Exercise 4.1.
Prove that if

$$P(D_1,\ldots,D_m|H_i)=\prod_j P(D_j|H_i),\quad 1\le i\le n, \tag{1}$$

and

$$P(D_1,\ldots,D_m|\overline{H_i})=\prod_j P(D_j|\overline{H_i}),\quad 1\le i\le n, \tag{2}$$

hold with $n>2$, then at most one of the factors

$$\frac{P(D_1|H_i)}{P(D_1|\overline{H_i})},\ldots,\frac{P(D_m|H_i)}{P(D_m|\overline{H_i})} \tag{3}$$

is different from unity, therefore at most one of the data sets D_j can produce any updating of the probability for H_i.

Solution
Let's consider the case for three hypotheses H_1,H_2,H_3 and two events D_1,D_2. We use the relation for mutually exclusive B,C:

$$P(A|B+C)=\frac{P(B)P(A|B)+P(C)P(A|C)}{P(B)+P(C)} \tag{4}$$

$\overline{H_3}=H_1+H_2$, so (2) becomes

$$P(D_1D_2|H_1+H_2)=P(D_1|H_1+H_2)P(D_2|H_1+H_2)$$

Using the formula (4), we get

$$\frac{P(H_1)P(D_1D_2|H_1)+P(H_2)P(D_1D_2|H_2)}{P(H_1)+P(H_2)}=$$
$$\frac{P(H_1)P(D_1|H_1)+P(H_2)P(D_1|H_2)}{P(H_1)+P(H_2)}\cdot\frac{P(H_1)P(D_2|H_1)+P(H_2)P(D_2|H_2)}{P(H_1)+P(H_2)}$$

Or, after using (1), cancelling denominators and rearranging terms:

$$(P(H_1)+P(H_2))(P(H_1)P(D_1|H_1)P(D_2|H_1)+P(H_2)P(D_1|H_2)P(D_2|H_2))=$$
$$(P(H_1)P(D_1|H_1)+P(H_2)P(D_1|H_2))(P(H_1)P(D_2|H_1)+P(H_2)P(D_2|H_2))$$

Expanding and cancelling terms (we can assume that $P(H_i)>0$) gives:

$$P(D_1|H_1)P(D_2|H_1)+P(D_1|H_2)P(D_2|H_2)=P(D_1|H_1)P(D_2|H_2)+P(D_1|H_2)P(D_2|H_1)$$

This equation is of the form $ab+cd=ac+bd$, which implies $a=c\vee b=d$, so

$$P(D_1|H_1)=P(D_1|H_2)\vee P(D_2|H_1)=P(D_2|H_2)$$

This is implied by (2) for $i=3$, but it must be true for any i. There are three possible values of i, so there must be a $j\in\{1,2\}$ such that the above equation is satisfied for D_j for two different values of i. WLOG let it be $j=1$; then we have

$$P(D_1|H_1) = P(D_1|H_2) = P(D_1|H_3)$$

This in turn implies, using (4):

$$P(D_1|\overline{H_3}) = P(D_1|H_1 + H_2) = P(D_1|H_3)$$