Meeting

Tuesday, July 8, 2014 12:34 PM

$F: X \longrightarrow X$. want to study fixed pt. Assume F has regular isolated fixed pts.

$\Gamma_F \subseteq X \times X$. $\Gamma_F = (x, F(x))$.

$\Gamma_F \cap \Delta = 0$ - manifold Δ

$= \{(x, x) \mid x \in X\}$

fixed pts of F = $\# \Gamma_F \cap \Delta$.

count each pt with multiplicity ± 1. (orientation).

if x is a fixed pt. $(x, x) \in \Gamma_F \cap \Delta$.

and orientation is determined by orientation of X, by DF_x

Intersection in $X \times X$.

$H_n(X \times X) \times H_n(X \times X) \longrightarrow H_0(X \times X)$.

bilinear.

If $Y \subseteq M$

$\overset{4}{\dim}$ $\quad$ $\overset{\wedge}{\dim}$.

$[Y] \in H_m(M)$.

$[Y] \cdot [Y'] \longrightarrow [Y \cap Y']$ For Y, Y'.

transversal, oriented n-dim.

sbmanifold.

Maybe Hutchens. Intersection parity.

$$H_n(X \times X) = \bigoplus_{k=0}^{n} H_k(X) \otimes H_{n-k}(X).$$

Hatcher. Kunneth. formula.

$\Delta = (\quad , \quad , \quad , \quad , \quad)$.

$\Gamma_F = (\; , - \quad - \quad , \;)$

Result. $[\Delta] \cdot [\Gamma_F] = \sum_{i=0}^{n} (-1)^i \operatorname{Tr} F_* \vec{C} H^i(X)$.

$X/\overline{\mathbb{F}_p}$.

$X(\overline{\mathbb{F}_p}) \longrightarrow$ Frobenius,

fixed pts. are. $X(\mathbb{F}_p)$.

Coh. perspective on intersection

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M n-dim. manifold.

Let $\omega \in \Omega^n(M)$. st. $\omega(p) \neq 0 \ \forall p$

"volume form".

$\int_M \omega = \mathrm{vol}_\omega(M)$.

· If such an ω exists.

Read chapter 4. sections

1 – 8.

Then ask for degree exercise.

· Kunneth theorem. (Hatcher).

· Prove homological Lef fixed pt

$L_f = \sum (-1)^i \operatorname{tr} f_* \mathbin{\circlearrowleft} H_i(M)$.

1) prove for simplicial complex.

2) for a manifold (Maybe easier to do coh'm)

Assume M = simplicial

wts $Lf = \sum (-1)^i \operatorname{tr} f^*_* H^i(M)$

Special case: $f = \mathrm{id}$, $\chi = I(\Delta, \Delta)$

$LF = T_f \cap \Delta$

$H_0(X) \xrightarrow{\deg} \mathbb{Z}$ $\quad \sum a_i [p_i] \longrightarrow \sum a_i$

$LF = \deg(T_f \cap \Delta)$

$$H^n(X \times X) = \bigoplus_{i=0}^{n} H^i(X) \otimes H^{n-i}(X)$$

$$\qquad\qquad \pi_1^* H^i(X) \qquad \pi_2^* H^{n-i}(X)$$

T_f gives an element $H^n(X \times X) = H_n(X \times X)^*$

by intersection

$C \longrightarrow \deg(T_f \cap C)$

$$H^n(X \times X) = \bigoplus H^i(X) \otimes H^{n-i}(X)$$

$= \bigoplus H^i(X) \otimes H^i(X)^*$

$H^{n-i}(X) \cong H^i(X)^*$?

$H^{n-i}(X) \cong H_{n-i}(X)^* \cong H_i(X)^{**}$ by intersection.

$= H^i(X)^*$

$\mathbin{/\!/}$

$= \bigoplus_{i=0}^{n} \operatorname{End} H^i(X)$.

$(H^n(X \times X) \cong H^n(X \times X)^*$ by same arg).

$= H_n(X \times X)$

$\Delta \in H^n(X \times X)^*$ is the map which

$\bigoplus^n (\operatorname{End} H^i(X))^*$

$(\varphi_0, \dots \varphi_n) \xrightarrow{\Delta} \sum (-1)^i \operatorname{Tr} \varphi_i$

$\varphi_i \in \operatorname{End} H^i(X)$

Then $\Gamma_f \in H_n(X \times X) = H^n(X \times X)^*$

$\Gamma_f = (f^*, f^*, f^*, \dots)$ pullback on each component.

$\Delta(f) = \sum (-1)^i \operatorname{Tr} f^* \circledcirc H^i(M)$.

Hatcher. PD duality, intersection

Hatcher. PD duality. understand statement

Poincaré Duality Theory

Hatcher p 238

coho. perspective on intersection

Lemma. $(\alpha\times\beta)\cdot(\gamma\times\delta) =$

$$\begin{cases} (-1)^{|\beta|}(\alpha\cdot\gamma)(\beta\cdot\delta) & |\beta|=|\gamma| \\ 0 & \text{o/w} \end{cases}$$

$(\alpha\times\beta)\cdot(\gamma\times\delta) = [X\times X]\cap((\alpha\times\beta)^*\cup(\gamma\times\delta)^*)$

$= [X\times X]\cap \pi_1^*\alpha^*\cup\pi_2^*\beta^*\cup\pi_1^*\gamma^*\cup\pi_2^*\delta^*$

$= [X\times X]\cap (-1)^{|\beta|}\pi_1^*(\alpha^*\cup\gamma^*)\,\pi_2^*(\beta^*\cup\delta^*)$

$= [X\times X]\cap (-1)^{|\beta|}\pi_1^*((\alpha\cdot\gamma)^*)\,\pi_2^*((\beta\cdot\delta)^*)$

$= [X\times X]\cap (-1)^{|\beta|}((\alpha\cdot\gamma)\times(\beta\cdot\delta))^*$

Use. $(c\times d)^* = \pi_1^*(c^*)\cup\pi_2^*(d^*)$

$[X]=\sum\sigma_i$. $[X\times X]=\sum\sigma_i\times\sigma_j$

$(X\_) = \Sigma\,\sigma, \ldots$ [illegible] $= \Sigma\,\sigma, x\,\sigma\}$

$$H^n(X \times X) = \bigoplus_{0}^{n} \mathrm{End}(H^k(X))$$
$$= \bigoplus_{0}^{n} H^k(X) \otimes H^k(X)^*$$

$\dim X = \dim Y$. then

$$H^n(X \times Y) = \bigoplus_{0}^{n} \mathrm{Hom}(H^k(X), H^k(Y))$$

Paper

Title: Lefschetz fixed pt theorem and ~~Solution~~ Counting Solution to polynomials over finite flds

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Introduction: (Grant ≤ 12 pages)
2-pages

- introduce idea of counting solns of polynomials finite fld $\mathbb{F}_q$
  = Counting pts on alg variety.

Examples: $\mathbb{P}^n$, $y^2 = x^3 + 1$ (elliptic curve), Grassmannians

$\xi^{n-1} + \cdots + \xi + 1$ nontrivial.

$\xi + 2\xi^{1/2} (0,1,1)$ table. Refer to later sections.

Interesting because # of solutions (almost) like ~~functions~~ polynomials of ξ.

- Observe: Coefficients of $(\xi^{1/2})^k = \beta_k$ = k^{th} Betti number of these varieties over $\mathbb{C}$.

Refer to later on

Solutions of $y^2 = x^3 + 1$ in $\mathbb{C}$ genus 1 surface with 1 pt removed.

- Explain briefly how Lef F.P theorem explains this connection.
- Outline remaining sections

- Background on manifold: alg top. intersection theory. Complex manifold

- Chain of examples pts intersection (D. the ring structure)
- start w/ $\mathbb{P}^n$ $\frac{1}{2}$ page.
- Gr_2^4 $\leq$ 3/4 pages
- $y^2 = x^3 + 1$ ≤ 1 page.

Describe $\cong \mathbb{C}/\Lambda$

Section 3. Lef F.P. theorem.

Section 4. Frobenius and Counting pts

Fixed pt of

$y^2 = x^3 + 1$ want to know #sol in $x, y \in \mathbb{F}_q$. get top. over $\overline{\mathbb{F}_q} = \bigcup \mathbb{F}_{q^n}$

Problem: finite sets are not geometric

Analogies would be to look at solutions $y^2 = x^3 + x$ in $\mathbb{Z}$.

there are many more pts over $\mathbb{Q}$ then over $\mathbb{Z}$.

$(x, y) \longmapsto (-x, iy)$

get topology in $\mathbb{C}$.

$x \in \overline{\mathbb{F}_q}$ and $x \in \mathbb{F}_q \iff \underline{x^{q-1} = 1}, x^q = x$.

$y^2 = x^3 + a$. $a \in \mathbb{F}_q$. $y, x \in \overline{\mathbb{F}_q}$

y^q. x^q is also a solution

$(y^2)^q = (x^3 + a)^q = (x^q)^3 + a^q = (x^q)^3 + a$.

$(x, y) \longmapsto (x^q, y^q)$. nice map.

Frob endo. map. on X = solutions over $\overline{\mathbb{F}_q}$

Fixed pts are $(x, y) \in \mathbb{F}_q \times \mathbb{F}_q$

Suppose in this geometry you have some coho theory and has the Lef F.P. theorem, then

$$\# X(\mathbb{F}_q) = \sum (-1)^i \operatorname{tr} \operatorname{Frob} \curvearrowright H_i(X).$$

Suppose furthermore this is a "nice" theory in the sense that, e.g., Betti numbers are the same and traces of f_* for F a polynomial map to be the same.

Then sometimes, we can actually compute $\# X(\mathbb{F}_q)$ using complex geometry.

$\mathbb{P}^n = (X_0 : \cdots : X_n)$

$\downarrow$

$(X_0^q : \cdots : X_n^q)$

$=$ Frob Endo.

Makes sense $/ \mathbb{C}$

- Ex. Compute the # fixed pts of this.
- Ex Compute the trace on homology gp on this map.

Examples of LFP

- Brouwer fixed pt.
- Compact Lie gp. home.

$\chi(h) = 0$

- Compact manifold with a non vanishing vector fd.

  $\chi(h) = 0$.

- Corollary. Hairy ball theorem.

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Intersection theory.

Statement of main theorem

× proof × definition.

refer the book.

$\mathbb{CP}^n$ on H^{2n}

trace H^{2n} = degree.

$= p^{n+1}/p = p^n$.

$2k$

H^{2k}: trace = degree

$= p^{k+1}/p = p^k$

$$\begin{tikzcd} H^{2k} \arrow[r, "\sim"] & H^{2k}(\mathbb{CP}^n) \\ H^{2k}(\mathbb{CP}^n) \arrow[u] & H^{2k}(\mathbb{CP}^n) \arrow[u] \end{tikzcd}$$

Example Intro.

$\mathbb{P}^n$ # pts.

Equation table small q

$3, 3^2, 3^3, 5, 5^2, 5^3, 7, 7^2, 7^3$

$- - - -$

q	$\# E(\mathbb{F}_q)$
3	$n = 3 + 2\sqrt{3}$ ___ $+1$

Grassmannian. # pts

$\cong \mathbb{C}/\langle 1, i\rangle$ Weierstrass

Complex analysis.

product structure.

$y^2 = x^3 + x. =: E.$ $i \mapsto i$ $x \mapsto -x$

$y \mapsto iy$

$E = \mathbb{C}/\langle 1, i\rangle = \Lambda.$

Fact 1 Any holomorphic maps

$\mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda$ sending $0 \to 0$.

is given by mult. by $c \in \mathbb{C}$.

$z \bmod \Lambda \longrightarrow c \cdot z \bmod \Lambda$.

need $c \in \Lambda$

well defined iff $c \in \Lambda$.

$\mathrm{End}(\mathbb{C}/\Lambda) = \mathbb{Z}[i]$

Fact 2.

Holomorphic maps $E(\mathbb{C}) \longrightarrow E(\mathbb{C})$

= algebraic maps.

i.e. $c \in \mathbb{Z}[i]$ gives a map.

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \longrightarrow & \mathbb{C}/\Lambda \\ \| & & \| \\ E(\mathbb{C}) & \longrightarrow & E(\mathbb{C}) \end{array}$$

AG theory of Eclipse curve.

$(x, y) \longmapsto (f(x, y), g(x, y))$

f, g polynomials and f, g are in $\mathbb{Z}[i, 1/2][x, y]$.

Frob. over $\mathbb{F}_q$ $(x, y) \longmapsto (x^q, y^q)$

Frob over $\overline{\mathbb{F}_q}$ $\quad (x, y) \longmapsto (x^q, y^q)$

$(x, y) \longmapsto (f(x,y), g(x,y))$

$q = 25$

$f(x,y) = x^{25} + 5\_\_$, $\quad g(x,y) = y^{25} + 5\_\_$

Fact 3

If $q \equiv 1 \mod 4$ $\quad (i \in \mathbb{F}_q)$

$\exists\, c \in \mathbb{Z}[i]$ such that $\|c\|^2 = q$

$\pm i \cdot c$ also satisfy this.

And one of $\pm i \cdot c$ as Endomorphism lifts Frobenius.

(Eclipse curve)

~~Sant~~

$\pm i \cdot c$ Frob. Suppose

$\mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda$

$(1+2i)z = z + \lambda$ $\qquad$ degree of this

$(1+2i)z = z + \lambda$

$\Rightarrow 2iz = \lambda \in \Lambda$

4 fixed pts.

degree of this map. $\|c\|^2 = q$.

- trace $H_2 = q$.
- $H_1(X;\mathbb{R}) \cong \mathbb{C}$.

$c = a+bi$ $x+yi$ trace $= 2 \operatorname{Re}$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Wt prime. Re part $\le \sqrt{q}$

since $\|c\|^2 = q$.

$\mathbb{C}/\langle 1, i\rangle$ $y^2 = x^3 - x$
