1.

(a) Disprove; trivial, skip.

(b)

(→)

$$\forall \epsilon > 0, \exists N \in \mathbb{N}\, s.t. |a_n - a| < \epsilon \Leftrightarrow \mathbb{I}(|a_n - a| \geq \epsilon) = 0$$

$$\mathbb{I}(|a_n - a| \geq \epsilon) = P(|a_n - a| \geq \epsilon) = 0$$

(←)

$P(|a_n - a| \geq \epsilon)$ is equivalent to

$$\forall \epsilon, \delta > 0, \exists N \in \mathbb{N}, s.t. \forall n \geq N, P(|a_n - a| \geq \epsilon) < \delta$$

Set $\delta = \frac{1}{4}$

$$P(|a_n - a| \geq \epsilon) = \mathbb{I}(|a_n - a| \geq \epsilon) < \frac{1}{4}$$

$\mathbb{I}(|a_n - a| \geq \epsilon) = 0,\ \forall \epsilon > 0, \exists N \in \mathbb{N}\, s.t. |a_n - a| < \epsilon$

Comment: δ doesn't to be $\frac{1}{4}$, just any number that is positive. If $P(|a_n - a| \geq \epsilon) = 1$, letting δ be $\frac{1}{2}$, then it contradicts with the assumption.

(c)

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \mu$$

$$Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\left\{Var\left(\sum_{i=1}^{n} X_i\right) + \sum_{i \neq j} Cov(X_i, X_j)\right\}$$

$$= \frac{1}{n^2}\left\{n\sigma^2 + \sum_{i \neq j} Cov(X_i, X_j)\right\}$$

Use Chevyshev's inequality:

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) \leq \frac{Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}{\epsilon^2} = \frac{1}{n^2\epsilon^2}\left\{n\sigma^2 + \sum_{i \neq j} Cov(X_i, X_j)\right\}$$

Note that

$$\frac{1}{n^2\epsilon^2}\left\{n\sigma^2 + \sum_{i\neq j} Cov(X_i, X_j)\right\} \leq \frac{1}{n^2\epsilon^2}\left\{n\sigma^2 + \frac{n(n-1)}{2}\max_{1\leq i\neq k\leq n}\left|Cov(X_i, X_j)\right|\right\}$$

As

$$\lim_{n\to\infty}\frac{1}{n^2\epsilon^2}\left\{n\sigma^2 + \frac{n(n-1)}{2}\max_{1\leq i\neq k\leq n}\left|Cov(X_i, X_j)\right|\right\} = 0$$

and

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) \geq 0$$

$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) = 0$, which is $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges to μ in probability.

*** Sandwich lemma ***

$a_n \to a, b_n \to b,\ a_n \leq x_n \leq b_n,\ x_n \to a$ as $n \to \infty$

2.

(a)

As $0 \le p < \frac{1}{2}$

$$n^{p-\frac{1}{2}} \to 0 \iff n^{p-\frac{1}{2}} \xrightarrow{p} 0$$

By Slutsky theorem,

$$n^{p-\frac{1}{2}} \times n^{\frac{1}{2}}(X_n - \theta) = n^p(X_n - \theta) \xrightarrow{D} 0$$

$$n^p(X_n - \theta) \xrightarrow{D} 0 \iff n^p(X_n - \theta) \xrightarrow{p} 0$$

(b)

By taylor expansion at $x = \theta$,

$$f(X_n) = f(\theta) + \frac{f^{(3)}(a)(X_n - \theta)^3}{3!}$$

Where $a \in (X_n, \theta)$, in (a), let $p = 0$, $X_n \xrightarrow{p} \theta$, therefore $a \xrightarrow{p} \theta$

$$n^{\frac{3}{2}}\big(f(X_n) - f(\theta)\big) = \frac{f^{(3)}(a)}{6}\left\{\sqrt{n}(X_n - \theta)\right\}^3$$

$$\left\{\sqrt{n}(X_n - \theta)\right\}^3 \xrightarrow{D} \sigma^3 Z^3 \ \cdots (1)$$

$$\frac{f^{(3)}(a)}{6} \xrightarrow{p} \frac{f^{(3)}(\theta)}{6} \cdots (2)$$

By $(1), (2)$, using Slutsky theorem, $n^{\frac{3}{2}}\big(f(X_n) - f(\theta)\big) \xrightarrow{D} \frac{f^{(3)}(\theta)}{6}\sigma^3 Z^3$

3.

$$L(\theta) = \left(\frac{2}{\theta^2}\right)^n \prod_{i=1}^{n} x_i \, \mathbb{I}(\max x_i \le \theta)$$

(a)

$$\hat{\theta}_{mle} = \max_{1 \le i \le n} X_i$$

(b)

Skip.

(c)

Median is shown as follows:

$$\xi = F^{-1}(0.5)$$

$$F(x) = \frac{x^2}{\theta^2}$$

$$F^{-1}(0.5) = \frac{\theta}{\sqrt{2}}$$

Use theorem 6.1.2(Invariance property of mle)

4.

(a)

$$\mathbb{E}(X^2 - 1) = \theta^2$$

$X^2 - 1$ is unbiased est. of θ^2

$$\begin{aligned} Var(X^2 - 1) &= Var(X^2) \\ &= \mathbb{E}(X^4) - \mathbb{E}(X^2)^2 \\ &= \theta^4 + 6\theta^2 + 3 - \theta^4 - 2\theta^2 - 1 \\ &= 4\theta^2 + 2 \end{aligned}$$

(b)

Use asymptotic distribution of mle:

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \to N\left(0, \frac{1}{I(\theta)}\right)$$

And do delta method.

But I will show different way, which is reparametrizing trick.

Let $\eta = \psi(\theta)$, then $\psi^{-1}(\eta) = \theta$. By reparameterization, $f(x;\theta) = g(x;\eta)$.

$$\begin{aligned} I(\eta) &= \mathbb{E}\left(\left(\frac{\partial}{\partial\eta} g(x;\eta)\right)^2\right) \\ &= \mathbb{E}\left(\left(\frac{\partial\theta}{\partial\eta}\frac{\partial}{\partial\theta} f(x;\theta)\right)^2\right) \\ &= \left(\frac{\partial\theta}{\partial\eta}\right)^2 I(\theta) = \left(\frac{\partial\psi^{-1}(\eta)}{\partial\eta}\right)^2 I(\theta) \end{aligned}$$

You can get $I(\theta^2) = \frac{1}{4\theta^2}$. Therefore, CRLB is $4\theta^2$