Takagi–Sugeno Fuzzy Observer Design for Nonlinear Descriptor Systems With Unmeasured Premise Variables and Unknown Inputs

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Abstract

This paper presents a new observer design framework for a class of nonlinear descriptor systems with unknown but bounded inputs. In the presence of *unmeasured* nonlinearities, *i.e.*, premise variables, designing nonlinear observers is known as particularly challenging. To solve this problem, we rewrite the nonlinear descriptor system in the form of a Takagi-Sugeno (TS) fuzzy model with nonlinear consequents. This model reformulation enables an effective use of the differential mean value theorem to deal with the *mismatching* terms involved in the estimation error dynamics. These nonlinear terms, issued from the unmeasured nonlinearities of the descriptor system, cause a major technical difficulty for TS fuzzy-model-based observer design. The descriptor form is treated through a singular redundancy representation. For observer design, we introduce into the Luenberger-like observer structure a virtual variable aiming at estimating the one-step ahead state. This variable introduction allows for free-structure decision variables involved in the observer design to further reduce the conservatism. Using Lyapunov-based arguments, the observer design is reformulated as an optimization problem under linear matrix inequalities with a single line search parameter. The estimation error bounds of both the state and the unknown input can be minimized by means of a guaranteed $\ell_\infty-$gain performance level. The interests of the new ℓ_∞ TS fuzzy observer design are clearly illustrated with two physically motivated examples.

Index Terms

Takagi-Sugeno models, fuzzy-model-based observers, nonlinear descriptor systems, unmeasured premise variables, unknown inputs, Lyapunov method.

I. Introduction

OBTAINING the real-time state information of dynamical systems is crucial either for feedback control design or for decision-making and supervision purposes [1]. Hence, state estimation for dynamical systems is a fundamental problem in control theory and applications. To this end, numerous approaches have been proposed to design full-order or reduced-order observers, with or without unknown inputs (UIs), for both linear and nonlinear systems, see [2]–[7] and references therein. Since the pioneer work on linear time-invariant systems [8], numerous extensions have been developed for nonlinear cases [2]. Although satisfactory solutions have been achieved for linear systems, finding a unified and systematic method for observer design of nonlinear systems is still widely open [9]. A promising idea to overcome this challenging issue is to represent highly nonlinear systems by Takagi-Sugeno (TS) fuzzy models [10]. Note that TS fuzzy models can also be considered as *polytopic* quasi-linear parameter varying (quasi-LPV) models whose scheduling variables correspond to premise variables in TS fuzzy case. The analogies between TS and LPV systems can be found in [11]. Since TS fuzzy systems are constructed from local linear models blending with nonlinear membership functions (MFs) [12], TS fuzzy observer design can be derived using Lyapunov method [10] and linear matrix inequality (LMI) formulation [13]. Moreover, from the viewpoint of estimation problem formulation, unmodeled dynamics, uncertain disturbances and faults in engineering systems, or attack signals in cyber-physical systems can be considered as unknown inputs [14], [15]. As a result, designing TS fuzzy observers in the presence of UIs has been an active research topic in the areas of fault estimation and diagnosis [16]–[20], fault-tolerant control [21], [22].

TS fuzzy-model-based observer design can be classified into the two following categories [9]:

- Category 1: TS fuzzy observer design with *measurable* premise variables, *i.e.*, system nonlinearities, and
- Category 2: TS fuzzy observer design with *unmeasurable* premise variables.

The first design category is a direct extension of the linear Luenberger-based method [10]. Then, the goal focuses on reducing the design conservatism via the use of various Lyapunov candidate functions and/or relaxation techniques [23]. However, assuming that all the premise variables can be measured from sensors, the TS fuzzy observer design in Category 1 can only be applied to a restrictive class of nonlinear systems. To overcome this drawback, research attention has been paid to the TS fuzzy observer design in Category 2. The technical challenge here is to deal with the *mismatching* nonlinear term involved

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in the estimation error dynamics, which is caused by unmeasured premise variables. A solution for this issue is to simply consider the mismatching term as a disturbance of the error dynamics [24], which should be minimized through an $\mathscr{L}_2-$gain performance. Note that this solution cannot guarantee an asymptotic convergence of the estimation errors. Moreover, since the disturbance *explicitly* depends on the system state and control input, its effects on the error dynamics strongly depend on the behaviors of the nonlinear system. The most common approach, guaranteeing the asymptotic estimation performance, is based on the Lipschitz property of the mismatching nonlinear term, see [25], [26] and related references. Despite its simplicity, this approach leads to conservative design results, especially for nonlinear systems with a large Lipschitz constant [27]. By decomposing the space of premise variables into crisp regions, an $\mathscr{H}_\infty$ method to design piecewise TS fuzzy observers has been proposed in [28]. Note that the proposed piecewise region decomposition may induce difficulties for real-time implementation of complex nonlinear systems. An immersion-based method has been proposed in [27] to avoid the issue of unmeasured premise variables. Unfortunately, this method can only be applied to a very restrictive class of nonlinear systems due to the non-convergence of the required dynamics-extension algorithm. Recently, using the differential mean value theorem (DMVT), the authors in [29] have proposed a new method to deal with the mismatch caused by unmeasurable MFs. The resulting TS fuzzy observer design requires a norm-bounded uncertainty approach to handle the time-varying terms stemmed from the use of DMVT. As a consequence, the proposed method leads to a complex design framework which may also induce conservative results. Note that most of the existing observer design results are usually based on the assumption that the premise variables are perfectly known. However, this is not the case in many practical situations, *e.g.*, due to inaccurate sensors. Within the LPV framework, significant advances have been achieved in designing LPV observers/filters with imperfect information on the scheduling parameters [30]–[33]. However, this topic is still widely open for TS fuzzy-model-based observer design [34], for which the premise variables directly depend on the state vector.

Descriptor models can naturally represent a large class of physical systems. The descriptor forms appear in various real-world applications including robotics [35], rehabilitation systems [36], [37], transportation and power systems [38], etc. Over the years, observer designs for linear descriptor systems have been extensively studied with significant advances, see for instance [38]–[40]. Various extensions have been proposed to LPV descriptor systems [41]–[43], and TS fuzzy descriptor systems [21], [23], [36], [44], [45]. However, dealing with the descriptor forms in LPV and/or TS fuzzy observer designs still remains a major challenge. Indeed, most of existing results can only be applied to systems with a linear descriptor matrix [21], [41]–[44]. In particular, up to now all results on TS fuzzy descriptor observers have mainly been concerned with the case of measured premise variables, which greatly restricts their real-world applicability.

Motivated by the above technical and practical issues, this paper provides a new framework to design discrete-time TS fuzzy observers for a large class of descriptor systems with unmeasurable nonlinearities and UIs. Note that existing TS fuzzy observer designs are *exclusively* based on the classical TS fuzzy modeling with linear consequent [10], which causes major difficulties in dealing with unmeasured MFs. To avoid this drawback, we rewrite the nonlinear descriptor system as a TS fuzzy descriptor model with nonlinear consequents [26], [46], called N-TS fuzzy model, whose MFs are measured. In particular, the unmeasured nonlinearities of the original descriptor system are isolated in local nonlinear consequents. These features of the proposed N-TS fuzzy reformulation allow for an effective use of the differential mean value theorem [47] to deal with the mismatching nonlinear term involved in the estimation error dynamics. To deal with the descriptor form, the singular redundancy approach [48] is exploited to design Luenberger-like fuzzy observers. Moreover, a virtual variable, representing the estimate of the one-step ahead system state, is introduced into the descriptor observer structure [45]. Together with special convexification techniques, this virtual variable allows for free-structure observer gains to reduce the design conservatism. Using Lyapunov stability arguments, the design conditions of N-TS fuzzy descriptor observers can be represented as LMI constraints, which can be effectively solved with available numerical solvers [13]. The estimation errors of both the state and the UI can be minimized by optimizing a guaranteed $\ell_\infty-$gain performance. Moreover, the proposed N-TS fuzzy observer design does not require any specific rank condition for UI decoupling as in [20], [21], [44], [49]. Hence, our *systematic* LMI-based observer design framework can be applied to a large class of descriptor systems with UIs and unmeasured nonlinearities. This main contribution has not been yet observed in any previous work in the open literature.

The paper is organized as follows. Section II formulates the observer design problem with useful tools for theoretical developments. Section III provides sufficient LMI-based conditions to design N-TS fuzzy-model-based observers for the class of nonlinear descriptor systems. The reconstruction of the unknown input with a guarantee on $\ell_\infty-$norm bounded error is also presented. In Section V, two physically motivated examples are given to show the interests of the proposed design framework. Section VI concludes the paper and discusses some future works.

Notation. $\mathbb{N}$ denotes the set of non-negative integers, and $\mathcal{I}_p = \{1, 2, ..., p\} \subset \mathbb{N}$. For a matrix X, $X^\top$ denotes its transpose, $X \succ 0$ (respectively $X \prec 0$) means that X is symmetric positive (respectively negative) definite, $\mathrm{He}X = X + X^\top$, and $\|X\|$ denotes its maximum singular value. For a vector $x \in \mathbb{R}^n$, we denote its 2-norm as $\|x\| = \sqrt{x^\top x}$. For a sequence of vectors $\{x_k\}_{k\in\mathbb{N}}$ (respectively matrices $\{X_k\}_{k\in\mathbb{N}}$), we denote $\|x\|_{\ell_\infty} = \sup_{k\geq 0} \|x_k\|$ (respectively $\|X\|_\infty = \sup_{k\geq 0} \|X_k\|$). Then, $\{x_k\}_{k\in\mathbb{N}} \in \ell_\infty$ if $\|x\|_{\ell_\infty} < \infty$ and $\{X_k\}_{k\in\mathbb{N}} \in \mathcal{L}_\infty$ if $\|X\|_\infty < \infty$. I is the identity matrix of appropriate dimension. "$\star$" stands for the terms deduced by symmetry. $\nabla_z f = \dfrac{\partial f}{\partial z}$ is the gradient of function f with relation to the variables in vector $z \in \mathbb{R}^\kappa$ and represents the row vector $\begin{bmatrix} \frac{\partial f}{\partial z_1} & \cdots & \frac{\partial f}{\partial z_\kappa} \end{bmatrix}$. $\mathrm{co}(\cdot)$ stands for the convex hull of a set of points. Consider multiple indices $(\kappa_1,\dots,\kappa_n)$ in $\mathcal{I}_{p_1}\times\cdots\times\mathcal{I}_{p_n}$. For simplicity, we can define a single index j in $\mathcal{I}_r$ with $r=\prod_{i=1}^n p_i$ representing these multiple indices by defining a one-to-one relationship between j and $(\kappa_1,\dots,\kappa_n)$. Whenever this multi-index notation is used, we consider $j\{i\}$ to represent the ith term of the n-tuple $(\kappa_1,\dots,\kappa_n)$ that corresponds to j. For example, consider an index $j\in\mathcal{I}_4$ that represents the tuple $(\kappa_1,\kappa_2)\in\mathcal{I}_2\times\mathcal{I}_2$ with $j=1\to(1,1)$, $j=2\to(1,2)$, $j=3\to(2,1)$ and $j=4\to(2,2)$. In this case, for $j=3$, $j\{1\}=2$ and $j\{2\}=1$. Throughout the paper, the argument of a function is omitted when its meaning is clear.

II. Problem Formulation

Based on a singular redundancy approach, this section formulates the ℓ_∞ observer design of nonlinear descriptor systems.

A. Description of N-TS Fuzzy Descriptor Systems

We consider the observer design problem of the following class of discrete-time nonlinear descriptor system:

$$\begin{aligned}\mathcal{E}(x_k)x_{k+1}&=\mathfrak{f}(x_k,u_k,d_k),\\ y_k&=\mathfrak{g}(x_k,u_k),\end{aligned}\tag{1}$$

where $x_k\in\mathscr{D}_x\subseteq\mathbb{R}^{n_x}$ is the state, $u_k\in\mathscr{D}_u\subseteq\mathbb{R}^{n_u}$ is the *known* input, $d_k\in\mathscr{D}_d\subseteq\mathbb{R}^{n_d}$ is the *unknown* input, and $y_k\in\mathbb{R}^{n_y}$ is the output. The nonlinear functions $\mathfrak{f}:\mathscr{D}_x\times\mathscr{D}_u\times\mathscr{D}_d\to\mathbb{R}^{n_x}$ and $\mathfrak{g}:\mathscr{D}_x\times\mathscr{D}_u\to\mathbb{R}^{n_y}$ are differentiable with respect to the state x_k. Here, the descriptor matrix $\mathcal{E}(x_k)$ is regular for $\forall x_k\in\mathscr{D}_x$ to guarantee a unique solution for system (1).

Remark 1. Various nonlinear systems in mechatronics and robotics engineering can be modeled in the form (1), for instance Lagrangian-Euler systems [35], [50], where $\mathcal{E}(x_k)$ represents the inertia matrix [36]. Due to the regularity of $\mathcal{E}(x_k)$, it is possible to transform the descriptor system (1) into the classical state-space representation, *i.e.*, $x_{k+1}=\mathcal{E}(x_k)^{-1}\mathfrak{f}(x_k,u_k,d_k)$. However, keeping the descriptor form for TS fuzzy-model-based designs allows for a significant reduction of both the number of fuzzy rules and that of LMI-based design conditions, see [45], [48] and related references.

Let us denote $\bar{x}_k=\begin{bmatrix}x_k^\top & x_{k+1}^\top\end{bmatrix}^\top$, and $z_k=z(y_k)\in\mathbb{R}^{n_z}$ a vector of measurable premise variables of system (1), which are bounded in a compact set $\mathscr{D}_x$ of the state space. Inspired by the N-TS fuzzy modeling [26], [51], we assume that system (1) can be reformulated in the form

$$\begin{aligned}E(z_k)x_{k+1}+J(z_k)\lambda(\bar{x}_k)&=A(z_k)x_k+f(z_k,u_k)+F(z_k)\phi(x_k,u_k)+B(z_k)d_k,\\ y_k&=C(z_k)x_k+g(z_k,u_k)+G(z_k)\psi(x_k,u_k),\end{aligned}\tag{2}$$

where the nonlinear vector-valued functions $f:\mathscr{D}_x\times\mathscr{D}_u\to\mathbb{R}^{n_x}$, $g:\mathscr{D}_x\times\mathscr{D}_u\to\mathbb{R}^{n_y}$, $\phi:\mathscr{D}_x\times\mathscr{D}_u\to\mathbb{R}^{n_\phi}$, $\psi:\mathscr{D}_x\times\mathscr{D}_u\to\mathbb{R}^{n_\psi}$ are differentiable with respect to x_k, and $\lambda:\mathscr{D}_x\times\mathscr{D}_x\to\mathbb{R}^{n_\lambda}$ is differentiable with respect to $\bar{x}_k$. Note that the elements of functions $f(z_k,u_k)$ and $g(z_k,u_k)$ are measurable, and those of $\phi(x_k,u_k)$, $\psi(x_k,u_k)$ and $\lambda(\bar{x}_k)$ cannot be directly accessible from the system output. The state-space matrices $A(z_k)$, $B(z_k)$, $C(z_k)$, $E(z_k)$, $F(z_k)$, $G(z_k)$ and $J(z_k)$ depend on the measurable premise variables. Assume that $B(z_k)$ is of full-column rank for $\forall x_k\in\mathscr{D}_x$, and $\{d_k\}_{k\in\mathbb{N}}\in\ell_\infty$. For observer design, we also consider the following assumption.

Assumption 1. The gradients of the continuously differentiable functions $\phi(x_k,u_k)$, $\psi(x_k,u_k)$ and $\lambda(\bar{x}_k)$ respectively belong to the following known bounded convex sets:

$$\begin{aligned}&\nabla_{x_k}\phi_i\in\mathrm{co}(\rho_{i1},\cdots,\rho_{in_{\phi_i}}),\quad \rho_{i\kappa}\in\mathbb{R}^{1\times n_x},\quad (i,\kappa)\in\mathcal{I}_{n_\phi}\times\mathcal{I}_{n_{\phi_i}},\\ &\nabla_{x_k}\psi_j\in\mathrm{co}(\zeta_{j1},\cdots,\zeta_{jn_{\psi_j}}),\quad \zeta_{j\kappa}\in\mathbb{R}^{1\times n_x},\quad (j,\kappa)\in\mathcal{I}_{n_\psi}\times\mathcal{I}_{n_{\psi_j}},\\ &\nabla_{\bar{x}_k}\lambda_\ell\in\mathrm{co}(\upsilon_{\ell1},\cdots,\upsilon_{\ell n_{\lambda_\ell}}),\quad \upsilon_{\ell\kappa}\in\mathbb{R}^{1\times 2n_x},\ (\ell,\kappa)\in\mathcal{I}_{n_\lambda}\times\mathcal{I}_{n_{\lambda_\ell}}.\end{aligned}\tag{3}$$

Remark 2. Note that the vertices constituting the convex hull of the bounded convex sets in (3) for the gradients can be determined in a similar way as the sector nonlinearity approach [10]. For illustrations, we consider a system with three states x_{1k},x_{2k},x_{3k} and a nonlinearity $\phi(x_k)=\sin(x_{1k}+x_{2k})$. The gradient of $\phi(x_k)$ is defined as

$$\nabla_{x_k}\phi=\begin{bmatrix}\cos(x_{1k}+x_{2k}) & \cos(x_{1k}+x_{2k}) & 0\end{bmatrix}.$$

Since $\cos(\cdot)\in[-1,1]$, then it follows that $\nabla_{x_k}\phi\in\mathrm{co}(\rho_1,\rho_2)$, with

$$\rho_1=\begin{bmatrix}-1 & -1 & 0\end{bmatrix},\quad \rho_2=\begin{bmatrix}1 & 1 & 0\end{bmatrix}.$$

Note that the proposed approach to form a convex bounded set from the whole gradient can lead to a smaller number of vertices than the element-by-element bounding approach in [52]. Indeed, this latter leads to a convex bounded set with 4 vertices for the considered example because it does not take advantage of the fact that both non-zero terms of the gradient $\nabla_{x_k}\phi$ are always the same.

Applying the sector nonlinearity approach [10, Chapter 2], the nonlinear descriptor system (2) can be *exactly* expressed by r fuzzy IF-THEN rules in the compact set $\mathscr{D}_x$ with local nonlinear consequents:

$$\begin{aligned}
&\text{RULE } R_i: \quad \text{IF } z_{1k} \text{ is } \mathcal{M}_1^i \text{ and } \dots \text{ and } z_{pk} \text{ is } \mathcal{M}_p^i \\
&\qquad\qquad \text{THEN} \quad \begin{cases} E_i x_{k+1} + J_i \lambda(\bar{x}_k) = A_i x_k + f(z_k, u_k) + F_i \phi(x_k, u_k) + B_i d_k \\ \qquad\qquad\qquad\quad y_k = C_i x_k + g(z_k, u_k) + G_i \psi(x_k, u_k) \end{cases}
\end{aligned} \tag{4}$$

where $(A_i, B_i, C_i, E_i, F_i, G_i, J_i)$ are known constant matrices with appropriate dimensions, R_i denotes the ith fuzzy inference rule. $\mathcal{M}_j^i$, with $i \in \mathcal{I}_r$ and $j \in \mathcal{I}_p$, is the fuzzy set. The fuzzy membership functions are given by

$$h_i(z) = \frac{\prod_{j=1}^p \mu_j^i(z_j)}{\sum_{i=1}^r \prod_{j=1}^p \mu_j^i(z_j)}, \quad \forall \in \mathcal{I}_r,$$

where $\mu_j^i(z_j)$ represents the membership grade of z_j in the respective fuzzy set $\mathcal{M}_j^i$. Note that the MFs satisfy the following convex sum property:

$$\sum_{i=1}^r h_i(z) = 1, \quad 0 \leq h_i(z) \leq 1, \quad \forall i \in \mathcal{I}_r. \tag{5}$$

Let Ω be the set of MFs satisfying (5). Using the center-of-gravity method for defuzzification, the N-TS fuzzy descriptor system (4) can be represented in the compact form

$$\begin{aligned}
E(h)x_{k+1} + J(h)\lambda(\bar{x}_k) &= A(h)x_k + f(z_k, u_k) + F(h)\phi(x_k, u_k) + B(h)d_k, \\
y_k &= C(h)x_k + g(z_k, u_k) + G(h)\psi(x_k, u_k),
\end{aligned} \tag{6}$$

where $h = \begin{bmatrix} h_1(z), h_2(z), \dots, h_r(z) \end{bmatrix}^\top \in \Omega$, and

$$\Pi(h) = \sum_{i=1}^r h_i(z)\Pi_i, \quad \Pi \in \{A, B, C, E, F, G, J\}.$$

Remark 3. Note that using the sector nonlinearity approach [10], the number of TS fuzzy subsystems increases *exponentially* according to the number of premise variables. Due to the retained nonlinearities $f(z_k, u_k)$, $g(z_k, u_k)$, $\phi(x_k, u_k)$ and $\psi(x_k, u_k)$ in the consequents of the N-TS fuzzy system (6), N-TS fuzzy modeling allows reducing significantly the number of fuzzy rules compared to the classical TS fuzzy modeling, especially for complex nonlinear systems [46], [53]. Hence, N-TS fuzzy-model-based approaches can lead to less conservative results and less computational burden compared to those based on the classical TS fuzzy modeling [26], [51].

For descriptor observer design, we rewrite system (6) in the following *equivalent* singular form:

$$\begin{aligned}
\bar{E}\bar{x}_{k+1} &= \bar{A}(h)\bar{x}_k - \bar{J}(h)\lambda(\bar{x}_k) + \bar{f}(z_k, u_k) + \bar{F}(h)\phi(x_k, u_k) + \bar{B}(h)d_k, \\
y_k &= \bar{C}(h)\bar{x}_k + g(z_k, u_k) + G(h)\psi(x_k, u_k),
\end{aligned} \tag{7}$$

with $\bar{f}(z_k, u_k) = \begin{bmatrix} 0 & f(z_k, u_k)^\top \end{bmatrix}^\top$ and

$$\begin{aligned}
\bar{A}(h) &= \begin{bmatrix} 0 & I \\ A(h) & -E(h) \end{bmatrix} \in \mathbb{R}^{2n_x \times 2n_x}, & \bar{B}(h) &= \begin{bmatrix} 0 \\ B(h) \end{bmatrix} \in \mathbb{R}^{2n_x \times n_d}, & \bar{E} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n_x \times 2n_x}, \\
\bar{C}(h) &= \begin{bmatrix} C(h) & 0 \end{bmatrix} \in \mathbb{R}^{n_y \times 2n_x}, & \bar{F}(h) &= \begin{bmatrix} 0 \\ F(h) \end{bmatrix} \in \mathbb{R}^{2n_x \times n_\phi}, & \bar{J}(h) &= \begin{bmatrix} 0 \\ J(h) \end{bmatrix} \in \mathbb{R}^{2n_x \times n_\lambda}.
\end{aligned}$$

Remark 4. The singular redundancy representation has been exploited for control and observer designs of TS fuzzy descriptor systems with linear consequents [23], [36], [48]. However, fuzzy observer design using the N-TS fuzzy form (7) has not yet been reported in the open literature. This singular model reformulation allows for an effective treatment of the nonlinear descriptor matrix $\mathcal{E}(x_k)$ in (1), especially in the context of TS fuzzy observer design with unmeasured premise variables.

B. N-TS Fuzzy Observer Structure and Useful Lemmas

For the state estimation of system (6), we consider a Luenberger-like observer of the N-TS fuzzy descriptor form

$$\begin{aligned}
\bar{E}\hat{\bar{x}}_{k+1} &= \bar{A}(h)\hat{\bar{x}}_k - \bar{J}(h)\lambda(\hat{\bar{x}}_k) + \bar{f}(z_k, u_k) + \bar{F}(h)\phi(\hat{x}_k, u_k) + \mathscr{L}(h)(y_k - \hat{y}_k), \\
\hat{y}_k &= \bar{C}(h)\hat{\bar{x}}_k + g(z_k, u_k) + G(h)\psi(\hat{x}_k, u_k),
\end{aligned} \tag{8}$$

where $\hat{\bar{x}}_k = \begin{bmatrix} \hat{x}_k^\top & \chi_k^\top \end{bmatrix}^\top$, $\hat{x}_k$ is an estimate of x_k and $\hat{x}_0 = 0$. The observer gain $\mathscr{L}(h) \in \mathbb{R}^{2n_x \times n_y}$ is to be determined.

Let us define the state estimation errors as

$$e_k = x_k - \hat{x}_k, \quad \bar{e}_k = \bar{x}_k - \hat{\bar{x}}_k = \begin{bmatrix} e_k \\ x_{k+1} - \chi_k \end{bmatrix}.$$

From (7) and (8), the estimation error dynamics is given by

$$\bar{E}\bar{e}_{k+1} = \mathcal{A}(h)\bar{e}_k - \bar{J}(h)\delta_\lambda + \bar{F}(h)\delta_\phi - \mathscr{L}(h)G(h)\delta_\psi + \bar{B}(h)d_k, \tag{9}$$

where

$$\begin{aligned} \mathcal{A}(h) &= \bar{A}(h) - \mathscr{L}(h)\bar{C}(h), \qquad &\delta_\phi &= \phi(x_k, u_k) - \phi(\hat{x}_k, u_k), \\ \delta_\psi &= \psi(x_k, u_k) - \psi(\hat{x}_k, u_k), \quad &\delta_\lambda &= \lambda(\bar{x}_k) - \lambda(\hat{\bar{x}}_k). \end{aligned}$$

Remark 5. Note that if $d_k = 0$, for $\forall k \in \mathbb{N}$, and the estimation error $\bar{e}_k$ converges to the origin, then $\hat{x}_k \to x_k$ and $\chi_k \to \hat{x}_{k+1}$ when $k \to \infty$. Hence, the variable χ_k can be viewed as an estimate of x_{k+1}. The key idea to introduce the *virtual* variable χ_k into the observer structure (8) is to guarantee the *consistency* between the proposed observer form and the singular redundancy representation (7), which allows avoiding any special structure of the observer gain $\mathscr{L}(h)$. Hence, a convex observer design framework with a reduced degree of conservatism can be achieved [45].

The nonlinear terms δ_ϕ, δ_ψ and δ_λ in (9) raise the technical challenges for TS fuzzy-model-based observer design with unmeasured premise variables. To deal with this major difficulty, we reformulate these mismatching terms as functions of the estimation error e_k through the following lemma.

Lemma 1 (Differential Mean Value Theorem [47])**.** Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ and $a, b \in \mathbb{R}^n$. If $f(x)$ is continuously differentiable on $\mathrm{co}(a, b)$, then $\exists c \in \mathrm{co}(a, b)$, such that

$$f(a) - f(b) = \nabla_x f(c)(a - b).$$

Applying Lemma 1 to each element of δ_ϕ, δ_ψ and δ_λ, these mismatching terms can be written as

$$\delta_\phi = \begin{bmatrix} \nabla_{x_k}\phi_1(t_1) \\ \vdots \\ \nabla_{x_k}\phi_{n_\phi}(t_{n_\phi}) \end{bmatrix} (x_k - \hat{x}_k), \quad \delta_\psi = \begin{bmatrix} \nabla_{x_k}\psi_1(s_1) \\ \vdots \\ \nabla_{x_k}\psi_{n_\psi}(t_{n_\psi}) \end{bmatrix} (x_k - \hat{x}_k), \quad \delta_\lambda = \begin{bmatrix} \nabla_{\bar{x}_k}\lambda_1(w_1) \\ \vdots \\ \nabla_{\bar{x}_k}\lambda_{n_\lambda}(w_{n_\lambda}) \end{bmatrix} (\bar{x}_k - \hat{\bar{x}}_k). \tag{10}$$

Moreover, since the gradients belong to convex bounded sets defined in (3), it follows that

$$\begin{aligned} \nabla_{x_k}\phi_i(t_i) &= \sum_{\kappa=1}^{n_{\phi_i}} \beta_{i\kappa}(t_i)\rho_{i\kappa}, \quad \beta_{i\kappa}(t_i) \geq 0, \quad \sum_{\kappa=1}^{n_{\phi_i}} \beta_{i\kappa}(t_i) = 1, \\ \nabla_{x_k}\psi_i(s_i) &= \sum_{\kappa=1}^{n_{\psi_i}} \theta_{i\kappa}(s_i)\zeta_{i\kappa}, \quad \theta_{i\kappa}(s_i) \geq 0, \quad \sum_{\kappa=1}^{n_{\psi_i}} \theta_{i\kappa}(s_i) = 1, \\ \nabla_{\bar{x}_k}\lambda_i(w_i) &= \sum_{\kappa=1}^{n_{\lambda_i}} \nu_{i\kappa}(w_i)\upsilon_{i\kappa}, \quad \nu_{i\kappa}(w_i) \geq 0, \quad \sum_{\kappa=1}^{n_{\lambda_i}} \nu_{i\kappa}(w_i) = 1. \end{aligned} \tag{11}$$

Let us define the following matrices:

$$K(\bar{\beta}) = \underbrace{\sum_{\kappa_1=1}^{n_{\phi_1}} \cdots \sum_{\kappa_{n_\phi}=1}^{n_{\phi_{n_\phi}}}}_{n_\phi \text{ times}} \left(\prod_{i=1}^{n_\phi} \beta_{i\kappa_i}(t_i) \right) \begin{bmatrix} \rho_{1\kappa_1} \\ \vdots \\ \rho_{n_\phi \kappa_{n_\phi}} \end{bmatrix} \in \mathbb{R}^{n_\phi \times n_x}, \tag{12a}$$

$$M(\bar{\theta}) = \underbrace{\sum_{\kappa_1=1}^{n_{\psi_1}} \cdots \sum_{\kappa_{n_\psi}=1}^{n_{\psi_{n_\psi}}}}_{n_\psi \text{ times}} \left(\prod_{i=1}^{n_\psi} \theta_{i\kappa_i}(s_i) \right) \begin{bmatrix} \zeta_{1\kappa_1} \\ \vdots \\ \zeta_{n_\psi \kappa_{n_\psi}} \end{bmatrix} \in \mathbb{R}^{n_\psi \times n_x}, \tag{12b}$$

$$N(\bar{\nu}) = \underbrace{\sum_{\kappa_1=1}^{n_{\lambda_1}} \cdots \sum_{\kappa_{n_\lambda}=1}^{n_{\lambda_{n_\lambda}}}}_{n_\lambda \text{ times}} \left(\prod_{i=1}^{n_\lambda} \nu_{i\kappa_i}(w_i) \right) \begin{bmatrix} \upsilon_{1\kappa_1} \\ \vdots \\ \upsilon_{n_\lambda \kappa_{n_\lambda}} \end{bmatrix} \in \mathbb{R}^{n_\lambda \times 2n_x}, \tag{12c}$$

with $r_\phi = \prod_{i=1}^{n_\phi} n_{\phi_i}$, $r_\psi = \prod_{i=1}^{n_\psi} n_{\psi_i}$ and $r_\lambda = \prod_{i=1}^{n_\lambda} n_{\lambda_i}$. For observer design, we make use of the multi-index notation to simplify the representation of matrices $K(\bar{\beta})$, $M(\bar{\theta})$ and $N(\bar{\nu})$. To this end, let $j \in \mathcal{I}_{r_\phi}$ represent the multiple indices

$(\kappa_1,\dots,\kappa_{n_\phi}) \in \mathcal{I}_{n_{\phi_1}} \times \cdots \times \mathcal{I}_{n_{\phi_{n_\phi}}}$ in (12a), $\ell \in \mathcal{I}_{r_\psi}$ represent the multiple indices $(\kappa_1,\dots,\kappa_{n_\psi}) \in \mathcal{I}_{n_{\psi_1}} \times \cdots \times \mathcal{I}_{n_{\psi_{n_\psi}}}$ in (12b), and $m \in \mathcal{I}_{r_\lambda}$ represent the multiple indices $(\kappa_1,\dots,\kappa_{n_\lambda}) \in \mathcal{I}_{n_{\lambda_1}} \times \cdots \times \mathcal{I}_{n_{\lambda_{n_\lambda}}}$ in (12c). We also denote $j\{q\}$, $\ell\{q\}$, $m\{q\}$ the κ_q subindex in j, ℓ and m, respectively. Then, the parameter-dependent matrices in (12) can be compactly represented in the following single-index form:

$$K(\bar\beta) = \sum_{j=1}^{r_\phi} \bar\beta_j(t) K_j, \quad M(\bar\theta) = \sum_{\ell=1}^{r_\psi} \bar\theta_\ell(s) M_\ell, \quad N(\bar\nu) = \sum_{m=1}^{r_\lambda} \bar\nu_m(w) N_m, \tag{13}$$

with $\bar\beta = \begin{bmatrix}\bar\beta_1(t) & \bar\beta_2(t) & \dots & \bar\beta_{r_\phi}(t)\end{bmatrix}^\top$, $\bar\theta = \begin{bmatrix}\bar\theta_1(s) & \bar\theta_2(s) & \dots & \bar\theta_{r_\psi}(s)\end{bmatrix}^\top$, $\bar\nu = \begin{bmatrix}\bar\nu_1(w) & \bar\nu_2(w) & \dots & \bar\nu_{r_\lambda}(w)\end{bmatrix}^\top$, and

$$\begin{aligned}
K_j &= \begin{bmatrix} \rho_{1j\{1\}} \\ \vdots \\ \rho_{n_\phi j\{n_\phi\}} \end{bmatrix}, \qquad \bar\beta_j(t) = \prod_{i=1}^{n_\phi} \beta_{ij\{i\}}\left(t_{j\{i\}}\right), \\
M_\ell &= \begin{bmatrix} \zeta_{1\ell\{1\}} \\ \vdots \\ \zeta_{n_\psi \ell\{n_\psi\}} \end{bmatrix}, \qquad \bar\theta_\ell(s) = \prod_{\ell=1}^{n_\psi} \theta_{i\ell\{i\}}\left(s_{j\{i\}}\right), \\
N_m &= \begin{bmatrix} \upsilon_{1m\{1\}} \\ \vdots \\ \upsilon_{n_\lambda m\{n_\lambda\}} \end{bmatrix}, \quad \bar\nu_m(w) = \prod_{i=1}^{n_\lambda} \nu_{im\{i\}}\left(w_{j\{i\}}\right).
\end{aligned} \tag{14}$$

Moreover, it follows from (11) and (14) that

$$\bar\beta_j(t) \ge 0, \quad \sum_{j=1}^{r_\phi} \bar\beta_j(t) = 1, \tag{15a}$$

$$\bar\theta_\ell(s) \ge 0, \quad \sum_{\ell=1}^{r_\psi} \bar\theta_\ell(s) = 1, \tag{15b}$$

$$\bar\nu_m(w) \ge 0, \quad \sum_{m=1}^{r_\lambda} \bar\nu_m(w) = 1. \tag{15c}$$

Let Ω_β, Ω_θ and Ω_ν be the sets of *unknown* time-varying parameters $\bar\beta$, $\bar\theta$ and $\bar\nu$ verifying the convex sum properties in (15a), (15b) and (15c), respectively. From the matrix definition (13), the mismatching nonlinear terms in (10) can be expressed by

$$\delta_\phi = K(\bar\beta)(x_k - \hat x_k), \quad \delta_\psi = M(\bar\theta)(x_k - \hat x_k), \quad \delta_\lambda = N(\bar\nu)(\bar x_k - \hat{\bar x}_k). \tag{16}$$

Then, from (9) and (16), the error dynamics is rewritten as

$$\bar E \bar e_{k+1} = \sum_{i=1}^{r} h_i(z) \left(\bar{\mathscr{A}}_i(\bar\beta,\bar\nu) - \mathscr{L}(h)\bar{\mathscr{C}}_i(\bar\theta)\right) \bar e_k + \bar B(h) d_k, \tag{17}$$

with $\bar{\mathscr{A}}_i(\bar\beta,\bar\nu) = \bar A_i - \bar J_i N(\bar\nu) + \bar F_i K(\bar\beta)\begin{bmatrix} I & 0\end{bmatrix}$ and $\bar{\mathscr{C}}_i(\bar\theta) = \bar C_i + \bar G_i M(\bar\theta)\begin{bmatrix} I & 0\end{bmatrix}$.

This paper provides an LMI-based procedure for the following N-TS fuzzy observer design problem.

Problem 1. Consider a nonlinear descriptor system (1), which can be reformulated in the form (2). Design a N-TS fuzzy observer (8) with the MF-dependent gain $\mathscr{L}(h)$ such that the error dynamics (17) verifies the following properties.

(P1) For $d_k = 0$, $\forall k \in \mathbb{N}$, the error dynamics (17) is globally uniformly exponentially stable with respect to the origin.

(P2) For zero initial error $e_0 = 0$ and any unknown input sequence $\{d_k\}_{k\in\mathbb{N}} \in \ell_\infty$, we have $\|e_k\| < \gamma \|d\|_{\ell_\infty}$, $\forall k \in \mathbb{N}$, for a predefined scalar $\gamma > 0$.

(P3) The estimation error e_k is uniformly bounded for any initial condition e_0 and any input sequence $\{d_k\}_{k\in\mathbb{N}} \in \ell_\infty$. That is, there exists a bound $\varphi(e_0, \|d\|_{\ell_\infty})$ such that $\|e_k\| \le \varphi(e_0, \|d\|_{\ell_\infty})$, for $\forall k \ge 0$. Moreover, we have

$$\lim_{k\to\infty} \sup \|e_k\| < \gamma \|d\|_{\ell_\infty}. \tag{18}$$

System (17) verifying properties (P1)–(P3) is said to be globally uniformly $\ell_\infty-$stable with a performance level γ. More details on the global uniform $\ell_\infty-$stablity can be found in [54, Chapter 4].

The following relaxation result is useful to convert a MF-dependent inequality into a finite set of LMI constraints.

Lemma 2 ([55])**.** Consider the MF-dependent inequality

$$\Upsilon_{hhh_+} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(z_k)h_j(z_k)h_l(z_{k+1})\Upsilon_{ijl} \prec 0, \tag{19}$$

where $h_+ = \left[h_1(z_{k+1}), h_2(z_{k+1}), \dots, h_r(z_{k+1})\right]^\top$, and $h, h_+ \in \Omega$. The symmetric matrices of appropriate dimensions Υ_{ijl}, with $i, j, l \in \mathcal{I}_r$, are linearly dependent on the unknown decision variables. Inequality (19) holds if

$$\begin{aligned}
&\Upsilon_{iil} \prec 0, \quad i, l \in \mathcal{I}_r,\\
&\frac{2}{r-1}\Upsilon_{iil} + \Upsilon_{ijl} + \Upsilon_{jil} \prec 0, \quad i, j, l \in \mathcal{I}_r, \quad i \neq j.
\end{aligned}$$

Lemma 2 provides a good tradeoff between numerical complexity and design conservatism [45]. Other relaxation results with different degrees of complexity and/or conservatism can be found in [56].

III. Fuzzy Descriptor Observer with Unmeasured Premise Variables and Unknown Input

This section presents an LMI-based solution to design an N-TS fuzzy descriptor observer for system (2). We provide sufficient conditions to guarantee the global uniform $\ell_\infty-$stability of the error dynamics (9) in the following theorem.

Theorem 1. Consider the estimation error dynamics (17). If there exist MF-dependent symmetric matrices $P(h) \in \mathbb{R}^{n_x \times n_x}$, $R(h) \in \mathbb{R}^{n_x \times n_x}$, a regular MF-dependent matrix $H(h) \in \mathbb{R}^{2n_x \times 2n_x}$, matrices $Q(h) \in \mathbb{R}^{n_x \times n_x}$, $L(h) \in \mathbb{R}^{2n_x \times n_y}$, and positive scalars $\tau \in (0,1)$, ϵ, γ such that

$$\Sigma(h, h_+) + \Phi(h, \bar{\beta}, \bar{\nu}, \bar{\theta}) + \Phi(h, \bar{\beta}, \bar{\nu}, \bar{\theta})^\top \prec 0, \tag{20}$$

$$\begin{bmatrix} P(h) & \star \\ I & \gamma I \end{bmatrix} \succ 0, \tag{21}$$

for $h, h_+ \in \Omega$, $\bar{\beta} \in \Omega_\beta$, $\bar{\nu} \in \Omega_\nu$, $\bar{\theta} \in \Omega_\theta$, where

$$\begin{aligned}
\bar{P}(h) &= \begin{bmatrix} P(h) & \star \\ Q(h) & R(h) \end{bmatrix},\\
\Sigma(h, h_+) &= \begin{bmatrix} (\tau-1)\bar{E}^\top \bar{P}(h)\bar{E} & \star & \star \\ 0 & \bar{P}(h_+) & \star \\ 0 & 0 & -\tau\gamma I \end{bmatrix},\\
\Phi(h, \bar{\beta}, \bar{\nu}, \bar{\theta}) &= \begin{bmatrix} \Gamma(h, \bar{\beta}, \bar{\nu}, \bar{\theta}) & -H(h) & H(h)\bar{B}(h) \\ \epsilon\Gamma(h, \bar{\beta}, \bar{\nu}, \bar{\theta}) & -\epsilon H(h) & \epsilon H(h)\bar{B}(h) \\ 0 & 0 & 0 \end{bmatrix},\\
\Gamma(h, \bar{\beta}, \bar{\nu}, \bar{\theta}) &= H(h)\bar{\mathscr{A}}(h, \bar{\beta}, \bar{\nu}) - L(h)\bar{\mathscr{C}}(h, \bar{\theta}),\\
\bar{\mathscr{A}}(h, \bar{\beta}, \bar{\nu}) &= \sum_{i=1}^{r} h_i(z)\bar{\mathscr{A}}_i(\bar{\beta}, \bar{\nu}), \quad \bar{\mathscr{C}}(h, \bar{\theta}) = \sum_{i=1}^{r} h_i(z)\bar{\mathscr{C}}_i(\bar{\theta}).
\end{aligned}$$

Then, system (17) is globally uniformly $\ell_\infty-$stable with performance level γ. Moreover, the N-TS fuzzy observer corresponding to the nonlinear descriptor system (2) is given by

$$\begin{aligned}
E(h)\chi_k + J(h)\lambda(\hat{x}_k, \chi_k) &= A(h)\hat{x}_k + f(z_k, u_k) + F(h)\phi(\hat{x}_k, u_k) + \mathscr{L}_2(h)(y_k - \hat{y}_k),\\
\hat{x}_{k+1} &= \chi_k + \mathscr{L}_1(h)(y_k - \hat{y}_k),\\
\hat{y}_k &= C(h)\hat{x}_k + g(z_k, u_k) + G(h)\psi(\hat{x}_k, u_k),
\end{aligned} \tag{22}$$

with $\mathscr{L}(h) \doteq \begin{bmatrix} \mathscr{L}_1(h)^\top & \mathscr{L}_2(h)^\top \end{bmatrix}^\top = H(h)^{-1}L(h)$.

Proof. For the observer design, we consider the following MF-dependent Lyapunov function:

$$\mathscr{V}_k = \bar{e}_k^\top \bar{E}^\top \bar{P}(h)\bar{E}\bar{e}_k. \tag{23}$$

Note that

$$\bar{E}^\top \bar{P}(h)\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P(h) & \star \\ Q(h) & R(h) \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P(h) & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, the Lyapunov function $\mathscr{V}_k$ in (23) can be rewritten as $\mathscr{V}_k = e_k^\top P(h)e_k$, which is positive definite for $\forall k \in \mathbb{N}$ since $P(h) \succ 0$. Inequality (20) can be represented in the form

$$\Sigma(h, h_+) + \Pi(h, \bar{\beta}, \bar{\nu}, \bar{\theta}) + \Pi(h, \bar{\beta}, \bar{\nu}, \bar{\theta})^\top \prec 0, \tag{24}$$

where

$$\Pi(h,\bar\beta,\bar\nu,\bar\theta) = \mathscr{H}(h)\begin{bmatrix}\Lambda(h,\bar\beta,\bar\nu,\bar\theta) & -I & \bar B(h)\end{bmatrix},$$
$$\mathscr{H}(h) = \begin{bmatrix}H(h)^\top & \epsilon H(h)^\top & 0\end{bmatrix}^\top,$$
$$\Lambda(h,\bar\beta,\bar\nu,\bar\theta) = \bar{\mathscr{A}}(h,\bar\beta,\bar\nu) - \mathscr{L}(h)\mathscr{C}(h,\bar\theta).$$

From the error dynamics equation (17), it follows that

$$2\mathscr{E}(h)\left(\Lambda(h,\bar\beta,\bar\nu,\bar\theta)\bar e_k - \bar E\bar e_{k+1} + \bar B(h)d_k\right) = 0,\ \forall k\in\mathbb{N}, \tag{25}$$

with $\mathscr{E}(h) = \bar e_k^\top H(h) + \epsilon\bar e_{k+1}^\top \bar E^\top H(h)$. Pre- and postmultiplying (24) with $\begin{bmatrix}\bar e_k^\top & \bar e_{k+1}^\top\bar E^\top & d_k^\top\end{bmatrix}$ and its transpose while taking into account equality (25), it follows that

$$\bar e_{k+1}^\top\bar E^\top\bar P(h_+)\bar E\bar e_{k+1} + (\tau-1)\bar e_k^\top\bar E^\top\bar P(h)\bar E\bar e_k - \tau\gamma d_k^\top d_k \prec 0,\quad \forall k\in\mathbb{N}. \tag{26}$$

Condition (26) can be rewritten in the form

$$\Delta\mathscr{V}_k + \tau(\mathscr{V}_k - \gamma d_k^\top d_k) < 0,\quad \forall k\in\mathbb{N}, \tag{27}$$

where $\mathscr{V}_k$ is defined in (23), and $\Delta\mathscr{V}_k = \mathscr{V}_{k+1} - \mathscr{V}_k$ is its difference along the trajectory of the error dynamics (17). It follows from (27) that

$$\mathscr{V}_k < (1-\tau)\mathscr{V}_{k-1} + \tau\gamma\|d_{k-1}\|^2,\quad \forall k\geq 1. \tag{28}$$

Since $\tau\in(0,1)$, by recursively it follows from (28) that

$$\begin{aligned}\mathscr{V}_k &< (1-\tau)^k\mathscr{V}_0 + \tau\gamma\sum_{i=0}^{k-1}(1-\tau)^i\|d_{k-1-i}\|^2\\ &< (1-\tau)^k\mathscr{V}_0 + \gamma\|d\|_{\ell_\infty}^2,\quad \forall k\geq 1.\end{aligned} \tag{29}$$

If $d_k = 0$, $\forall k\in\mathbb{N}$, it follows from (29) that $\mathscr{V}_k < (1-\tau)^k\mathscr{V}_0$, which guarantees property (P1). By Schur complement lemma [13], it follows from (21) and (29) that

$$\|e_k\|^2 < \gamma\mathscr{V}_k < \gamma(1-\tau)^k\mathscr{V}_0 + \gamma^2\|d\|_{\ell_\infty}^2,\quad \forall k\geq 1. \tag{30}$$

Note that inequality (30) implies (18), and $\mathscr{V}_0 = 0$, for $e_0 = 0$. Then, it follows from (30) that

$$\|e_k\|^2 < \gamma^2\|d\|_{\ell_\infty}^2,\quad \forall k\geq 1. \tag{31}$$

Properties (P2)–(P3) follow directly from (31). Moreover, the N-TS fuzzy observer structure (22) is straightforwardly recovered from (8). This concludes the proof. □

Remark 6. Note from (18) that the estimation error e_k can be minimized by minimizing the $\ell_\infty-$gain performance γ.

Remark 7. The positive scalar ϵ and the MF-dependent matrix $H(h)$ are introduced in the descriptor observer design through equality (25) to reduce the conservatism. This null-term relaxation approach has been effectively exploited for TS fuzzy-model-based control and observer design, see for instance [23], [57].

Remark 8. Theorem 1 can directly be applied to the classical TS fuzzy systems with unmeasured premise variables and unknown input by setting $E(h) = I$ and $\lambda(\bar x_k) = 0$. Moreover, if $\phi(x_k,u_k) = 0$, $\psi(x_k,u_k) = 0$ and $\lambda(\bar x_k) = 0$, then we recover the class of nonlinear descriptor systems considered in [23]. Hence, the proposed N-TS fuzzy observer design is more general than the existing results in the open literature.

Theorem 1 cannot be directly applied to design ℓ_∞ N-TS fuzzy observer (8) due to the MF-dependent nature of conditions (20)–(21). On the basis of Theorem 1, the following theorem provides a finite set of tractable design conditions while minimizing the state estimation error.

Theorem 2. Consider the estimation error dynamics (17) and a positive scalar $\tau\in(0,1)$. If there exist symmetric matrices $P_i\in\mathbb{R}^{n_x\times n_x}$, $R_i\in\mathbb{R}^{n_x\times n_x}$, regular matrices $H_i\in\mathbb{R}^{2n_x\times 2n_x}$, matrices $Q_i\in\mathbb{R}^{n_x\times n_x}$, $L_i\in\mathbb{R}^{2n_x\times n_y}$, $i\in\mathcal{I}_r$, and positive scalars ϵ, γ such that the following optimization problem is feasible:

$$\underset{\zeta_i=(\epsilon,\gamma,P_i,R_i,Q_i,L_i,H_i),\ i\in\mathcal{I}_r}{\text{minimize}}\quad \gamma \tag{32}$$

subject to

$$\Xi_{iil\kappa\ell m} \prec 0, \tag{33}$$

$$\frac{2}{r-1}\Xi_{iil\kappa\ell m} + \Xi_{ijl\kappa\ell m} + \Xi_{jil\kappa\ell m} \prec 0, \tag{34}$$

$$\begin{bmatrix}P_i & \star\\ I & \gamma I\end{bmatrix} \succ 0, \tag{35}$$

for $i, j, l \in \mathcal{I}_r$, $i \neq j$, $\kappa \in \mathcal{I}_{r_\phi}$, $\ell \in \mathcal{I}_{r_\psi}$, $m \in \mathcal{I}_{r_\lambda}$. The quantity $\Xi_{ijl\kappa\ell m}$ in (34) is given by

$$\Xi_{ijl\kappa\ell m} = \Sigma_{il} + \Phi_{ij\kappa\ell m} + \Phi_{ij\kappa\ell m}^\top,$$

where

$$\Sigma_{il} = \begin{bmatrix} (\tau-1)\bar{E}^\top \bar{P}_i \bar{E} & \star & \star \\ 0 & \bar{P}_l & \star \\ 0 & 0 & -\tau\gamma I \end{bmatrix},$$
$$\Phi_{ij\kappa\ell m} = \begin{bmatrix} \Gamma_{ij\kappa\ell m} & -H_i & H_i\bar{B}_j \\ \epsilon\Gamma_{ij\kappa\ell m} & -\epsilon H_i & \epsilon H_i \bar{B}_j \\ 0 & 0 & 0 \end{bmatrix},$$
$$\Gamma_{ij\kappa\ell m} = H_j\left(\bar{A}_i - \bar{J}_i N_m + \bar{F}_i K_\kappa \begin{bmatrix} I & 0 \end{bmatrix}\right) - L_j\left(\bar{C}_i + \bar{G}_i M_\ell\right).$$

Then, the estimation error dynamics (17) is globally uniformly $\ell_\infty-$stable with performance level γ. Moreover, the N-TS fuzzy observer structure corresponding to the initial descriptor system (6) is given in (22), with $\mathscr{L}(h) = H(h)^{-1}L(h)$ and

$$\begin{bmatrix} H(h) & L(h) \end{bmatrix} = \sum_{i=1}^{r} h_i(z_k)\begin{bmatrix} H_i & L_i \end{bmatrix}.$$

Proof. The proof is based on the convex combination form of the state-space matrices $A(h)$, $B(h)$, $C(h)$, $E(h)$, $F(h)$, $G(h)$, $J(h)$ and the decision matrices $P(h)$, $R(h)$, $Q(h)$, $L(h)$, $H(h)$. Indeed, using Lemma 2 and the convexity property of the bounded sets for the gradients, inequalities (33)–(34) imply clearly (20) in Theorem 1. Moreover, condition (21) can also be rewritten as

$$\sum_{i=1}^{r} h_i(z_k)\begin{bmatrix} P_i & \star \\ I & \gamma I \end{bmatrix} \succ 0. \tag{36}$$

Remark that condition (35) guarantees (36). Then, the proof can be concluded by following the results of Theorem 1. □

Remark 9. The design conditions in Theorem 2 are expressed in terms of LMIs with a line search over the scalar ϵ. A gridding method can be performed to search for ϵ. Then, the optimization problem (32) under LMI constraints (33)–(35) can be effectively solved with available numerical solvers [13].

IV. Unknown Input Estimation with ℓ_∞ Norm-Bounded Guarantee

This section presents a method to reconstruct the unknown input of system (2). Especially, the ultimate norm upper-bound of the UI estimation error is also characterized. To this end, we derive from (6) that

$$B(h)d_k = E(h)x_{k+1} + J(h)\lambda(\bar{x}_k) - A(h)x_k - \mathscr{F}(x_k, z_k, u_k), \tag{37}$$

with $\mathscr{F}(x_k, z_k, u_k) = f(z_k, u_k) + F(h)\phi(x_k, u_k)$. Since $B(h)$ is of full-column rank for $\forall h \in \Omega$, there exists $B(h)^\dagger$ such that $B(h)^\dagger B(h) = I$. Premultiplying (37) with $B(h)^\dagger$ yields

$$d_k = B(h)^\dagger\left(E(h)x_{k+1} - A(h)x_k - \mathscr{F}(x_k, z_k, u_k)\right). \tag{38}$$

We propose the estimate $\hat{d}_k$ of d_k of the form

$$\hat{d}_k = B(h)^\dagger\left(E(h)\hat{x}_{k+1} + J(h)\lambda(\hat{\bar{x}}_k) - A(h)\hat{x}_k - \mathscr{F}(\hat{x}_k, z_k, u_k)\right). \tag{39}$$

Let us define the UI estimation error as $\varepsilon_k = d_k - \hat{d}_k$. It follows from (38) and (39) that

$$\varepsilon_k = B(h)^\dagger\left(E(h)e_{k+1} + J(h)\delta_\lambda - A(h)e_k - F(h)\delta_\phi\right), \tag{40}$$

where δ_ϕ and δ_λ are given in (16). We decompose $N(\bar{\nu})$, defined in (13), as $N(\bar{\nu}) = \begin{bmatrix} N_1(\bar{\nu}) & N_2(\bar{\nu}) \end{bmatrix}$. Expression (40) can be compactly rewritten as

$$\varepsilon_k = B(h)^\dagger\left(E_d(h,\bar{\nu})e_{k+1} - A_d(h,\bar{\beta},\bar{\nu})e_k\right), \tag{41}$$

with $E_d(h,\bar{\nu}) = E(h) + J(h)N_2(\bar{\nu})$ and $A_d(h,\bar{\beta},\bar{\nu}) = A(h) + F(h)K(\bar{\beta}) - J(h)N_1(\bar{\nu})$. It follows from (41) that

$$\begin{aligned} \|\varepsilon_k\| &\leq \|B(h)^\dagger E_d(h,\bar{\nu})\|\|e_{k+1}\| + \|B(h)^\dagger A_d(h,\bar{\beta},\bar{\nu})\|\|e_k\| \\ &\leq \|B(h)^\dagger\|(\|E_d(h,\bar{\nu})\|\|e_{k+1}\| + \|A_d(h,\bar{\beta},\bar{\nu})\|\|e_k\|. \end{aligned} \tag{42}$$

If the optimization problem (32) is feasible, it follows that

$$\lim_{k\to\infty}\sup\|e_k\| < \gamma\|d\|_{\ell_\infty}, \quad \lim_{k\to\infty}\sup\|e_{k+1}\| < \gamma\|d\|_{\ell_\infty}. \tag{43}$$

From (42) and (43), the ultimate norm upper-bound of the estimation error ε_k is given by

$$\lim_{k\to\infty} \sup \|\varepsilon_k\| \leq \gamma\eta\|d\|_{\ell_\infty}, \tag{44}$$

with $\eta = \|B(h)^\dagger\|_\infty \left(\|E_d(h,\bar{\nu})\|_\infty + \|A_d(h,\bar{\beta},\bar{\nu}\|_\infty\right)$. The following result summarizes the ℓ_∞ norm-bounded UI estimation.

Corollary 1. Consider the error dynamics (9). If the optimization problem (32) in Theorem 2 is feasible, then there exists a N-TS fuzzy descriptor observer (22) such that the following properties are verified.

- The error dynamics (9) is globally uniformly $\ell_\infty-$stable with a guaranteed performance level γ.
- With the UI estimate $\hat{d}_k$ defined in (39), the UI estimation error ε_k is ultimately norm-bounded as in (44).

Remark 10. Assume that the optimization problem (32) is feasible with an *arbitrarily* small $\ell_\infty-$gain γ. Then, for a bounded η, both the state and the UI can be estimated with an *arbitrary* degree of accuracy as respectively shown in (43) and (44).

The UI observer design is summarized in Algorithm 1.

Algorithm 1: Observer Design Procedure

Input: Nonlinear descriptor system (1) and its equivalent N-TS fuzzy form (6).
Output: N-TS fuzzy observer (22) and UI estimator (39) such that

$$\lim_{k\to\infty} \sup \|x_k - \hat{x}_k\| < \gamma\|d\|_{\ell_\infty}, \quad \lim_{k\to\infty} \sup \|d_k - \hat{d}_k\| < \gamma\eta\|d\|_{\ell_\infty},$$

with γ and η respectively specified in Theorem 2 and inequality (44).

1 Check the regularity of matrix $\mathcal{E}(x_k)$ and the boundedness condition (3) in Assumption 1.
- If YES, then go to Step 2.
- If NO, then the proposed method is unapplicable to the considered system.

2 Solve the LMI-based optimization problem (32) to get H_i and L_i, for $i \in \mathcal{I}_r$.
3 Construct the N-TS fuzzy observer (22) to estimate x_k.
4 Construct the UI estimator (39) to estimate d_k.

The proposed fuzzy UI observer design is illustrated in the following section.

V. Illustrative Examples

This section presents two real-world examples to illustrate the interests of the proposed N-TS fuzzy observer design. The first example is concerned with the estimation problem of a chaotic Lorenz system with an external input. For this example, the unmeasured nonlinearities are involved in the descriptor matrix $\mathcal{E}(x_k)$. In the second example, we focus on estimating the *angle of attack* and *aerodynamical forces and moments* of a simulated aircraft, for which unmeasurable nonlinearities are involved in the right-hand side of the system state dynamics and in the system output. It is important to note that existing TS fuzzy-model-based observer designs in the literature cannot be applied to these two examples.

All LMI-based optimizations were solved using YALMIP toolbox [58] and MOSEK 8.1.0.67 solver within MATLAB R2018b environment. For the gridding line search, we consider a logarithmically spaced grid as $\epsilon \in [10^{-1}, 10^3]$.

Example 1 (Chaotic dynamical system)**.** We revisit the following Lorenz system [27], widely used in secured communications, whose dynamics is given by

$$\begin{aligned}\dot{x}_1 &= -px_1 + px_2,\\ \dot{x}_2 &= qx_1 - x_1x_3 - x_2,\\ \dot{x}_3 &= x_1x_2 - sx_3 + d,\end{aligned} \tag{45}$$

where $x_1 \in [-21, 21]$, $x_2 \in [-25, \frac{\pi}{12}]$, $x_3 \in [0, 50]$ are the system states, d is the external input, and $p = 10$, $q = \frac{8}{3}$, $s = 28$ are the system parameters. Assume that signals x_2 and x_3 can be measured and the goal is to estimate x_1 and d. For this example, we consider a backward-Euler-like discretization with a time step $T = 0.01$s to transform system (45) into its discrete-time representation of the following form:

$$\begin{aligned}E(z_k)x_{k+1} + J\lambda(\bar{x}_k) &= Ax_k + Bd_k,\\ y_k &= Cx_k,\end{aligned} \tag{46}$$

with $x_k = \begin{bmatrix} x_{1k} & x_{2k} & x_{3k}\end{bmatrix}^\top$, $y_k = \begin{bmatrix} x_{2k} & x_{3k}\end{bmatrix}^\top$, $z_k = x_{2k}$, $A = I$, and

$$E(z_k) = \begin{bmatrix} 1+pT & -pT & 0\\ -qT & 1+T & 0\\ -Tx_{2k} & 0 & 1+sT\end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0\\ T & 0\\ 0 & T\end{bmatrix}, \quad \lambda(\bar{x}_k) = \begin{bmatrix} x_{1k}x_{3(k+1)}\\ x_{1k}x_{2(k+1)}\end{bmatrix}, \quad B = \begin{bmatrix} 0\\ 0\\ T\end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Note that $\lambda(\bar{x}_k)$ is continuously differentiable with respect to $\bar{x}_k$, and its gradients are given by

$$\nabla_x \lambda_1 = \begin{bmatrix} x_{3(k+1)} & 0 & 0 & 0 & 0 & x_{1k} \end{bmatrix}, \qquad \nabla_x \lambda_2 = \begin{bmatrix} x_{2(k+1)} & 0 & 0 & 0 & x_{1k} & 0 \end{bmatrix}. \tag{47}$$

Using the sector nonlinearity approach [10, Chapter 2], an equivalent N-TS fuzzy descriptor representation (6) with two fuzzy rules ($r = 2$) can be obtained for the nonlinear descriptor system (46). Moreover, with the gradient definitions (47), a convex bounded set representation is obtained for $N(\bar{\nu})$ in (13) with 16 vertices. The state-space matrices and the nonlinear MFs of the corresponding N-TS fuzzy representation, as well as the vertex matrices for the convex bounded set are omitted for brevity.

Applying Theorem 2 to system (46), the suboptimal $\ell_\infty-$gains found for different values of the decay rate τ are depicted in Fig. 1. Note that γ is infinite when the optimization problem (32) is not feasible. Remark that a larger value of τ can improve the estimation performance, however too large value of τ may lead to infeasibility. To highlight the interest of introducing the virtual variable χ_k into the observer structure (8), we compare the feasibility regions obtained with the conditions in Theorem 2 and those derived without the introduction of χ_k, *i.e.*, similar design conditions as in Theorem 2 obtained by setting $\mathscr{L}_1(h) = 0$. The comparison result is shown in Fig. 2. Observe that Theorem 2 leads to less conservative results in this case because the corresponding feasibility region is clearly larger than that obtained without using the virtual variable χ_k.

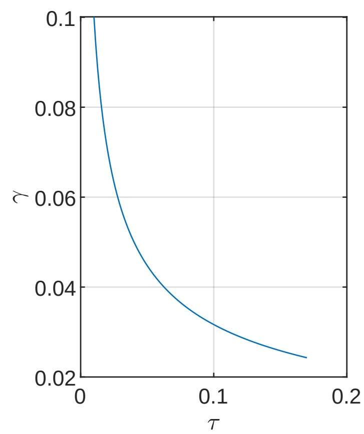

Fig. 1. Effect of the decay rate parameter τ on the $\ell_\infty-$gain bound γ for the Lorenz dynamical system in Example 1.

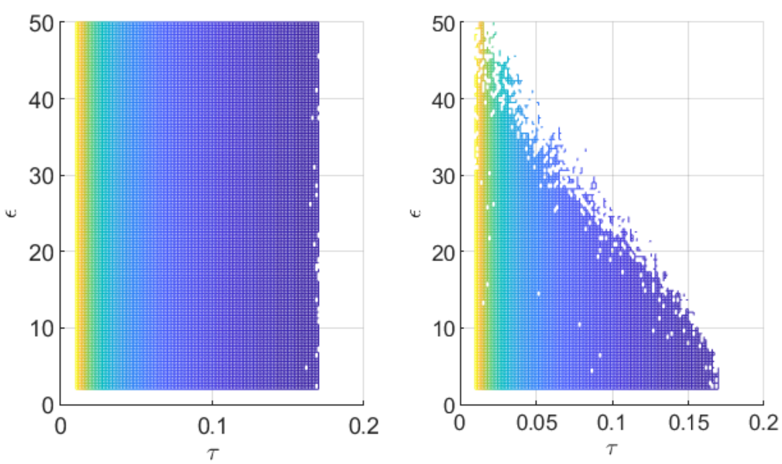

Fig. 2. Feasibility regions for $\tau \in [0.01, 0.17]$ and $\epsilon \in [2, 50]$ obtained the Lorenz dynamical system in Example 2. The colored areas indicate points where LMI-based design conditions were feasible, whereas the white areas represent the infeasibility areas. The left plot represents the conditions in Theorem 2, while the right plot represents the results with $\mathscr{L}_1(h) = 0$.

For illustrations, the estimation results obtained from Theorem 2 with $\tau = 0.17$ and $\epsilon = 2$ are summarized in Figs. 3 and 4. We can see that the proposed N-TS fuzzy descriptor observer provides an accurate estimation performance for both the state x_{1k} and the unknown input d_k.

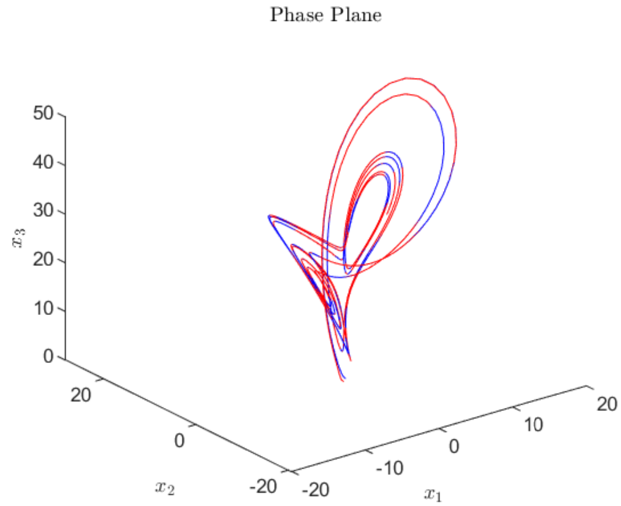

Fig. 3. System trajectory of Lorenz system in Example 1. The blue solid line indicates the trajectory of the nonlinear system (46), whereas the red line indicates the trajectory given by the proposed N-TS fuzzy observer.

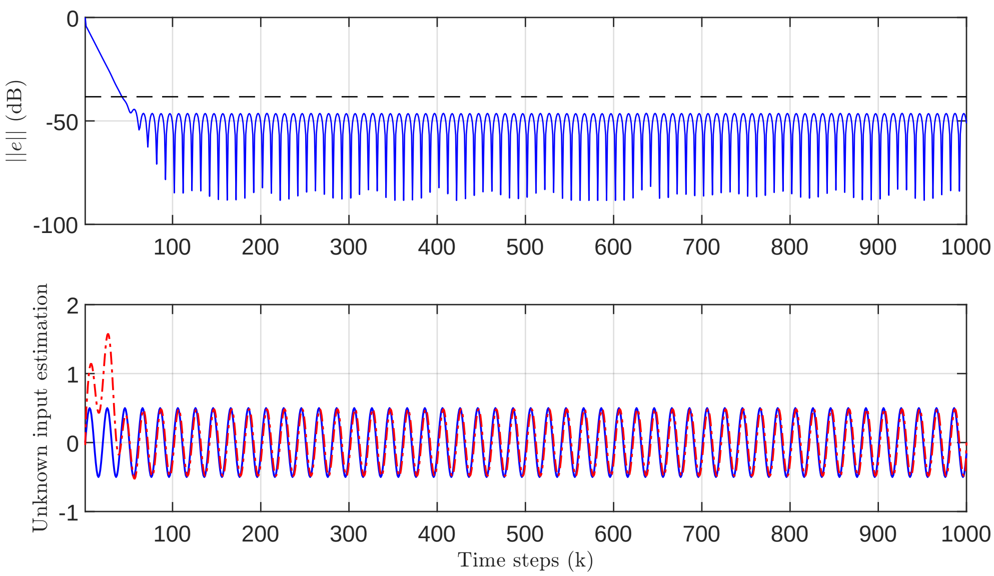

Fig. 4. Error in the state estimation and unknown input estimation for Lorenz dynamical system in Example 1. The blue solid line indicates simulated unknown input, whereas the red dash-dotted line indicates its estimate.

Example 2 (Aircraft Longitudinal Dynamics)**.** The longitudinal movement dynamics of an aircraft in wind axis can be expressed as follows [50, eq. (2.5-32)]:

$$\begin{aligned} m\dot{v} &= F_t\cos(\alpha) - D - mg\sin(\theta-\alpha), \\ mv\dot{\alpha} &= -F_t\sin(\alpha) - L + mg\cos(\theta-\alpha) + mvq, \\ J_y\dot{q} &= M_y, \\ \dot{\theta} &= q, \end{aligned} \tag{48}$$

where v is the speed relative to the wind, α is the angle of attack of the aircraft, q is the pitch rate, θ is the pitch angle, m is the mass of the aircraft, J_y is the pitch moment of inertia, g is the acceleration of gravity, F_t is the thrust force, D is the drag force, L is the lift force and M_y the total pitch moment (torque). We consider that $m = 9357.61\text{kg}$, $J_y = 75673.62\text{kgm}^2$ and $g = 9.81\text{m/s}^2$, which correspond to the simulation of an F16 aircraft longitudinal model taken from the FlightGear open-source flight simulator [59]. The signals v, q, θ, F_t and $v_h = v\sin(\theta-\alpha)$, with v_h the rate at which the aircraft is changing its altitude, can be measured by sensors. The goal is to estimate the remaining state α and the unknown inputs D, L and M_y.

Using the forward-Euler discretization with a time step $T = 0.1$s, the discrete-time counterpart of system (48) can be obtained as

$$\begin{aligned} E(z_k)x_{k+1} &= A(z_k)x_k + F(z_k)\phi(x_k, u_k) + Bd_k, \\ y_k &= Cx_k + G(z_k)\psi(x_k), \end{aligned} \tag{49}$$

with $x_k = \begin{bmatrix} v_k & \alpha_k & q_k & \theta_k \end{bmatrix}^\top$, $u_k = F_{tk}$, $d_k = \begin{bmatrix} D & L & M_y \end{bmatrix}^\top$, $z_k = \begin{bmatrix} v_k & \sin(\theta_k) & \cos(\theta_k) \end{bmatrix}^\top$, $y_k = \begin{bmatrix} v_k & q_k & \theta_k & v_{h_k} \end{bmatrix}^\top$, $\phi(x_k, u_k) = \begin{bmatrix} F_{tk}\cos(\alpha_k) & F_{tk}\sin(\alpha_k) & \cos(\alpha_k) & \sin(\alpha_k) \end{bmatrix}^\top$, $\psi(x_k) = \begin{bmatrix} \cos(\alpha_k) & \sin(\alpha_k) \end{bmatrix}^\top$ and

$$E(z_k) = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & mz_{1k} & 0 & 0 \\ 0 & 0 & J_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -T & 0 & 0 \\ 0 & -T & 0 \\ 0 & 0 & T \\ 0 & 0 & 0 \end{bmatrix},$$

$$A(z_k) = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & mz_{1k} & Tmz_{1k} & 0 \\ 0 & 0 & J_y & 0 \\ 0 & 0 & T & 1 \end{bmatrix},$$

$$F(z_k) = \begin{bmatrix} T & 0 & -Tmgz_{2k} & Tmgz_{3k} \\ 0 & -T & Tmgz_{3k} & Tmgz_{2k} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G(z_k) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ z_{1k}z_{2k} & -z_{1k}z_{3k} \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The value ranges of the variables of system (49) are given by [50]

$$v_k \in [90, 150], \quad \alpha_k \in [-\frac{\pi}{18}, \frac{\pi}{12}], \quad \theta_k \in [-\frac{\pi}{18}, \frac{\pi}{12}], \quad F_{tk} \in [0, 2.66 \times 10^4]. \tag{50}$$

Since both nonlinear functions $\phi(x_k, u_k)$ and $\psi(x_k)$ are continuously differentiable with respect to x_k, whose gradients are respectively defined as

$$\begin{aligned} \nabla_x\phi_1 &= \begin{bmatrix} 0 & -F_{tk}\sin(\alpha_k) & 0 & 0 \end{bmatrix}, & \nabla_x\phi_2 &= \begin{bmatrix} 0 & F_{tk}\cos(\alpha_k) & 0 & 0 \end{bmatrix}, \\ \nabla_x\phi_3 &= \begin{bmatrix} 0 & -\sin(\alpha_k) & 0 & 0 \end{bmatrix}, & \nabla_x\phi_4 &= \begin{bmatrix} 0 & \cos(\alpha_k) & 0 & 0 \end{bmatrix}, \\ \nabla_x\psi_1 &= \begin{bmatrix} 0 & -\sin(\alpha_k) & 0 & 0 \end{bmatrix}, & \nabla_x\psi_2 &= \begin{bmatrix} 0 & \cos(\alpha_k) & 0 & 0 \end{bmatrix}. \end{aligned}$$

Moreover, it follows from (50) that $\sin(\theta_k) \in [-0.1736, 0.2588]$, $\cos(\theta_k) \in [0.9659, 1]$, $\sin(\alpha_k) \in [-0.1736, 0.2588]$, $\cos(\alpha_k) \in [0.9659, 1]$, $F_{tk}\sin(\alpha_k) \in [-6884, 4618]$ and $F_{tk}\cos(\alpha_k) \in [0, 2.66 \times 10^4]$. Using the sector nonlinearity approach [10, Chapter 2], an equivalent N-TS fuzzy descriptor representation (6) with eight fuzzy rules ($r = 8$) can be obtained for the nonlinear descriptor system (49). A convex bounded set representation can be determined for $K(\bar{\beta})$ and $M(\bar{\theta})$ in (13) with 16 and 4 vertices, respectively. The state-space matrices and the nonlinear MFs corresponding to this N-TS fuzzy representation, as well as the vertex matrices for the convex bounded sets are omitted for brevity.

By Theorem 2, an observer solution can be obtained with $\tau = 0.27$, $\epsilon = 117.6812$, and a suboptimal $\ell_\infty-$gain level of $\gamma = 2.432 \times 10^{-4}$. The corresponding estimation results are summarized in Figs. 5 and 6. We can see that the designed N-TS fuzzy observer can provide a satisfactory estimation performance for this real-world application, except for a small bias on the estimation of α_k and D_k.

VI. Concluding Remarks

A new ℓ_∞ observer design has been proposed for a large class of nonlinear descriptor systems with UIs. To deal with the challenging issue on unmeasured premise variables, the nonlinear system is rewritten in a specific N-TS fuzzy form which permits isolating unmeasurable nonlinearities. Together with Lyapunov-based arguments and a judicious use of the DMVT, this allows for an effective N-TS fuzzy observer design framework guaranteeing the global uniform $\ell_\infty-$stablity of the estimation error dynamics. Moreover, for observer design, a special singular redundancy representation is exploited for the treatment of the nonlinear descriptor form. As a result, all involved decision variables for the observer design can be of free structure to further reduce the design conservatism. The estimation accuracy of both the state and the UI can be improved by minimizing a suboptimal $\ell_\infty-$gain level. The interests of the proposed N-TS fuzzy observer design are highlighted with two real-world applications. Future works focus on exploiting the proposed results for N-TS fuzzy observer-based control of descriptor systems with unmeasured nonlinearities. Another interesting future direction is to extend the proposed observer design method to the case where the descriptor matrix $\mathcal{E}(x_k)$ in (1) is singular and/or the premise variables cannot be precisely obtained from sensors.

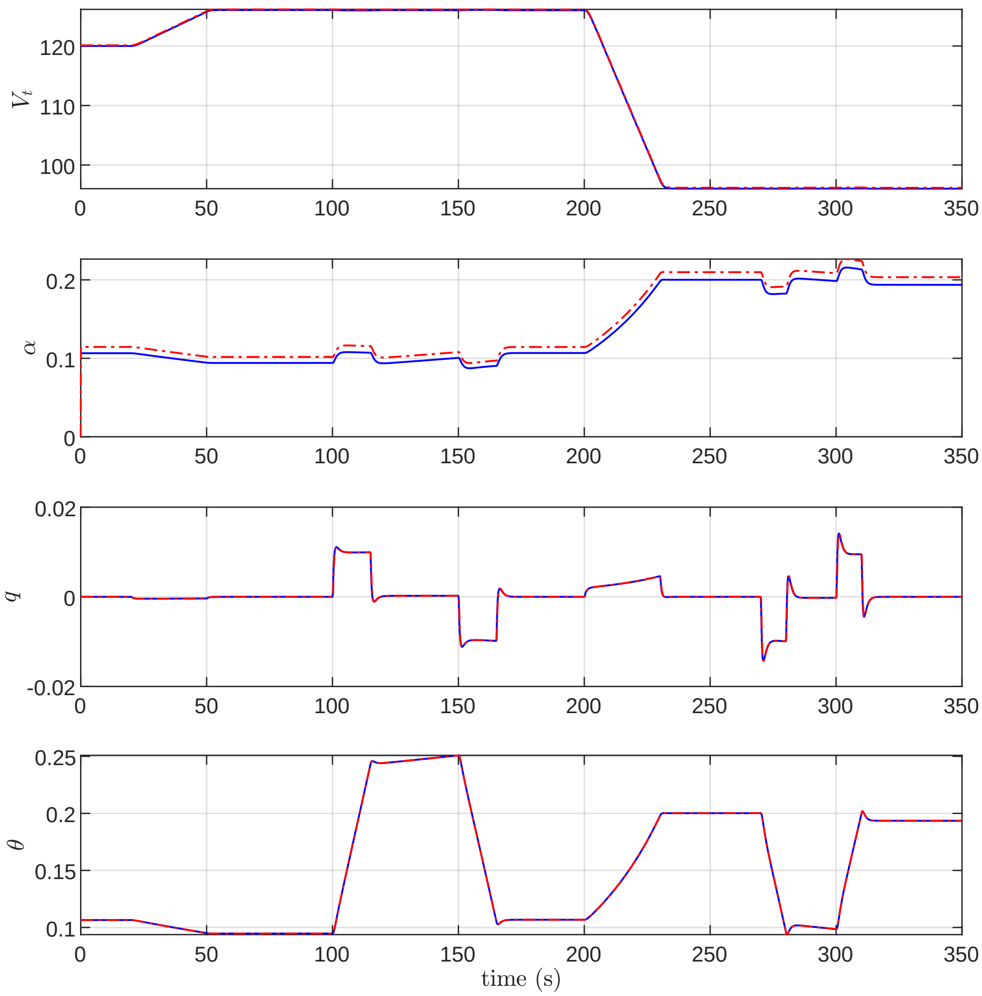

Fig. 5. State estimation for the aircraft in Example 2. The blue solid lines indicate the original state, whereas the red dash-dotted lines indicate their estimates.

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

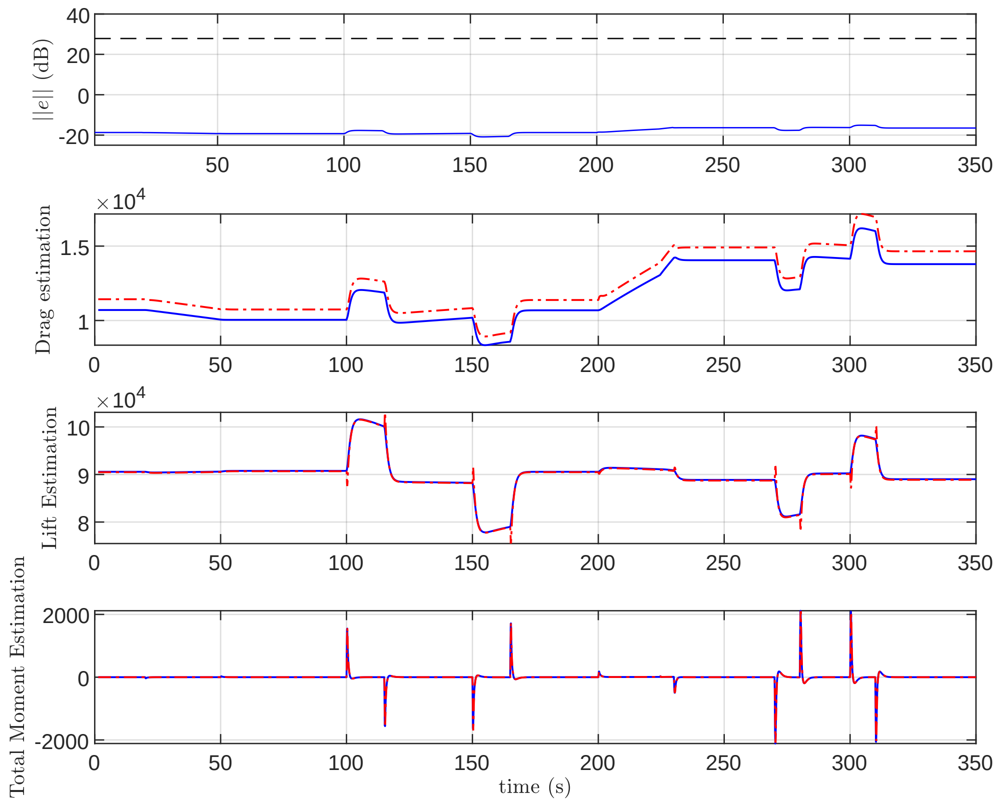

Fig. 6. Error in the state estimation and unknown input estimation for the aircraft in Example 2. The blue solid lines indicate the original signal, whereas the red dash-dotted lines indicate their estimates.

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