Lecture 2

Thursday, October 2, 2014 9:01 AM

Lemma. μ Borel measure on $\mathbb{R}^n$, B Borel set

1) if $\mu(B) < \infty$ then $\forall \varepsilon > 0$ $\exists C$ closed st. $C \subset B$ and $\mu(B \setminus C) < \varepsilon$.

2) μ Radon measure $\forall \varepsilon > 0$ $\exists U$ open st. $B \subset U$ $\mu(U \setminus B) < \infty$.

Example. $\mu : 2^{\mathbb{R}^n} \longrightarrow [0, \infty]$. measure that leads to Leb meas in $\mathbb{R}$.

$\mu(A) = \inf \{ \sum l(I_k)$. I_k open. $A \subset \cup I_k \}$ length.

$\nu = \mu \llcorner B$. finite Borel measure.

$\mathcal{F} = \{ A \subset \mathbb{R}^n$, A μ-meas and $\forall \varepsilon > 0$ $\exists C$ closed st. $C \subset A$, and $\nu(A \setminus C) < \varepsilon \}$

prove this is a σ-alg containing Borel sets.

1) contains all closed sets.

2) $\{A_i\} \subset \mathcal{F}$ wts $\cap A_i \in \mathcal{F}$ $*$

3) $\{A_i\} \subset \mathcal{F}$ wts $\cup A_i \in \mathcal{F}$. $**$

$\mathcal{G} = \{ A \in \mathcal{F}$ st. $A^c \in \mathcal{F} \}$ $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$.

$***$ $\{A_i\} \subset \mathcal{G} \Rightarrow \cup A_i \in \mathcal{G} \Rightarrow \mathcal{G}$ σ-algebra.

since closed sets $\in \mathcal{F}$ by 2) open sets $\in \mathcal{F} \Rightarrow$ both in $\mathcal{G} \Rightarrow \mathcal{G}$ contains all Borel sets $\Rightarrow \mathcal{F}$ contains all Borel sets.

$*$

$A_i \in \mathcal{F} \Rightarrow \exists C_i \subset A_i$ closed $\nu(A_i \setminus C_i) < \varepsilon/2^i$ let $C = \cap C_i$ closed

$\Rightarrow C \subset \cap A_i$. wts $\nu(A \setminus C) < \varepsilon$

$\nu(A \setminus C) = \nu(A \setminus \cap C_i) = \nu(\cap A_i \setminus \cap C_i) \le \nu(\cup (A_i \setminus C_i))$.

$\le \sum \nu(A_i \setminus C_i) < \varepsilon$.

$**$

exactly the same choice

$\nu(A) < \infty \Rightarrow A \setminus \overset{m}{\cup} C_i$ decreasing family

$\Rightarrow \lim \nu(A \setminus \overset{m}{\cup} C_i) = \nu(A \setminus \cup C_i) = \nu(A \setminus C) \le \varepsilon$.

$\Rightarrow \exists m$ st. $\nu(A \setminus \overset{m}{\cup} C_i) < 2\varepsilon$ and $\overset{m}{\cup} C_i$ closed.

$***$ $A_i \in \mathcal{G} \Rightarrow A_i^c \in \mathcal{G} \Rightarrow \cap A_i^c \in \mathcal{G} \Rightarrow \cup A_i \in \mathcal{G}$

$U_m = B(0, m)$ open ball $U_m \setminus B$ Borel and $\mu(U_m \setminus B) < \infty$.

(μ Radon)

$\exists C_m \subset U_m \setminus B$ st $\mu((U_m \setminus B) \setminus C_m) < \varepsilon/2^m$

$(U_m \setminus B) \setminus C_m = (U_m \setminus C_m) \setminus B \Rightarrow \mu((U_m \setminus C_m) \setminus B) < \varepsilon/2^m$.

$U_m \setminus C_m$ open since C_m closed.

let $U = \bigcup (U_m \setminus C_m)$ open and $U_m \cap B \subset U_m \setminus C_m$.
$B = \bigcup_1^\infty (U_m \cap B) \subset \bigcup_1^\infty (U_m \setminus C_m) = U \Rightarrow \mu(U \setminus B) < \varepsilon$.

- μ Radon measure. Then
  1) $\forall A \subset \mathbb{R}^n$; $\mu(A) = \inf \{\mu(U) \mid A \subset U \text{ open}\}$
  2) $\forall$ μ-meas A. $\mu(A) = \sup \{\mu(K) \mid K \subset A.\ K \text{ compact}\}$.

pf of 1): wlog, assume $\mu(A) < \infty$.
Step 1: assume A Borel. from Lemma $\exists U$ st. $A \subset U$.
and $\mu(U \setminus A) < \varepsilon$ $\mu(U) = \mu(A) + \mu(U \setminus A)$.
$\Rightarrow \mu(A) = \mu(U) - \varepsilon$
Step 2: General A (not Borel). μ Radon $\Rightarrow \exists B$ Borel
st. $A \subset B$ and $\mu(B) = \mu(A)$
by step 1. $\exists U$ st. $\mu(B) = \mu(U) - \varepsilon$ $A \subset B \subset U$.
$\Rightarrow \mu(A) = \mu(B) \geq \inf \{\mu(U) \mid B \subset U \text{ open}\}$
$\geq \inf \{\mu(U) \mid A \subset U \text{ open}\}$.

Task: Real measure theory.

Measurable functions
$f: X \to Y$. X μ measure. Y top space. f is μ-meas.
if $f^{-1}(B)$ is μ-meas for $B \subset Y$ open.
$f \cdot g$. $\Rightarrow$ replace by Borel

- $f_n \cdot g_n: X \to \mathbb{R}$ meas. $f \pm g$. fg. $|f|$. f/g. $\max\{f, g\}$.
  $\min\{f, g\}$. $\inf f_n$. $\sup f_n$. $\underline{\lim} f_n$. $\overline{\lim} f_n$ all meas.

$f: X \to [0, \infty]$ meas. Then $\exists$ μ-meas sets $\{A_n\} \subset X$ st.
$f = \sum \frac{1}{k} \chi_{A_k}$.
Pf: $A_1 = \{x \in X \mid f(x) \geq 1\}$. $A_k = \{x \in X \mid f(x) \geq \frac{1}{k} + \sum_1^{k-1} \frac{1}{j} \chi_{A_j}\}$.
$f \geq \sum \frac{1}{k} \chi_{A_k}$. Trivial.
$f(x) = \infty$. $\Rightarrow x \in A_k$ $\forall k$. for infinitely many A_k. $x \notin A_k$

PDE.
Equation involving derivatives of an unknown fn.
$F(D^k u, D^{k-1} u, \dots u, x) = 0$. in $U \subset \mathbb{R}^n$.
+ boundary condition.
Solve the equation.
Solve means prove the solution exists and study the qualitative property of the solution.

classical solution.

$$\begin{cases} |u'| = 1 & \text{in } (-1,1). \\ u(\pm 1) = 0. \end{cases}$$

Classical solution? X.

relax the notion of solution. Lip cont function.

infinitely many solutions.

relax even more the notion of solution, week solution.

uniqueness, existence, continuous dependence.

Hamilton-Jacobi equations

first-order equation. method of characteristics.

$$\begin{cases} u_t = H(u_x). & \mathbb{R} \times (0,\infty). \\ u|_{t=0} = u_0. & (2n+1). \end{cases}$$

$\exists$ system of odes.

$t \to x(t)$. $\quad t \longrightarrow u_x(x(t),t)$. $\quad t \longrightarrow u_t(x(t),t)$.

$\qquad\qquad\qquad p(t)$. $\qquad\qquad\qquad z(t)$.

$$\begin{cases} \dot{x} = \\ \dot{p} = \\ \dot{z} = \end{cases}$$

Construct odes.

$\dot{p}(t) = u_{xx}\dot{x} + u_{xt}$. $\qquad u_t = H(u_x) \Rightarrow u_{xt} = H'(u_x)u_{xx}$.

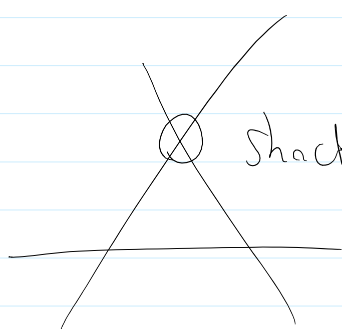
$\qquad = u_{xx}\underbrace{(\dot{x} + H'(u_x))}_{=0}$.

$$\begin{cases} \dot{x} = -H'(p). \\ \dot{p} = 0. \\ \dot{z} = \dot{u}(x(t),t) = p\dot{x} + u_t = H - pH'. \end{cases}$$

$\Rightarrow \dot{z} = H(p) - pH'(p)$.

$x(0) = 0$. $\quad p(0) = u_{0x}(x)$ $\quad z(0) = u_0(x)$

$\Rightarrow x(t) = x - H'(u_{0x}(x))t$.



Shock.