Cherenkov Radiation

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1.1 Cherenkov History Introduction

Everyone is familiar with Einstein's postulate that the speed of light in a vacuum c is an absolute limit on velocity, however, it is also true that light travels through any medium at a slower rate. Heaviside was the first to realize in the late nineteenth century that a particle passing through a medium faster than the speed of light in that medium ought to emit radiation. Although, not yet having Einstein's postulate he also calculated what this would be for a particle traveling through the aether. The angle away from the particle's path with which that light will be emitted is

$$cos\theta_C = 1/\beta n \tag{1}$$

where θ_C is the angle, β is the velocity of the particle relative to the speed of light in a vacuum (β = v/c), and n is the index of refraction of the medium through which the particle is passing. Unaware of Heaviside's theory, in 1937 Cherenkov and Vavilov working in Russia discovered this radiation while studying nuclear decay. Pavel Cherenkov won the 1958 Nobel Prize for the discovery of the effect, and surely would have shared it with Vavilov if he was still alive.

1.2 Frank-Tamm Theory

Frank and Tamm were collegues of Vavilov and Cherenkov at the Lebedev Physical Institute in Moscow, so they naturally had a head start in formulating the theory. In 1938 they published their theory of Cherenkov radiation [citation?] for which they would receive the Nobel Prize along with Cherenkov in 1958. Pedagogical introductions to their theory can be found in the books by Jelley [1] and Zrelov [2].

Developing along lines which might have been familiar to Heaviside, the starting point is the relation of the electric field **E** and its associated polarization vector **P** in a medium with refractive index n:

$$\mathbf{P} = (n^2 - 1)\mathbf{E} \tag{2}$$

Note also that there is a frequency dependence of this equation arising from the implicit Fourier expansions

$$\mathbf{P} = \int_{-\infty}^{+\infty} \mathbf{P}(\omega) e^{i\omega t} d\omega, n = \int_{-\infty}^{+\infty} n(\omega) e^{i\omega t} d\omega, etc. \tag{3}$$

of all the variable quantities. Now to write Maxwell's equations in this medium we convert to

$$\mathbf{D} = n^2 \mathbf{E} \tag{4}$$

which yields

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 4\pi\rho & (5) \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial H}{\partial t} & (6) \\ \nabla \cdot \mathbf{H} &= 0 & (7) \\ \nabla \times \mathbf{H} &= \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial D}{\partial t}. & (8) \end{aligned}$$

Then one can reduce Maxwell's equations to a more suitable form using the potentials

$$\begin{aligned} \mathbf{E} &= -\nabla\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} & (9) \\ \mathbf{H} &= \nabla \times \mathbf{A} & (10) \end{aligned}$$

which are given in the Lorentz gauge,

$$\nabla \cdot \mathbf{A} = -\frac{\varepsilon}{c}\frac{\partial \varphi}{\partial t}. \tag{11}$$

Note that the two potential equations are equivalent to (6) and (7). Taking the gradient of (11) and adding it to $\frac{\varepsilon}{c}\frac{\partial}{\partial t}$ operated on (9) yields one

$$\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \nabla(\nabla \cdot \mathbf{A}) - \frac{\varepsilon}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{12}$$

and plugging (10) into (8) yields

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \tag{13}$$

which combined with the previous gives the final result

$$\nabla^2 \mathbf{A} - \frac{\varepsilon}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} \tag{14}$$

for the vector potential and current. Similarly, taking the partial derivative of (11) with respect to time and adding it to $\varepsilon c \nabla\cdot$ operated on (9) yields

$$c\nabla \cdot \mathbf{D} = -\varepsilon c \nabla^2 \varphi + \frac{\varepsilon^2}{c}\frac{\partial^2 \varphi}{\partial t^2}. \tag{15}$$

Combining this with (5) finally gives

$$\nabla^2 \varphi - \frac{\varepsilon}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{4\pi}{\varepsilon}\rho. \tag{16}$$

Now since we are interested in only the optical properties of our medium, we recall from (4) that $\varepsilon = n^2$ and that Fourier transforming the equations simply means $\frac{\partial}{\partial t} \to i\omega$. Thus if we apply this knowledge to (14) and (16) we get

$$\nabla^2 \mathbf{A} - \frac{n^2\omega^2}{c^2}\mathbf{A} = -\frac{4\pi}{c}\mathbf{j} \tag{17}$$

$$\nabla^2 \varphi - \frac{n^2\omega^2}{c^2}\varphi = -\frac{4\pi}{n^2}\rho \tag{18}$$

It is at this juncture where we part from Heaviside so as to study the simplest possible case of a single electron passing through the medium in question. To the resolution of current experiments the electron is a point particle so we use Dirac delta functions,

$$\mathbf{j} = ev\delta(x)\delta(y)\delta(z - vt)\hat{z} \tag{19}$$

where v is the electron's velocity and the z-axis corresponds to the beamline. Now to Fourier transform this equation we recall that $\delta(ax) = \frac{1}{|a|}\delta(x)$ which means that

$$\mathbf{j} = e\delta(x)\delta(y)\int_{-\infty}^{+\infty} \delta(\frac{z}{v} - t)e^{i\omega t}d\omega\hat{z} \tag{20}$$

$$= e\delta(x)\delta(y)e^{\frac{i\omega z}{v}}\hat{z} \tag{21}$$

assuming, as we have, a positive velocity v. Hopefully it is clear that the first e is the electron's charge whereas the second is 2.718... If we now plug (21) into (17) to get an equation for the vector potential in the medium through which the electron passes, this yields

$$\nabla^2 \mathbf{A} - \frac{n^2\omega^2}{c^2}\mathbf{A} = -\frac{4e\pi}{c}\delta(x)\delta(y)e^{\frac{i\omega z}{v}}\hat{z}. \tag{22}$$

Clearly in the x and y directions there are identical Helmholtz equations, but we are only interested in the trivial solutions since ????. Thus we concentrate on the beam direction, where

$$\nabla^2 A_z - \frac{n^2\omega^2}{c^2} A_z = -\frac{4e\pi}{c}\delta(x)\delta(y)e^{\frac{i\omega z}{v}} \tag{23}$$

and with prescient forsight we guess a solution of the form $A_z = u(\rho)e^{\frac{i\omega z}{v}}$, switching to cylindrical coordinates as well. Simplifying,

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\omega^2}{v^2}(\beta^2 n^2 - 1)u = -\frac{2e}{c\rho}\delta(\rho) \tag{24}$$

since $\delta(x)\delta(y) = \frac{1}{2\pi\rho}\delta(\rho)$. This Bessel equation's solution will have different characteristics depending on whether $\beta^2 n^2 = 1$, $\beta^2 n^2 < 1$ or $\beta^2 n^2 > 1$. Letting

$$\frac{\omega^2}{v^2}(\beta^2 n^2 - 1) = k^2 \tag{25}$$

to simplify matters, one sees that the solutions to the homogeneous part of (24) are the usual Bessel functions

$$J_0(k\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\rho sin\theta} d\theta \tag{26}$$

$$Y_0(k\rho) = -\frac{2}{\pi}\int_1^{\infty} \frac{cos(k\rho t)}{\sqrt{t^2-1}} dt. \tag{27}$$

Returning to the inhomgeneaity in (24) we multiply through by ρ and integrate,

$$\int_0^{\infty} (\rho\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial u}{\partial \rho} + k^2 \rho u) d\rho = -\int_0^{\infty} \frac{2e}{c}\delta(\rho) d\rho \tag{28}$$

$$\int_0^{\infty} (-\frac{\partial u}{\partial \rho} + \frac{\partial u}{\partial \rho} + k^2 \rho u) d\rho + \left(\rho\frac{\partial u}{\partial \rho}\right)\Bigg|_0^{\infty} = -\frac{2e}{c} \tag{29}$$

$$k^2 \int_0^{\infty} \rho u d\rho + \rho \overset{lim}{\to} \infty \left(\rho\frac{\partial u}{\partial \rho}\right) - \rho \overset{lim}{\to} 0 \left(\rho\frac{\partial u}{\partial \rho}\right) = -\frac{2e}{c} \tag{30}$$

$$\rho \overset{lim}{\to} 0 \left(\rho\frac{\partial u}{\partial \rho}\right) = \frac{2e}{c} \tag{31}$$

where one can eliminate the first two terms in the penultimate equation because ?????. (Actually, it seems to me they ought to diverge since J, J', Y and Y' all go like $\frac{1}{\sqrt{\rho}}$ at large ρ.)

Thus far we have followed the usual procedure to solve a differential equation of known form. Herefollows the gimmick that earned Frank and Tamm their Nobel prize. Instead of continuing on to find a particular solution of the inhomogeneous equation and then using initial conditions to find the coeffecients of (26) and (27) in the general solution, they used (31) to get the asymptotic coefficients of the solution $u(\rho)$. Since the length scale of the electromagnetic force is microscopic, it is not unreasonable to use an asymptotic approximation of the Bessel functions for large ρ,

$$\begin{aligned} u(\rho) &= AJ_0(k\rho) + BY_0(k\rho) && (32) \\ &\approx \sqrt{\frac{2}{k\pi\rho}}\left(Acos(k\rho - \frac{\pi}{4}) + Bsin(k\rho - \frac{\pi}{4})\right), && (33) \end{aligned}$$

as we are interested in macroscopic effects. Now it is clear that the $\beta^2 n^2 = 1$ is nonphysical since $k = 0$ in this case. For the second case it makes sense to rotate the coefficients to

$$u(\rho) \approx \sqrt{\frac{2}{k\pi\rho}}\left(Ce^{i(k\rho-\frac{\pi}{4})} + De^{-i(k\rho-\frac{\pi}{4})}\right). \tag{34}$$

Considering the $\beta^2 n^2 < 1$ case, k is imaginary which means that D must be zero to avoid divergences. All that's left is a decaying exponential which will leave no appreciable vector potential at large distances. This means that low velocity particles don't emit Cherenkov radiation. The final case involves those particles with high enough energy for which $\beta^2 n^2 > 1$. Here the Bessel functions naturally yield a nice outgoing cylindrical wave.

skipping ahead....

Finally this yields the Frank-Tamm equation,

$$\frac{dW}{dl} = \frac{e^2}{c^2}\int_{\beta n>1}\left(1 - \frac{1}{\beta^2 n^2}\right)\omega d\omega \tag{35}$$

1.3 Dispertion

The refractive index of a given substance is actually a function of the wavelength λ of the light passing through it.

1.3.1 Complex Indices of Refraction

$$\epsilon(\omega) = \epsilon_R(\omega) + i\epsilon_I(\omega) \tag{36}$$

Which modifies the Frank-Tamm equation to

$$\frac{dW}{dl} = \frac{e^2}{c^2}\int_{\beta n>1}\left(1 - \frac{\epsilon_R(\omega)}{\beta^2\left|\epsilon(\omega)\right|^2}\right)\omega d\omega \tag{37}$$

and leads to damping.

1.4 Diffraction

Main cause of width of Cherenkov Ring:

$$\Delta\theta_C \approx \frac{\lambda}{Lsin\theta} \tag{38}$$

where L is the length of the particle's path in the radiator and λ is the wavelength of light being emitted.

1.5 Scattering

1.6 Edge Effects

1.7 Radiation Below the β Threshold

1.8 Quantum Mechanical Modifications

There is a small correction to (1), namely

$$cos\theta_C = \frac{1}{\beta n} + \frac{\Lambda}{\lambda}\left(\frac{n^2-1}{2n^2}\right) \tag{39}$$

1.9 Cherenkov Radiation is not Bremsstrahlung!

Cherenkov radiation is a macroscopic effect of the whole medium on a charged particle, whereas Bremsstrahlung is the interaction of a charged particle with a individual screened nucleus.

References

[1] J. V. Jelley, *Cherenkov Radiation and Its Applications*, 1958.

[2] V. P. Zrelov, *Cherenkov Radiation in High-Energy Physics*, 1970.