Lecture 1

Tuesday, September 30, 2014 9:14 AM

X (typically $\mathbb{R}^n$ or a subset) $\mu : 2^X \longrightarrow [0, \infty]$ st.

- $\mu(\phi) = 0$
- $\mu(\bigcup_1^\infty A_i) = \sum \mu(A_i)$ outer measure

$\Rightarrow$ ($A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$)

- μ measure on X. $A \subset X$ $\mu \llcorner A (B) = \mu(A \cap B)$

$A \subset X$ μ-measurable iff $\forall B \subset X$ we have

$\mu(B) = \mu(A \cap B) + \mu(B \setminus A)$.

then $\mu(A) = 0 \Rightarrow A$ meas

A meas iff $X \setminus A$ meas

$A \subset X$ any μ-meas set is $\mu \llcorner A$ meas.

Thm. $(A_k)_{k=1}^\infty$ μ-measurable

1) $\bigcup A_k$, $\bigcap A_k$ is μ-meas

2) if (A_k) disj, then $\mu(\bigcup A_k) = \sum \mu(A_k)$

3) $A_1 \subset A_2 \subset \cdots$ then $\mu(\bigcup A_k) = \lim \mu(A_k)$

4) $A_1 \supset A_2 \supset \cdots \supset$ $\mu(A_1) < \infty$, then $\mu(\bigcap A_k) = \lim \mu(A_k)$.

- A_1, A_2 μ-meas $\Rightarrow$ $A_1 \cup A_2$ meas.
- countable additivity

$B_j = \bigcup_1^j A_k$ μ-measurable.

$\mu(B_{j+1}) = \mu(B_{j+1} \cap A_{j+1}) + \mu(B_{j+1} \setminus A_j)$

$= \mu(A_{j+1}) + \mu(B_j)$

$\Rightarrow \mu(A_k) = \sum \mu(A_k)$ $\mu(\bigcup A_k) \geq \sum^j \mu(A_j)$.

$\Rightarrow \mu(\bigcup A_k) \geq \sum \mu(A_k)$. also $\mu(\bigcup A_k) \leq \sum \mu(A_k)$

σ algebra

$\mathcal{A} \subset 2^X$ σ algebra

1) $\phi, X \in \mathcal{A}$

2) $A \in \mathcal{A}$ iff $X \setminus A \in \mathcal{A}$.

3) $A_k \in \mathcal{A} \Rightarrow \bigcup A_k \in \mathcal{A}$.

Borel σ-algebra in $\mathbb{R}^n$ smallest σ-alg containing open sets

measure μ on X is regular if $\forall A \subset X$, $\exists$ μ-meas B st. $A \subset B$ and $\mu(A) = \mu(B)$

measure μ on $\mathbb{R}^n$ is Borel if all Borel sets are meas.

μ on $\mathbb{R}^n$ is B-regular if $A \subset X$ $\exists B$ Borel set st.

$A \subset B$ $\quad \mu(A) = \mu(B)$

Thm $\Rightarrow$ Set of μ-meas sets form a σ-alg

- μ-regular measure on X. $\quad A_1 \subset A_2 \subset \cdots A_k \subset \cdots \subset X$.

$\Rightarrow \lim \mu(A_k) = \mu(\cup A_k)$

μ B-regular meas on $\mathbb{R}^n$. A μ-meas. $\mu(A) < \infty$

$\Rightarrow \mu \lfloor A$ is Radon

Radon B-reg. finite meas for compact sets

Pf: B Borel st. $A \subset B$ $\quad \mu(A) = \mu(B) \Rightarrow \mu(B \setminus A) = 0$.

$\forall C \subset \mathbb{R}^n$. $\quad (\mu \lfloor B)(C) = \mu(B \cap C)$

$$= \mu(B \cap C \cap A) + \mu(B \cap C \setminus A)$$

$$\leq \mu(C \cap A) + \mu(B \setminus A). = (\mu \lfloor A)(C)$$

μ B-reg $\Rightarrow \exists E$ st $A \cap C \subset E$. $\mu(A \cap C) = \mu(E)$

- μ Borel reg meas on $\mathbb{R}^n$. B Borel set $\Rightarrow$
  1) if $\mu(B) < \infty$, then $\forall \varepsilon < 0$. $\exists C$ closed $\subset \mathbb{R}^n$ st $C \subset B$ and $\mu(B \setminus C) < \varepsilon$
  2) μ-Radon-meas $\Rightarrow \forall \varepsilon > 0$. $\exists U$ open $\subset \mathbb{R}^n$ st $B \subset U$ and $\mu(U \setminus B) < \varepsilon$

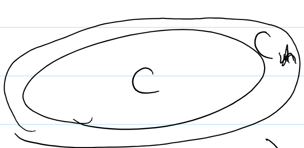
- μ meas on $\mathbb{R}^n$. if $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \subset \mathbb{R}^n$ st. $d(A,B) > 0$ $\Rightarrow \mu$ Borel
- μ Radon meas on $\mathbb{R}^n$. Then for all $A \subset \mathbb{R}^n$ (not necc meas).

$\mu(A) = \inf\{\mu(U) \mid A \subset U.\ U \text{ open}\}$

for all μ-meas $A \subset \mathbb{R}^n$. $\mu(A) = \sup\{\mu(K) \mid K \subset A \text{ compact}\}$

Pf: Any closed sets are meas is sufficient. C closed

$\forall A \subset \mathbb{R}^n$ $\quad \mu(A) \geq \mu(A \cap C) + \mu(A \setminus C)$. $\quad \mu(A) < \infty$.



$C_n = \{x \in \mathbb{R}^n \mid d(x, C) < 1/n\}$

$d(A \setminus C_n, A \cap C) > 0$

$\Rightarrow \mu((A \cap C) \cup (A \setminus C_n)) = \mu(A \cap C) + \mu(A \setminus C_n)$

$\Rightarrow \mu(A) \geq \mu(A \cap C) + \mu(A \setminus C_n) \quad \forall n$.

$A \setminus C_1 \subset A \setminus C_2 \subset \cdots \quad \mu(A \setminus C_n) < \infty$.

$\Rightarrow \mu(A \setminus C) = \lim_{n \to \infty} \mu(A \setminus C_n)$

$\Rightarrow \mu(A) \geq \mu(A \cap C) + \mu(A \setminus C) \Rightarrow$ All closed sets are measurable $\Rightarrow$ Borel sets measurable.

$R_k = \{x \in A \mid \frac{1}{k+1} < \text{dist}(x, C) < \frac{1}{k}\}$

$A \setminus C = (A \setminus C_n) \cup (\bigcup_{n}^{\infty} R_k) \Rightarrow \mu(A \setminus C_n) \leq \mu(A \setminus C) \leq \mu(A \setminus C_n) + \sum \mu(R_k)$

$d(R_i, R_j) > 0$ if $j \geq i+2$

$\sum \mu(R_{2k}) = \mu(\bigcup R_{2k}) \leq \mu(A)$

$\sum \mu(R_{2k+1}) = \mu(\bigcup R_{2k+1}) \leq \mu(A)$

$\Rightarrow \sum \mu(R_k) \leq 2\mu(A)$.