Lecture 2

Thursday, October 2, 2014 12:06 PM

Ordinals.

Problem We consider the following game. Consider the lattice. $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. We know that there is a bouncing ball which at time $t=0$ begins at some point. (a_0, b_0, c_0). (fixed but unknown to us) and at each sbsequent moment (second). the bounces according to the rule. $(x, y, z) \longmapsto (x+d, y+e, z+f)$. where d, e, f are fixed but unknown to us.

Every hour, we look at a particular pt. if the ball is there at precisely that moment, we win. Can we win in finite time?

Q: What determines the ball's position at time t?

(a_0, b_0, c_0). d, e, f.

The ball has countably many possible "strategies".

$((a_0, b_0, c_0), d, e, f)$.

We have countably opportunities to guess. As long as I use my n^{th} guess to determine the ball is using that strategy. I'm guaranteed to win in a finite time.

"Infinite analogue of classical number theory".

Key property of $\mathbb{N}$.

1) Well ordered. every nonempty sbset has a minimum element

2) $\mathbb{N}$ captures "cardinality of finite set"

What are natural numbers? Sets are determined by its element.

$0 = \phi$.

$1 = \{0\}$.

$2 = \{0, 1\}$ $3 = \{0, 1, 2\}$, $\cdots$ $n+1 = \{0, \cdots n\}$.

$\in$ = set relationship.

$a \in X$ a is an element of X.

The set of natural numbers is well ordered by $\in$

Note that: 1) each $n \in \mathbb{N}$ is well ordered by $\in$.

2) each $n \in \mathbb{N}$ is transitive, i.e. has the property that $\bigcup n \subseteq n$.

Note: This property "X is a set which is transitive and is well ordered by $\in$" defines "X is an ordinal".

Let's think about what ordinals look like.

General facts:

1) every element of an ordinal is an ordinal

Pf: Suppose X is an ordinal and $y \in X$. Then $y \subseteq X$. If $z \in y$, then $z \in X$, then $z <_\in y$.

Every element is $<_\in$ than z, and z is $<_\in$ than y, hence is $<_\in$ than y. so such element belongs to y. Then y is transitive.

Now y is well ordered simply because X is well ordered.

2) If X is an ordinal, then $X \cup \{x\}$ is an ordinal.

Pf: $y = X \cup \{x\}$. pick $z \in y$ if $z = X$, then $\forall z' \in z$. $z \in x \Rightarrow z \in y$. if $z \neq X$, then X transitive, done.

given $\forall$ subset of y A. if $X \in A$, then since any other elt belongs to $x \Rightarrow X$ is not the smallest elt

$\Rightarrow A \setminus \{x\}$ ~~has sm~~ $\subset X$ has smallest elt z.

$z <_\in x \Rightarrow z$ is the smallest elt of A. done.

3) If X is an ordinal, Z is an ordinal $X \in Z$, then $X \cup \{x\} \subseteq Z$.

Pf: Z ordinal. $X \in Z \Rightarrow X \subseteq Z$, $X \in Z \Rightarrow \{x\} \in Z$

$\Rightarrow X \cup \{x\} \subseteq Z$.

4) X ordinal $y \in X$. then y is an initial segment of X.

Def. Z linear ordered. For any $a \in Z$. $\{x \in Z, x < a\}$ is a linear segment.

Pf:

5) X is an ordinal $y \subseteq X$. initial segment. then y is an ordinal. Moreover, either $y \in X$ or $y = X$.

Pf: $\Rightarrow y = \{z \in X, z < z_0\}$ well ordered ✓.

$\forall \alpha \in \beta \in y \Rightarrow \alpha <_\in \beta$. y initial segment. $\beta \in y$

$\Rightarrow \beta < z_0$ ~~also~~ $\alpha \in \beta \Rightarrow \alpha \subseteq \beta \subseteq X \Rightarrow \alpha \in X$

$\Rightarrow \alpha < z_0$ and $\alpha \in X \Rightarrow \alpha \in y$.

If $y \neq X$ then given $\forall z \in y \Rightarrow z <_\in \alpha \Leftrightarrow z \in \alpha$.

$\Rightarrow y = \alpha \Rightarrow$ done. $y \in X$.

6) X, y ordinals Then $x = y$. $X \in y$ or $y \in X$.

Pf: if $X \neq y$ let $z = X \cap y$. z initial segment in both X and y.

So either $\begin{cases} z = X, & z \in X \\ z = y, & z \in y \end{cases}$

[illegible] all $z = x \cap y$. [illegible] initial segment in both x and y.

So either $\begin{cases} z = x. & z \in x. \\ z = y. & z \in y. \end{cases}$

Cond. For any two ordinals α, β. Write "$\alpha < \beta$" if $\alpha \in \beta$. Then $<$ is irreflexive, transitive and tricho. on the class of all ordinals. Moreover it's a well ordering.

Let $\alpha \in B$. look at $\alpha \cap B$. if is empty, then α is a least elt of B. If not the least elt of α is the least elt of B.