Lecture 4

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$S_k^{(2)} \leq S_{\binom{k}{2}}$ primitive.

$G \leq S_n$ primitive if G_x is max in G.

Stab of $\{1,2\}$ in S_k is $S_2 \times S_{k-2}$.

$(\rho_g \lambda_h \rho_g^{-1} \lambda_h^{-1})(x) =$

Centralizers of R_G is exactly L_G. prove no other perm centralize R_G.

$C(R_G) = L_G$. $\forall \varphi \in Sym(G)$ centralize R_G.

$\Rightarrow \forall \lambda_g, \exists \lambda_h$ st. $\varphi \circ \lambda_g \circ \varphi^{-1} = \lambda_h$

$\Rightarrow \varphi(\varphi^{-1}(x) g^{-1}) = x h^{-1} \Rightarrow \varphi^{-1}(x) g^{-1} = \varphi^{-1}(x h^{-1})$.

G primitive solvable $\Rightarrow n = p^k$.

pf. G solvable $\Rightarrow \exists N \neq 1$. $N \trianglelefteq G$, $N' = 1$. $\Rightarrow N$ transitive.

and N transitive abelian $\Rightarrow$ regular.

$\Rightarrow N$ minimal

Claim: N normal sbgp of G. N abelian, $N \neq 1$

$\Rightarrow N$ minimal normal sbgp of G No proper sbgp satisfies the same requirement.

N min $\trianglelefteq G$, then N is char simple

$\Rightarrow N$ char simple $\Rightarrow N$ elementary abelian group.

$$AGL(d,q) = \{\text{affine transformations of } \mathbb{F}_q^d\}$$
$$= \{x \longrightarrow Ax + b \mid A \in M_d(\mathbb{F}_q), \det(A) \neq 0, b \in \mathbb{F}_q^d\}$$

$T = \{x \longrightarrow x + b$ translations in $\mathbb{F}_q^d\} \cong (\mathbb{F}_q^d, +)$

$\cong \mathbb{Z}_p^{td}$ where $q = p^t$.

$T \trianglelefteq AGL(d,q)$ $AGL(d,q)/T \cong GL(d,q)$.

In fact. $AGL = T \rtimes GL$.

Conjugation action.

$AGL = T \cdot GL$. $T \trianglelefteq AGL$. $T \cap GL = 1$.

Thm If G has an elementary abelian normal sbgp N and is primitive, then $G \leq AGL(k,p)$. where $p^k = |N|$.

pf. N elementary abelian $N = \mathbb{Z}_p^k$. view it as a vspace over $\mathbb{F}_p$.

Action of G on N. $G \xrightarrow{C_G(N)} \longrightarrow Aut(N) = GL(k,p)$.

N regular $\Rightarrow C_{Sym}(N)$ also regular

N abelian $\Rightarrow$ $N \leq C_{\mathrm{Sym}}(N) \Rightarrow C_{\mathrm{Sym}}(N) = N$.
$\Rightarrow C_G(N) = N$.
$\Rightarrow G/N \leq GL(k,p)$.

$|\Omega| = n = |N|$. N regular.

$x_1 \quad x_1^n$

given $n \in N$. $x_n = x_1^n$.
$\Omega = N$.

Claim: $G_{x_1} \leq \mathrm{Aut}(N) = GL(k,p)$.

Lemma: N regular normal sbgp of G. Then G is a semi direct product of N and G_x.
Thm. If $G \leq \mathrm{Sym}(\Omega)$, N regular normal sbgp, then $G = N \rtimes G_x$ and $G_x \leq \mathrm{Aut}(N)$.

$|GL(d,q)| =$
$|GL(d,q)| \leq |M_d(q)| = q^{d^2}$.
$|GL(d,q)| = (q^d - 1) \cdots (q^d - q^{d-1})$.
quotient $= q^{d^2} \left(1 - \frac{1}{q^d}\right) \cdots \left(1 - \frac{1}{q}\right)$.

$\alpha(q) = \prod_{t=1}^{\infty} \left(1 - \frac{1}{q^t}\right)$

$\sum \log(1 - a_n) < n \log\left(\frac{n - \sum a_n}{n}\right)$

Ex. $0 \leq a_n < 1$. $\prod (1 - a_n) \neq 0 \Leftrightarrow \sum a_n < \infty$

$= \log\left(1 - \frac{\sum a_n}{n}\right)$.

$(n = q^d)$.

$|GL(d,q)| \leq q^{d^2} = n^d = n^{\log_q n}$.

Cor. If $G \leq S_n$ ~~then~~ has a abelian regular normal sbgp, then $|G| \leq n^{1 + \log_2 n}$.

Ex Prove that exact same bound if drop abelian

Palf-Wolf-Thm.
If G is primitive and solvable, then $|G| \leq n^c$ $c = 3.4\ldots$

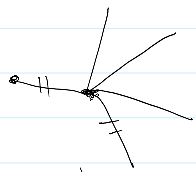
Ex T tournament, then $\mathrm{Aut}(T)$ odd order.
Ex. Prove equivalent:
1) $\mathrm{Aut}\, T$ solvable.
2) odd order. Thm.

Thm. (Luks) If X is a graph of degree at most d and connected, $e \in E(X)$, then $(\mathrm{Aut}(X))_e \in \Gamma_{d-1}$.
$\Gamma_k = \{$finite gps st. every composition factor is a sbgp of $S_k\}$

Ex.
Lem. $G \in \Gamma_k$ iff G has a sbgp chain $G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$

st. for $(\forall i)$ $(|G_{i-1} : G_i| \overset{d-1}{\le} k)$.

$|G : G_x| = |x^G|$.



fix edges 1 by 1.
connected $\Rightarrow$ reach all.

This characterizes the Stabilizers of Aut X in a graph of degree d.

B-Cameron-Palfy.

If $G \le S_n$, primitive. $G \in \Gamma_k$. then $|G| \le n^{O(k)}$.

$\tilde{\Gamma}_k$ all non abelian comp. k factors $\le S_k$. Same conclusion

$\overline{\Gamma}_k$: BCP-groups. A_{k+1} is not involved in G. (as a quotient of sbgps).

Ex $\tilde{\Gamma}_k \subset \overline{\Gamma}_k$. Still the same conclusion.