Probability

Tuesday, July 1, 2014 1:27 PM

Polymer Chains.

"Monomers" attached randomly.

$\mathbb{Z}^d = \mathbb{Z} \times \cdots \times \mathbb{Z}$. $d=3$.

Random walk path.

Polymer of length n (in $\mathbb{Z}^d$) starting 0 is a sequence of pts.

$\bar{x} = [x_0, \cdots x_n]$ $x_0 = 0$ $x_i \in \mathbb{Z}^d$ $|x_j - x_{j-1}| = 1$.

How many paths are there of length n? $(2d)^n$.

What does a "typical" path look like?

Give unif prob for each path of length n, $\frac{1}{(2d)^n}$.

How far from origin? Expected distance from origin.

$\approx \sqrt{n}$.

Flory. Self-avoiding walks

Def. A SAW of length n (in $\mathbb{Z}^d$) starting at 0 is a sequence of pts $[x_0 = 0, x_1, \cdots x_n]$ st. $x_j \in \mathbb{Z}^d$. $|x_j - x_{j-1}| = 1$. $\forall j \geq 1$. And $x_j \neq x_k$ if $j<k$.

How many SAWs are there? of length n.

What does a "typical" look like?

How do I generate SAW on the computing?

$d=1$. 2. typical walk is a straight line.

generating easy.

$d=2$. $C_n = \#$ SAWs starting at 0. $C_n \leq (2d)^n$. ✓

$C_n \leq (2n)(2d-1)^{n-1}$.

$C_n \geq d^n$, only moves in positive direction

$C_n \geq 2^d d^n$.

Conjecture: $\exists$ number β such that $C_n = \beta^n$.

e.g. $d=2$ $2 \leq \beta \leq 3$.

Thm. There exists a number μ_d such that $\lim_{n\to\infty} \frac{\log C_n}{n} = \log \mu_d$.

Def. A function $f: \mathbb{R}^+ \to \mathbb{R}$ is called subadditive if for $\forall n, m$.

$f(n+m) \leq f(n) + f(m)$.

Subadditivity Lemma: if f subadditive, then $\lim\limits_{n\to\infty} \frac{f(n)}{n}$ exists and $\lim\limits_{n\to\infty} \frac{f(n)}{n} = \inf \frac{f(n)}{n}$

pf: Let $N \in \mathbb{N}^+$. If n is any integer, $n = kN + r$ for $k \in \mathbb{N}^+$, $0 \le r \le N-1$.

$$\lim_{n\to\infty} \frac{f(n)}{n} = \lim_{n\to\infty} \frac{f(kN+r)}{kN+r} \le \lim_{n\to\infty} \frac{kf(N)}{kN+r} + \frac{f(r)}{kN+r}$$

$$= \lim_{n\to\infty} \frac{kf(N)+f(r)}{kN+r}$$

$$= \lim_{n\to\infty} \frac{k}{kN+r} \cdot f(N) + \frac{f(r)}{kN+r}$$

$$f(r) \le \max_{1\le j\le N-1} f(j) = M_N \Rightarrow \lim_{n\to\infty} \frac{f(n)}{n} \le \liminf_{k\to\infty} \frac{k}{kN+r} f(N) + \frac{M_N}{kN+r}$$

$$= \liminf_{k\to\infty} \frac{k}{kN+r} f(N)$$

$$= \frac{f(N)}{N}$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{n} \le \frac{f(N)}{N} \quad \text{for } \forall N \in \mathbb{N}^+$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{n} = \inf \frac{f(N)}{N} = \inf \frac{f(n)}{n}$$

Observation: $C_{n+m} \le C_n C_m \Rightarrow \log C_{n+m} \le \log C_n + \log C_m$

$\Rightarrow \lim\limits_{n\to\infty} \frac{\log C_n}{n} = \inf\limits_n \frac{\log C_n}{n} = \log \beta$ (define)

we know $C_n \ge 2^d \cdot d^n \Rightarrow \log C_n \ge \log 2^d + n \log d$

$\Rightarrow \frac{\log C_n}{n} \ge \log d$

$C_n \le (2d-1)^n 2d \Rightarrow \frac{\log C_n}{n} \le \log(2d-1)$

Exercise: Prove for $d=2$ $d < \beta < 2d-1$.

$\exists\, g(n)$ not SAW of strength n. $\Rightarrow C_n \le 2d(2d-1)^{n-1} - g(n)$,

$\beta = \inf\limits_n C_n^{\frac{1}{n}}$ $\quad 2 < \beta < 3$ $\quad \forall n\ C_n^{\frac{1}{n}} > 2$ $\quad$ ~~$n=4$ $C_n \le 4\times 3^3 - 4^2$~~

$\exists n\ C_n^{\frac{1}{n}} < 3$

Def $d=2$. β is called the Connective constant.

4.	4×3.	4×3^3	
5.	$4\times 4+4\times 4=32$.	4×3^4	
6.	$12\times 4+4\times 12+4\times 4\times 4\times 3$. $48+48+64$. $96+64=160$.	4×3^5	81×3.
7.	$4\times 3^2\times 4\times 2+12\times 4\times 3\times 2$.		

$$\frac{\log 2^d \cdot n^d \cdot g(n)}{n}.$$

$$\frac{\log g(n)}{n} \to c.$$

$g(n)=$ exponent $=c^n$.

Really: slow algo $\Rightarrow$ Choose each SAW of length n w/ prob $\frac{1}{C_n}$.

typical: Choose uniformly among C_n SAW of length n.

What is expected value of distance from origin?

Def. The exponent $\nu=\nu_d$ is defined by expected distance $\approx n^{\nu_d}$.

e.g. $\nu_1=1$.

Flory conjecture: $\nu_d=\begin{cases}\frac{3}{2+d} & d=1.2.3.4.\\ 1/2. & d=4.5.6....\end{cases}$

$d=2$. $d=3$ nothing. $d=5$ thm. $d=4$. close to thm.

$d=2$. believe true $\nu_2=3/4$.

$d=3$. $\nu=.588$. numerical.

In the ball of radius n, how many pts does RW meet?

RWs $\to$ 2 dim set.

for $d\geq 2\times 2=4$. RWs don't tend to intersect.

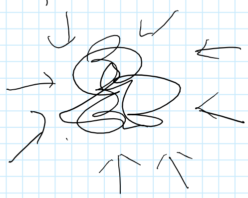
4 is the cut.

B. Mandelbrot

R.W. in dim 2. outer boundary / coast line. $\frac{4}{3}$ dim. of the RW.

Precise Statement:

Brownian motion. Frontier $\to$ boundary of the unbounded comp of the complement. (Everything can be seen from infinity).



Fractal dim $= \frac{4}{3}$.

(Complex analysis + probability)