the definition of subtyping, we have $\langle P, C\rangle \implies \langle Q, C\rangle$ and $P'' \leq Q$ for some Q.

  We need to find P' and C' such that $\Theta; \Gamma \vdash s : P'$ and $\langle P, C\rangle \to \langle P', C'\rangle$ and $OK_n(P', C')$. Take Q as P' and C as C'. Then $\langle P, C\rangle \implies \langle P', C'\rangle$ holds, and then $OK_n(P', C')$ and $\mathit{consistency}(H, R, C')$ hold. We also have $\Theta; \Gamma \vdash s : P'$ from T-Sub, $\Gamma \vdash s : P''$ and $P'' \leq Q$.

- Case: $\langle H, R, *x \leftarrow y, n, C\rangle \to \langle H', R, \mathbf{skip}, n, C\rangle$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, *x \leftarrow y, n, C\rangle : \langle P, C\rangle$, we have $\Theta; \Gamma \vdash *x \leftarrow y : P$, $OK_n(P, C)$ and $\mathit{consistency}(H, R, C)$. From the inversion of typing rules, we have $0 \leq P$.

We need to find P' and C' such that $\Theta;\Gamma \vdash \textbf{skip} : P'$, $\langle P, C\rangle \Longrightarrow \langle P', C'\rangle$ and $OK_n(P', C')$. Take P as P' and C as C'. Then $\langle P, C\rangle \Longrightarrow \langle P', C'\rangle$ holds, and then $OK_n(P', C')$ and $consistency(H', R, C')$ hold from Lemma 4.1 and Lemma 4.2. We also have $\Theta;\Gamma \vdash \textbf{skip} : P'$ from T-SKIP, $0 \leq P$ and T-SUB.

- Case: $\langle H, R, \textbf{let}\ x = y\ \textbf{in}\ s, n, C\rangle \to \langle H, R', [x'/x]s, n, C\rangle$

  From the assumption $\Theta;\Gamma \vdash \langle H, R, \textbf{let}\ x = y\ \textbf{in}\ s, n, C\rangle : \langle P, C\rangle$, we have $\Theta;\Gamma, y \vdash \textbf{let}\ x = y\ \textbf{in}\ s : P$, $OK_n(P, C)$ and $consistency(H, R, C)$. From the inversion of typing rules, we have $\Theta;\Gamma, x, y \vdash s : P''$ and $\textbf{let}\ x = y\ \textbf{in}\ P'' \leq P$ for some P''. By subtying, we have $\langle P, C\rangle \to \langle Q, C\rangle$ and $[x'/x]P'' \leq Q$ for some Q.

  We need to find P' and C' such that $\Theta;\Gamma, x', y \vdash [x'/x]s : P'$, $\langle P, C\rangle \to \langle P', C'\rangle$, $OK_n(P', C')$ and $consistency(H, R', C')$. Take Q as P' and C as C'. Then $\langle P, C\rangle \Longrightarrow \langle P', C'\rangle$ and $OK_n(P', C')$ hold. From $\Theta;\Gamma, x, y \vdash s : P''$ and $\textbf{let}\ x = y\ \textbf{in}\ P'' \leq P$, we have $\Theta;\Gamma, x'', y \vdash [x''/x]s : [x''/x]P''$ and $\textbf{let}\ x'' = y\ \textbf{in}\ [x''/x]P'' \leq P$, and then by subtying we have $[x''/x]P'' \leq Q'$ for some Q'. Therefore, we have $\Theta;\Gamma, x'', y \vdash [x''/x]s : Q'$. Take x'' as x' and Q' as P', then $\Theta;\Gamma, x', y \vdash [x'/x]s : P'$ holds.

- Case: $\langle H, R, \textbf{let}\ x = \textbf{null}\ \textbf{in}\ \ s, n\rangle \to \langle H, R', [x'/x]s, n\rangle$

  Similar to the above.

- Case: $\langle H, R, \textbf{let}\ x = *y\ \textbf{in}\ \ s, n\rangle \to \langle H, R', [x'/x]s, n\rangle$

  Similar to the above.

- Case: $\langle H, R, \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ \ s_2, n, C\rangle \xrightarrow{\textbf{null}(*x)} \langle H, R, s_1, n, C\rangle$ if $H(R(x)) = \textbf{null}$ and $\textbf{const}(*x) \notin C$

  From assumption $\Theta;\Gamma \vdash \langle H, R, \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ \ s_2, n, C\rangle : \langle P, C\rangle$, we have $\Theta;\Gamma \vdash \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ \ s_2 : P$, $OK_n(P, C)$ and $consistency(H, R, C)$. From the inversion of typing rules, we have $\Theta;\Gamma \vdash s_1 : P_1$, $\Theta;\Gamma \vdash s_2 : P_2$ and $(*x)(P_1, P_2) \leq P$. According to the rule Tr-NotConst1 and $\textbf{const}(*x) \notin C$, we have $\langle (*x)(P_1, P_2)\rangle \xrightarrow{\textbf{null}(*x)} \langle P_1, C\rangle$, and then by definition of subtyping, we get $\langle P, C\rangle \xRightarrow{\textbf{null}(*x)} \langle Q, C\rangle$ and $P_1 \leq Q$ for some Q.

  We need to find P' and C' such that $\Theta;\Gamma \vdash s_1 : P'$, $\langle P, C\rangle \xRightarrow{\textbf{null}(*x)} \langle P', C'\rangle$ and $OK_n(P', C')$. Take Q as P' and C as C'. Then $\langle P, C\rangle \xRightarrow{\textbf{null}(*x)} \langle P', C'\rangle$ and $OK_n(P', C')$ hold. We also have $\Theta;\Gamma \vdash s_1 : P'$ from T-SUB, $\Theta;\Gamma \vdash s_1 : P_1$ and $P_1 \leq Q$.

- Case: $\langle H, R, \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ \ s_2, n, C\rangle \xrightarrow{\neg\textbf{null}(*x)} \langle H, R, s_1, n, C\rangle$ if $H(R(x)) \neq \textbf{null}$ and $\textbf{const}(*x) \notin C$

  Similar to the above.

- Case: $\langle H, R, \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ \ s_2, n, C\rangle \xrightarrow{\textbf{null}(*x)} \langle H, R, s_1, n, C'\rangle$ if $H(R(x)) = \textbf{null}$, $\textbf{const}(*x) \in C$ and $C' = C \cup \{\textbf{null}(*x)\}$

  From assumption $\Theta;\Gamma \vdash \langle H, R, \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ \ s_2, n, C\rangle : \langle P, C\rangle$, we have $\Theta;\Gamma \vdash \textbf{ifnull}\ (*x)\ \textbf{then}\ s_1\ \textbf{else}\ \ s_2 : P$, $OK_n(P, C)$ and $consistency(H, R, C)$. From the inversion of typing rules, we have $\Theta;\Gamma \vdash s_1 : P_1$, $\Theta;\Gamma \vdash s_2 : P_2$ and $(*x)(P_1, P_2) \leq P$. According to rule Tr-NNullNotIn1, $\textbf{const}(*x) \in C$ and $C' = C \cup \textbf{null}(*x)$, we have $\langle (*x)(P_1, P_2)\rangle \xrightarrow{\textbf{null}(*x)} \langle P_1, C \cup$

$\mathbf{null}(*x)\rangle$, and then by the definition of subtyping, we get $\langle P, C\rangle \xRightarrow{\mathbf{null}(*x)} \langle Q, C \cup \{\mathbf{null}(*x)\}\rangle$ and $P_1 \leq Q$ for some Q.

We need to find P' and C' such that $\Theta; \Gamma \vdash s_1 : P'$, $\langle P, C\rangle \xRightarrow{\mathbf{null}(*x)} \langle P', C'\rangle$, $OK_n(P', C')$ and *consistency*(H, R, C'). Take Q as P' and $C \cup \{\mathbf{null}(*x)\}$ as C'. Then $\langle P, C\rangle \xrightarrow{\mathbf{null}(*x)} \langle P', c'\rangle$ holds, and then $OK_n(P', C')$ and *consistency*(H, R, C')hold by Lemma 4.1 and Lemma 4.2. We also have $\Theta; \Gamma \vdash s_1 : P'$ from T-SUB, $\Theta; \Gamma \vdash s_1 : P_1$ and $P_1 \leq Q$.

- Case: $\langle H, R, \textbf{ifnull } (*x) \textbf{ then } s_1 \textbf{ else } s_2, n, C\rangle \xrightarrow{\neg\mathbf{null}(*x)} \langle H, R, s_2, n, C'\rangle$ if $H(R(x)) \neq \mathbf{null}$, $\mathbf{const}(*x) \in C$ and $C' = C \cup \{\neg\mathbf{null}(*x)\}$

  Similar to the above proof.

- Case: $\langle H, R, s_1; s_2, n, C\rangle \to \langle H', R', s_1'; s_2, n', C'\rangle$

  From the assumption $\Theta; \Gamma \vdash \langle H, R, s_1; s_2, n, C\rangle : \langle P, C\rangle$, we have $\Theta; \Gamma \vdash s_1; s_2 : P$, $OK_n(P, C)$ and *consistency*(H, R, C). By inversion of typing rules, we have $\Theta; \Gamma \vdash s_1 : P_1$, $\Theta; \Gamma \vdash s_2 : P_2$ and $P_1; P_2 \leq P$ for some P_1 and P_2.

  By IH on $\langle H, R, s_1, n, C\rangle$ with derivation $\langle H, R, s_1, n, C\rangle \xrightarrow{\rho} \langle H', R', s_1', n', C'\rangle$, we have $\exists P_1', C_1'$ s.t. $\Theta; \Gamma \vdash \langle H', R', s_1', n', C'\rangle : \langle P_1', C_1'\rangle$ and $\langle P_1, C\rangle \xrightarrow{\rho} \langle P_1', C_1'\rangle$.

  By subtyping we have $\langle P, C\rangle \xrightarrow{\rho} \langle Q, C_1'\rangle$ and $P_1'; P_2 \leq Q$ for some Q.

  We need to find P' and C' s.t. $\langle P, C\rangle \xrightarrow{\rho} \langle P', C'\rangle$, $OK_n(P', C')$ and $\Theta; \Gamma \vdash s_1'; s_2 : P'\rangle$. Take Q as P' and C_1' as C', $\langle P, C\rangle \xrightarrow{\rho} \langle P', C'\rangle$ and $OK_n(P', C')$ hold. By T-Sub, $\Theta; \Gamma \vdash s_1'; s_2 : P_1'; P_2$ and $P_1'; P_2 \leq Q$, we have $\Theta; \Gamma \vdash s_1'; s_2 : P'$ holds.

□

We write $\langle H, R, s, n, C\rangle \xrightarrow{\rho}$ if there is a transition $\xrightarrow{\rho}$ from $\langle H, R, s, n, C\rangle$.

Lemma 4.3. *If* $\Theta \vdash \langle H, R, s, n, C\rangle : \langle P, C\rangle$ *and* $\langle H, R, s, n, C\rangle \xRightarrow{\rho}$ *and* $\rho \in \{\mathbf{malloc}, \mathbf{free}, \mathbf{null}(*x), \neg\mathbf{null}(*x)\}$, *then there exists* P' *and* C' *such that* $\langle P, C\rangle \xRightarrow{\rho} \langle P', C'\rangle$.

Proof. Induction on the derivation of $\Theta; \Gamma \vdash \langle H, R, s, n, C\rangle : \langle P, C\rangle$. □

Proof of Lemma 2.3:
By contradiction. Assume $\langle H, R, s, n, C\rangle \xrightarrow{\rho}$ **OutOfMemory**. Then, n is 0 and $\rho = \mathbf{malloc}$ from SEM-OUTOFMEM. From the assumption we have $\Theta; \Gamma \vdash s : P$ and $OK_0(P, C)$. From Lemma 4.3, there exists P' and C' such that $\langle P, C\rangle \xRightarrow{\mathbf{malloc}} \langle P', C'\rangle$. However, this contradicts $OK_0(P, C)$.

□

Proof of Theorem 2.1:
We have $\Theta; \emptyset \vdash s : P, \vdash D : \Theta$, $OK_n(P, C)$ and *consistency*(H, R, C).
Suppose that there exists σ such that $\langle \emptyset, \emptyset, s, n, C\rangle \xrightarrow{\sigma} \langle H', R', s', n', C'\rangle \xrightarrow{\rho}$ **OutOfMemory**. Then, $n' = 0$ and $\rho = \mathbf{malloc}$. From Lemma 2.2, there exists P' and C' such that $\Theta; \Gamma' \vdash s' : P'$, $\langle P, C\rangle \xRightarrow{\sigma} \langle P', C'\rangle$, and $OK_0(P', C')$; hence $\langle H', R', s', 0\rangle \xrightarrow{\mathbf{malloc}}$. However, this contradicts Lemma 2.3.

□ ¿¿¿¿¿¿¿ cbf9f98628ddf24a08c69bd0bc0f8a8c90557969