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Cardinal invariants of the continuum.

Cantor. We say two sets A. B are equinumerous (equivalent). if $\exists$ a 1-1 and onto mapping from A to B.

Rk. This is equivalence relation

The number of elements of A is the equivalence class of A. denoted $|A|$, the cardinality of A.

Rk. A set is infinite precisely when it can be put in bijection with a strict subset.

The continuum hypothesis is the statement:
Every $A \subseteq \mathbb{R}$ is either equi. w/ $\mathbb{Q}$ or is equi w/ the set $\mathbb{R}$ if A is infinite.
i.e. $\nexists$ infinite set w/ card between $\mathbb{Q}$ and $\mathbb{R}$.

Arithmetic on cardinals.

1) What's an order on cardinals?
First, A. B sets and there exists $f: A \longrightarrow B$ inj. $g: B \longrightarrow A$ inj. then $\exists$ bij. $h: A \longrightarrow B$.
This, the cardinals are linearly ordered
In fact, cardinals are well-ordered. (every nonempty subset has a least element).
Notice: for any cardinal λ, there is a unique successor λ^+ in the linear order on cardinals.

κ, λ cardinals

1) $\kappa+\lambda$ is the cardinality $|A \cup B|$ where $|A| = \kappa$, $|B| = \lambda$. where $A \cap B = \phi$.
2) $\kappa\lambda$ is the card $|A \cdot B|$, where $A \cap B = \phi$.
3) $\kappa^\lambda = |A|^{|B|} = |{}^B A| = |\{f : B \longrightarrow A \mid f \text{ function}\}|$.

Fact 1) These agree w/ usual arithmetic for finite cardinalities
2) distrib. assoc. transitivity. equalities hold.

2) $(\lambda^\mu)^\kappa = \lambda^{\mu\kappa}$. $\lambda^\mu \cdot \lambda^\kappa = \lambda^{\mu+\kappa}$.

3) Inequalities are more elegant.

Thm (Fundamental Thm of Cardinal Arith)
For cardinals κ, λ, at least one of which is infinite.
Then $\kappa+\lambda = \kappa * \lambda = \max\{\kappa, \lambda\}$.

Q: So everything becomes trivial?

Claim: There exists arbitrarily large cardinals?

Thm (Cantor) For any cardinal λ, $2^\lambda > \lambda$

Pf: Note first that we may identify 2^λ w/ $\mathcal{P}(\lambda)$ power set of all sbsets of λ by associating each sbset to its character fn.
Second note that $\lambda \le 2^\lambda$.
Remains to show that there $\nexists$ bij between λ and 2^λ
Suppose there $\exists\, f: \lambda \to 2^\lambda$ 1-1 onto. define $Y = \{y \in \lambda \mid y \notin f(y)\}$.
Y possibly empty. $Y \subseteq \lambda$. $Y \in \mathcal{P}(\lambda)$ so there exists a preimg z_* i.e. $f(z_*) = Y$. So is $z_* \in Y$?
if $z_* \in Y$ then $z_* \in f(z_*) = Y$ $\Rightarrow$ contradiction
if $z_* \notin Y$ $\Rightarrow$ $z_* \notin f(z_*)$ $\Rightarrow$ $z_* \in Y$ $\Rightarrow$ contradiction.

Q: We have two ways of increasing cardinals: λ^+, 2^λ.
Are they the same? i.e. Is $2^\lambda = \lambda^+$?
In particular, is $\aleph_1 = 2^{\aleph_0}$?

Cardinal Invariant of Continuum.

Program: Look at properties of infinite families, which hold if the family is countable and fail for some family of size continuum ($|\mathbb{R}|$). And name the first card, where this property may fail.

Def 1) ${}^\omega\omega$, ${}^{\mathbb{N}}\mathbb{N}$: set of all functions $\mathbb{N} \to \mathbb{N}$.

2) we say $g: \mathbb{N} \to \mathbb{N}$ eventually dominates $f \in {}^{\mathbb{N}}\mathbb{N}$, $g \geq^* f$ if $g(n) \geq f(n)$ for all but finitely many n.

3) We define $\mathfrak{b}$ the boundary number to be the smallest size of an unbounded family of fns $F \subset {}^{\mathbb{N}}\mathbb{N}$ st.

no $g \in {}^{\mathbb{N}}\mathbb{N}$ eventually dominates all $f \in \mathcal{F}$.

Rk. $\mathfrak{b} \le 2^{\aleph_0}$ just take $\mathcal{F} = {}^{\mathbb{N}}\mathbb{N}$.

$\aleph_0 \le \mathfrak{b}$. Suppose $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$. define g by

$g(k) \geq \max_n f_n(k)$.

Def. The dominating number $\mathfrak{d}$ is the smallest size of a family $\mathcal{F} \subseteq {}^{\mathbb{N}}\mathbb{N}$, which is dominating meaning ~~that~~ for any $g \in {}^{\mathbb{N}}\mathbb{N}$. There is $f \in \mathcal{F}$, which eventually dominates g.