Subtask A - Estimation of σ_0^2

Prove that the MLE for variance σ_0^2 is:

$$\hat{\sigma}_n^2 = \frac{1}{N}\sum_{i=1}^{N}\left(y_i - x_i\hat{\beta}_N\right)^2$$

Introduction, planning the task:

Let us start by restating log-likelihood function for linear regression (where symbols have same meaning as in task introduction)

$$\log L(\beta_0, \sigma_0^2 \mid y, X) = -\frac{N}{2}\ln(2\pi\sigma_0^2) - \frac{1}{2\sigma_0^2}\sum_{i=1}^{N}(y_i - X_i\beta_0)^2$$

To prove the equation from task definition, we will take the derivative of the log-likelihood with respect to σ_0^2 and find it's maximum. (Equation from task introduction).

$$\frac{\partial}{\partial\sigma^2} l\left(\beta, \sigma^2 | y, X\right) = 0$$

Calculations:

We calculate the derivative and look for the maximum:

$$\frac{\partial}{\partial\sigma_0^2} l(\beta_0, \sigma_0^2 \mid y, X) = -\frac{N}{2\sigma_0^2} + \frac{1}{2(\sigma_0^2)^2}\sum_{i=1}^{N}\left(y_i - x_i\beta_0\right)^2$$

$$-\frac{N}{2\sigma_0^2} + \frac{1}{2(\sigma_0^2)^2}\sum_{i=1}^{N}\left(y_i - x_i\beta_0\right)^2 = 0$$

Now we determine σ_0^2:

$$\frac{1}{2(\sigma_0^2)^2}\sum_{i=1}^{N}\left(y_i - x_i\beta_0\right)^2 = \frac{N}{2\sigma_0^2}$$

$$\sum_{i=1}^{N}\left(y_i - x_i\beta_0\right)^2 = N(\sigma_0^2)$$

$$\frac{1}{N}\sum_{i=1}^{N}\left(y_i - x_i\beta_0\right)^2 = \sigma_0^2$$

Conclusion:

$$\hat{\sigma}_n^2 = \frac{1}{N}\sum_{i=1}^{N}\left(y_i - x_i\hat{\beta}_N\right)^2$$

We have reached desired form, which concludes the proof.