Durham Year 1 Differential Equations Handout

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Problem sheets will be found in a separate document on the same webpage.
This document is not complete. It only contains Michaelmas content and lacks two non-examinable explanations.

1 Michaelmas

1.1 First order separable ODEs

If we have an ODE of the form:

$$f(y)y' = g(x)$$

Then we can integrate both sides to get

$$\int f(y)\mathrm{d}y = \int g(x)\mathrm{d}x.$$

Example 1.

$$y' = \frac{\cos x}{\sin y}$$

$$(\sin y)y' = \cos x$$

$$\int \sin y \mathrm{d}y = \int \cos x \mathrm{d}x$$

$$\cos y = C - \sin y$$

Example 2.

$$\sec x(\log y)y' = y$$

$$\frac{\log y}{y}y' = \cos x$$

$$\int \frac{\log y}{y}\mathrm{d}y = \int \cos x \mathrm{d}x$$

$$(\log u)^2 = 2\sin x + C$$

1.2 First order homogenous ODEs

A first order ODE is homogenous if we have that for

$$y' = f(x, y),$$

$$f(x, y) = f(tx, ty).$$

In this case, we use the substitution $y = ux$.

Example 1.

$$y' = \frac{x+y}{x-y}$$

$$y = ux \implies y' = u + xu'$$

$$u + xu' = \frac{x+ux}{x-ux} = \frac{1+u}{1-u}$$

$$\frac{1-u}{1+u^2}u' = \frac{1}{x}$$

$$\int \frac{1-u}{1+u^2}\mathrm{d}u = \int x\mathrm{d}x$$

$$2\arctan u - \log\left(1+u^2\right) = 2\log Ax$$

$$2\arctan\left(yx^{-1}\right) - \log\left(1+y^2x^{-2}\right) = 2\log Ax$$

Example 2.

$$y' = \frac{x^2 + y^2 \sin\frac{y}{x}}{xy\sin\frac{y}{x}}$$

$$y = ux \implies y' = u + xu'$$

$$u + xu' = \frac{x^2 + u^2x^2\sin u}{x^2 u \sin u}$$

$$u + xu' = \frac{1}{u\sin u} + u$$

$$(u\sin u)u' = \frac{1}{x}$$

$$\int u\sin u\mathrm{d}u = \int \frac{1}{x}\mathrm{d}x$$

$$\sin u - u\cos u = \log Ax$$

$$\sin yx^{-1} - \frac{y\cos yx^{-1}}{x} = \log Ax$$

1.3 Integrating factor

In the case that we have:

$$y' + p(x)y = q(x)$$

We can multiply the whole equation by

$$\exp\left(\int p(x)\mathrm{d}x\right).$$

Example 1.

$$y' + y\cot x = \cos x$$
$$\exp\left(\int \cot x\mathrm{d}x\right) = \sin x$$
$$y'\sin x + y\cos x = \cos x\sin x$$
$$(y\sin x)' = \frac{1}{2}\sin 2x$$
$$y\sin x = \int \frac{1}{2}\sin 2x\mathrm{d}x = -\frac{1}{4}\cos 2x + C$$
$$y = \frac{1}{2}\sin x + A\csc x$$

Example 2.

$$y' = \frac{y}{x} - y\cot x + \frac{x^2}{\sin x}$$
$$y' + (\cot x - x^{-1}))y = \frac{x^2}{\sin x}$$
$$\exp\left(\int \cot x - \frac{1}{x}\mathrm{d}x\right) = \exp\left(\log(\sin x) - \log x\right) = \frac{\sin x}{x}$$
$$\frac{\sin x}{x}y' + \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)y = x$$
$$\left(\frac{\sin x}{x}y\right)' = x$$
$$\frac{\sin x}{x}y = \int x\mathrm{d}x = \frac{1}{2}x^2 + C$$
$$y = \frac{Cx + x^3}{2\sin x}$$

1.4 Bernoulli equations

If we have an ODE of the form:

$$y' + p(x)y = q(x)y^n$$

Then we use the substitution

$$u = y^{1-n}$$

Example 1.

$$y' + y = y^{101}$$

$$u = y^{-100} \implies y = u^{-\frac{1}{100}} \implies y' = -\frac{1}{100}u^{-\frac{101}{100}}u'$$

$$-\frac{1}{100}u^{-\frac{101}{100}}u' + u^{-\frac{1}{100}} = u^{-\frac{101}{100}}$$

$$\frac{u'}{1+100u} = 1$$

$$\log(1+100u) = 100x + C$$

$$y = \left(Ae^{100x} - \frac{1}{100}\right)^{-\frac{1}{100}}$$

Example 2.

$$y' - \frac{y}{2(x^2+1)\arctan x} = xy^3$$

$$u = y^{-2} \implies y = u^{-\frac{1}{2}} \implies y' = -\frac{1}{2}u^{-\frac{3}{2}}$$

$$-\frac{1}{2}u^{-\frac{3}{2}}u' - \frac{u}{2(x^2+1)\arctan x} = xu^{-\frac{3}{2}}$$

$$u' + \frac{u}{(x^2+1)\arctan x} = -2x$$

$$u' \arctan x + \frac{u}{x^2+1} = -2x \arctan x$$

$$\int (u \arctan x)' = \int -2x \arctan x \mathrm{d}x$$

$$u \arctan x = x - \arctan x - x^2 \arctan x + C$$

$$y^{-2} = \frac{x - \arctan x - x^2 \arctan x + C}{\arctan x} \implies y^2 = \frac{\arctan x}{x - \arctan x - x^2 \arctan x + C}$$

1.5 First order exact ODEs

Suppose we have $M(x,y)+N(x,y)y'=0$. Then if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

our differential equation is exact. For an exact differential equation we have $g=c$ where

$$M=\frac{\partial g}{\partial x},N=\frac{\partial g}{\partial y}.$$

Example 1.

$$e^x+\log y+xy+\left(\frac{x}{y}+\frac{1}{2}x^2+e^y\right)y'=0$$

$$M(x,y)=e^x+\log y+xy,N(x,y)=\frac{x}{y}+\frac{1}{2}x^2+e^y$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=\frac{1}{y}+x$$

$$\frac{\partial g}{\partial x}=M \implies g=e^x+x\log y+\frac{1}{2}x^2y+\varphi(y)$$

$$\frac{\partial g}{\partial y}=N \implies N=\frac{x}{y}+\frac{1}{2}x^2+\varphi'(y)$$

$$\textit{So } \varphi'(y)=e^y \implies \varphi(y)=e^y$$

$$g=c \implies e^x+x\log y+\frac{1}{2}x^2y+e^y=c$$

Example 2.

$$y^2e^y + \left(xy^2e^y + xy\sin(xy) + \cos(xy)\right)\frac{dy}{dx} = y^2\sin(xy)$$

$$e^y - \sin(xy) + \left(xe^y - \frac{x\sin xy}{y} - \frac{\cos(xy)}{y^2}\right)\frac{dy}{dx} = 0$$

$$M(x,y) = e^y - \sin(xy), N(x,y) = xe^y - \frac{x\sin xy}{y} - \frac{\cos(xy)}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^y - x\cos(xy)$$

$$\frac{\partial g}{\partial x} = M \implies g = xe^y + \frac{\cos(xy)}{y} + \varphi(y)$$

$$\frac{\partial g}{\partial y} = N \implies N = xe^y - \frac{x\sin xy}{y} - \frac{\cos(xy)}{y^2} + \varphi'(y)$$

$$So\ \varphi'(y) = 0 \implies \varphi(y) = 0$$

$$g = c \implies xe^y + \frac{cos(xy)}{y} = c$$

1.6 Second order homogenous ODEs

If a second order ODE is of the form:

$$ay'' + by' + cy = 0,$$

Then it is homogenous. Letting $a\lambda^2 + b\lambda + c = 0$ with solutions λ_1 and λ_2,

$$b^2 - 4ac > 0 \implies y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

$$b^2 - 4ac = 0 \implies y = (A + Bx)e^{\lambda_1 x}$$

$$b^2 - 4ac < 0 \implies y = e^{\operatorname{Re}\lambda_1}[A\cos(|\operatorname{Im}\lambda_1|x) + B\sin(|\operatorname{Im}\lambda_1|x)]$$

Where $a, b, c \in \mathbb{R}$ and for $z = a + bi, \operatorname{Re} z = a, \operatorname{Im} z = b$. Note that in any case, λ_1 and λ_2 can be swapped.

Example 1.

$$y'' + 4y = 0$$

$$\lambda^2 + 4 = 0 \implies \lambda = \pm 2i$$

$$y = A\cos(2x) + B\sin(2x)$$

Example 2.

$$y'' - 5y + 6 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \implies \lambda = 2, 3$$

$$y = Ae^{2x} + Be^{3x}$$

1.7 Particular integral 'guessing'

For a non-homogenous ODE of the form:

$$ay'' + by' + cy = f(x),$$

The solution is given by $y_{CF} + y_{PI}$. y_{CF} is given by the solution to

$$ay'' + by' + cy = 0$$

And the particular integral may be 'guessed' according to $f(x)$. To get a grasp of how to perform a good 'guess', visit section 3.X

Example 1.

$$y'' - 6y' + 9y = x^3$$

$$y_{CF} = (A + Bx)e^{3x}$$

$$y_{PI} = ax^3 + bx^2 + cx + d$$

Notice that the only coefficient we need is of x^3 so we need not differentiate.

$$a = \frac{1}{9}, b = c = d = 0$$

$$y = (A + Bx)e^{3x} + \frac{1}{9}x^3$$

Example 2.

$$y'' - 4y' + 3 = 2e^{3x}$$

$$y_{CF} = Ae^{3x} + Be^{x}$$

$$y_{PI} = axe^{3x}$$

$$y'_{PI} = a(3x + 1)e^{3x}$$

$$y''PI = a(9x + 6)e^{3x}$$

$$6a - 4a = 2 \implies a = 1$$

$$y = Ae^{3x} + Be^{x} + e^{3x}$$

1.8 Variation of parameters

Define the Wronskian, $W(y_1, y_2) := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. For a differential equation of the form:

$$ay'' + by' + cy = \varphi(x)$$

With $y_{CF} = Ay_1 + By_2$ Then $y_{PI} = u_1 y_1 + u_2 y_2$ where:

$$u_1 = -\int \frac{y_2 \varphi(x)}{aW(y_1, y_2)}, u_2 = \int \frac{y_1 \varphi(x)}{aW(y_1, y_2)}$$

A derivation and explanation of this is given in section 3.X

Example 1.

$$y'' + 7y' + 10y = e^x \sin x$$

$$y_{CF} = Ae^{-5x} + Be^{-2x} = Ay_1 + By_2$$

$$W(y_1, y_2) = 3e^{-7x}$$

$$u_1 = -\frac{1}{3} \int e^{6x} \sin x \mathrm{d}x = \frac{e^{6x}(6 \sin x - \cos x)}{111}$$

$$u_2 = \frac{1}{3} \int e^{3x} \sin x \mathrm{d}x = \frac{e^{3x}(3 \sin x - \cos x)}{30}$$

$$\implies y_{PI} = \frac{e^x(6 \sin x - \cos x)}{111} + \frac{e^x(3 \sin x - \cos x)}{30} = \frac{171 \sin x - 47 \cos x}{1110} e^x$$

$$y = Ae^{-5x} + Be^{-2x} + \frac{171 \sin x - 47 \cos x}{1110} e^x$$

Example 2.

$$y'' + y = \tan x$$

$$y_{CF} = A \cos x + B \sin x = Ay_1 + By_2$$

$$W(y_1, y_2) = \cos^2 x + \sin^2 x = 1$$

$$u_1 = -\int \frac{\cos x \tan x}{1 \cdot 1} \mathrm{d}x = \cos x$$

$$u_2 = \int \frac{\sin x \tan x}{1 \cdot 1} \mathrm{d}x = \int \sec x - \cos x \mathrm{d}x = \log|\sec x + \tan x| - \sin x$$

$$y_{PI} = \cos^2 x - \sin^2 x + \sin(x) \log|\sec x + \tan x|$$

$$y = A \cos x + B \sin x + \cos^2 x - \sin^2 x + \sin(x) \log|\sec x + \tan x|$$

1.9 Systems of DEs

For the scope of this course, we only need to know how to solve coupled differential equations of the form:

$$z' = p_1(x)z + q_1(x)y$$

$$y' = p_2(x)z + q_2(x)y$$

With **constant** p_1, p_2, q_1, q_2. We may therefore relabel them as a_1, a_2, b_1, b_2 respectively. For a brief introduction to non-constant functions, check section 3.X. To solve coupled DEs of this form, we differentiate to get:

$$z'' = a_1 z' + b_1 y'$$

$$y'' = a_2 z' + b_2 y'$$

And then we may just substitute to get a second order differential equation which we may solve using one of the methods we've learned above.

Example 1.

$$y' = 2y + z$$

$$z' = y + 2z$$

$$y'' = 2y' + z', z'' = y' + 2z'$$

$$z'' = 2y + z + 2z' = 2z' - 4z + z + 2z' \implies z'' - 4z' + 3 = 0$$

$$z = Ae^x + Be^{3x}$$

$$But\ z' = y + 2z\ so\ Ae^x + 3Be^{3x} = y + 2Ae^x + 2Be^{3x}$$

$$y = Be^3x - Ae^x, z = Be^3x + Ae^x$$

Example 2.

$$y' = z - y + \log(\sec x + \tan x)$$

$$z' = y - z$$

$$y'' = z' - y' + \sec x, z'' = y' - z'$$

$$z'' + z = \sec x$$

Variation of parameters...

$$z_{CF} = A\cos x + B\sin x = Az_1 + Bz_2$$

$$W(z_1, z_2) = \sin^2 x + \cos^2 x = 1$$

$$u_1 = -\int \frac{\sec x \cos x}{1 \cdot 1} \mathrm{d}x = -\int \mathrm{d}x = -x$$

$$u_2 = \int \frac{\sec x \sin x}{1 \cdot 1} \mathrm{d}x = \int \tan x \mathrm{d}x = \log \sec x$$

$$z_{PI} = -x\cos x + \log(\sec x)\sin x$$

$$z = A\cos x + B\sin x - x\cos x + \log(\sec x)\sin x$$

$$\textit{But } z' = y - z \textit{ so}$$

$$y = (B - A + x + \tan x + \log \sec x)\sin x + (A + B - 1 - x + \log \sec x)\cos x$$

$$z = A\cos x + B\sin x - x\cos x + \log(\sec x)\sin x$$

2 Epiphany

2.1 Series solutions for second order linear ODEs

Example 1.

Example 2.

2.2 Frobenius' method

Example 1.

Example 2.

2.3 Fourier Transform application

Example 1.

Example 2.

2.4 Separation of variables for PDEs

Example 1.

Example 2.

3 Explanations and non-examinable content

3.1 1.1 Explanation

When we have some $f(y) = g(x)$ then differentiating we get $f'(y)y' = g'(x)$. Working backwards, we can infer that we may integrate the second equation to get what we started with also. This is the principle behind separating first order ODEs.

3.2 1.2 Explanation

If we have that:

$$y' = f(x, y)$$
$$f(x, y) = f(tx, ty)$$

Then

$$y = ux \implies y' = u + xu'$$

Subtituting back in, we yield

$$u + xu' = f(x, ux)$$

However, we know that $f(x, y) = f(tx, ty)$ so $f(x, ux) = f(1, u)$ which we can just denote by $g(u)$ so we obtain

$$u + xu' = g(u)$$

Which is now separable as

$$\frac{u'}{g(u) - u} = \frac{1}{x}$$

3.3 1.3 Explanation

Suppose we have:

$$y' + p(x)y = q(x)$$

We know that $\frac{\partial}{\partial x} e^{p(x)} = p'(x)e^{p(x)}$ So multiplying by $\exp\left(\int p(x)\mathrm{d}x\right)$ we yield

$$y' \exp\left(\int p(x)\mathrm{d}x\right) + p(x)\exp\left(\int p(x)\mathrm{d}x\right) y = q(x)\exp\left(\int p(x)\mathrm{d}x\right)$$

However, since $\frac{\partial}{\partial x} e^{p(x)} = p'(x)e^{p(x)}$ and also $(uv)' = u'v + uv'$,

$$\left(y \exp\left(\int p(x)\mathrm{d}x\right)\right)' = q(x)\exp\left(\int p(x)\mathrm{d}x\right).$$

So we can just find the antiderivative of both sides now to get

$$y \exp\left(\int p(x)\mathrm{d}x\right) = \int q(x)\exp\left(\int p(x)\mathrm{d}x\right)\mathrm{d}x.$$

3.4 1.4 Explanation

Suppose we have

$$y' + p(x)y = q(x)y^n$$

Then,

$$u = y^{1-n} \implies u' = (1-n)y^{-n}y' = (1-n)u^{\frac{-n}{1-n}}y'$$

Substituting this, we get

$$(1-n)^{-1}u^{\frac{n}{1-n}}u' + p(x)u^{\frac{1}{1-n}} = q(x)u^{\frac{n}{1-n}}.$$

Now we may multiply by $u^{\frac{-n}{1-n}}$ to get

$$(1-n)^{-1}u' + p(x)u = q(x).$$

Which can now be solved using an integrating factor.

3.5 1.5 Explanation

For a function $g(\mathbf{x})$ where $\mathbf{x} = x_1 + x_2 + ...x_n$, we define the total differential dg to be:

$$dg = \frac{\partial g}{\partial x_1}dx_1 + \frac{\partial g}{\partial x_2}dx_2 + ...\frac{\partial g}{\partial x_n}dx_n$$

Now, we only need to thinking of this two dimensions for this course, so we may write that for $g(x, y)$,

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy$$

Now note that

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx}$$

We say that a differential form is *exact* if it is equal to the total differential. In other words,

$$M(x, y) + N(x, y)y' = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}y'$$

. From this, we can gather that for exactness, we need a function $g(x, y)$ that satisfies:

$$M = \frac{\partial g}{\partial x}, N = \frac{\partial g}{\partial y}$$

Now, if we have an equation of the form $M(x, y) + N(x, y)y' = 0$ then clearly $dg = 0$ too, implying $g = c$ for some constant c.
Due to the equality of mixed partial derivatives, namely $g_{xy} = g_{yx}$ and also $M = g_x, N = g_y$, we require

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Which results in an ODE of the form $M(x, y)+N(x, y)y' = 0$ being exact iff $M_y = N_x$. Using these definitions all together, we can start solving differential equations which are exact. Note that when we integrate with respect to one variable, instead of having a $' + C'$, we instead gain a $' + \varphi(y)'$ due to the fact that any function of y is treated as a constant when taking the partial derivative with respect to x.

3.6 1.6 Explanation

A natural guess for the solution of a second order differential equation would be one of the form $y = Ae^{\lambda x}$ due to the nature of its derivatives. But how do we know that in particular, for

$$ay'' + by' + cy = 0,$$

the solutions are going to include the roots of

$$a\lambda^2 + b\lambda + c?$$

First let us note that if we guess a solution of the form $y = e^{\lambda x}$, we get that

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0.$$

Now, since e^{Bx} can never be equal to 0, we have that

$$a\lambda^2 + b\lambda + c = 0,$$

which is exactly what we wanted to get! Now, you may be wondering why the A has been omitted. Notice that if we did keep it in, it would only scale our equation by a factor of A, which we would be able to cancel. For the two cases where the $b^2 - 4ac = 0$ and $b^2 - 4ac < 0$, the work done is similar. In fact, while it may seem that the negative discriminant case seems the most irregular, note that the repeated roots case is the only "strange" case. As a reader, if you cannot see why this is true, try to prove that these are indeed the solutions to this differential equation.

3.7 1.7 Explanation

Let us explore this in two parts. Firstly, lets tackle the complimentary function y_{CF}. Note that if we have

$$ay'' + by' + cy = f(x)$$

and our complimentary function is the solution of $ay'' + by' + cy = 0$, then having our complimentary function added to our particular integral, nothing changes due to the fact that we're just adding on 0. As for the particular integral, also sometimes called the particular solution, this is a solution that **is not** the complimentary function but *also* solves the given differential equation. There are various tactics of guessing this particular solution. Mainly, the thing to keep in mind is "what function can we differentiate once or twice and then possibly retain our function". This may sound confusing, but going back to the examples, you may see that this is exactly what's been happening. Now, in addition to this, if our particular solution shares a term with with the complimentary function, then we need to add an "x" to the front of our particular solution. This is due to the fact that with this shared term, its possible that we "cancelled" a term with an x in front with out complimentary function and the function itself. Below is a table of some common particular integrals.

$f(x)$	y_{PI}
$a_0 + a_1 x + ... + a_n x$	$b_0 + b_1 x + ... + b_n x$
$ae^{\gamma x}$	$be^{\gamma x}$
$a_0 \cos \gamma x + a_1 \sin \gamma x$	$b_0 \cos \gamma x + b_1 \sin \gamma x$
$e^{\gamma x}(a_0 \cos \delta x + a_1 \sin \delta x)$	$e^{\gamma x}(b_0 \cos \delta x + b_1 \sin \delta x)$

3.8 1.8 Explanation

Sometimes, guessing the particular integral may not be easy. Or in fact, sometimes our particular integral may be non elementary. A method to work out our particular integral instead of guessing is using the *method of variation of parameters.* Suppose that we have

$$ay'' + by' + cy = \varphi x$$

Now, we have that $y_C F = Ay_1 + By_2$ as the solution to the homogenous version of the given ODE. We look to replace A and B with functions of x which will help solve the equation. In other words, we are slightly adjusting our complimentary function to match our differential equation with intent of finding the particular solution. Let us call those functions u_1, u_2 such that

$$y = u_1 y_1 + u_2 y_2$$

What we're looking for is to make this satisfy our differential equation. With this, we need to find expressions for both y'' and y'.

$$y' = u_1' y + u_1 y' + u_2' y + u_2 y'$$

Now, we may be tempted to differentiate again and plug it into our differential equation, but we also notice that this is getting messy. Additionally, notice that we only have one condition and we have two arbitrary functions so we need to impose one further condition. Now, it would be useful to not create another second order differential equation, so lets impose the condition $u_1' y_1 + u_2' y_2 = 0$. It's almost too good to be true, it simplifies down our equations nicely and we no longer have a huge mess. You may be thinking "*but how do we know that's true?*". The thing is, we don't. At least not yet. The proof of this *ansatz* is beyond the scope of the course, however will be given for curious readers at the end. With our simplification, we now have

$$y' = u_1 y_1' + u_2 y_2'$$

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Substituting back in, we get

$$a(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2') + b(u_1 y_1' + u_2 y_2') + c(u_1 y_1 + u_2 y_2) = \varphi(x)$$

This does not look very nice, but we know that both y_1 and y_2 satify our **homogenous** differential equation. After grouping some terms we get

$$u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1' y_1' + u_2' y_2') = \varphi(x)$$

$$\implies a(u_1' y_1' + u_2' y_2') = \varphi(x)$$

Let us remind ourselves that $u_1' y_1 + u_2' y_2 = 0$. With this, we get simulatenous equations which solve to give us:

$$u_1' = -\frac{y_2 \phi(x)}{a(y_1 y_2' - y_2 y_1')}, u_2' = -\frac{y_1 \phi(x)}{a(y_1 y_2' - y_2 y_1')}$$

But since $W(y_1, y_2) = (y_1 y_2' - y_2 y_1')$,

$$u_1' = -\frac{y_2 \phi(x)}{aW(y_1, y_2)}, u_2' = -\frac{y_1 \phi(x)}{aW(y_1, y_2)}$$

We can now integrate and get what we require.

Ansatz - NON-EXAMINABLE

Why $u_1' y_1 + u_2' y_2 = 0$ [will add later]

3.9 1.9 Explanation

We are now looking coupled differential equations. In this course, we only look at simple coupled differential equations of the form

$$y' = ay + bz$$

$$z' = cy + dz$$

The essence of solving these comes to differentiating. The reason why is because we can freely rearrange either one of the equations. For example, we can simply obtain

$$bz = y' - ay$$

$$bz' = y'' - ay'$$

and then substitute into the second equation to get a second order differential equation. Now, the thing to note is that it doesnt actually matter which way around we substitute. If we wanted to get a second order differential equation in z instead of y, thats completely alright and you'll end up getting the same answer as shown in the two examples.
Harder systems - NON-EXAMINABLE
This will contain an explanation on harder systems, systems with more than two equations and more than two dependent variables. It will also cover an alternate method to solving these types of differential equations, providing some more reasoning for the ansatz provided in the previous section [will add later]