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Twin-T Notch sensor for the measurement of the complex permittivity of snow

Transfer function calculation

Stefan Dierer, AEE15, April 2017

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FH JOANNEUM Gesellschaft mbH
Alte Poststraße 149, A-8020 Graz, AUSTRIA, T.: +43 (0)316 5453-8800, E.: info@fh-joanneum.at, www.fh-joanneum.at

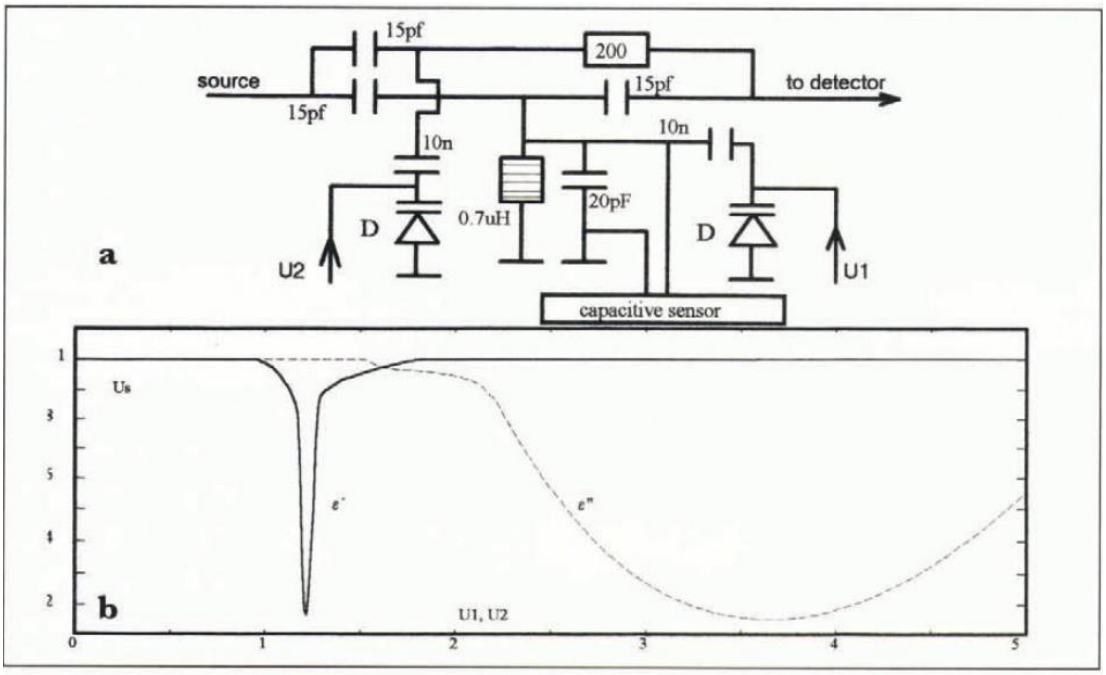

Figure 1: Tunable Twin-T bridge, used for permittivity measurement of snow (A. Denoth, 1994).

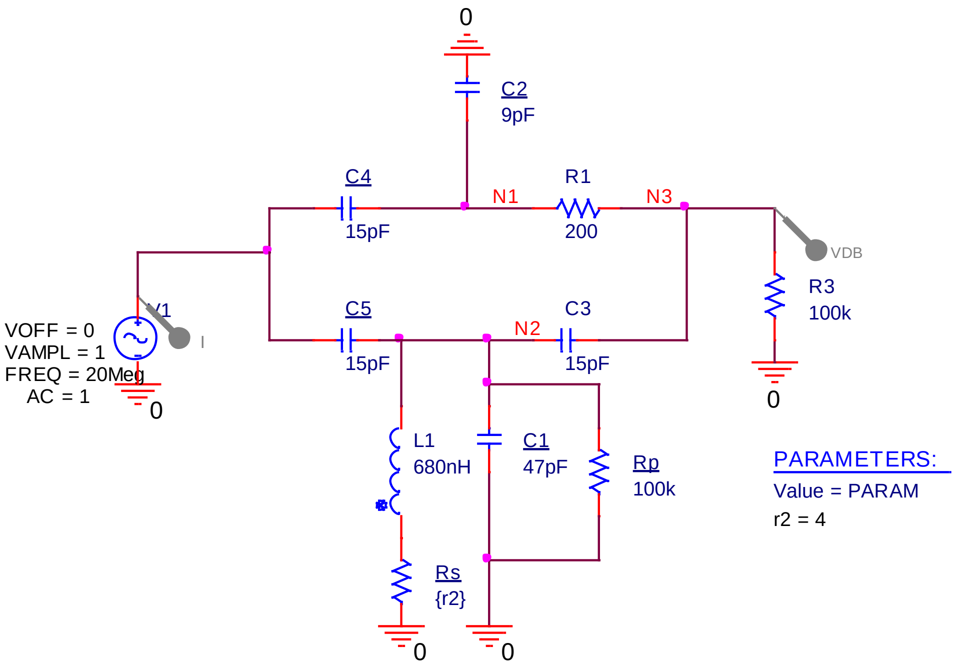

Figure 2:Twin-T bridge PSpice simulation schematic. The 10nF coupling capacitors were neglected.

$$C_1 = C_{varicap} + C_{fix} + C'_{sensor} = 47\,pF \quad \text{with} \quad \underline{C}_{sensor} = \varepsilon_0 \left(\varepsilon_r' - j\varepsilon_r''\right) k$$

$$\underline{Y}_{sensor} = j\omega\varepsilon_0 \left(\varepsilon_r' - j\varepsilon_r''\right) k = j\omega C_0 \varepsilon_r' + \omega C_0 \varepsilon_r'' = j\omega \underbrace{C_0 \varepsilon_r'}_{C'_{sensor}} + \frac{1}{R_p}$$

$$R_p = \frac{1}{\omega C_0 \varepsilon_r''} \qquad \Delta C_1 = C_0\varepsilon_r' - C_0 \Rightarrow \varepsilon_r' = 1 + \frac{\Delta C_1}{C_0}$$

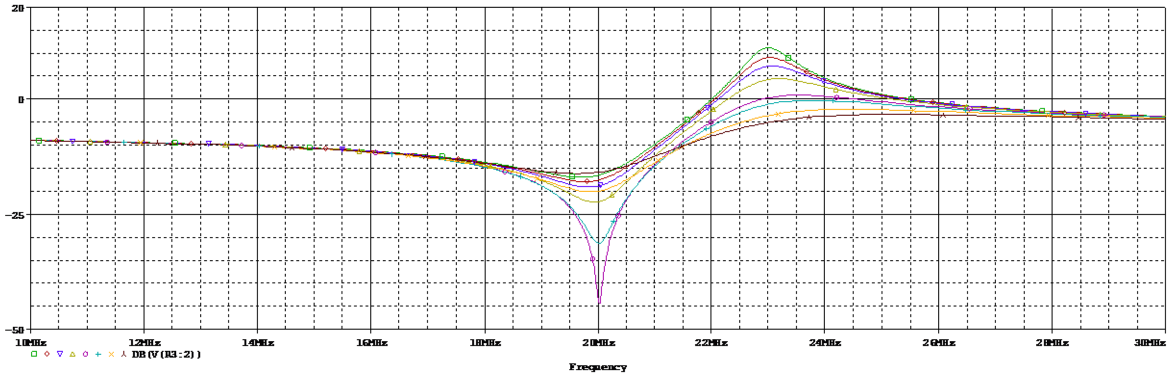

Figure 3: Bode plot of schematic in Figure 2 with different Rs resistors (0.1, 1, 2, 4, 8, 10, 15, 20 Ω).

Calculation of the transfer function

Node equations: $C_3 = C_4 = C_5 = C$, $R_1 = R$

N1: $$(V_{in} - V_1)sC + \frac{V_{out} - V_1}{R} - V_1 sC_2 = 0$$

N2: $$(V_{in} - V_2)sC + (V_{out} - V_2)sC - \frac{V_2}{Z} = 0$$

N3: $$\frac{V_{out} - V_1}{R} + (V_{out} - V_2)sC = 0$$ with neglected $R_3 = R_{load}$ for $R_{load} \geq 100k\Omega$

N_1 – isolate V_1: $$V_{in}sC - V_1 sC + \frac{V_{out}}{R} - \frac{V_1}{R} - V_1 sC_2 = 0$$

$$V_{in}sC + \frac{V_{out}}{R} = V_1 sC + \frac{V_1}{R} + V_1 sC_2 = 0$$

$$\boxed{V_1 = \frac{V_{in}sC + \frac{V_{out}}{R}}{sC + \frac{1}{R} + sC_2}}$$

N_2 – isolate V_2: $$\boxed{V_2 = \frac{V_{in}sC + V_{out}sC}{2sC + \frac{1}{Z}}}$$

N_3 – insert V_1 and V_2: $$\frac{V_{out}}{R} - \frac{V_1}{R} + V_{out}sC - V_2 sC = 0$$

$$\frac{V_{out}}{R}-\frac{V_{in}sC+\frac{V_{out}}{R}}{sC+\frac{1}{R}+sC_2}\frac{1}{R}+V_{out}sC-\frac{V_{in}sC+V_{out}sC}{2sC+\frac{1}{Z}}sC=0$$

Separate V_{in} and V_{out}:

$$\frac{V_{in}sC}{\frac{1}{R}+s(C+C_2)}\frac{1}{R}+\frac{V_{in}s^2C^2}{2sC+\frac{1}{Z}}=\frac{V_{out}}{R}+V_{out}sC-\frac{V_{out}}{R^2}\frac{1}{sC+\frac{1}{R}+sC_2}-\frac{V_{out}s^2C^2}{2sC+\frac{1}{Z}}$$

$$V_{in}\left[\frac{sC}{1+sR\left(C+C_2\right)}+\frac{s^2C^2}{2sC+\frac{1}{Z}}\right]=V_{out}\left[\frac{1}{R}+sC-\frac{1}{R}\frac{1}{1+sR\left(C+C_2\right)}-\frac{s^2C^2}{2sC+\frac{1}{Z}}\right]$$

$$H(s)=\frac{V_{out}}{V_{in}}=\frac{\frac{sC}{1+sR\left(C+C_2\right)}+\frac{s^2C^2}{2sC+\frac{1}{Z}}}{\frac{1}{R}+sC-\frac{1}{R}\frac{1}{1+sR\left(C+C_2\right)}-\frac{s^2C^2}{2sC+\frac{1}{Z}}}$$

$$H(s)=\frac{V_{out}}{V_{in}}=sCR\frac{\frac{1}{1+sR(C+C_2)}+\frac{sC}{2sC+\frac{1}{Z}}}{1+sRC-\frac{1}{1+sR(C+C_2)}-\frac{s^2RC^2}{2sC+\frac{1}{Z}}}$$

$$H(s)=sCR\frac{2sC+\frac{1}{Z}+sC+s^2RC(C+C_2)}{\left(1+sRC\right)\left(1+sR(C+C_2)\right)\left(2sC+\frac{1}{Z}\right)-2sC-\frac{1}{Z}-s^2RC^2\left(1+sR(C+C_2)\right)}$$

$$H(s)=sCR\frac{3sC+\frac{1}{Z}+s^2RC(C+C_2)}{\left(1+sRC\right)\left(1+sR(C+C_2)\right)\left(2sC+\frac{1}{Z}\right)-2sC-\frac{1}{Z}-s^2RC^2\left(1+sR(C+C_2)\right)}$$

$$H(s)=sCR\frac{3sC+\frac{1}{Z}+s^2RC(C+C_2)}{\left(1+sR(C+C_2)\right)\left(2sC+\frac{1}{Z}+s\frac{RC}{Z}+s^2RC^2\right)-2sC-\frac{1}{Z}}$$

$$H(s)=sCR\frac{3sC+\frac{1}{Z}+s^2RC(C+C_2)}{s\frac{RC}{Z}+s^2RC^2+sR(C+C_2)\left(2sC+\frac{1}{Z}+s\frac{RC}{Z}+s^2RC^2\right)}$$

$$H(s)=\frac{3sC+\frac{1}{Z}+s^2RC(C+C_2)}{\frac{1}{Z}+sC+\left(1+\frac{C_2}{C}\right)\left(2sC+\frac{1}{Z}+s\frac{RC}{Z}+s^2RC^2\right)}$$

$$\boxed{H(s)=\frac{3sC+\frac{1}{Z}+s^2RC(C+C_2)}{\frac{1}{Z}+sC\left(3+2\frac{C_2}{C}\right)+\left(1+\frac{C_2}{C}\right)\left(\frac{1}{Z}(1+sRC)+s^2RC^2\right)}}$$

Coupling C = 10nF are neglected (10nF at 20MHz → short)

$$X_c=\frac{1}{\omega C}=\frac{1}{2\pi fC}=\frac{1}{2\pi\cdot 2\cdot 10^7\cdot 10^{-8}}=\frac{1}{0.4\pi}=0.8\Omega$$

$$\rightarrow Z_2=\frac{1}{sC_2}$$

$$Z=\frac{1}{\frac{1}{R_s+sL}+sC_1+\frac{1}{R_p}}=R_p\frac{R_s+sL}{R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p}$$

$$\frac{1}{Z}=\frac{1}{R_p}\frac{R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p}{R_s+sL}$$

with R_s.... inductor series resistance, R_p.... parallel resistance from dielectric loss

Ideal case with $R_p\rightarrow\infty$ $\qquad \frac{1}{Z}=\frac{1+sC_1R_s+s^2LC_1}{R_s+sL}$

Insert Z within H(s) for the ideal case:

$$H(s)=\frac{3sC+\frac{1+sC_1R_s+s^2LC_1}{R_s+sL}+s^2RC(C+C_2)}{\frac{1+sC_1R_s+s^2LC_1}{R_s+sL}+sC\left(3+2\frac{C_2}{C}\right)+\left(1+\frac{C_2}{C}\right)\left(\frac{1+sC_1R_s+s^2LC_1}{R_s+sL}(1+sRC)+s^2RC^2\right)}$$

$$H(s)=\frac{3sC(R_s+sL)+1+sC_1R_s+s^2LC_1+s^2RC(C+C_2)(R_s+sL)}{1+sC_1R_s+s^2LC_1+sC\left(3+2\frac{C_2}{C}\right)(R_s+sL)+\left(1+\frac{C_2}{C}\right)\left((1+sC_1R_s+s^2LC_1)(1+sRC)+s^2RC^2(R_s+sL)\right)}$$

$$H(s)=\frac{1+sR_s\left(C_1+3C\right)+s^2\left(L\left(C_1+3C\right)+RC(C+C_2)R_s\right)+s^3RC(C+C_2)L}{1+sR_s\left(C_1+C\left(3+2\frac{C_2}{C}\right)\right)+s^2L\left(C_1+C\left(3+2\frac{C_2}{C}\right)\right)+\left(1+\frac{C_2}{C}\right)\left(\left(1+sC_1R_s+s^2LC_1\right)\left(1+sRC\right)+s^2RC^2\left(R_s+sL\right)\right)}$$

$$H(s)=\frac{1+sR_s\left(C_1+3C\right)+s^2\left(L\left(C_1+3C\right)+RC(C+C_2)R_s\right)+s^3RC(C+C_2)L}{1+\left(1+\frac{C_2}{C}\right)+s\left[R_s\left(C\left(3+2\frac{C_2}{C}\right)+C_1\left(2+\frac{C_2}{C}\right)\right)+RC\left(1+\frac{C_2}{C}\right)\right]+s^2\left[L\left(C_1\left(2+\frac{C_2}{C}\right)+C\left(3+2\frac{C_2}{C}\right)\right)+RR_sC\left(1+\frac{C_2}{C}\right)\left(C+C_1\right)\right]+s^3LCR\left(1+\frac{C_2}{C}\right)\left(C_1+C\right)}$$

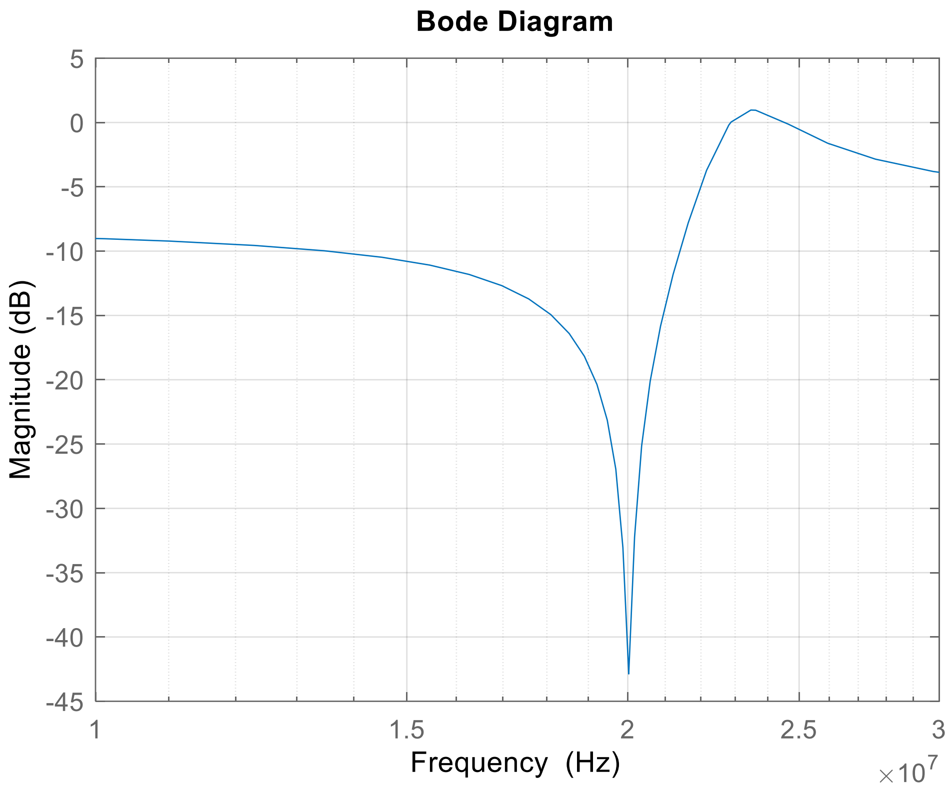

Figure 4: Bode plot of the Notch sensor with Rs = 8 Ω, using above transfer function.

For the non-ideal case:

$$H(s)=\frac{3sC+\frac{1}{R_p}\frac{R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p}{R_s+sL}+s^2RC(C+C_2)}{\frac{1}{R_p}\frac{R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p}{R_s+sL}+sC+\left(1+\frac{C_2}{C}\right)\left(2sC+\frac{1}{R_p}\frac{R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p}{R_s+sL}\left(1+sRC\right)+s^2RC^2\right)}$$

$$H(s)=\frac{3sC\left(R_s+sL\right)+\frac{1}{R_p}\left(R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p\right)+s^2RC(C+C_2)\left(R_s+sL\right)}{\frac{1}{R_p}\left(R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p\right)+sC\left(R_s+sL\right)+\left(1+\frac{C_2}{C}\right)\left(2sC\left(R_s+sL\right)+\frac{1}{R_p}\left(R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p\right)\left(1+sRC\right)+s^2RC^2\left(R_s+sL\right)\right)}$$

$$H(s)=\frac{3sCR_p\left(R_s+sL\right)+R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p+s^2RR_pC(C+C_2)\left(R_s+sL\right)}{R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p+sCR_p\left(R_s+sL\right)+\left(1+\frac{C_2}{C}\right)\left(2sCR_p\left(R_s+sL\right)+\left(R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p\right)\left(1+sRC\right)+s^2RR_pC^2\left(R_s+sL\right)\right)}$$

$$H(s)=\frac{R_p+R_s+s\left(L+C_1R_sR_p+3CR_pR_s\right)+s^2\left(LC_1R_p+3CR_pL+RR_pC(C+C_2)\left(R_s+sL\right)\right)}{R_p+R_s+s\left(L+C_1R_sR_p+CR_pR_s\left(3+2\frac{C_2}{C}\right)\right)+s^2\left(LC_1R_p+CR_pL\left(3+2\frac{C_2}{C}\right)+RR_pR_sC^2\left(1+\frac{C_2}{C}\right)\right)+\left(1+\frac{C_2}{C}\right)\left(\left(R_p+R_s+sL+sC_1R_sR_p+s^2LC_1R_p\right)\left(1+sRC\right)+s^2RR_pC^2\left(sL\right)\right)}$$

Numerator: $num=R_p+R_s+s\left(L+R_sR_p\left(C_1+3C\right)\right)+s^2\left(LR_p\left(C_1+3C\right)+RR_pC(C+C_2)R_s\right)+s^3RR_pC(C+C_2)L$

$$num=\left(R_p+R_s\right)\left(1+s\left(\frac{L+R_sR_p\left(C_1+3C\right)}{R_p+R_s}\right)+s^2\left(\frac{LR_p\left(C_1+3C\right)+RR_pC(C+C_2)R_s}{R_p+R_s}\right)+s^3\frac{RR_pC(C+C_2)L}{R_p+R_s}\right)$$

Denominator:

$$\begin{aligned}denom=&\left(R_p+R_s\right)\left(2+\frac{C_2}{C}\right)+s\left(L\left(2+\frac{C_2}{C}\right)+C_1R_sR_p\left(2+\frac{C_2}{C}\right)+CR_pR_s\left(3+2\frac{C_2}{C}\right)+RC\left(R_p+R_s\right)\left(1+\frac{C_2}{C}\right)\right)+\\&+s^2\left(LC_1R_p\left(2+\frac{C_2}{C}\right)+CR_pL\left(3+2\frac{C_2}{C}\right)+RR_pR_sC^2\left(1+\frac{C_2}{C}\right)+LRC\left(1+\frac{C_2}{C}\right)+RCC_1R_sR_p\left(1+\frac{C_2}{C}\right)\right)+\\&+s^3\left(1+\frac{C_2}{C}\right)\left(LC_1R_pRC+RR_pC^2L\right)\end{aligned}$$

$$\begin{aligned}denom=&\left(R_p+R_s\right)\left(2+\frac{C_2}{C}\right)+s\left(L\left(2+\frac{C_2}{C}\right)+C_1R_sR_p\left(2+\frac{C_2}{C}\right)+CR_pR_s\left(3+2\frac{C_2}{C}\right)+RC\left(R_p+R_s\right)\left(1+\frac{C_2}{C}\right)\right)+\\&+s^2\left(LR_p\left(C_1\left(2+\frac{C_2}{C}\right)+C\left(3+2\frac{C_2}{C}\right)\right)+RR_pR_sC\left(1+\frac{C_2}{C}\right)\left(C+C_1\right)+LRC\left(1+\frac{C_2}{C}\right)\right)+\\&+s^3LCRR_p\left(1+\frac{C_2}{C}\right)\left(C_1+C\right)\end{aligned}$$

$$denom=\left(R_p+R_s\right)\left[\begin{aligned}&\left(2+\frac{C_2}{C}\right)+s\left(\frac{L\left(2+\frac{C_2}{C}\right)}{R_p+R_s}+\frac{C_1R_sR_p}{R_p+R_s}\left(2+\frac{C_2}{C}\right)+\frac{CR_pR_s}{R_p+R_s}\left(3+2\frac{C_2}{C}\right)+RC\left(1+\frac{C_2}{C}\right)\right)+\\&+s^2\left(\frac{LR_p}{R_p+R_s}\left(C_1\left(2+\frac{C_2}{C}\right)+C\left(3+2\frac{C_2}{C}\right)\right)+\frac{RR_pR_sC}{R_p+R_s}\left(1+\frac{C_2}{C}\right)\left(C+C_1\right)+\frac{LRC}{R_p+R_s}\left(1+\frac{C_2}{C}\right)\right)+\\&+s^3\frac{LCRR_p}{R_p+R_s}\left(1+\frac{C_2}{C}\right)\left(C_1+C\right)\end{aligned}\right]$$

For $R_p>>R_s$ => $\frac{R_p}{R_p+R_s}\approx 1$

$$num=\left(R_p+R_s\right)\left(1+s\left(\frac{L}{R_p+R_s}+R_s\left(C_1+3C\right)\right)+s^2\left(L\left(C_1+3C\right)+RC(C+C_2)R_s\right)+s^3RC(C+C_2)L\right)$$

$$denom = \left(R_p + R_s\right)\left[\begin{array}{l}\left(2+\frac{C_2}{C}\right)+s\left(\frac{L\left(2+\frac{C_2}{C}\right)}{R_p+R_s}+C_1R_s\left(2+\frac{C_2}{C}\right)+CR_s\left(3+2\frac{C_2}{C}\right)+RC\left(1+\frac{C_2}{C}\right)\right)+\\ \qquad +s^2\left(L\left(C_1\left(2+\frac{C_2}{C}\right)+C\left(3+2\frac{C_2}{C}\right)\right)+RR_sC\left(1+\frac{C_2}{C}\right)\left(C+C_1\right)+\frac{LRC}{R_p+R_s}\left(1+\frac{C_2}{C}\right)\right)+\\ \qquad +s^3LCR\left(1+\frac{C_2}{C}\right)\left(C_1+C\right)\end{array}\right]$$

$$\boxed{H(s)=\frac{1+s\left(\frac{L}{R_p+R_s}+R_s\left(C_1+3C\right)\right)+s^2\left(L\left(C_1+3C\right)+RC(C+C_2)R_s\right)+s^3RC(C+C_2)L}{\left(2+\frac{C_2}{C}\right)+s\left(\frac{L\left(2+\frac{C_2}{C}\right)}{R_p+R_s}+C_1R_s\left(2+\frac{C_2}{C}\right)+CR_s\left(3+2\frac{C_2}{C}\right)+RC\left(1+\frac{C_2}{C}\right)\right)+s^2\left(L\left(C_1\left(2+\frac{C_2}{C}\right)+C\left(3+2\frac{C_2}{C}\right)\right)+RR_sC\left(1+\frac{C_2}{C}\right)\left(C+C_1\right)+\frac{LRC}{R_p+R_s}\left(1+\frac{C_2}{C}\right)\right)+s^3LCR\left(1+\frac{C_2}{C}\right)\left(C_1+C\right)}}$$

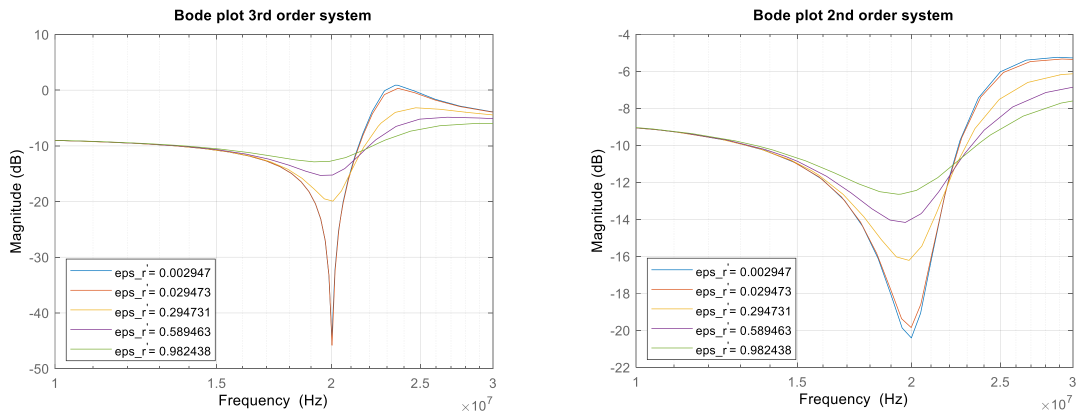

Figure 5: Bode plots of the non-ideal transfer function (left) and with neglection of the 3th order terms (right), for Rp values of 100k, 10k, 1k, 500Ω, 300Ω.

For Rp = 100k: sys =

```
 4.896e-26 s^3 + 6.314e-17 s^2 + 7.428e-10 s + 1
-----------------------------------------------------------------
2.024e-25 s^3 + 1.283e-16 s^2 + 6.299e-09 s + 2.6
```

For Rp = 10k:

```
 4.896e-26 s^3 + 6.314e-17 s^2 + 8.039e-10 s + 1
-----------------------------------------------------------------
2.024e-25 s^3 + 1.286e-16 s^2 + 6.458e-09 s + 2.6
```

For Rp = 1k:

```
 4.896e-26 s^3 + 6.314e-17 s^2 + 1.411e-09 s + 1
-----------------------------------------------------------------
2.024e-25 s^3 + 1.316e-16 s^2 + 8.036e-09 s + 2.6
```

For Rp = 500:

4.896e-26 s^3 + 6.314e-17 s^2 + 2.075e-09 s + 1

2.024e-25 s^3 + 1.347e-16 s^2 + 9.762e-09 s + 2.6

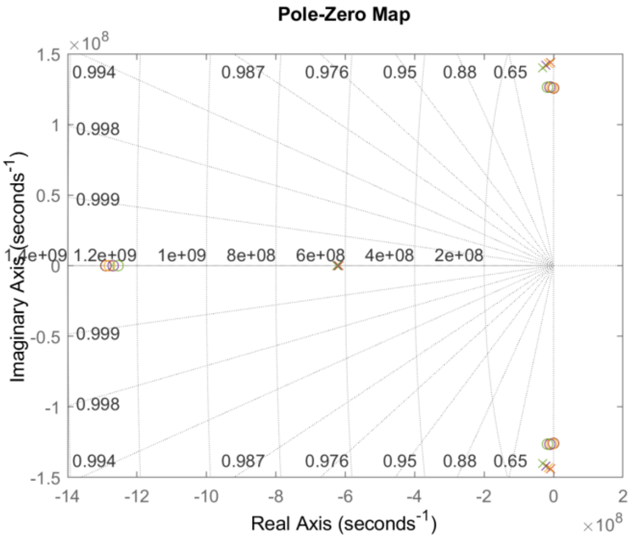

Figure 6: Pole-zero map of the 3rd order transfer functions. The zeroes dominate the notch center frequencies.

In the transfer function the second order numerator term defines the center frequency of the notch.

$$6.134\cdot 10^{-17}s^2 => \frac{\omega^2}{\omega_0^2} => \omega_0 = \frac{1}{\sqrt{6.134\cdot 10^{-17}}} => f_0 = \frac{1}{2\pi\sqrt{6.134\cdot 10^{-17}}} = 20.321 MHz$$

Therefore, the center frequency is defined by:

$$\boxed{f_0 = \frac{1}{2\pi\sqrt{L\left(C_1+3C\right)+RC(C+C_2)R_s}} \cong \frac{1}{2\pi\sqrt{L\left(C_1+3C\right)}}}$$

$$f_0 = \frac{1}{2\pi\sqrt{\underbrace{680\cdot 10^{-9}\left(47\cdot 10^{-12}+45\cdot 10^{-12}\right)}_{`62.56\cdot 10^{-18}}+\underbrace{200\cdot 15\cdot 10^{-12}(15\cdot 10^{-12}+9\cdot 10^{-12})8}_{5.76\cdot 10^{-19}}}}$$

Magnitude at center frequency (real part of numerator equals zero):

$$|H(j\omega)| = \frac{\left|1+j\omega\left(\frac{L}{R_p+R_s}+R_s\left(C_1+3C\right)\right)-\omega^2\left(L\left(C_1+3C\right)+RC(C+C_2)R_s\right)-j\omega^3RC(C+C_2)L\right|}{\left|\left(2+\frac{C_2}{C}\right)+j\omega\left(\frac{L\left(2+\frac{C_2}{C}\right)}{R_p+R_s}+C_1R_s\left(2+\frac{C_2}{C}\right)+CR_s\left(3+2\frac{C_2}{C}\right)+RC\left(1+\frac{C_2}{C}\right)\right)-\omega^2\left(L\left(C_1\left(2+\frac{C_2}{C}\right)+C\left(3+2\frac{C_2}{C}\right)\right)+RR_sC\left(1+\frac{C_2}{C}\right)(C+C_1)+\frac{LRC}{R_p+R_s}\left(1+\frac{C_2}{C}\right)\right)-j\omega^3LCR\left(1+\frac{C_2}{C}\right)\left(C_1+C\right)\right|}$$

$$|H(\omega_0)| = \frac{\left|\omega_0\left(\frac{L}{R_p+R_s}+R_s\left(C_1+3C\right)\right)-\omega_0^3RC(C+C_2)L\right|}{\sqrt{\left[\left(2+\frac{C_2}{C}\right)-\omega_0^2\left(L\left(C_1\left(2+\frac{C_2}{C}\right)+C\left(3+2\frac{C_2}{C}\right)\right)+RR_sC\left(1+\frac{C_2}{C}\right)(C+C_1)+\frac{LRC}{R_p+R_s}\left(1+\frac{C_2}{C}\right)\right)\right]^2+\left[\omega_0\left(\frac{L\left(2+\frac{C_2}{C}\right)}{R_p+R_s}+C_1R_s\left(2+\frac{C_2}{C}\right)+CR_s\left(3+2\frac{C_2}{C}\right)+RC\left(1+\frac{C_2}{C}\right)\right)-\omega_0^3LCR\left(1+\frac{C_2}{C}\right)\left(C_1+C\right)\right]^2}}$$

$$\left|H(\omega_0)\right| = \frac{2\pi\cdot 20\cdot 10^6 \left|\left(\frac{6.8\cdot 10^{-7}}{R_p+8}+7.36\cdot 10^{-7}\right)-7.73\cdot 10^{-10}\right|}{\sqrt{\left[2.6-\omega_0^2\left(1.2594\cdot 10^{-16}+2.38\cdot 10^{-18}+\frac{3.264\cdot 10^{-15}}{R_p+R_s}\right)\right]^2+\omega_0^2\left[\left(\frac{1.768\cdot 10^{-6}}{R_p+R_s}+9.776\cdot 10^{-10}+5.04\cdot 10^{-10}+4.8\cdot 10^{-9}\right)-3.196\cdot 10^{-9}\right]^2}}$$

$$\left|H(\omega_0)\right| = \frac{2\pi\cdot 20\cdot 10^6 \left|\left(\frac{6.8\cdot 10^{-7}}{R_p+8}+7.36\cdot 10^{-7}\right)-7.73\cdot 10^{-10}\right|}{\sqrt{\left[2.6-1.99+0.0376+\frac{51.543}{R_p+R_s}\right]^2+\left[\frac{222.17}{R_p+R_s}+0.123+0.063+0.603-0.4\right]^2}}$$

With `Rp = [1E5 1E4 1E3 5E2 3E2] in Ω =>`

`εr'' = [0.0029 0.0295 0.2947 0.5895 0.9824] (assuming C_sensor = 27 pF)`

`Mag_numerator = [0.0041  0.0036  0.0799  0.1635  0.2729]`

`Mag_denomintator = [0.6879  0.6957  0.7976  0.9476  1.17949]`

`Total magnitude:`

`mag =  [0.0060  0.0052  0.1002  0.1725  0.2314]`

mag^{db} `=  [-44.4706  -45.7610  -19.9810  -15.2622  -12.7136] dB`

separate C2:

Numerator = $\left|\omega_0\left(\frac{L}{R_p+R_s}+R_s\left(C_1+3C\right)\right)-\omega_0^3 RLC^2-\omega_0^3 RLCC_2\right|$

(Denominator)^2 =

$$\left[2-\omega_0^2\left(L\left(2C_1+3C\right)+RR_sC\left(C+C_1\right)+\frac{LRC}{R_p+R_s}\right)+C_2\left(\frac{1}{C}-\omega_0^2\left(\frac{LR}{R_p+R_s}+RR_s\left(C+C_1\right)+2L+L\frac{C_1}{C}\right)\right)\right]^2$$

$$\left[\omega_0\left(\frac{2L}{R_p+R_s}+2C_1R_s+3CR_s+RC\right)-\omega_0^3 LCR\left(C_1+C\right)+C_2\left(\omega_0\left(\frac{\frac{L}{C}}{R_p+R_s}+R_s\frac{C_1}{C}+2R_s+R\right)-\omega_0^3 LR\left(C_1+C\right)\right)\right]^2$$

$$H\left(\omega_0\right)=\frac{\left|\frac{85,58}{R_p+R_s}+0.0926-0.061-4.0664\cdot 10^9 C_2\right|}{\sqrt{\left[0.4793+\frac{32.3112}{R_p+R_s}+C_2\left(9.8074\cdot 10^{-9}-\frac{2.1541\cdot 10^{12}}{R_p+R_s}\right)\right]^2+\left[0.2654+\frac{171.16}{R_p+R_s}+C_2\left(1.3531\cdot 10^{10}+\frac{5.7\cdot 10^{12}}{R_p+R_s}\right)\right]^2}}$$

With C2 in pF:

$$H(\omega_0) = \frac{\left| \frac{85,58}{R_p + R_s} + 0.0316 - 4.0664 \cdot 10^{-3} C_2 \right|}{\sqrt{\left[0.4793 + \frac{32.3112}{R_p + R_s} - C_2 \frac{2.1541}{R_p + R_s} \right]^2 + \left[0.2654 + \frac{171.16}{R_p + R_s} + C_2 \left(1.3531 \cdot 10^{-2} + \frac{5.7}{R_p + R_s} \right) \right]^2}}$$

The absolute minimum at the notch center frequency is defined by the numerator:

$$\left| \omega_0 \left(\frac{L}{R_p + R_s} + R_s (C_1 + 3C) \right) - \omega_0^3 RLC^2 - \omega_0^3 RLCC_2 \right| = 0$$

$$\left| \left(\frac{L}{R_p + R_s} + R_s (C_1 + 3C) \right) - \omega_0^2 RLC^2 \right| = \omega_0^2 RLCC_2$$

$$\frac{1}{R_p + R_s} = \omega_0^2 RC (C_2 + C) - \frac{R_s}{L} (C_1 + 3C) = \frac{4.0664 \cdot 10^{-3} C_2 - 0.0316}{58,58}$$

$$R_p = \frac{1}{\omega_0^2 RC (C_2 + C) - \frac{R_s}{L} (C_1 + 3C)} - R_s = \frac{1}{\omega_0 C_0 \varepsilon_r''}$$

For $R_p >> R_s$ => $\frac{1}{R_p} = \omega_0 C_0 \varepsilon_r'' = \omega_0^2 RC (C_2 + C) - \frac{R_s}{L} (C_1 + 3C)$

⇨ $\boxed{\varepsilon_r'' = \omega_0 R \frac{C}{C_0} (C_2 + C) - \frac{R_s}{\omega_0 L C_0} (C_1 + 3C) = 1.3984 \cdot 10^{-2} C_2 - 0.1088}$ with C2 in pF

Calibration in dry air: $\varepsilon_r'' = 0$ => $\omega_0 R \frac{C}{C_0} (C_2 + C) = \frac{R_s}{\omega_0 L C_0} (C_1 + 3C)$

$$\boxed{C_{2_{\min}} = \frac{0.1088}{1.3984 \cdot 10^{-2}} = 7.78 \text{ pF}}$$

Measurement range with 20pF Varicap:

$$\varepsilon_{r_{\max}}'' = 1.3984 \cdot 10^{-2} C_2 - 0.1088 = 0.17$$

Resolution with 12 Bit DAC (assuming linear Varicap): $\Delta C_2 = \frac{20}{4096} = 0.00488 \text{pF}$

$$\Delta \varepsilon_r'' = 68.24 \cdot 10^{-6}$$

Lowest detectable minimum with 12 Bit ADC: $V_{pp_{\min}} = \frac{V_{ref}}{2048} = \frac{2.5}{2048} = 1.22\, mV$

$$V_{pp_{\min}} = -66.23\, dBV$$

Table 1: Permittivity range and loss resistance

C2 / pF	eps_r''	Rp / Ω
7,781	9,504E-06	41865252,3
8	0,003072	129520,6
8,5	0,010064	39535,7
9	0,017056	23328,3
9,5	0,024048	16545,5
10	0,03104	12818,5
10,5	0,038032	10461,9
11	0,045024	8837,2
11,5	0,052016	7649,3
12	0,059008	6742,9
12,5	0,066	6028,6
13	0,072992	5451,1
13,5	0,079984	4974,6
14	0,086976	4574,7
14,5	0,093968	4234,3
15	0,10096	3941,0
15,5	0,107952	3685,8
16	0,114944	3461,6
16,5	0,121936	3263,1
17	0,128928	3086,1
17,5	0,13592	2927,4
18	0,142912	2784,1
18,5	0,149904	2654,3
19	0,156896	2536,0
19,5	0,163888	2427,8
20	0,17088	2328,5