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# Robust observer-based controller design for Takagi–Sugeno systems with nonlinear consequent parts

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**Abstract**

This paper considers the design of robust observer-based controller for a class of continuous-time nonlinear systems with disturbances. The system is presented by Takagi–Sugeno (TS) model with nonlinear consequents. The proposed TS structure reduces the number of rules in the Sugeno model by using local nonlinear rules. It is assumed that the nonlinearities satisfy a sector condition as well as an incremental quadratic constraint. These constraints are fuzzified to reduce the conservativeness. The proposed controller guarantees $H_\infty$ performance of the system by using Linear Matrix Inequality (LMI) formulation. Moreover, the proposed observer benefits from a nonlinear injection term to add more degrees of freedom to the LMIs. Numerical examples illustrate effectiveness of the proposed method as compared to the existing methods in literature in terms of performance and enlarging the class of system.


*Keywords:* Takagi–Sugeno model; State estimation; Robust control; Nonlinear local model; Nonlinear injection

## 1. Introduction

Takagi–Sugeno (TS) fuzzy models are well-known tools for nonlinear system modeling with increasing interest in recent years. Since the TS model is a universal approximator, it can model any smooth nonlinear system with desired accuracy [1]. Furthermore, the convex structure of the whole TS model allows one to use powerful linear system tools, such as Linear Matrix Inequalities (LMIs), to analyze the TS fuzzy systems. However, as complexity of the system increases, the number of rules in the model increases as well. This will enlarge the dimensions of the LMIs, and hence becomes harder to solve. One possible solution is to reduce the accuracy in the model, which decreases the model complexity. However, in general, the convergence of the fuzzy controller or the observer is not guaranteed in this case. Another approach is to use nonlinear local subsystems for the TS model. This will decrease the number of rules while increasing the model accuracy.

A very simple form of these nonlinear TS models is used in [2], where the authors have employed a linear form for the consequent parts plus a sinusoidal term. A more advanced method is introduced by Dong in [3–5], where

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the author has used sector-bounded functions in the subsystems. Another more popular form of the TS system with nonlinear consequent parts uses polynomial subsystems, which has been the center of interest for many researchers in recent years, e.g. [6–8]. In this manuscript, in order to have nonlinear local subsystems in the TS model, a method similar to the Dong's work (i.e., adding a sector bounded nonlinear term to the linear consequent part of the TS model) is presented and it is tried to reduce the conservativeness in his works.

For the TS systems with such structure, very limited results on the observer design or the observer-based controller design exist. The output feedback controller for the discrete-time systems with nonlinear TS model with disturbances is discussed in [3], where the observer and controller gains are designed using a two-step algorithm. The observer design for the continuous-time systems with unmeasured premise variables, with and without uncertainties, is proposed in [9] and [10] respectively. The observer-based controller design for such systems is also considered in papers like [11] and [12].

In this paper, a robust observer-based controller for the TS systems with nonlinear consequent parts is designed. The novel ideas are proposed to reduce the conservativeness in the design procedure. First, a nonlinear injection term is added to the observer structure. This idea has been introduced in [13]. However, to the best of the authors' knowledge, this is the first use of this kind of injection term in the fuzzy TS systems. This is a direct benefit of having nonlinear subsystems versus the linear ones. Moreover, the Lipschitz condition on the nonlinear term in the consequent parts, which has been assumed in Dong's works, is replaced by the Incremental Quadratic Constraint ($\delta$QC) [13]. It will be shown that how such constraint can significantly relax the Lipschitz condition and hence, encompasses a larger class of systems, yielding a more robust closed-loop system. Finally, both the sector condition and the $\delta$QC are fuzzified to increase the degrees of freedom in the LMIs.

This paper is organized as follows: In Section 2, the nonlinear TS model and the $\delta$QC are described. The proposed controller and some definitions are discussed in Section 3. The stability analysis for the observer-based controller with external disturbances is presented in Section 4. In Section 5, numerical examples are given. Section 6 concludes the paper.

## 2. Preliminaries and notations

In this section, some preliminaries and mathematical definitions are given. First, the TS model with nonlinear subsystems, based on the model introduced in [3], is described. Then, the incremental quadratic constraint that is used in the proposed observer design is described.

### 2.1. TS model with nonlinear subsystems

Consider a class of nonlinear systems described by

$$\begin{aligned} \dot{x}(t) &= f_a\big(x(t)\big) + f_b\big(x(t)\big)\bar{\varphi}\big(x(t)\big) + g_1\big(x(t)\big)u(t) + g_2\big(x(t)\big)v_1(t) \\ y(t) &= f_{ya}\big(x(t)\big) + f_{yb}\big(x(t)\big)\bar{\varphi}\big(x(t)\big) + g_3\big(x(t)\big)v_2(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $y(t) \in \mathbb{R}^{n_y}$ is the measurable output, $g_i(x(t))$ $(i = 1, 2, 3)$ and $f_n(x(t))$ $n \in \{a, b, ya, yb\}$ are nonlinear functions that will be linearized, $v_i(t) \in L_2$ $(i = 1, 2)$ represent the noise and disturbances on the states and outputs, respectively, and $\bar{\varphi}(x(t)) \in \mathbb{R}^{n_\varphi}$ is a vector of nonlinear functions that are sector bounded in the following cone:

$$\bar{\varphi}_i\big(x(t)\big) \in \mathrm{co}\big\{\mathbf{E}_{Li}x(t), \mathbf{E}_{ui}x(t)\big\}, \quad i = 1, \dots, n_\varphi \tag{2}$$

where $\mathbf{E}_{Li}$ and $\mathbf{E}_{ui}$ are constant vectors with appropriate dimensions. For simplicity, a transformation is performed on the nonlinear term to make the lower sector bound equal to zero; that is $\varphi_i(x) = \bar{\varphi}_i(x) - \mathbf{E}_{Li}x(t)$, so that

$$\varphi_i\big(x(t)\big) \in \mathrm{co}\big\{0, \mathbf{E}_i x(t)\big\}, \quad i = 1, \dots, n_\varphi \tag{3}$$

where $\mathbf{E}_i = \mathbf{E}_{ui} - \mathbf{E}_{Li}$.

The system in (1) can be represented by a TS fuzzy system with local nonlinear models as follows:

Plant Rule $i$: IF $z_1(t)$ is $\mu_{i1}(z)$, ..., and $z_p(t)$ is $\mu_{ip}(z)$,

$$\begin{aligned} \text{THEN} \quad \dot{x}(t) &= \mathbf{A}_i x(t) + \mathbf{G}_{xi}\varphi\big(x(t)\big) + \mathbf{B}_i u(t) + \mathbf{D}_{1i}v_1(t) \\ y(t) &= \mathbf{C}_i x(t) + \mathbf{G}_{yi}\varphi\big(x(t)\big) + \mathbf{D}_{2i}v_2(t) \end{aligned} \tag{4}$$

where $i = 1, \dots, r$ is the number of rules, $z_1(t), \dots, z_p(t)$ are the premise variables, $\mu_{ij}$ $(j = 1, \dots, p)$ denote the fuzzy sets, and $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{G}_{xi}$, $\mathbf{G}_{yi}$ and $\mathbf{D}_{2i}$ are matrices with appropriate dimensions selected for the best representation of the system model. In this case, the entire fuzzy system can be represented as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \omega_i(z)\big[\mathbf{A}_i x(t) + \mathbf{G}_{xi}\varphi\big(x(t)\big) + \mathbf{B}_i u(t) + \mathbf{D}_{1i} v_1(t)\big] \\ y(t) &= \sum_{i=1}^{r} \omega_i(z)\big[\mathbf{C}_i x(t) + \mathbf{G}_{yi}\varphi\big(x(t)\big) + \mathbf{D}_{2i} v_2(t)\big] \end{aligned} \tag{5}$$

where

$$\omega_i(z) = \frac{h_i(z)}{\sum_{k=1}^{r} h_k(z)}, \qquad \sum_{i=1}^{r} \omega_i(z) = 1, \qquad h_i(z) = \prod_{j=1}^{p} \mu_{ij}(z). \tag{6}$$

### 2.2. Incremental quadratic constraint

For the nonlinear function $\varphi(x(t))$ in (1), suppose that the following relations exist:

$$\begin{aligned} &\varphi\big(x(t), t\big) = \phi(s, q) \\ &q = \mathbf{C}_q x(t) + \mathbf{D}_q \varphi\big(x(t), t\big) \end{aligned} \tag{7}$$

where $\phi$ is a continuous function, $q \in \mathbb{R}^{n_q}$, $\mathbf{C}_q$ and $\mathbf{D}_q$ are constant matrices with proper dimensions, $t$ is the time variable and $s = (t, y(t))$. Note that $\mathbf{D}_q$ term is included to treat the systems where the nonlinear term depends on the derivative of some of the state variables. To have a better representation of the aforementioned variables, consider a system described by $\dot{x}_1 = x_2, \dot{x}_2 = 0.5\sin(x_1 + \dot{x}_2)$ [13]. Let $\varphi = \sin(x_1 + \dot{x}_2)$. Then $\dot{x}_1 = x_2$ and $\dot{x}_2 = 0.5\varphi$. Now, consider $q = x_1 + \dot{x}_2$. Hence, $\varphi = \phi(q) = \sin(q)$, where $q = \mathbf{C}_q x + \mathbf{D}_q \varphi$, in which $\mathbf{C}_q = [\,1 \quad 0\,]$ and $\mathbf{D}_q = 0.5$.

Characterization of the nonlinear element $\phi(s, q)$ is based on a set of symmetric matrices $\mathcal{M}$, which is referred to as the "incremental multiplier matrix" [13]. Specifically, for all $\mathbf{\Lambda} \in \mathcal{M}$ the following incremental quadratic constraint ($\delta$QC) holds:

$$\begin{pmatrix} q_2 - q_1 \\ \phi(s, q_2) - \phi(s, q_1) \end{pmatrix}^T \mathbf{\Lambda} \begin{pmatrix} q_2 - q_1 \\ \phi(s, q_2) - \phi(s, q_1) \end{pmatrix} \geq 0. \tag{8}$$

It should be mentioned that $\mathcal{M}$ provides a characterization of $\phi(s, q)$ in an incremental sense. Now, defining $\sigma := \mathbf{C}_q x$, it gives

$$\begin{aligned} &\phi\big(s, \sigma + \mathbf{D}_q \varphi\big(x(t), t\big)\big) = \psi(s, \sigma) \\ &\varphi\big(x(t), t\big) = \psi\big(s, \mathbf{C}_q x(t)\big) \end{aligned} \tag{9}$$

For the previous example, for every $\sigma = \mathbf{C}_q x$ there exists a unique solution for $\varphi = \sin(v + 0.5\varphi)$ that can be written as $\varphi = \psi(\sigma)$.

Note that $\psi(s, \sigma)$ satisfies the incremental quadratic constraint in (8). That is,

$$\begin{pmatrix} \delta\sigma \\ \delta\psi^T \end{pmatrix} \mathbf{N} \begin{pmatrix} \delta\sigma \\ \delta\psi \end{pmatrix} \geq 0 \tag{10}$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{I} & \mathbf{D}_q \\ 0 & \mathbf{I} \end{pmatrix}^T \mathbf{\Lambda} \begin{pmatrix} \mathbf{I} & \mathbf{D}_q \\ 0 & \mathbf{I} \end{pmatrix}. \tag{11}$$

Generally, when $\mathbf{D}_q \neq 0$ it is easier to obtain a $\delta$QC characterization for a nonlinear term using $\phi$ rather than $\psi$. Sometimes, calculating $\psi$ is only possible using numerical methods, while for $\phi$ this condition can be easily shown. More examples of such nonlinearities are given in [13]. It is shown that this condition embraces several nonlinearities including the Lipschitz ones.

## 3. Controller representation

The controller used in this paper is based on a fuzzy Luenberger-type observer, given as follows:

$$\begin{aligned}
&\text{Controller Rule } i\text{: IF } z_1(t) \text{ is } \mu_{i1}(z), \ldots, \text{ and } z_p(t) \text{ is } \mu_{ip}(z),\\
&\qquad\text{THEN } \dot{\hat{x}}(t) = \mathbf{A}_i\hat{x}(t) + \mathbf{G}_{xi}\hat{\varphi}\big(\hat{x}(t)\big) + \mathbf{B}_i u(t) + \mathbf{L}_i\big[\hat{y}(t) - y(t)\big]\\
&\qquad\qquad\quad \hat{y}(t) = \mathbf{C}_i\hat{x}(t) + \mathbf{G}_{yi}\hat{\varphi}\big(\hat{x}(t)\big)\\
&\qquad\qquad\quad u(t) = -\mathbf{K}_{ai}\hat{x}(t) - \mathbf{K}_{bi}\hat{\varphi}\big(\hat{x}(t)\big)\\
&\qquad\qquad\quad \hat{\varphi}\big(\hat{x}(t)\big) = \varphi\big(\hat{x}(t) + \mathbf{L}_\varphi\big[\hat{y}(t) - y(t)\big]\big)
\end{aligned} \tag{12}$$

where $\hat{x}$ is the estimation of $x$. Unlike other Sugeno observers, here the nonlinear injection term $\mathbf{L}_\varphi[\hat{y}(t) - y(t)]$ [13] is used to have a better estimation of $\varphi(x(t))$, which in turn provides better estimation of all states in the system. For the analysis of the error convergence, two cases can be distinguished: 1) the scheduling vector $z(t) = [z_1(t) \ldots z_p(t)]^T$ does not depend on the estimated states and 2) the scheduling vector $z(t)$ depends on some of the estimated states. In this study, the first case is considered. The observer design for the second case is performed in [10] that can be extended to the observer-based controller design using the ideas in this paper. For the first case, controller (12) becomes

$$\begin{aligned}
\dot{\hat{x}}(t) &= \sum_{i=1}^{r}\omega_i(z)\big\{\mathbf{A}_i\hat{x}(t) + \mathbf{G}_{xi}\hat{\varphi}\big(\hat{x}(t)\big) + \mathbf{B}_i u(t) + \mathbf{L}_i\big[\hat{y}(t) - y(t)\big]\big\}\\
\hat{y}(t) &= \sum_{i=1}^{r}\omega_i(z)\big[\mathbf{C}_i\hat{x}(t) + \mathbf{G}_{yi}\hat{\varphi}\big(\hat{x}(t)\big)\big]\\
u(t) &= -\sum_{i=1}^{r}\omega_i(z)\big[\mathbf{K}_{ai}\hat{x}(t) + \mathbf{K}_{bi}\hat{\varphi}\big(\hat{x}(t)\big)\big]
\end{aligned} \tag{13}$$

For brevity, the following definition will be used:

$$\mathbf{X}_z := \sum_{i=1}^{r}\omega_i(z)\mathbf{X}_i \tag{14}$$

for $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{G}_x, \mathbf{G}_y, \mathbf{L}, \mathbf{K}_a, \mathbf{K}_b\}$. Hence, the augmented system and the controller dynamic can be rewritten as

$$\begin{aligned}
\dot{x}_a = &\begin{bmatrix} \mathbf{A}_z - \mathbf{B}_z\mathbf{K}_{az} & \mathbf{B}_z\mathbf{K}_{az} \\ 0 & \mathbf{A}_z + \mathbf{L}_z\mathbf{C}_z \end{bmatrix} x_a + \begin{bmatrix} \mathbf{G}_{xz} - \mathbf{B}_z\mathbf{K}_{bz} & \mathbf{B}_z\mathbf{K}_{bz} \\ 0 & \mathbf{G}_{xz} + \mathbf{L}_z\mathbf{G}_{yz} \end{bmatrix}\varphi_a\big(x(t)\big)\\
&+ \begin{bmatrix} \mathbf{D}_{1z} & 0 \\ \mathbf{D}_{1z} & \mathbf{L}_z\mathbf{D}_{2z} \end{bmatrix} v(t)
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
&x_a(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \qquad \varphi_a\big(x(t)\big) = \begin{bmatrix} \varphi(x(t)) \\ \varphi_e(x(t)) \end{bmatrix}, \qquad v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}\\
&e(t) = x(t) - \hat{x}(t), \qquad \varphi_e\big(x(t)\big) = \varphi\big(x(t)\big) - \hat{\varphi}\big(\hat{x}(t)\big)
\end{aligned}. \tag{16}$$

In this study, the aim is to design the control input $u(t)$ in (12) such that the following control performance is achieved [14]:

$$\int_{t_0}^{t_f} x_a^T(t)\mathbf{\Upsilon}_1 x_a(t)dt \le x_a^T(0)\mathbf{\Upsilon}_2 x_a(0) + \eta\int_{t_0}^{t_f} v^T(t)v(t)dt \tag{17}$$

where matrices $\mathbf{\Upsilon}_1$ and $\mathbf{\Upsilon}_2$ and scalar $\eta$ are selected to satisfy the desired performance.

For further simplification in notations, let define $\mathbf{\Gamma}_i := \mathrm{diag}[\gamma_{1i} \cdots \gamma_{ki}]$ $(i = 1, \ldots, r)$ and $\mathbf{E} := [\,\mathbf{E}_1^T\ \mathbf{E}_2^T\ \cdots\ \mathbf{E}_k^T\,]$. Then,

$$\varphi\big(x(t)\big)^T(\boldsymbol{\Gamma}_z)^{-1}\mathbf{E}x(t)-\varphi\big(x(t)\big)^T(\boldsymbol{\Gamma}_z)^{-1}\varphi\big(x(t)\big)=\big[\gamma_{1z}^{-1}\varphi_1\big(x(t)\big)\cdots\gamma_{kz}^{-1}\varphi_k\big(x(t)\big)\big]\begin{bmatrix}\mathbf{E}_1x(t)\\ \mathbf{E}_2x(t)\\ \vdots\\ \mathbf{E}_kx(t)\end{bmatrix}$$
$$-\sum_{i=1}^{k}\gamma_{iz}^{-1}\varphi_i^2\big(x(t)\big)$$
$$=\sum_{i=1}^{k}\gamma_{iz}^{-1}\big\{\varphi_i\big(x(t)\big)\big(\mathbf{E}_ix(t)-\varphi_i\big(x(t)\big)\big)\big\}. \tag{18}$$

where $\boldsymbol{\Gamma}_\mathrm{z} := \sum_{i=1}^{r}\omega_i(z)\boldsymbol{\Gamma}_\mathrm{i}$. According to (2), it can be written

$$\varphi_i\big(x(t)\big)\big[\mathbf{E}_ix(t)-\varphi_i\big(x(t)\big)\big]\geq 0 \quad \text{for } i=1,\dots,n_\varphi. \tag{19}$$

Combining (18) and (19), it yields

$$\varphi\big(x(t)\big)^T\boldsymbol{\Gamma}_z^{-1}\mathbf{E}x(t)-\varphi\big(x(t)\big)^T\boldsymbol{\Gamma}_z^{-1}\varphi\big(x(t)\big)\geq 0. \tag{20}$$

From now on, (20) will be used as the sector nonlinearity condition on the nonlinear term $\varphi(x(t))$. When there is no uncertainty in the model, it follows from (13) that $\hat{y}(t)-y(t)=\sum_{i=1}^{r}\omega_i(z)(\mathbf{C}_ie(t)+\mathbf{G}_{yi}\varphi_e(x(t)))$ and by defining two variables $\sigma_1$ and $\sigma_2$ as

$$\begin{aligned}&\sigma_1 := \mathbf{C}_qx(t)\\ &\sigma_2 := \mathbf{C}_q\hat{x}(t)+\mathbf{L}_\varphi\left[\sum_{i=1}^{r}\omega_i(z)\big(\mathbf{C}_ie(t)+\mathbf{G}_{yi}\varphi_e\big(x(t)\big)\big)\right]\end{aligned} \tag{21}$$

and based on (10), it yields

$$\begin{pmatrix}e(t)\\ \varphi_e(x(t))\end{pmatrix}^T\boldsymbol{\Phi}^T\boldsymbol{\Lambda}_z\boldsymbol{\Phi}\begin{pmatrix}e(t)\\ \varphi_e(x(t))\end{pmatrix}\geq 0 \tag{22}$$

where $\boldsymbol{\Lambda}_z=\sum_{i=1}^{r}\omega_i(z)\boldsymbol{\Lambda}_i$ and

$$\boldsymbol{\Phi} := \sum_{i=1}^{r}\omega_i(z)\begin{pmatrix}\mathbf{C}_q+\mathbf{L}_\varphi\mathbf{C}_i & \mathbf{D}_q+\mathbf{L}_\varphi\mathbf{G}_{xi}\\ 0 & \mathbf{I}\end{pmatrix}. \tag{23}$$

In order to analyze the system using LMIs, it is assumed that matrix $\boldsymbol{\Lambda}_i$ has the following form:

$$\boldsymbol{\Lambda}_i=\begin{pmatrix}\boldsymbol{\Lambda}_{ai} & 0\\ 0 & -\boldsymbol{\Lambda}_{bi}\end{pmatrix} \tag{24}$$

where $\boldsymbol{\Lambda}_{ai}=\boldsymbol{\Lambda}_{ai}^T>0$ and $\boldsymbol{\Lambda}_{bi}=\boldsymbol{\Lambda}_{bi}^T>0$ are matrices with appropriate dimensions. It should be noted that the results obtained here could be easily extended for other possible forms of $\boldsymbol{\Lambda}_i$. Based on (24), it can be written

$$\boldsymbol{\Phi}^T\boldsymbol{\Lambda}_z\boldsymbol{\Phi}=\begin{pmatrix}(\mathbf{C}_q+\mathbf{L}_\varphi\mathbf{C}_z)^T\\ (\mathbf{D}_q+\mathbf{L}_\varphi\mathbf{G}_{yz})^T\end{pmatrix}\boldsymbol{\Lambda}_{az}\begin{pmatrix}\mathbf{C}_q+\mathbf{L}_\varphi\mathbf{C}_z & \mathbf{D}_q+\mathbf{L}_\varphi\mathbf{G}_{yz}\end{pmatrix}-\begin{pmatrix}0\\ \mathbf{I}\end{pmatrix}\boldsymbol{\Lambda}_{bz}\begin{pmatrix}0 & \mathbf{I}\end{pmatrix}. \tag{25}$$

This representation is useful in converting the analysis results into LMIs.

## 4. Main results

In this section, the conditions for asymptotic convergence of the observer states in (12) to the system states in (4) and convergence of the system states to zero, based on the performance introduced in (17), will be given. The following lemma is used in the sequel.

**Lemma 1.** *(See [15].) If the following conditions hold:*

$$\begin{aligned}&\mathbf{M}_{ii} < 0 && 1 \le i \le r\\ &\frac{1}{r-1}\mathbf{M}_{ii} + \frac{1}{2}(\mathbf{M}_{ij} + \mathbf{M}_{ji}) < 0 && 1 \le i \ne j \le r\end{aligned} \tag{26}$$

*then, the following inequality holds:*

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i\omega_j\mathbf{M}_{ij} < 0 \tag{27}$$

*where* $\omega_i$ $(1 \le i \le r)$ *satisfies* $0 \le \omega_i \le 1$ *and* $\sum_{i=1}^{r}\omega_i = 1$.

The following theorem provides the main result of this paper.

**Theorem 1.** *The augmented system* (15) *is asymptotically stable in the case of* $v(t) = 0$ *and the* $H_\infty$ *control performance in (17) is guaranteed for a prescribed* $\eta > 0$ *in the case of* $v(t) \ne 0$ *if there exist matrices* $\mathbf{P}_2 = \mathbf{P}_2^T > 0$, $\mathbf{Q}_1 = \mathbf{Q}_1^T > 0$, $\mathbf{X}_{ai}, \mathbf{X}_{bi}, \mathbf{Y}_i$ $(1 \le i \le r)$, $\mathbf{\Gamma}_i = \mathrm{diag}[\gamma_{1i} \cdots \gamma_{ki}]$, $\gamma_{ij} > 0$, $\mathbf{\Lambda}_{\mathrm{ai}} = \mathbf{\Lambda}_{\mathrm{ai}}^T > 0$, $\mathbf{\Lambda}_{\mathrm{bi}} = \mathbf{\Lambda}_{\mathrm{bi}}^T > 0$, *and scalars* $\alpha_j > 0$ $(1 \le j \le 3)$ *such that*

$$\begin{aligned}&\mathbf{\Omega}_{ii} < 0, \qquad \mathbf{\Theta}_{ii} < 0 \quad 1 \le i \le r\\ &\frac{1}{r-1}\mathbf{\Theta}_{ii} + \frac{1}{2}(\mathbf{\Theta}_{ij} + \mathbf{\Theta}_{ji}) < 0 \qquad 1 \le i \ne j \le r\\ &\frac{1}{r-1}\mathbf{\Omega}_{ii} + \frac{1}{2}(\mathbf{\Omega}_{ij} + \mathbf{\Omega}_{ji}) < 0 \qquad 1 \le i \ne j \le r\end{aligned} \tag{28}$$

*where*

$$\mathbf{\Theta}_{ij} = \begin{bmatrix} \mathbf{\Theta}_{11}^{ij} & * & * & * & * \\ \mathbf{\Theta}_{21}^{ij} & -2\mathbf{\Gamma}_j & * & * & * \\ \mathbf{X}_{aj}^T\mathbf{B}_i^T & 0 & -\alpha_2\mathbf{Q}_1 & * & * \\ \mathbf{X}_{bj}^T\mathbf{B}_i^T & 0 & 0 & -\alpha_3\mathbf{\Gamma}_j & * \\ \begin{pmatrix}\mathbf{D}_{1i} & 0\end{pmatrix}^T & 0 & 0 & 0 & -\frac{\eta}{2}\mathbf{I} \end{bmatrix} \tag{29}$$

$$\mathbf{\Omega}_{ij} = \begin{bmatrix} \mathbf{\Omega}_{11}^{ij} & * & * & * & * & * \\ \mathbf{G}_{xi}^T\mathbf{P}_2^T + \mathbf{G}_{yj}^T\mathbf{Y}_i^T & -\mathbf{\Lambda}_{bi} & * & * & * & * \\ \begin{pmatrix}\mathbf{P}_2\mathbf{D}_{1i} & \mathbf{Y}_i\mathbf{D}_{2j}\end{pmatrix}^T & 0 & -\frac{\eta}{2}\mathbf{I} & * & * & * \\ \mathbf{\Lambda}_{ai}\mathbf{C}_q + \mathbf{L}_n\mathbf{C}_j & \mathbf{\Lambda}_{ai}\mathbf{D}_q + \mathbf{L}_n\mathbf{G}_{yj} & 0 & -\mathbf{\Lambda}_{ai} & * & * \\ \mathbf{I} & 0 & 0 & 0 & -\frac{1}{\alpha_2}\mathbf{Q}_1 & * \\ 0 & \mathbf{I} & 0 & 0 & 0 & -\frac{1}{\alpha_3}\mathbf{\Gamma}_j \end{bmatrix} \tag{30}$$

*in which*

$$\begin{aligned}&\mathbf{\Theta}_{11}^{ij} = \mathbf{A}_i\mathbf{Q}_1 - \mathbf{B}_i\mathbf{X}_{aj} + \mathbf{Q}_1^T\mathbf{A}_i^T - \mathbf{X}_{aj}^T\mathbf{B}_i^T + \mathbf{Q}_1\\ &\mathbf{\Theta}_{21}^{ij} = \mathbf{\Gamma}_j\mathbf{G}_{xi}^T - \mathbf{X}_{bj}\mathbf{B}_i^T + \mathbf{E}\mathbf{Q}_1\\ &\mathbf{\Omega}_{11}^{ij} = \mathbf{P}_2\mathbf{A}_i + \mathbf{A}_i^T\mathbf{P}_2^T + \mathbf{Y}_i\mathbf{C}_j + \mathbf{C}_j^T\mathbf{Y}_i^T + \alpha_1\mathbf{P}_2\end{aligned} \tag{31}$$

*where* $\eta > 0$ *is the* $H_\infty$ *performance criterion. Then, the observer and controller gains are*

$$\mathbf{L}_i = \mathbf{P}_2^{-1}\mathbf{Y}_i, \qquad \mathbf{L}_\varphi = \mathbf{\Lambda}_{az}^{-1}\mathbf{L}_n, \qquad \mathbf{K}_{ai} = \mathbf{X}_{ai}\mathbf{Q}_1^{-1}, \qquad \mathbf{K}_{bi} = \mathbf{X}_{bi}\mathbf{\Gamma}_z^{-1} \quad 1 \le i \le r \tag{32}$$

**Proof.** Based on Lemma 1, (28) results in

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i\omega_j\boldsymbol{\Theta}_{ij}<0 \tag{33}$$

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i\omega_j\boldsymbol{\Omega}_{ij}<0 \tag{34}$$

Pre- and post-multiplying (33) by diag([ $\mathbf{Q}_1^{-1}$ $\boldsymbol{\Gamma}_z^{-1}$ $\mathbf{Q}_1^{-1}$ $\boldsymbol{\Gamma}_z^{-1}$ $\mathbf{I}$ ]) and its transpose yield

$$\begin{bmatrix} \mathbf{S}_{11}^{ij} & * & * & * & * \\ \mathbf{S}_{21} & -2\boldsymbol{\Gamma}_z^{-1} & * & * & * \\ (\mathbf{B}_z\mathbf{K}_{az})^T\mathbf{P}_1^T & 0 & -\alpha_2\mathbf{P}_1^T & * & * \\ (\mathbf{B}_z\mathbf{K}_{bz})^T\mathbf{P}_1^T & 0 & 0 & -\alpha_3\boldsymbol{\Gamma}_z^{-1} & * \\ \begin{pmatrix}\mathbf{P}_1\mathbf{D}_{1z} & 0\end{pmatrix}^T & 0 & 0 & 0 & -\frac{\eta}{2}I \end{bmatrix}<0 \tag{35}$$

where $\mathbf{P}_1:=\mathbf{Q}_1^{-T}$ and

$$\begin{aligned} &\mathbf{S}_{11}:=\mathbf{P}_1(\mathbf{A}_z-\mathbf{B}_z\mathbf{K}_{az})+(\mathbf{A}_z-\mathbf{B}_z\mathbf{K}_{az})^T\mathbf{P}_1^T+\mathbf{P}_1^T \\ &\mathbf{S}_{21}:=(\mathbf{G}_{xz}-\mathbf{B}_z\mathbf{K}_{bz})^T\mathbf{P}_1^T+\boldsymbol{\Gamma}_z^{-1}\mathbf{E} \end{aligned} \tag{36}$$

Pre- and post-multiplying (35) by [ $x^T(t)$ $\varphi^T(x(t))$ $e^T(t)$ $\varphi_e^T(x(t))$ $v(t)$ ] and its transpose yield

$$\begin{aligned} &x^T(t)\big(\mathbf{P}_1(\mathbf{A}_z-\mathbf{B}_z\mathbf{K}_{az})+(\mathbf{A}_z-\mathbf{B}_z\mathbf{K}_{az})^T\mathbf{P}_1^T+\mathbf{P}_1^T\big)x(t) \\ &\quad+\mathrm{He}\big(\varphi^T\big(x(t)\big)\boldsymbol{\Gamma}_z^{-1}\mathbf{E}x(t)\big)-2\varphi^T\big(x(t)\big)\boldsymbol{\Gamma}_z^{-1}\varphi\big(x(t)\big) \\ &\quad-\alpha_3\varphi_e^T\big(x(t)\big)\boldsymbol{\Gamma}_z^{-1}\varphi_e\big(x(t)\big)-\alpha_2e^T(t)\mathbf{P}_1^Te(t)-\frac{\eta}{2}v^T(t)v(t) \\ &\quad+\mathrm{He}\big(\varphi^T\big(x(t)\big)\big((\mathbf{G}_{xz}-\mathbf{B}_z\mathbf{K}_{bz})^T\mathbf{P}_1^T\big)x(t)\big) \\ &\quad+\mathrm{He}\big(e^T(t)(\mathbf{B}_z\mathbf{K}_{az})^T\mathbf{P}_1^Tx(t)\big)+\mathrm{He}\left(v^T(t)(\,\mathbf{D}_{1z}\quad 0\,)^T\mathbf{P}_1^Tx(t)\right) \\ &\quad+\mathrm{He}\big(\varphi_e^T\big(x(t)\big)(\mathbf{B}_z\mathbf{K}_{bz})^T\mathbf{P}_1^Tx(t)\big)<0 \end{aligned} \tag{37}$$

where $\mathrm{He}(\mathbf{A})=\mathbf{A}^T+\mathbf{A}$. Moreover, by applying the Schur complement on (34), it yields

$$\begin{bmatrix} \mathbf{T}+\begin{bmatrix}(\mathbf{C}_q+\mathbf{L}_\varphi\mathbf{C}_j)^T \\ (\mathbf{D}_q+\mathbf{L}_\varphi\mathbf{G}_{yj})^T\end{bmatrix}\boldsymbol{\Lambda}_{az}[\,\mathbf{C}_q+\mathbf{L}_\varphi\mathbf{C}_j\quad \mathbf{D}_q+\mathbf{L}_\varphi\mathbf{G}_{yj}\,] & * \\ \begin{pmatrix}\mathbf{P}_2\mathbf{D}_{1z} & \mathbf{Y}_z\mathbf{D}_{2z}\end{pmatrix}^T\quad 0 & -\frac{\eta}{2}\mathbf{I} \end{bmatrix}<0 \tag{38}$$

where

$$\begin{aligned} &\mathbf{T}=\begin{bmatrix}\mathbf{T}_{11} & * \\ \mathbf{T}_{21} & \mathbf{T}_{22}\end{bmatrix} \\ &\mathbf{T}_{11}=\mathbf{P}_2\mathbf{A}_i+\mathbf{A}_i^T\mathbf{P}_2^T+\mathbf{Y}_i\mathbf{C}_j+\mathbf{C}_j^T\mathbf{Y}_i^T+\alpha_1\mathbf{P}_2+\alpha_2\mathbf{P}_1 \\ &\mathbf{T}_{21}=\mathbf{G}_{xi}^T\mathbf{P}_2+\mathbf{G}_{yj}^T\mathbf{Y}_i^T \\ &\mathbf{T}_{22}=-\boldsymbol{\Lambda}_{bi}+\alpha_3\boldsymbol{\Gamma}^{-1}. \end{aligned} \tag{39}$$

Pre- and post-multiplying (38) by [ $e^T(t)$ $\varphi_e^T(x(t))$ $v^T(t)$ ] and its transpose, it gives

$$\begin{aligned} &\big[e^T(t)(\mathbf{A}_z+\mathbf{L}_z\mathbf{C}_z)^T+\varphi_e^T\big(x(t)\big)(\mathbf{G}_{xz}+\mathbf{L}_z\mathbf{G}_{yz})^T\big]\mathbf{P}_2e(t) \\ &\quad+e^T(t)\mathbf{P}_2\big[(\mathbf{A}_z+\mathbf{L}_z\mathbf{C}_z)e(t)+(\mathbf{G}_{xz}+\mathbf{L}_z\mathbf{G}_{yz})\varphi_e\big(x(t)\big)\big] \\ &\quad+\alpha_2e^T(t)\mathbf{P}_1^Te(t)+\begin{pmatrix}e(t)\\ \varphi_e(x(t))\end{pmatrix}^T\boldsymbol{\Phi}^T\mathbf{M}_z\boldsymbol{\Phi}\begin{pmatrix}e(t)\\ \varphi_e(x(t))\end{pmatrix} \end{aligned}$$

$$+\alpha_1 e^T(t)\mathbf{P}_2 e(t) - \frac{\eta}{2}\upsilon^T(t)\upsilon(t) + \alpha_3\varphi_e^T\big(x(t)\big)\mathbf{\Gamma}_z^{-1}\varphi_e\big(x(t)\big)$$
$$\upsilon^T(t)(\mathbf{D}_{1z}\quad \mathbf{L}_z\mathbf{D}_{2z})^T\mathbf{P}e(t) + e^T(t)\mathbf{P}_2(\mathbf{D}_{1z}\quad \mathbf{L}_z\mathbf{D}_{2z})\upsilon(t) < 0. \tag{40}$$

For the augmented dynamic (15), consider the Lyapunov function as $\mathbf{V}(x(t)) = x_a^T\bar{\mathbf{P}}x_a$, where $\bar{\mathbf{P}} = \mathrm{diag}([\,\mathbf{P}_1\ \mathbf{P}_2\,])$. Then, the time derivative of this Lyapunov function is

$$\dot{\mathbf{V}}\big(x(t)\big) = 2x_a^T\bar{\mathbf{P}}\dot{x}_a = \sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i(z)\omega_j(z)\Phi_{ij} \tag{41}$$

where

$$\begin{aligned}\Phi_{ij} = {}& x^T(t)\big(\mathbf{P}_1(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_{aj}) + (\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_{aj})^T\mathbf{P}_1^T\big)x(t)\\ &+ \mathrm{He}\big(\varphi^T\big(x(t)\big)\big((\mathbf{G}_{xi} - \mathbf{B}_i\mathbf{K}_{bj})^T\mathbf{P}_1^T\big)x(t)\big)\\ &+ \mathrm{He}\big(e^T(t)(\mathbf{B}_i\mathbf{K}_{aj})^T\mathbf{P}_1^T x(t)\big) + \mathrm{He}\Big(\upsilon^T(t)(\,\mathbf{D}_{1z}\quad \mathbf{L}_z\mathbf{D}_{2z}\,)^T\mathbf{P}_2 e(t)\Big)\\ &+ \mathrm{He}\big(\varphi_e^T\big(x(t)\big)(\mathbf{B}_i\mathbf{K}_{bj})^T\mathbf{P}_1^T x(t)\big) + \mathrm{He}\Big(\upsilon^T(t)(\,\mathbf{D}_{1i}\quad 0\,)^T\mathbf{P}_1^T x(t)\Big)\\ &\times\big[e^T(t)(\mathbf{A}_z + \mathbf{L}_z\mathbf{C}_z)^T + \varphi_e^T\big(x(t)\big)(\mathbf{G}_{xz} + \mathbf{L}_z\mathbf{G}_{yz})^T\big]\mathbf{P}_2 e(t)\\ &+ e^T(t)\mathbf{P}_2\big[(\mathbf{A}_z + \mathbf{L}_z\mathbf{C}_z)e(t) + (\mathbf{G}_{xz} + \mathbf{L}_z\mathbf{G}_{yz})\varphi_e\big(x(t)\big)\big].\end{aligned} \tag{42}$$

By adding (37) and (40), and based on Lemma 1, (20) and (22), (41) becomes

$$\dot{\mathbf{V}}(t) + \alpha_1 e^T(t)\mathbf{P}_2 e(t) + x^T(t)\mathbf{P}_1 x(t) - \eta\upsilon^T(t)\upsilon(t) < 0 \tag{43}$$

which implies that the error dynamic is asymptotically stable in the disturbance-free case. Furthermore, when disturbances exist in the model, integrating (43) in time yields

$$\mathbf{V}\big(x(t)\big) - \mathbf{V}\big(x(0)\big) + \int_0^t \big(\alpha_1 e^T(t)\mathbf{P}_2 e(t) + x^T(t)\mathbf{P}_1 x(t)\big)dt - \eta\int_0^t \upsilon^T(t)\upsilon(t)dt < 0. \tag{44}$$

When $\mathbf{V}(x(t)) > 0$, then performance criteria (17) is achieved with

$$\mathbf{\Upsilon}_1 = \begin{bmatrix}\mathbf{P}_1 & 0\\ 0 & \alpha_1\mathbf{P}_2\end{bmatrix} \quad \text{and} \quad \mathbf{\Upsilon}_2 = \bar{\mathbf{P}}. \qquad \square \tag{45}$$

**Remark 1.** In order to reach the best disturbance rejection performance, the minimum value of $\eta$ and the maximum value of $\mathbf{P}_1$ and $\mathbf{P}_2$ should be found based on (17). Hence, the problem of the observer design can be stated as

$$\begin{aligned}&\text{Minimize } \eta\\ &\text{subject to (29)},\quad \mathbf{P}_2 > \frac{1}{\varepsilon}I,\quad \text{and}\quad \mathbf{Q}_1 < \varepsilon I\end{aligned} \tag{46}$$

where $\varepsilon > 0$ is a parameter that should be selected as small as possible. Formulating as (46), the value of $\eta$ can be determined by an LMI solver and hence, there is no need to determine it in advance. However, some inequalities in (28) are not strictly LMIs and parameters $\alpha_1$, $\alpha_2$, and $\alpha_3$ should be properly selected by the user. By changing parameters $\alpha_2$ and $\alpha_3$, feasible solutions from LMIs can be obtained while $\alpha_1$ can be changed to satisfy the desired performance for disturbance rejection.

Note that based on performance criteria (17), there is no constraint on the inputs. Hence, this method may result in high values of the control signal. To prevent this, the following theorem is used.

**Theorem 2.** *Assume that the initial condition $x(0)$ is known. Then, the constrained input $\|\mathbf{u}(t)\| \le \eta_2$ $\forall t > 0$ is enforced if there exists $\alpha_4$ such that the following LMIs hold:*

$$\begin{bmatrix}\alpha_4 & x(0)^T\\ x(0) & \mathbf{Q}_1\end{bmatrix} \ge 0 \tag{47}$$

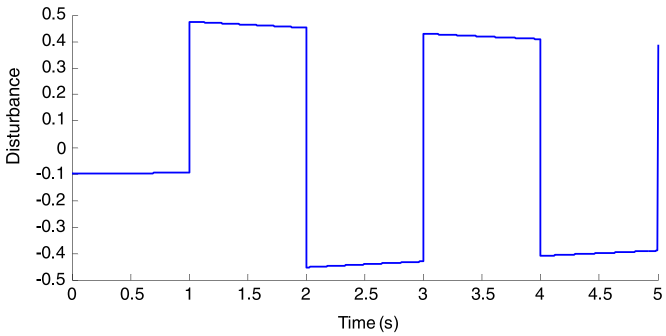


Fig. 1. Disturbances $v_i(t)$ applied to inverted pendulum.

$$\begin{bmatrix} \mathbf{Q}_1 & * & * \\ -\mathbf{E}\mathbf{Q}_1 & 2\boldsymbol{\Gamma}_i & * \\ \mathbf{X}_{ai} & \mathbf{X}_{bi} & \frac{\eta_2^2}{2\alpha_4}\mathbf{I} \end{bmatrix} \geq 0 \tag{48}$$

**Proof.** See [11].

Note that $\alpha_4$ should be properly selected by the user before solving the LMIs in (47) and (48).

## 5. Simulation examples

### 5.1. Example 1

To illustrate performance of the proposed fuzzy control approach and compare the results with the traditional linear TS model, the control problem of balancing an inverted pendulum is considered here. The state equations of the inverted pendulum are given as follows [16]:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{-dx_2 - a(mlx_2)^2 \sin x_1(t)\cos x_1(t) + mgl\sin x_1(t)}{(J + ml^2) - a(ml\cos x_1(t))^2} \\ &\quad + \frac{-aml\cos x_1(t)}{(J + ml^2) - a(ml\cos x_1(t))^2} u(t) + v_1(t) \\ y(t) &= x_1(t) + 0.01 v_2(t) \end{aligned} \tag{49}$$

where $a = 1/(M + m)$, $x_1$ denotes the angle (rad) of the pendulum from the vertical position, $x_2$ is the angular velocity (rad/s), $g = 9.8\ \mathrm{m/s^2}$ is the gravity constant, $m = 0.3$ kg is the mass of the pendulum, $M = 15$ kg is the mass of the cart, $d = 0.007$ N/rad/s is the friction coefficient of the pendulum, $l = 0.3$ m is the length from the center of mass of the pendulum to the shaft axis, $J = 0.005\ \mathrm{kg\,m^2}$ is the moment of inertia of the pendulum, $u(t)$ is the force (N) applied to the cart, and $v_1(t) = v_2(t)$ are disturbances shown in Fig. 1.

For this system, a fuzzy observer-based controller with four rules is designed in [16] to control the pendulum in the range of $[-\pi/3, \pi/3]$. Here, two types of modeling are considered to control the system in the same range. The first one is a nonlinear TS model with only three rules and triangular type membership functions while the second one is an exact model with the sector-nonlinearity approach. The first model is as follows:

$$\begin{aligned} &\text{Plant Rule } i\text{: IF} x_i \text{ is about } M_i \\ &\qquad \text{THEN } \dot{x} = \mathbf{A}_i x(t) + \mathbf{G}_{xi}\varphi\big(x(t)\big) + \mathbf{B}_i u(t) + \mathbf{D}_{1i}\omega(t) \end{aligned} \tag{50}$$

where $M_1 = 0$, $M_2 = \pm\pi/4$, $M_3 = \pm\pi/3$ and

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ 24.9555 & -0.0311 \end{bmatrix}, \qquad \mathbf{B}_1 = \begin{bmatrix} 0 \\ -0.2222 \end{bmatrix}, \qquad \mathrm{G}_{x1} = \begin{bmatrix} 0 \\ 39.2 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 22.4645 & -0.0280 \end{bmatrix}, \qquad \mathbf{B}_2 = \begin{bmatrix} 0 \\ -0.1414 \end{bmatrix}, \qquad \mathrm{G}_{x2} = \begin{bmatrix} 0 \\ 35.28 \end{bmatrix}$$
$$\mathbf{A}_3 = \begin{bmatrix} 0 & 1 \\ 21.3904 & -0.0267 \end{bmatrix}, \qquad \mathbf{B}_3 = \begin{bmatrix} 0 \\ -0.0952 \end{bmatrix}, \qquad \mathrm{G}_{x3} = \begin{bmatrix} 0 \\ 33.6 \end{bmatrix}$$
$$\mathbf{C}_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \mathbf{G}_{y_i} = 0, \qquad \mathbf{D}_{1i} = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T, \qquad \mathbf{D}_{2i} = 0.01$$
$$\varphi\big(x(t)\big) = \sin x_1 - \frac{2}{\pi} x_1, \qquad \mathbf{C}_q = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \mathbf{D}_q = 0$$
$$\mathbf{E} = \begin{bmatrix} 1 - \frac{2}{\pi i} & 0 \end{bmatrix}, \qquad \mathbf{\Lambda}_{ai} = 0.6366 \mathbf{\Lambda}_{bi}, \qquad \mathbf{\Lambda}_{bi} > 0 \tag{51}$$

The membership functions are of the triangular type. Based on Theorem 1, the observer and controller gains are obtained using the YALMIP software [17] as follows:

$$\mathbf{K}_{a1} = [-215.5408 \quad -62.5975], \qquad \mathbf{K}_{a2} = [-336.9484 \quad -103.6139],$$
$$\mathbf{K}_{a3} = [-431.5498 \quad -133.6282],$$
$$\mathbf{X}_{b1} = -0.8700, \qquad \mathbf{X}_{b2} = -2.0340, \qquad \mathbf{X}_{b3} = -1.4653,$$
$$\mathbf{\Gamma}_1 = 0.0068, \qquad \mathbf{\Gamma}_2 = 0.0111, \qquad \mathbf{\Gamma}_3 = 0.0047, \qquad \mathbf{L}_\varphi \approx 1$$
$$\mathbf{L}_1 = [-41.6557 \quad -209.7805], \qquad \mathbf{L}_2 = [-41.7777 \quad -210.3411], \qquad \mathbf{L}_3 = [-41.8066 \quad -210.4721]$$
$$\mathbf{P}_1 = \begin{bmatrix} 0.0357 & -0.0332 \\ -0.0332 & 0.4976 \end{bmatrix}, \qquad \mathbf{P}_2 = \begin{bmatrix} 2749.3 & -535.6 \\ -535.6 & 106.4 \end{bmatrix}. \tag{52}$$

For the sector-nonlinearity model, let $x_2 \in [-\delta, \delta]$. Then

$$\dot{x} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} E_i(z_1)N_j(z_2)S_k(z_3)\left(\begin{bmatrix} 0 & 1 \\ \frac{2}{\pi}gq_i & -\frac{aml}{2}q_i c_j \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -aq_i d_k \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ gq_i \end{bmatrix} \varphi\big(x(t)\big)\right)$$
$$z_1 = 1\bigg/\left(\frac{4l}{3} - aml\cos^2(x_1)\right), \qquad z_2 = x_2 \sin(2x_1), \qquad z_3 = \cos(x_1)$$
$$q_1 = 1\bigg/\left(\frac{4l}{3} - aml\cos^2\left(\frac{\pi}{3}\right)\right), \qquad q_2 = 1\bigg/\left(\frac{4l}{3} - aml\right), \qquad c_1 = \delta, \qquad c_2 = -\delta, \qquad d_1 = 1,$$
$$d_2 = \cos\left(\frac{\pi}{3}\right)$$
$$E_1 = \frac{z_1 - q_2}{q_1 - q_2}, \qquad E_2 = 1 - E_1, \qquad N_1 = \frac{z_2 - c_2}{c_1 - c_2}, \qquad N_2 = 1 - N_1, \qquad S_1 = \frac{z_3 - d_2}{d_1 - d_2},$$
$$S_2 = 1 - S_1$$
$$\mathbf{C}_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \mathbf{G}_{y_i} = 0, \qquad \mathbf{D}_{1i} = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T, \mathbf{D}_{2i} = 0.01$$
$$\varphi\big(x(t)\big) = \sin x_1 - \frac{2}{\pi} x_1, \qquad \mathbf{C}_q = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \mathbf{D}_q = 0$$
$$\mathbf{E} = \begin{bmatrix} 1 - \frac{2}{\pi i} & 0 \end{bmatrix}, \qquad \mathbf{\Lambda}_{ai} = 0.6366 \mathbf{\Lambda}_{bi}, \qquad \mathbf{\Lambda}_{bi} > 0 \tag{53}$$

The sector-nonlinearity model has eight rules in contrast to the 16 rules in [18]. Based on Theorem 1, the observer and controller gains are obtained as follows:

$$\mathbf{K}_{a1} = -\begin{bmatrix} 226.2201 & 89.0512 \end{bmatrix}, \qquad \mathbf{K}_{a2} = -\begin{bmatrix} 411.5069 & 161.9519 \end{bmatrix},$$
$$\mathbf{K}_{a3} = -\begin{bmatrix} 224.8568 & 87.6486 \end{bmatrix},$$
$$\mathbf{K}_{a4} = -\begin{bmatrix} 409.3688 & 159.7238 \end{bmatrix}, \qquad \mathbf{K}_{a5} = -\begin{bmatrix} 221.1601 & 85.0652 \end{bmatrix},$$
$$\mathbf{K}_{a6} = -\begin{bmatrix} 406.9201 & 156.9493 \end{bmatrix},$$
$$\mathbf{K}_{a7} = -\begin{bmatrix} 220.2382 & 84.1741 \end{bmatrix}, \qquad \mathbf{K}_{a8} = -\begin{bmatrix} 410.2477 & 158.6944 \end{bmatrix},$$
$$\mathbf{X}_{b1} = -2.8967, \qquad \mathbf{X}_{b2} = -3.9457, \qquad \mathbf{X}_{b3} = -2.8829, \qquad \mathbf{X}_{b4} = -3.9371,$$
$$\mathbf{X}_{b5} = -2.7891, \qquad \mathbf{X}_{b6} = -3.9049, \qquad \mathbf{X}_{b7} = -2.7763, \qquad \mathbf{X}_{b8} = -5.0156,$$

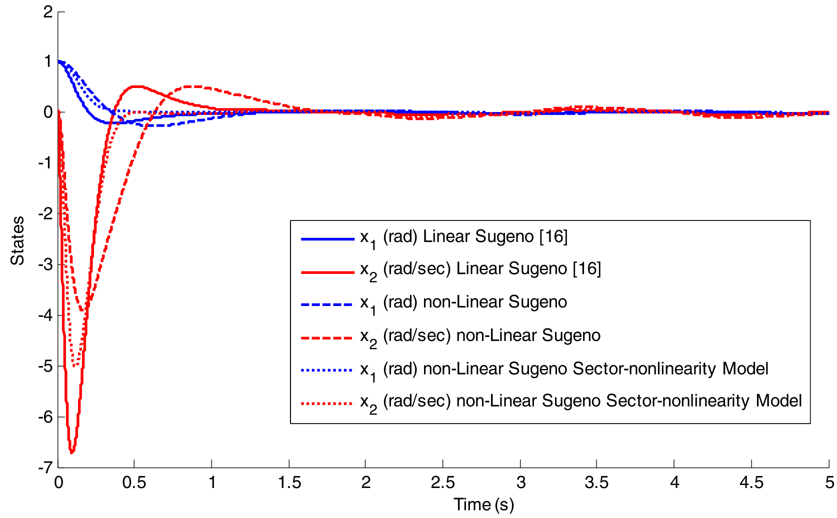


Fig. 2. States of inverted pendulum for linear TS (solid), nonlinear TS model (dashed), and nonlinear TS with sector-nonlinearity model (dotted). (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

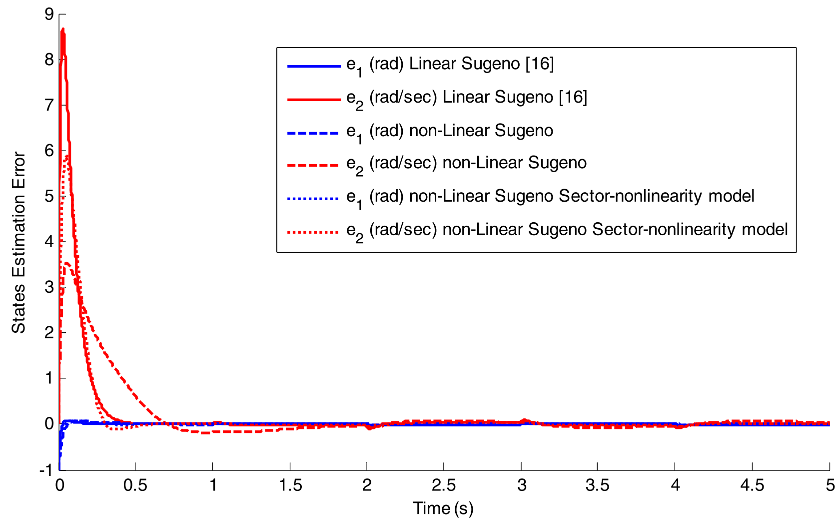


Fig. 3. States estimation error of inverted pendulum for linear TS (solid), nonlinear TS model (dashed), and nonlinear TS with sector-nonlinearity model (dotted). (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

$$\begin{aligned}
&\boldsymbol{\Gamma}_1 = 0.0269, \quad \boldsymbol{\Gamma}_2 = 0.0184, \quad \boldsymbol{\Gamma}_3 = 0.0269, \quad \boldsymbol{\Gamma}_4 = 0.0183,\\
&\boldsymbol{\Gamma}_5 = 0.0262, \quad \boldsymbol{\Gamma}_6 = 0.0181, \quad \boldsymbol{\Gamma}_7 = 0.0261, \quad \boldsymbol{\Gamma}_8 = 0.0258, \quad \mathbf{L}_\varphi \approx -1,\\
&\mathbf{L}_1 = -\begin{bmatrix} 13.8110 & 230.9142 \end{bmatrix}, \quad \mathbf{L}_2 = -\begin{bmatrix} 72.2504 & 695.6329 \end{bmatrix}, \quad \mathbf{L}_3 = -\begin{bmatrix} 58.1794 & 542.2110 \end{bmatrix},\\
&\mathbf{L}_4 = -\begin{bmatrix} 70.6118 & 622.8001 \end{bmatrix}, \quad \mathbf{L}_5 = -\begin{bmatrix} 58.7462 & 489.9482 \end{bmatrix}, \quad \mathbf{L}_6 = -\begin{bmatrix} 70.6993 & 548.6078 \end{bmatrix},\\
&\mathbf{L}_7 = -\begin{bmatrix} 13.5088 & 230.2542 \end{bmatrix}, \quad \mathbf{L}_8 = -\begin{bmatrix} 13.3847 & 226.8678 \end{bmatrix}
\end{aligned} \tag{54}$$

Figs. 2, 3 and 4 show the states of the system, their estimation errors, and the control signals, respectively, using the methods proposed here and in [16]. The defined parameters for both models are $\alpha_1 = 1, \alpha_2 = 15, \alpha_3 = 5$ and $\varepsilon = 0.5$. For the first model, the resulting $H_\infty$ parameter is $\eta = 0.448$. If the variations on the conditions of $\varphi(x(t))$ are omitted, then the $H_\infty$ parameter obtained from Theorem 1 will increase to $\eta = 0.4488$; and if the nonlinear injection term is also omitted, then the $H_\infty$ parameter will increase to $\eta = 1.0834$. Note that for this comparison the parameters are kept constant.

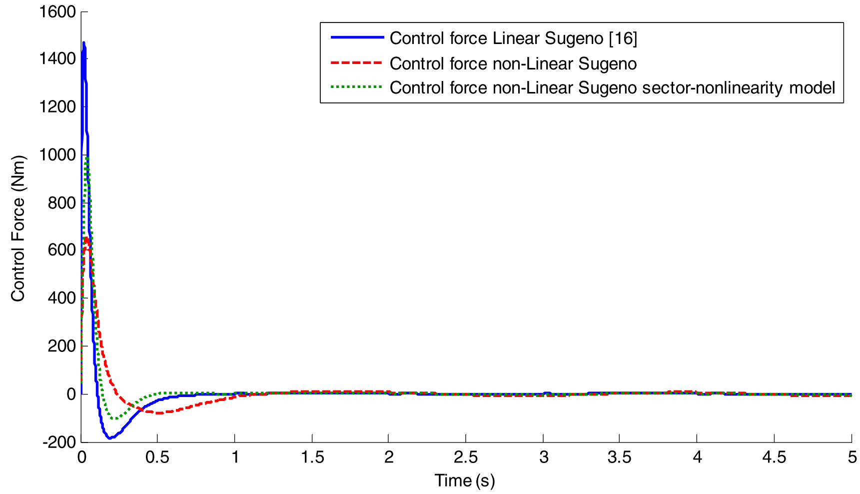


Fig. 4. Control input for linear TS model (solid), nonlinear TS model (dashed), and nonlinear TS with sector-nonlinearity model (dotted).

The simulation results show that in addition of using fewer fuzzy rules, drastic reduction in the control effort is also achieved. Moreover, the effect of adding the nonlinear injection term and using fuzziness on the constraints of the nonlinear term can result in substantial reduction of the $H_\infty$ bound.

### 5.2. Example 2

Another example is illustrated to highlight the benefits of the novelties introduced in this paper; i.e., using the nonlinear injection term, fuzzifying the sector condition, and using a fuzzy incremental quadratic constraint instead of the Lipschitz condition [3]. Consider the following fuzzy system (a modified version of Example 2 in [19]):

$$
\begin{aligned}
&\mathbf{A}_1=\begin{pmatrix}2 & -10\\ 2 & 0\end{pmatrix}, \qquad \mathbf{A}_2=\begin{pmatrix}a & -5\\ 1 & 2\end{pmatrix}\\
&\mathbf{B}_1=\begin{pmatrix}1 & 1\end{pmatrix}^T, \qquad \mathbf{B}_2=\begin{pmatrix}b & 2\end{pmatrix}^T, \qquad \mathbf{C}=\begin{pmatrix}1 & 0\end{pmatrix}\\
&\mathbf{G}_{x1}=\mathbf{G}_{x2}=\begin{pmatrix}0.1b & 0\\ 0 & 0.1a\end{pmatrix}, \qquad \mathbf{G}_y=\begin{pmatrix}0 & 0\end{pmatrix}\\
&\mathbf{E}=\mathbf{C}_q=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \mathbf{D}_q=0, \qquad \mathbf{\Lambda}_{ai}=\mathbf{\Lambda}_{bi}, \qquad \mathbf{\Lambda}_{ai}>0\\
&\mathbf{D}_1=\mathbf{D}_2=\mathbf{D}_3=\begin{pmatrix}0 & 0.01\end{pmatrix}^T, \qquad \mathbf{D}_y=0.01
\end{aligned}
\tag{55}
$$

Fig. 5 compares the feasible area for Theorem 1 (stars) and Theorem 1 with $\mathbf{L}_\varphi=0$ and constant matrixes $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ (circles), for different values of parameters $a$ and $b$ for this system. Parameters $\alpha_1=0.01$ and $\varepsilon=0.2$ are considered constant and parameters $\alpha_2$ and $\alpha_3$ are determined using the genetic algorithm with initial population of size ten and equal to $[\,0.01\ 0.1\ 1\ 2\ 5\ 10\ 15\ 20\ 50\ 100\,]$ for both $\alpha_2$ and $\alpha_3$. The number of generations is limited to 20 and the fitness function is the value of $\eta$ to be minimized. The values of $\eta$ less than one hundred are considered as feasible points because an $H_\infty$ index above this value is certainly not satisfactory. As Fig. 5 shows, the proposed approach has considerably increased the feasible area. Moreover, the mean value of $\eta$ for these feasible points for Theorem 1 is equal to 0.1194 while for the same theorem with $\mathbf{L}_\varphi=0$ and constant matrices $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ is equal to 33.1865.

## 6. Conclusion

A Sugeno-type fuzzy observer-based controller with nonlinear local subsystems for a class of continuous-time nonlinear systems with disturbances was proposed in this paper. The use of the nonlinear consequent parts for the TS system reduces the number of rules in the model. The observer and controller gains are calculated simultaneously using

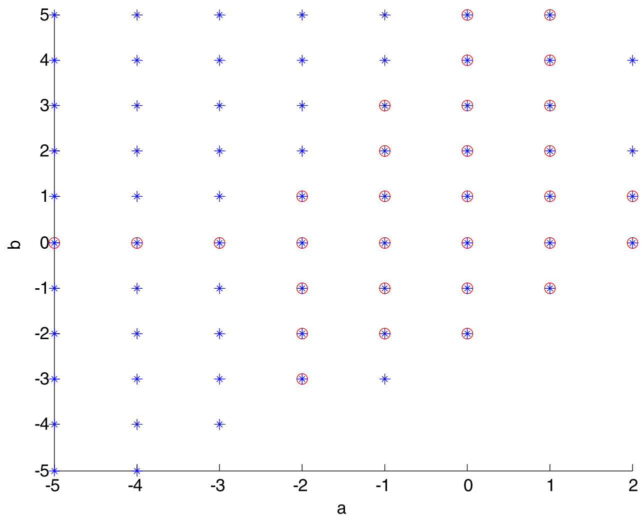


Fig. 5. Feasible area for Theorem 1 (*) and Theorem 1 without the nonlinear injection term and fuzzified conditions (o).

the LMI formulation. To reduce the conservativeness, constraints on the nonlinear terms are fuzzified. Moreover, a nonlinear injection term is added to the Luenberger observer with linear injection term. This can provide more degrees of freedom to the LMI solver. It was shown through simulating examples that the proposed method yields better performances as well as extending the class of systems as compared to the recently proposed methods in literature.

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