Lecture 4

Thursday, October 9, 2014 12:07 PM

Leehre4. Transfinite. Induction. and. Recursion.

Last time:

Claim: Every well-ordered set is isomorphic to an ordinal.
Consider a given $(A,<)$. Consider the class of ordinals isomorphic to some initial seg of $(A,<)$.
Show this class B is well-ordered. Show it's transitive.

Now let's return to the big picture. We had said $\mathbb{N}$ has the key property:

1) Natural numbers represent order types of finite well ordered sets.

2) They represent sizes of finite set.

- Ordinals.
- Cardinals: $\alpha \in \mathrm{Ord}$ is a cardinal if it cannot be put into bijection with any strictly smaller ordinal.
  i.e. for $\beta < \alpha$. $|\beta| < |\alpha|$.

Note that even though $n \in \mathbb{N} \Rightarrow n$ is a cardinal. Not all ordinals are cardinals.

How to get the theory of size?
We will use without further comments.

Principle: Every set can be well ordered.
Accepting this, let X be any set. Let $(X,<)$ be a well ordering. Then $(X,<)$ is order-isom. to some ordinal α. let $|X| = |\beta|$, where β is the least such that.
Thus every set has a size.

Q: Is this controversial?
A: Interesting to consider axiomatic strength.

We can define arithmetic on cardinals by $\kappa + \lambda = |A \cup B|$.
where $A \cap B = \phi$, $|A| = \kappa$ and $|B| = \lambda$.
$\kappa \cdot \lambda = |A \times B|$. $|A| = \kappa$. $|B| = \lambda$.

Fundamental Thm of Cardinal Arithmetic

Suppose both κ, λ are cardinals and both non-zero and at least one is infinite, then

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

Rmk: This shows that for any card κ, we may generalize results on $\mathbb{N}$ from Lecture 1.

Example: Suppose we consider a point P in the plane and we have a set X of points and $|X| < |\mathbb{R}|$. $P \notin X$. Then there exists a circle of radius 1 through P which does not intersect X.

Pf: Let $c = |\mathbb{R}|$.

For any pt x, there are 2 circles of radius 1 through x and P. There are $\kappa < c$ pts in X and c many good circles.

Recalls that induction works on $\mathbb{N}$.

Thm. Principle of Transfinite Induction.

Let $\mathcal{C}$ be a class of ordinals, Suppose.

1) $0 \in \mathcal{C}$.

2) if $\alpha \in \mathcal{C}$ then $\alpha + 1 \in \mathcal{C}$.

3) If α is a nonzero limit ordinal and $\beta < \alpha \Rightarrow \beta \in \mathcal{C}$, then $\alpha \in \mathcal{C}$.

Then the class $\mathcal{C}$ is the class of all ordinals.

Pf: Suppose not. Let $\alpha \in \mathrm{Ord}$ be the least ordinal $\notin \mathcal{C}$. Then apply 1, 2 or 3.

More subtly, we'll want to define by induction trans-finite recursion

Thm. $\mathbb{R}^3$ may be written as the disjoint union of circles of radius 1.

Pf: Let $c = |\mathbb{R}|$, by FTCA $c = |\mathbb{R}^3|$.

Let's enumerate the points of $\mathbb{R}^3$ as $\langle P_\alpha : \alpha < c \rangle$.

Let's construct, by induction on $w < c$, a sequence $\langle C_\alpha : \alpha < c \rangle$ of circles of the property that:

- For each α, C_α is either ϕ or a circle of radius 1.
- For each α, the point $P_\alpha \in \bigcup C_\beta$.
- For each α and each $\beta < \alpha$ $C_\alpha \cap C_\beta = \phi$.

This would suffice. Let's carry out the induction.
When $\alpha = 0$, let C_α be any circle of radius 1 through P_0.
When $\alpha > 0$, we have defined a sequence $\langle C_\beta, \beta < \alpha \rangle$ satisfies the induction hypothesis.
If $P_\alpha \in \bigcup C_\beta$ done. Let $C_\alpha = \phi$.
If not, then choose a plane through P_α not containing any of the circles $\langle C_\beta, \beta < \alpha \rangle$.
I have $|\alpha| < c$ circles so far. I have c planes through P_α. Each circle defines a unique plane, so there are plenty to choose from.
Now consider the plane P and the point P_α. Let X be the set of points in P which lie in some circle in $\langle C_\beta, \beta < \alpha \rangle$.
The set ~~of~~ X has size $< c$. then $\exists$ plane P contains no pt in $\langle C_\beta, \beta < \alpha \rangle$. Pick the desired circle in this plane P.