Exercise 1.
If $z = x + iy$ is a complex number with $x, y \in \mathbb{R}$, define

$$|z| = \sqrt{x^2 + y^2},$$

and call this the *modulus* or *absolute value* of z.

(a) What is the geometric interpretation of $|z|$?

(b) Show that if $|z| = 0$, then $z = 0$.

(c) Show that if $\lambda \in \mathbb{R}$, then $|\lambda z| = |\lambda|\,|z|$, where $|\lambda|$ is the ordinary absolute value.

(d) If z_1, z_2 are two complex numbers, prove that

$$|z_1 z_2| = |z_1||z_2|, \qquad |z_1 + z_2| \leq |z_1| + |z_2|.$$

(e) Show that if $z \neq 0$, then $\left|\frac{1}{z}\right| = \frac{1}{|z|}$.

Solution

(a) The distance from $(0, 0)$ to (x, y) on the Euclidian plane.

(b) $|z| = \sqrt{x^2 + y^2} = 0$, so $x^2 + y^2 = 0$, but squares of real numbers are nonnegative, so $x = y = 0$.

(c) $|\lambda z| = |\lambda x + \lambda y i| = \sqrt{(\lambda x)^2 + (\lambda y)^2} = \lambda\sqrt{x^2 + y^2} = \lambda|z|$.

(d) The inequality arises from the triangle inequality in the geometric interpretation. The equality is derived as follows:

$$|z_1 z_2| = |(x_1 + y_1 i)(x_2 + y_2 i)| = |x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1)i| =$$
$$\sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = |z_1||z_2|$$

(e) $\left|\frac{1}{z}\right| = \left|\frac{x - yi}{x^2 + y^2}\right| = \frac{1}{x^2+y^2}|x - yi| = \frac{1}{x^2+y^2}\sqrt{x^2 + y^2} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{|z|}$

Exercise 2.
If $z = x + iy$ with $x, y \in \mathbb{R}$, define the complex conjugate

$$\overline{z} = x - iy.$$

(a) What is the geometric interpretation of $\overline{z}$?

(b) Show that $|z|^2 = z\overline{z}$.

(c) Prove that if z lies on the unit circle, then $\frac{1}{z} = \overline{z}$.

Solution

(a) Reflection of z by the x axis on the complex plane.

(b) $|z|^2 = x^2 + y^2 = (x + yi)(x - yi)$.

(c) $\frac{1}{z} = \frac{1}{x+yi} = \frac{x-yi}{x^2+y^2} = x - yi = \overline{z}$.

Exercise 3.
A sequence $\{w_n\}_{n=1}^{\infty}$ of complex numbers is said to converge if there exists $w \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |w_n - w| = 0.$$

We then call w the limit of the sequence.

(a) Show that a convergent sequence of complex numbers has a unique limit.

A sequence $\{w_n\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists N such that

$$|w_n - w_m| < \varepsilon \quad \text{whenever } n, m > N.$$

(b) Prove that a sequence of complex numbers converges if and only if it is a Cauchy sequence. [Hint: Recall the analogous result for real numbers.]

(c) Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $\sum_n a_n$ converges. Show that if $\{z_n\}$ is a sequence of complex numbers satisfying $|z_n| \leq a_n$ for all n, then $\sum_n z_n$ converges. [Hint: Use the Cauchy criterion.]

Solution

(a) If there where two limits l_1, l_2, then there must be an n such that $|l_1 - w_n| < \frac{1}{2}|l_1 - l_2|$ and $|w_n - l_2| < \frac{1}{2}|l_1 - l_2|$, but then by the triangle inequality $|l_1 - l_2| > |l_1 - w_n| + |w_n - l_2| \geq |l_1 - l_2|$.

(b) If $\{w_n\}$ is a Cauchy sequence, then there exists an n such that for $m \geq n$ we have $|w_n - w_m| < 1$, so the sequence is bounded in both the real and imaginary parts. The sequence is infinite, so there must be an infinite subsequence with monotonic real parts, and that subsequence must have an infinite subsequence with monotonic imaginary parts. By the monotone convergence theorem, both the real parts x_i and imaginary parts y_i of w_i must converge to some limits, say x and y, respectively. Then for every ε there must be an n such that $|x_m - x_n| < \frac{\sqrt{2}}{2}\epsilon$ and $|y_m - y_n| < \frac{\sqrt{2}}{2}\epsilon$ for all $m \geq n$ in the subsequence. Then $|w_m - w| = |w_m - (x + yi)| \leq \varepsilon$ for such m. This means that for all ε there is an w_n from the subsequence such that $|w_n - w| < \frac{\varepsilon}{2}$ and $|w_m - w_n| < \frac{\varepsilon}{2}$ for all $m \geq n$ (because of the Cauchy property), so by the triangle inequality $|w_m - w| < \varepsilon$ for all m. The implication in the other direction is trivial.

(c) Since $\sum_n a_n$ converges, it is a Cauchy sequence, meaning that for every ε there exists an n such that for all $m \geq n$ we have (from the triangle inequality) $|(\sum_1^m z_i) - (\sum_1^n z_i)| = |\sum_n^m z_i| \leq \sum_n^m |z_i| \leq \sum_n^m a_i = |\sum_n^m a_i| < \varepsilon$, so by the Cauchy criterion we have that $\sum_n z_n$ converges.

Exercise 4.
For $z \in \mathbb{C}$, define the complex exponential by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(a) Prove that this series converges for every complex z, and that the convergence is uniform on every bounded subset of $\mathbb{C}$.

(b) Show that for any $z_1, z_2 \in \mathbb{C}$, we have $e^{z_1} e^{z_2} = e^{z_1+z_2}$.

(c) If $z = iy$ with $y \in \mathbb{R}$, show that $e^{iy} = \cos y + i \sin y$ (Euler's identity).

(d) Show that for $x, y \in \mathbb{R}$,

$$e^{x+iy} = e^x(\cos y + i \sin y), \quad \text{and} \quad |e^{x+iy}| = e^x.$$

(e) Prove that $e^z = 1$ if and only if $z = 2\pi k i$ for some integer k.

(f) Show that every complex number $z = x+iy$ can be written as $z = re^{i\theta}$, with $r = |z|$ and θ unique up to multiples of 2π.

(g) In particular, $i = e^{i\pi/2}$. What is the geometric meaning of multiplication by i, or by $e^{i\theta}$?

(h) Show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

(i) Use the exponential form to derive trigonometric identities such as

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$

and also

$$2\sin\theta\sin\phi = \cos(\theta - \phi) - \cos(\theta + \phi), \quad 2\sin\theta\cos\phi = \sin(\theta + \phi) + \sin(\theta - \phi).$$

Solution

(a) Let a be the supremum of all $|z|$ on the bounded subset. Then we may denote $z_i = \frac{z^n}{n!}$, $a_i = \frac{a^n}{n!}$. The previous exercises' solution shows that $\sum_{n=0}^{\infty} z_i$ converges pointwise, but we can trace the proof for the Cauchy criterion to get a bound. Let $w_n = \sum_{i=0}^{n} z_i$, and let w be the pointwise limit. If $|w_m - w_n| \leq \sum_{i=n-1}^{m} a_i < \varepsilon$ for all $m > n$, then $|w_n - w| < \varepsilon$. This means that w_i converge as ε converges, and ε is dependent only on a, so the series converges uniformly.

(b) We have

$$e^{z_1+z_2} = \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{z_1^m z_2^{n-m}}{m!(n-m)!} =$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z_1^n z_2^m}{n!m!} = \left(\sum_{n=0}^{\infty} \frac{z_1^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{z_2^n}{n!}\right) = e^{z_1}e^{z_2}$$

(c) We have

$$\cos y + i\sin y = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n}\right) + i\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} y^{2n+1}\right) =$$

$$\left(\sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!}\right) + \left(\sum_{n=0}^{\infty} \frac{(yi)^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} \frac{(yi)^n}{n!} = e^{yi}$$

(d) $|e^{x+yi}| = |e^x||e^{yi}| = e^x|\cos y + i\sin y| = e^x\sqrt{\cos^2 y + \sin^2 y} = e^x\sqrt{1} = e^x$.

(e) From exercise 4(d) we have that since the absolute value is 1, then the real part of the exponent is 0. We are left with $e^{yi} = \cos y + i\sin y = 1$, which only holds when $\cos y = 1$ and $\sin y = 0$, which is only true when y is of the form $2\pi k$.

(f) .

(g) It means rotating it by the angle θ, because when multiplying, the angles in the exponents sum up.

(h) $\frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta + \cos\theta + i\sin(-\theta)}{2} = \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} = \cos\theta$. A similar argument holds for the sine function.

(i) I'm skipping this.

Exercise 5.
Verify that $f(x) = e^{inx}$ is periodic with period 2π and that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Use this to show that for $n, m \geq 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \begin{cases} 0, & n \neq m, \\ 1, & n = m, \end{cases}$$

and similarly for $\sin nx, \sin mx$. Finally, show that

$$\int_{-\pi}^{\pi} \sin nx \cos mx\,dx = 0.$$

Solution
$f(x)$ is periodic because $f(x+2\pi) = e^n i(x+2\pi) = f(x)e^n i2\pi = f(x)$. When $n = 0$, then $e^i nx = 1$ for all x, so that case is trivial.