Reading

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$V = \{1, \ldots n\} = [n]$. $H = \{H_1, \ldots H_n\}$. $H_i \subseteq V$.

$\chi : [V] \to \{\pm 1\}$.

$disc.(H) = \min_{\chi} \max_{i \in [n]} |\sum_{j \in H_i} \chi(j)|$.

$disc(H) = O(\sqrt{n \log n})$

$disc(H) = O(\sqrt{n})$ Spencer.

Bounded degree Setting.

$deg(j) \le t$. B-F Setting.

$disc(H) \le 2t-1$. $(2t-3)$

Beck-Fiala Conjecture.

If $deg(j) \le t$ then $disc(H) = O(\sqrt{t})$.

Restatement.

$A \to$ incidence matrix $m \times n$ $\{0,1\}$.

$disc(A) = \min_{x \in \{\pm 1\}} \|A \cdot x\|_\infty = \min_{x \in \mathbb{R}^n} \max_i |\sum a_{ij} x_j|$.

For $\forall A \in \mathbb{R}^{m \times n}$, $disc(A) = \min_{x \in \{\pm 1\}^n} \|A x\|_\infty$.

$A_{*j} \to j$ th col of A. $\min \|\sum A_{*j} \cdot x_j\|_\infty$.

Beck and Fiala.

For $A \in \mathbb{R}^{m \times n}$ with $\|A_{*j}\|_1 \le 1$ we have $disc(A) \le 2$.

Komlos Conjecture.

Let $A \in \mathbb{R}^{m \times n}$ with $\|A_{*j}\|_2 \le 1$, for $\forall j \in [n]$. Then $disc(A)$. $\le K$ for some constant K.

Vector Discrepancy.

$vec\ disc(A) = \min_{u_1 \ldots u_n \in S^{n-1}} \max_{i \in [n]} \|\sum a_{ij} \bar{u}_j\|_2$. $vecdisc(A) \le disc(A)$.

Main Theorem

For any $A \in \mathbb{R}^{m \times n}$ with $\|A_{*j}\|_2 \leq 1$, we have that $\mathrm{vecdisc}(A) \leq 1$. The result is tight ($I_n$)

Theorem (Matousek)

For any $A \in \mathbb{R}^{m \times n}$ $\mathrm{vecdisc}(A) \geq D$ iff "There exists a distribution over $[n]$ and a $w \in \mathbb{R}^n$ with $\sum w_j \geq D$ such that $\forall z \in \mathbb{R}^n$ $\mathbb{E}\left(\sum_{j=1}^{n} a_{ij} z_j\right)^2 \geq \sum w_j z_j^2$ ".

$D = \sqrt{1+\varepsilon}$ arbitrary $\varepsilon \geq 0$. We'll show that for $\forall p \in \mathbb{R}^m_n$, $\forall w \in \mathbb{R}^n$, that satisfies the condition, there always exists some z such that

$$\mathbb{E}_{i \sim p} \left(\sum a_{ij} w_j\right)^2 < \sum w_j z_j^2$$

$\mathrm{vecdisc}(A) < \sqrt{1+\varepsilon}$ $\forall \varepsilon > 0 \Rightarrow \mathrm{vecdisc}(A) \leq 1$

$w \in \mathbb{R}^n$, $\sum w_j \geq D^2 = 1+\varepsilon$ $w_j > 0$

if $w_j < 0$ for some j we set $z_j = 0$

$\forall w \in \mathbb{R}^n$ $\exists z \in \mathbb{R}^m$ $z^T A^T P A z < z^T W z$, where

$$P = \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_m \end{pmatrix} \quad W = \begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_n \end{pmatrix}$$

Let's assume that $\forall z \in \mathbb{R}^n$; $z^T A^T P A z \geq z^T W z$ $k \leq n$ $z_i = 0$ $\forall i > k$, $\forall k \leq n$, $\forall u \in \mathbb{R}^k$ $u^T A_{[k]}^T P A_{[k]} u \geq u^T W_k u$

$A_{[k]} = (A_{*1}, \cdots A_{*k})$

Ⓐ $X \in \mathbb{R}^{m \times m}$, $Y \in \mathbb{R}^{m \times n}$ X, Y p.s.d. Suppose that $\forall u \in \mathbb{R}^n$ $u^T X u \geq u^T Y u$. Then $\det(X) \geq \det(Y)$

Ⓑ $\forall k \leq n$ $\det(A_{[k]}^T P A_{[k]}) \leq p_1 \cdots p_k$

Ⓒ $x_1 \geq \cdots \geq x_n \geq 0$

$y_1 \geq \cdots \geq y_n \geq 0$ $\forall k \leq n$ $x_1 \cdots x_k \geq y_1 \cdots y_k$ Then $\forall i$ $x_i \cdots x_k \geq y_i \cdots y_k$

Pf of Ⓐ. $E(M) = \{u \mid u^T M u \le \rho\}$. ~~ellip~~ i.e. ellipsoid

$$\mathrm{vol}(E(M)) = \frac{\mathrm{vol}(B^n)}{\sqrt{\det(M)}}$$

If $\det(m) = 0 \Rightarrow$ ellipsoid is unbounded.

$E(X) \le E(Y)$.

- if $\det(Y) = 0$ ✓
- if $\det(Y) \ne 0$, $\det(X) \ne 0$. $\mathrm{vol}(E(X)) \le \mathrm{vol}(E(Y))$.

$\Rightarrow \det(X) \ge \det(Y)$

Pf of Ⓑ: $A_{[k]}$ Let $u_1 \cdots u_k$ be ONB of the space spanned by the cols of $A_{[k]}$

$U_k = (u_1, \cdots u_k)$. $\qquad A_{[k]} = U_k U_k^T A_{[k]}$

$\det(A_{[k]}^T U_k) = \det(U_k^T A_{[k]}) \le 1$ by Hadamard's ineq.

$\det(B) \le \prod \|B_{*j}\|_2$.

$$\det(A_{[k]}^T P A_{[k]}) = \det(A_{[k]}^T U_k U_k^T P U_k U_k^T A_{[k]})$$
$$= \det(A_{[k]}^T U_k) \cdot \det(U_k^T P U_k) \cdot \det(U_k^T A_{[k]})$$
$$\le \det(U_k^T P U_k)$$

Cauchy interlace Thm.

- $X \in \mathbb{R}^{n\times n}$ Symm with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$.
- $U \in \mathbb{R}^{n\times k}$ with mutually ON columns.
- $U^T X U$ — eigenvalues $\mu_1, \cdots \mu_k$.

Then $\forall i \le k$, $\lambda_{n-k+i} \le \mu_i \le \lambda_i$.

⊛ Corollary. $X \in \mathbb{R}^{n\times n}$, psd. with eigenvalues $\delta_1 \ge \cdots \ge \delta_n \ge 0$

$U \in \mathbb{R}^{n\times n}$ mutually ON cols. Then $\det(U^T X U) \le \delta_1 \cdots \delta_k$.

Pf of Ⓒ

$X \succeq Y$. $\quad X = (x_1, \cdots x_n)$. $x_1 \ge \cdots \ge x_n$.

$\qquad Y = (y_1, \cdots y_n)$ $y_1 \ge \cdots \ge y_n$.

$\forall k \le n.\ \sum_1^k x_i \ge \sum_1^k y_i$. f is Schur convex if $X \succ Y \Rightarrow$.
$f(X) \ge f(Y)$.

f convex. symm $\Rightarrow$. f Schur convex.

$f(x_1, \cdots x_k) = \sum_1^k e^{x_i}$ $\quad \sum \log x_i \ge \sum \log y_i$.

$x_1 + \cdots + x_k \ge y_1 + \cdots + y_k$.