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Well-Quasi-Orderings and the Robertson-Seymour Theorems

7.1 Basic WQO Theory

As we will see, well-quasi-orderings (WQO's) provide a powerful engine for demonstrating that classes of problems are *FPT*. In this section, we will look at the rudiments of the theory of WQO's, and in subsequent sections, we will examine applications to combinatorial problems.

A *quasi-ordering* on a set S is a reflexive and transitive relation on S. We will usually represent a quasi-ordering as $\leq_S$ or simply $\leq$ when the underlying set S is clear. Let $\langle S, \leq\rangle$ be a quasi-ordered set. We will write $x < y$ if $x \leq y$ and $y \not\leq x$, $x \equiv y$ if $x \leq y$ and $y \leq x$, and, finally, $x|y$ if $x \not\leq y$ and $y \not\leq x$. Note that if $\langle S, \leq\rangle$ is a quasi-ordered set, then $S/\equiv$ is partially ordered by the quasi-order induced by $\leq$.[1]

Definition 7.1 (Ideal and Filter) Let $\langle S, \leq\rangle$ be a quasi-ordered set. Let S' be a subset of S.

(i) We say that S' is a *filter* if it is closed under $\leq$ upward; that is, if $x \in S'$ and $y \geq x$, then $y \in S'$. The *filter generated by* S' is the set $F(S') = \{y \in S : \exists x \in S'(x \leq y)\}$.[2]

(ii) We say that S' is a *(lower) ideal* if S' is $\leq$ closed downward; that is, if $x \in S'$ and $y \leq x$, then $y \in S'$. The *ideal generated by* S' is the set $I(S') = \{y \in S : \exists x \in S'(x \geq y)\}$.

[1]Recall that a *partial ordering* is a quasi-ordering that is also antisymmetric.
[2]Sometimes, filters are called *upper ideals*.

(iii) Finally, if S' is a filter (an ideal) that can be generated by a finite subset of S', then we say that S' is *finitely generated.*

We will need some distinguished types of sequences of elements.

Definition 7.2 Let $\langle S, \leq \rangle$ be a quasi-ordered set. Let $A = \{a_0, a_1, \ldots\}$ be a sequence of elements of S. Then, we say the following.

(i) A is *good* if there is some $i < j$ with $a_i \leq a_j$.

(ii) A is *bad* if it is not good.

(iii) A is an ascending *chain* if for all $i < j, a_i \leq a_j$.

(iv) A is an *antichain* if for all $i \neq j$ $a_i | a_j$.

(v) $\langle S, \leq \rangle$ is *Noetherian* if S contains no infinite (strictly) descending sequences (i.e., there is no sequence $b_0 > b_1 > b_2 \cdots$).

(vi) $\langle S, \leq \rangle$ has the *finite basis property* if for all subsets $S' \subseteq S$, $F(S')$ is finitely generated.

Theorem 7.3 (Folklore, after Higman [268]) *Let $\langle S, \leq \rangle$ be a quasi-ordered set. The following are equivalent:*

(i) *$\langle S, \leq \rangle$ has no bad sequences.*

(ii) *Every infinite sequence in S contains an infinite chain.*

(iii) *$\langle S, \leq \rangle$ is Noetherian and S contains no infinite antichain.*

(iv) *$\langle S, \leq \rangle$ has the finite basis property.*

Proof. (i) $\rightarrow$ (ii) and (ii) $\rightarrow$ (iii) are easy. (See Exercise 1.)
(iii) $\rightarrow$ (iv). Suppose that $\langle S, \leq \rangle$ is Noetherian and S contains no infinite antichain. Let F be a filter of S with no finite basis. We generate a sequence from F in stages as follows. Let f_0 be any element from F. As F is not finitely generated, there is some $f_1 \in F$ with $f_1 \notin F(\{f_0\})$. For step $i + 1$, having chosen $\{f_0, \ldots, f_i\}$, we find some $f_{i+1} \in F$ with $f_{i+1} \notin F(\{f_0, \ldots f_i\})$. Since F is not finitely generated, this process will not terminate. Notice that for all $i < j$, it cannot be that $f_i \leq f_j$ since, in particular, $f_j \notin F(\{f_i\})$. For $i < j$, color the pairs f_i, f_j *red* if $f_i > f_j$ and color the pair *blue* if $f_i | f_j$. (Every pair is colored either red or blue by the construction of the sequence.) By Ramsey's theorem,[3] there is an infinite homogeneous set; that is there is a sequence $f_{i_0} f_{i_1} \ldots$ and a color $\chi \in \{red, blue\}$, such that for all $i_k < i_t$, the pair f_{i_k}, f_{i_t} is colored χ. If χ is red, then we have an infinite de-

[3]Ramsey's Theorem (Ramsey [381]) states that if B is an infinite set and k is a positive integer, then if we colour the subsets of B of size k with colours chosen from $\{1, \ldots, m\}$, then there is an infinite subset $B' \subseteq B$ such that all the size k subsets of B' *have the same colour*. B' is referred to a *homogeneous* subset.

scending sequence. If χ is blue, we have an infinite antichain. In either case, we contradict (iii).
(iv) $\rightarrow$ (i) Suppose that $\langle S, \leq \rangle$ has the finite basis property. Let $B = \{b_0, b_1, \ldots\}$ be a sequence of elements of S. Let $B' \subseteq B$ be a finite basis for $F(B)$. Let $m = \max\{i : b_i \in B'\}$. Choose $b_j \in B - B'$ with $j > m$. Then, for some $b_i \in B'$, we must have $b_i \leq b_j$ since $b_j \in F(B')$, and $i < j$ by construction. Hence, B is good.

The several equivalent characterizations above lead to an important class of quasi-orderings.

Definition 7.4 (Well-Quasi-Ordering) Let $\langle S, \leq \rangle$ be a quasi-ordering. If $\langle S, \leq \rangle$ satisfies any of the characterizations of Theorem 7.3, then we say that $\langle S, \leq \rangle$ is a *well-quasi-ordering* (WQO).

The reader might well wonder what any of this abstract pure mathematics has to do with algorithmic considerations. The key is provided by the finite basis characterizations of a WQO. The answer is provided by the WQO principle below.

Well-Quasi-Ordering Principle Suppose that $\langle S, \leq \rangle$ is a WQO. Suppose further that for any x, the parameterized problem ABOVE(x) below is polynomial time computable. Then, for any filter F of $\langle S, \leq \rangle$, the decision problem "Is $y \in F$?" is polynomial time.

ABOVE(x)

Input: $y \in S$.
Parameter: $x \in S$.
Question: Is $x \leq y$?

The proof of the Well-Quasi-Ordering Principle is clear: If F is a filter, then F has a finite basis $\{b_1, \ldots, b_n\}$. Then, to decide if $y \in F$, we need only ask "$\exists i \leq n(b_i \leq_S y)$?" Since for each fixed i, ABOVE(b_i) is in P, the question "$\exists i \leq n(b_i \leq_S y)$?" is in P too.

Actually, the Well-Quasi-Ordering Principle is often stated in a dual form for ideals. If F is a filter of $\langle S, \leq \rangle$, then $S - F$ is an ideal, and vice versa.