How far away can two lines intersect?

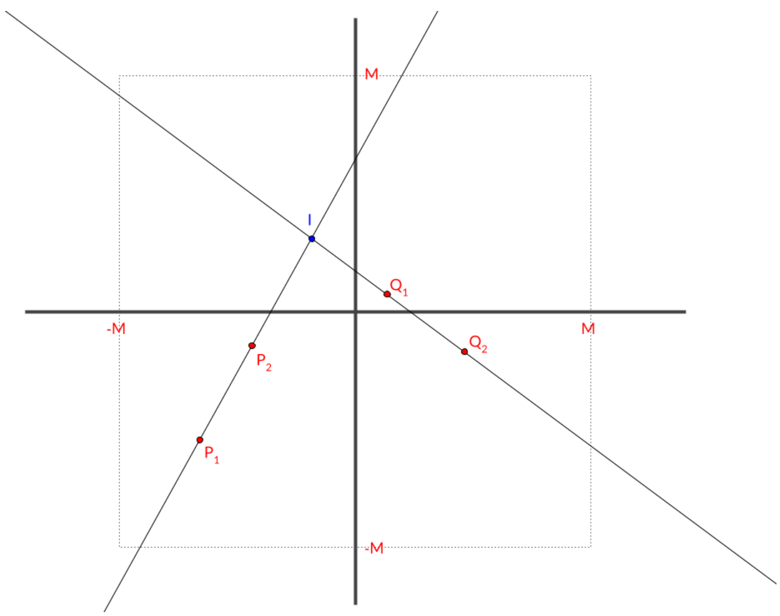
Oct 6, 2024

Remark: This issue occurred in Problem C of ICPC ECNA 2023—the authors got this wrong! You can find more details (warning: spoilers) here and here.

Problem Statement

You are given four points P_1, P_2, Q_1, Q_2, each with integer coordinates bounded by some constant M. Suppose the lines P_1P_2 and Q_1Q_2 intersect (uniquely) at some point I.

What's the maximum distance I can be from the origin?



Here are some multiple-choice options. In each case, $|I|$ is the distance from I to the origin.

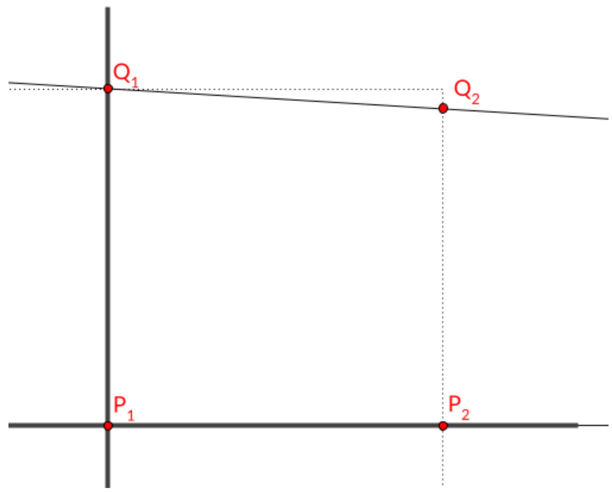
1. $|I|$ is $\Theta(M)$ in the worst case.
2. $|I|$ is $\Theta(M^2)$ in the worst case.
3. $|I|$ is $\Theta(M^3)$ in the worst case.
4. $|I|$ can be arbitrarily large in the worst case.

Try it for yourself! My answers start on the next page.

Wrong answer 1: $|I|$ is $\Theta(M)$ in the worst case.

We can construct a counterexample by choosing two lines far apart and with a small slope difference.

$$\begin{aligned} P_1 &= (0,\, 0) \qquad & Q_1 &= (0,\, M) \\ P_2 &= (M,\, 0) \qquad & Q_2 &= (M,\, M-1) \end{aligned}$$



The line P_1P_2 is given by the equation $y = 0$ and the line Q_1Q_2 is given by the equation $y = M - \frac{1}{M}x$.

The two lines have slope difference $\frac{1}{M}$ and are initially M units apart. So, we expect the lines to intersect at some point with distance M^2 from the origin. Indeed, we can calculate their intersection point as $I = (M^2,\, 0)$.

Wrong answer 2: $|I|$ is $\Theta(M^2)$ in the worst case.

If you thought this was the correct answer, you're in good company—the problem authors thought so too!

How can we make $|I|$ even larger? We can't do much to worsen the initial distance—no matter what we do, the two lines are at most $2M$ units apart in the initial box around the origin.

But their slopes can differ by a tiny amount—even smaller than $\frac{1}{M}$! Do you see how?

Let's write the two slopes as $\frac{p_y}{p_x}$ and $\frac{q_y}{q_x}$, where p_x is the distance between P_1 and P_2 in the x-direction, and p_y is the distance in the y-direction. q_x and q_y are defined similarly.

Then, the difference in slopes is:

$$\frac{p_y}{p_x} - \frac{q_y}{q_x} = \frac{p_y q_x - p_x q_y}{p_x q_x}.$$

To make this as small as possible, we want $p_y q_x - p_x q_y$ to be small and $p_x q_x$ to be large.

These values work,

$$\begin{aligned} p_x &= M - 1 \qquad & q_x &= M \\ p_y &= M \qquad & q_y &= M + 1 \end{aligned}$$

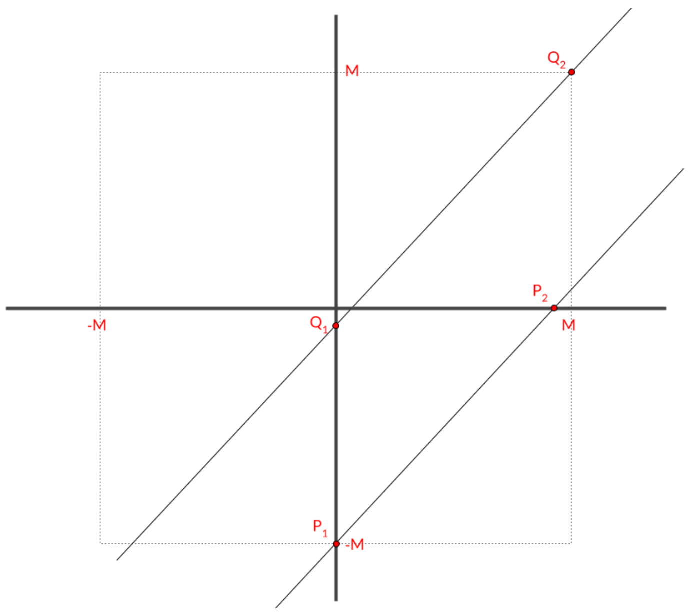
because the slope difference becomes

$$\frac{p_y q_x - p_x q_y}{p_x q_x} = \frac{M^2 - (M-1)(M+1)}{(M-1)(M)} = \frac{1}{M(M-1)}.$$

Woah! We achieved a slope difference of about $\frac{1}{M^2}$, an order of magnitude smaller than $\frac{1}{M}$.

All that's left is to choose points so that the lines are about M apart:

$$\begin{aligned} P_1 &= (0, -M) \qquad & Q_1 &= (0, -1) \\ P_2 &= (M-1, 0) \qquad & Q_2 &= (M, M) \end{aligned}$$



The lines look parallel—but P_1P_2 has slope $\frac{M}{M-1}$ and Q_1Q_2 has slope $\frac{M+1}{M}$. Since the lines are $M-1$ units vertically apart at $x = 0$, they intersect at

$$x = \frac{\text{initial } y\text{-distance}}{\text{slope difference}} = M(M-1)^2.$$

We can also calculate the y-coordinate as $(M+1)(M-1)^2 - 1$, so we have $|I| \in \Theta(M^3)$.

Great! We've discovered a way to construct pairs of lines that intersect at distance $\Theta(M^3)$ from the origin. But, could there be something even larger?

$\Theta(M^3)$ is the correct worst-case bound.

Firstly, a crude lower bound for the magnitude of the slope difference is

$$\left|\frac{p_y q_x - p_x q_y}{p_x q_x}\right| \geq \frac{1}{4M^2},$$

because the numerator is at least 1 and each term in the denominator is at most $2M$. We can use this to derive bounds on $|x_I|$ and $|y_I|$ for the intersection point $I = (x_I, y_I)$.

As before, the lines are at most $2M$ units vertically apart at $x = 0$, so $|x_I|$ is at most $8M^3$. To obtain the identical bound $|y_I| \leq 8M^3$, we can swap x and y-coordinates and repeat the same analysis. So, $|I|$ is indeed $\Theta(M^3)$ in the worst case.

Aside: You may notice that I've missed some edge cases: I implicitly assumed none of p_x, p_y, q_x, q_y, and $p_y q_x - p_x q_y$ are zero. Fortunately, this is not a problem. If any of the first four quantities are zero, then we are dealing with horizontal/vertical lines—these are easy to check separately. If the last quantity is zero, then P_1P_2 and Q_1Q_2 are parallel. We ignore this case because we want a unique intersection point.