Lecture 3

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Lusin.

μ Borel-reg on $\mathbb{R}^n$. $f: \mathbb{R}^n \to \mathbb{R}^m$. μ-meas. $A \subset \mathbb{R}^n$ μ meas and $\mu(A) < \infty$. For every $\varepsilon > 0$. $\exists K \subset A$ compact st.
1) $\mu(A \setminus K) < \varepsilon$ 2) $f|_K$ cont.

Pf: for each $i \in \mathbb{N}$. $(B_{ij})_{j=1}^{\infty} \subset \mathbb{R}^m$ disj Borel $\mathbb{R}^m = \bigcup_j B_{ij}$.
$\operatorname{diam} B_{ij} < 1/i$

$A_{ij} = A \cap f^{-1}(B_{ij})$ μ-meas. disj $A = \bigcup A_{ij}$. $\mu(A_{ij}) < \infty$.
let $\nu = \mu \lfloor A$. Radon measure. $\exists K_{ij} \subset A_{ij}$ comp. st.
$\mu(A_{ij} \setminus K_{ij}) < \varepsilon / 2^{i+j}$
$\Rightarrow \mu(A \setminus \bigcup K_{ij}) = \nu(A \setminus \bigcup K_{ij}) \leq \sum \nu(A_{ij} \setminus K_{ij}) < \varepsilon/2^i$.
$\lim_{m \to \infty} \mu(A^j \setminus \bigcup_j^m K_{ij}) = \mu(A \setminus \bigcup K_{ij})$

$\Rightarrow \exists M(i)$ st $\mu(A \setminus \bigcup_j^{M(i)} K_{ij}) < \varepsilon/2^i$. $\bigcup^{M(i)} K_{ij}$ compact.
let $D_i = \bigcup_j^{M(i)} K_{ij}$. K_{ij} compact and have positive distance from each other.
fix $b_{ij} \in B_{ij}$. $g_i : D_i \to \mathbb{R}^m$ st. $g_i(x) = b_{ij}$ if $x \in K_{ij}$
claim that g_i continuous. also $|f(x) - g_i(x)| < 1/i$ if $x \in D_i$.
Let $K = \bigcap_i D_i$. Compact set and non empty.

$\mu(A \setminus K) < \sum_i \mu(A \setminus D_i) < \varepsilon$. $g_i \to f$ unif on K.
$\Rightarrow f|_K$ cont.

Coro. μ B-reg on $\mathbb{R}^n$. $f: \mathbb{R}^n \to \mathbb{R}^m$ μ-meas. $A \subset \mathbb{R}^n$ μ meas and $\mu(A) < \infty$. For each $\varepsilon > 0$. $\exists \tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ cont such that $\mu(\{x \in A \mid f(x) \neq \tilde{f}(x)\}) < \varepsilon$.
just extend $f|_K$.

Thm. (Egroff's thm) μ meas on $\mathbb{R}^n$. $f_k : \mathbb{R}^n \to \mathbb{R}^m$. μ meas $A \subset \mathbb{R}^n$ μ-meas. $\mu(A) < \infty$ and $f_k \to g$ μ. a.e. on A.
then $\forall \varepsilon > 0$. $\exists$ μ-meas $B \subset A$ where
1) $\mu(A \setminus B) < \varepsilon$ 2) $f_k \to g$ unif on B

Pf: $C_{ij} = \bigcup_{k=j}^{\infty} \{x \in A \mid |f_k(x) - g(x)| > 2^{-i}\}$ measurable set

monotone increasing in j and i. $C_{i(j+1)} \subset C_{ij}$. $\mu(A) < \infty$
$\lim \mu(A \cap \bigcap^m C_{ij}) = \mu(A \cap \bigcap^{\infty} C_{ij}) = 0$
$\Rightarrow \exists M(i)$ st. $\mu(A \cap \bigcap^{M(i)} C_{ij}) < \varepsilon/2^i$.
$B = A \setminus \bigcup_i C_{i, M(i)}$. claim. $\mu(A \setminus B) < \varepsilon$.

$\mu(A \setminus B) \le \sum \mu(A \setminus C_{i,m(i)}) < \varepsilon$
$\forall i.\ x \in B.\ k \ge m(i).\quad |f_k(x) - g(x)| \le 2^{-i}$

Integrals and Limits.
- μ meas on X, $g: X \to [-\infty, +\infty]$. simple if $R(g)$ is countable
- g simple, nonnegative. $\int g\, d\mu = \sum_y y \cdot \mu(\{g^{-1}(y)\})$.
- g simple, μ-meas and either $\int g^+ d\mu < \infty$ or $\int g^- d\mu < \infty$
  $\int g\, d\mu = \sum_{-\infty < y < \infty} y\, \mu(\{g^{-1}(y)\})$
- $f: X \to [-\infty, \infty]$ meas. upper integral. $\int^* f\, d\mu = \inf\{\int g\, d\mu,\ g \text{ meas simple } f \le g\}$
  lower integral $\int_* f\, d\mu = \sup \int g\, d\mu.\ g$ meas. simple. $g \le f$.

$f: X \to [-\infty, \infty]$ is μ-integrable if $\int^* f\, d\mu = \int_* f\, d\mu = \int f\, d\mu$.

- f is μ summable if $\int |f|\, d\mu < \infty$.

f is locally μ-sum if $\int_K |f|\, d\mu < \infty$ for $\forall K$ compact.

$X = \mathbb{R}^n$. μ Borel.

Every nonnegative μ-meas function is μ-integrable.

Integrals are linear.

Fatou's Lemma.
$f_n: X \to [-\infty, \infty]$ μ-meas. $\int \liminf f_k\, d\mu \le \liminf \int f_k\, d\mu$.

Q1: $f_k: [0,1] \to \mathbb{R}$. $\int \underline{\lim} f_k \le \underline{\lim} \int f_k$.

Q2: What is the inequality satisfied by $\overline{\lim}$.

Q3: is $=$ true in general?

Pf: $g = \sum a_j \chi_{A_j}$. A_j disj. $a_j \ge 0$. $g \le \underline{\lim} f_k$. $0 < t < 1$.
$B_{j,k} = A_j \cap \{f_i(x) > t a_j,\ \forall i \ge k\}$
$\int f_k\, d\mu \ge \sum_j \int_{A_j} f_k\, d\mu$. $\quad X = \bigcup_j A_j \cup \{g = 0\}$.

$\ge \sum \int \quad f\, d\mu$

$$\int f_k \, d\mu \geq \sum_j \int_{A_j} f_k \, d\mu$$

$$\geq \sum_j \int_{B_{j,k}} f_k \, d\mu$$

$$\geq t \cdot \sum_j a_j \, \mu(B_{j,k})$$

$\Rightarrow \underline{\lim} \int f_k \, d\mu \geq t \cdot \sum_j a_j \, \mu(A_j) = t \int g \, d\mu$ let $t \to 1$.

Monotone convergence

$f_k : X \to [0, \infty]$ monotone. $f_k \leq f_{k+1}$.

$\Rightarrow \lim \int f_k \, d\mu = \int \lim f_k \, d\mu$.

DCT. $f_k : X \to \mathbb{R}$. $g : X \to \mathbb{R}$. $f_k \to f$ a.e. $|f_k| \leq g$.

$\Rightarrow \int |f_k - g| \, d\mu \to 0$.

sDCT. $f_n : X \to \mathbb{R}$. $g_n : X \to [0, \infty]$. Summable. $f_n \to f$ a.e.

$g_n \to g$ a.e. and $\int g_n \to \int g$. $|f_n| \leq g_n$.

$\Rightarrow \int |f - f_n| \, d\mu \to 0$.