# Lecture 4.

Thursday, October 9, 2014 9:07 AM

- Assume that $f, f_k$ are $\mu$-summable. $\int |f_k - f| \, d\mu \to 0$ $k \to \infty$
Then $\exists k_j$ st. $f_{k_j} \to f$.

pf: $\exists k_j$ st. $\sum \int |f_{k_j} - f| \, d\mu < \infty$. $\exists b_j$ st. $\int |f_{k_j} - f| \, d\mu < 2^{-j}$

fix $\varepsilon > 0$. $\{\overline{\lim} |f_j - f| \geq \varepsilon\} \subset \bigcap_i \bigcup_j \{|f_{i+j} - f| \geq \varepsilon\}$.

$\mu(\{\overline{\lim} |f_{i+j} - f| \geq \varepsilon\}) \leq \sum_{j=1}^{\infty} \mu(\{|f_j - f| \geq \varepsilon\})$.

$\leq \frac{1}{\varepsilon} \cdot \sum \int |f_j - f| \to 0$.

$\Rightarrow \mu(\{\overline{\lim} |f_{i+j} - f| \geq \varepsilon\}) = 0$. - - - - - -

a.e. conv $\not\Rightarrow$ conv in $L^1$.
Conv in $L^1$ $\Rightarrow$ a.e. along sbsequence.

Convergence in Measure.
$f_k, f$ $\mu$-meas. $|f_k|, |f| < \infty$ a.e. $f_k \xrightarrow{m} f$ if $\forall \delta > 0$.
$\lim_{k \to \infty} \mu(\{x \mid |f_k(x) - f(x)| > \delta\}) = 0$.

- $f_k \to f$ $\mu$-a.e. in $E$. St. $\mu(E) < \infty$. $\mu$ Borel reg $\Rightarrow f_k \xrightarrow{m} f$.

Pf: Egoroff $\Rightarrow \forall \eta, \varepsilon \exists k$ st. $\exists F \subset E$ closed st. $\mu(E \setminus F) < \eta$ and $|f_n(x) - f(x)| < \varepsilon$ for $n \geq k$. write $F_\eta$. $k_{\eta, \varepsilon}$.
$\{x \in E \mid |f_k(x) - f(x)| \geq \varepsilon\} \subset E \setminus F$.
$\Rightarrow \mu(\{x \in E \mid |f_k(x) - f(x)| \geq \varepsilon\}) \leq \mu(E \setminus F) < \eta$
let $\eta \to 0$. we get $\lim_{k \to \infty} \mu(\text{---}) = 0$

$f_k = \chi_{\{|x| \leq k\}} \to 1$. a.e. but not in measure.

Converge in $L^1$ $\Rightarrow$ converges in measure.
Conv in measure $\Rightarrow$ a.e in sbsequence

$|I_k| \to 0$ st. each point belongs to infinitely many of the $I_k$'s $\chi_{I_k} \xrightarrow{m} 0$.

$f_k \to f$. $\exists k_j$ st. $\mu(\underbrace{\{|f - f_{k_j}| > \tfrac{1}{j}\}}_{E_j}) < 1/2^j$.
$H_m = \bigcup_m^{\infty} E_j$. $\mu(H_m) \leq 2^{-m+1}$ and $|f - f_{k_j}| < 1/j$ in $E \setminus E_j$

measure in product spaces. $(X, \mu)$ $(Y, \nu)$ $X \times Y$. $\mu \times \nu$

- what sbsets of $X\times Y$ are $\mu\times\nu$ measurable.
- $S = A\times B$. $A, B$. $\mu, \nu$ meas. $(\mu\times\nu)(S) = \mu(A)\nu(B)$
- $\int_{X\times Y} f(x,y)\, d(\mu\times\nu) = \int_Y [\int_X f(x,y)\, d\mu]\, d\nu = \int_X [\int_Y f(x,y)\, d\nu]\, d\mu.$

$\mu\times\nu : 2^{X\times Y} \to [0,\infty]$

$(\mu\times\nu)(S) = \inf \{ \sum \mu(A_i)\nu(B_i) : S \subset \bigcup A_i\times B_i \}$.

Thm. $\mu$ meas on $X$. $\nu$ meas on $Y$.

1) $\mu\times\nu$ regular on $X\times Y$. true even if neither $\mu, \nu$ is reg.

2) $A\subset X$ meas. $B\subset Y$. meas, then $A\times B$ is $\mu\times\nu$ meas and

$(\mu\times\nu)(A\times B) = \mu(A)\nu(B)$

3) $S\subset X\times Y$. $\sigma$-finite set in $\mu\times\nu$, $S$ $\mu\times\nu$ meas. then

$S_y = \{x \mid (x,y)\in S\}$ is $\mu$-meas. for $\nu$ a.e $y$.

$S_x = \{y \mid (x,y)\in S\}$ is $\nu$-meas for $\mu$ a.e. $x$.

$(\mu\times\nu)(S) = \int_Y \mu(S_y)\, d\nu = \int_X \nu(S_x)\, d\mu.$

$= \int_Y \int_X \chi_{S_y}\, d\mu\, d\nu = \int_X \int_Y \chi_{S_x}\, d\mu.$

4) $f$ is $\mu\times\nu$ int. and $\sigma$-finite $\Rightarrow$ $y \to \int_X f(x,y)\, d\mu.$ is $\nu$-int. $x \to \int_Y f(x,y)\, d\nu$ is $\mu$-int.

$\int_{X\times Y} f(x,y) = \int_X \int_Y = \int_Y \int_X.$

$\sigma$-finite : $\{x \mid f(x) \neq 0\}$ is $\sigma$-finite.

$\mathcal{F} = \{ S\subset X\times Y \mid x \to \int_Y \chi_S(x,y).\ \mu\text{-int for } \forall y.\ ;\ y \to \int_X \chi_S(x,y).\ \nu \text{ int for } \forall x \}.$

$S\in\mathcal{F}$. $\rho(S) = \int_Y [\int_X \chi_S(x,y)\, d\mu]\, d\nu.$

PDE.

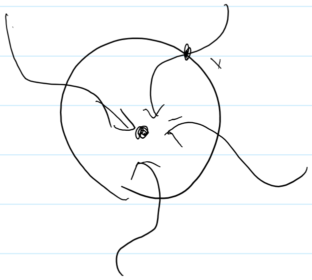
$\begin{cases} H(Du, u, x) = 0 & \text{in } U. \\ u = g & \text{on } \Gamma\subset\partial U. \end{cases}$

$\begin{cases} |u'| = 1 & \text{in } (-1,+1). \\ u(\pm 1) = 0 \end{cases}$
- don't have global classical solution.
- Some loose uniquely a.e.

- "correct" class of weak solutions

$^\wedge$ unique.

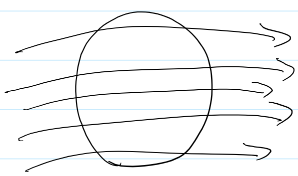
$b(x)$ vector fd. $b(x)\cdot Du = 0$ in $U$. $b$ nice.
let $\dot{x} = b(x)$, then $t \longrightarrow u(x(t))$ constant. $u$ smooth.



$\dot{\underline{X}} = b(\underline{X})$. $\underline{X}(0,x) = x$. $\underline{X}(x,t) \longrightarrow 0$. $t \to \infty$.
for all $x$.
$u = g$ on $\partial B$.
$u(x) = g(\underline{X}^{-1}(x,t))$.

Method of char.
$t \longrightarrow \underline{X}(t)$. along which we have. $p(t) = Du(\underline{X}(t))$.
$z(t) = u(\underline{X}(t))$.

the origin can't be traced back to same pt on boundary for finite time.
We can't have a $C^1$ solution for whole unit ball.



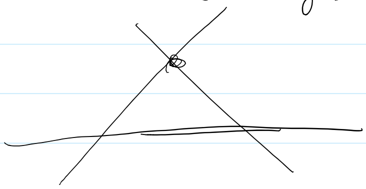
$u(x) = g(\underline{X}^{-1}(x,t))$.
assign $T$. if $T = \partial B$. $g$ can't be general since two pts on projectory have the same value.

$u_t + f(u)_x = 0$. conservation law Burger's equation.
$u_t + (u^2)_x = 0$. quasilinear.
$z(t) = u(X(t))$.
$p(t) = Du(X(t))$.
$p(t) = u_x(X(t))$. $\Rightarrow \dot{p} = u_{xx}\dot{x}$.
$H(p(t), z(t), x(t)) = 0$.
$H_p u_{xx} + H_z u_x + H_x = 0$ guess what's $\dot{x}$

$$\begin{cases} \dot{x} = H_p \\ \dot{p} = -H_z p - H_x \\ \dot{z} = H_p \cdot p - H \end{cases}$$

when $\begin{cases} u_t + H(u_x) = 0 \\ u = g \quad t = 0 \end{cases}$, then $\begin{cases} \dot{x} = -H'(p) \\ \dot{p} = 0 \\ \dot{z} = -H'(p)\cdot p - H \end{cases}$

Invert ODE. go ~~backward~~ backward in the flow. uniqueness of soln.



Next time: weak solutions. defn. unique.