Lecture 3 ordinals and cardinals.

A set is called an ordinal if it is transitive and well ordered by $\in$.

We proved that ordinals satisfy various crucial properties.

Last time:

- Every element of an ordinal is an ordinal.
- If x is an ordinal, then $x \cup \{x\}$ is an ordinal.
- If x is an ordinal and $y \in x$, then y is an initial segment of x.
- If x is an ordinal, then any initial seg of x is an ordinal either $= x$ or an elt of x.
- If x, y are ordinals, then either $x \in y$, $y \in x$ or $x = y$

Writing "$<$" instead of "$\in$" since this is a well ordering

Obs. If $(A, <)$ is any well ordered set and $f : A \to A$ is any monotonic map that for $\forall a \in A$. Then we have $a \le f(a)$

Pf: Suppose $\exists a$ st. $f(a) < a$. by monotonicity $f(f(a)) < f(a) < a$. we get infinite descending seq. $a, f(a), f(f(a)), f^3(a) \dots$

Since the set is well ordered, contradiction.

Cor. No w-o set is order-isomorphic to a strict initial segment of itself

Pf: Suppose $(A, <)$ w-o set $(X, <)$ initial seg and $\exists f : (A, <) \to (X, <)$ order iso. let $a \in A \setminus X$. then $f(a) \in X$. $f(a) < a$. contradiction.

Cond. If x, y are distinct ordinals then x, y are not order iso.

Pf: $x \in y$ or $y \in x$.

Lemma. Every w-o set is order iso to precisely one ordinal!

Pf: $(A, <)$ w-o set. $(B, <)$ the collection of all ordinals, which are order iso to some initial seg of A.

For each $x \in B$, let a_x be the unique elt of A st the initial seg of A determined by a_x is order iso to x. And let g_x be the order iso corresponds to this. Now if $y \in x$, then y is order iso to the initial seg of A determined by

$g_x(y)$. Hence $y \in B$. We've shown that $x \in B$, $y \in x \Rightarrow y \in B$.
So B is transitive.

Def. Let Θ be an ordinal, then Θ is a cardinal iff
$\forall x \in \Theta$ $\nexists$ bijection $\Theta \longrightarrow x$.
$|X| = |Y| \iff \exists$ bijection $X \longrightarrow Y$.
X is a set, R is a binary relation on X. R is a partial order iff
1) reflexive aRa.
2) transitive aRb, $bRc \Rightarrow aRc$.
3) antisymmetric aRb, $bRa \Rightarrow a = b$.

CSB. Suppose X, Y are sets and $\exists$ injection $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ then there exists a bijection $X \longrightarrow Y$
Pf: If f injective, then $\forall b \in Y$ has at most one preimage and in this case we write $f^{-1}(b)$ for the unique preimage
Define $X_n = \{a \in X \mid \ell(a) = n\}$ where ℓ is the length of a, define by ~~the~~ length of sequence $a, g^{-1}(a), f^{-1}(g^{-1}(a)) \dots$
$X_\infty \{\ell(a) = \infty\}$.

$X = X_1 \cup X_2 \cup \dots \cup X_\infty$ Let $a \in X_n$. $a, g^{-1}(a), \dots$ length n.
$g \uparrow \downarrow f$
$Y = Y_1 \cup Y_2 \cup \dots \cup Y_\infty$.

$$h(a) = \begin{cases} f(a) & \text{if } f(a) \text{ is odd or } \infty \\ g^{-1}(a) & \text{if } f(a) \text{ is even.} \end{cases} \Rightarrow |X| = |Y|.$$

$|\mathbb{R}| = 2^{\aleph_0} = |P(\mathbb{N})|$. Every $x \in [0,1)$ has a unique decimal expansion that does not end with a infinite string of 9's.

If $\exists$ surjection $X \longrightarrow Y \Rightarrow \exists$ injection $Y \longrightarrow X$.

$|2^{\mathbb{N}}| = |[0,1]| = |\mathbb{R}|$. $[0,1] \xrightarrow{\text{incl}} \mathbb{R}$
$\mathbb{R} \xrightarrow{\arctan} (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow (0,1)$.