Complexity of Boolean functions

Sunday, June 29, 2014 9:49 PM

Def. Boolean function. $f: \{0,1\}^n \longrightarrow \{0,1\}$.
Boolean circuit $\vee, \wedge, \neg$.
sensitivity = # x_i changed changes value of f
Size of B circuit = # gates.

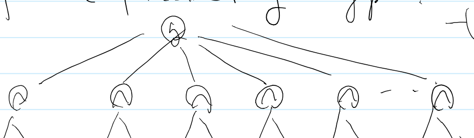
Def $\text{parity}_n = \sum^n x_i$
$\text{Majority}_n = \begin{cases} 1 & \sum x_i \geq n/2 \\ 0 & o/w \end{cases}$

Def. $s(f,x) = \#\{i \mid f(x) \neq f(x \oplus e_i)\}$.
$s(f) = \max_x s(f,x)$.

Def. depth: longest path input to output
Ex1. Every boolean function representable by depth 2 circuit
AND of ORs (CNF)
DNF OR of ANDs.
Ex2. parity requires $\Omega(n 2^n)$ size in depth 2.
Ex3. parity requires $\Omega(n 2^{n/d-1})$ in depth $(d-1)$ circuits.
Ex4. parity in depth $O(\log n)$ works in poly size?
Ex5. If T be polytime decidable predicate.
$T: \{0,1\}^* \longrightarrow \{0,1\}$
n^{th} slice $T_n: \{0,1\}^n \longrightarrow \{0,1\}$ boolean fn.
$\Rightarrow T_n$ has poly size circuit
Ex6. (Shannon) A.a. Boolean fns requires circuit size $\Omega(2^n/n)$
Ex7. Prove that addition of n-digit numbers is in depth 3, poly size.
Thm: (Yao, Hastad). Parity in depth requires circuits of size at least $2^{\frac{1}{10} n^{d-1}}$

Ex 8. Multiplication in fixed depth requires large circuit.
parity gate $\oplus$
$\text{Mod}_m = \begin{cases} 1 & m \mid \sum x_i \\ 0 & o/w \end{cases}$

k-party Communication Complexity
NoF. number of forehead model.
$c(f) = \min_{p \text{ protocol}} \max_{x,y,z} \#\text{Steps on } x,y,z$.
$GIP_k(x,y,z) = \sum x_i y_i z_i \mod 2$
Thm. $c(GIP_k) = \Omega(n/4^k)$
Thm. If f has super poly log comm complexity for super poly log players then f cannot be computable by a bdd depth mod m gate.
Def AC^0 class of boolean fns that can be computed by bdd depth polysize circuits.
ACC^0 — — w/ mod m gates
Thm (Yao). ACC^0 circuits can be simulated by depth 2 circuits of the following type. g is a symm fn of quasipolytime $\leq \exp(c(\log n)^c)$



Ex 9. Adelmann Hastad.

Ex 9. Goldmann Hastad.
Ex 10. $CC(GIP_{2+\log_2 n}) = \Theta(\log_2 n)$.

Block sensitivity. $B \subset [n]$. $x^B = x$ w/ $i \in B$ flipped.
$B_1 \cdots B_k \subset [n]$ st. $B_i \cap B_j = \phi$.
$bs(f, x, B_1, \cdots B_k) = |\{i \mid f(x) \neq f(x^{B_i})\}|$
$bs(f, x) = \max\limits_{\{B_i\}} bs(f, x, B_1 \cdots B_k)$. $bs(f) = \max\limits_{x} bs(f, x)$.

$s(f) \le bs(f) \overset{?}{\le} s(f)^{O(1)}$. Sensitivity Conjecture.
Ex 1. $bs(f) \le C^{s(f)}$ for some C.
Ex 2. Find f st. $bs(f) = \Omega(n)$. $s(f) = \Theta(\sqrt{n})$.
Ex 3. monotone boolean fn f, then $s(f) = bs(f)$.

Graph properties.
$v = \#$ vertices. $n = \binom{v}{2}$ $\quad S_v \longrightarrow S_{\binom{v}{2}} = S_n$.

boolean fns invariant under action of S_v.
Ex 4. (Turan's thm) If f is a graph property, then $s(f) = \Omega(v) = \Omega(\sqrt{n})$.
Ex 5. Find graph property w/ sensitivity $\theta(v)$.

How to construct g.p. w/ sensitivity $\theta(v)$

Ex 6. Find graph property w/ a gap between $s(f)$ and $bs(f)$
i.e. $\lim\limits_{n\to\infty} \frac{s(f)}{bs(f)} = 0$.
Ex 7. f 3-graph property, then $s(f) = \Omega(v)$.
Ex 8. Find f, st. $s(f) = \Theta(v^2)$.

Problem: characterize g.p with sensitivity $\Theta(v)$.
Problem: Shrink the gap ↑ lower bound or ↓ upper bound.

Ex 9. k-graph properties. $s(f) = \Omega(v)$
Ex 10. $\exists$ k-graph prop. f. st. $s(f) = O(n^{\lceil \frac{k}{2} \rceil})$.

$G \le S_n$. G permutation gp of degree n. G transitive if $(\forall i, j)(\exists \sigma \in G)(\sigma(i) = j)$.
Ex 11. Aut gp of Petersen's graph is $S_5^{(2)}$. write vertices as pairs of 5 elements.
Ex 12. $S_v^{(k)}$ transitive gp
Thm. If f is invariant under a transitive gp. $\Rightarrow bs(f) \ge n^{1/3}$.

Problem: If f is a k-graph prop. then is it true that $bs(f) = \Omega(\sqrt{n})$?

Lemma If $G \le S_n$ transitive. $A \subset [n]$. $|A| = t$. then $\exists \ge \frac{n}{t^2}$ disj G-translates of A.
Def (G-translates). $A^\sigma = \{i^\sigma \mid i \in A\}$

Ex 12. Use Lemma to prove thm. $t = n^{1/3}$.
Ex 13. $\mathcal{H}$. t-unif regular hypergraph. $\Rightarrow \exists$ at least $\frac{n}{t^2}$.

Ex12. Use Lemma to prove thm. $t = n^{1/3}$.

Ex13. $\mathcal{H}$. t-unif regular hypergraph. $\Rightarrow \exists$ at least $\frac{n}{t^2}$ disj edges

Ex14. Use Ex13 to prove Lemma.

Ex15. for $\forall f$ depends on n vars. smallest possible. $s(f) \geq \log n - \log\log n - 1$.

Explain Sanven.
$\exists$ cyclically invariant. fn st. $s(f) = \Theta(n^{1/3})$.

Fourier Analysis on the boolean cube.

$\{f: \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2\}$.

vspace. $\langle f, g\rangle = \mathbb{E}(fg) = \frac{1}{2^n} \sum_x f(x) g(x)$

$\|f\|_2 = \sqrt{\mathbb{E}(f^2)} = \sqrt{\langle f, f\rangle}$

Def (character). $\chi: G \longrightarrow \mathbb{C}^\times$

Ex. $\chi: \mathbb{Z}_2^n \longrightarrow \mathbb{C}^\times$ $\quad \chi(a)\chi(b) = \chi(a+b)$.

E.g. $\chi(a) = 1$.

$\chi(x) = p(x) = \begin{cases} -1 & \text{odd parity} \\ 1 & \text{even parity} \end{cases}$

E.g. $\chi(x) = \begin{cases} 1 & x_i = 1 \\ -1 & x_i = 0 \end{cases}$

E.g. $S \subset [n]$. $\chi_S(x) = \begin{cases} -1 & \{x_i = 0 \mid i \in S\} \text{ ~~even~~ odd parity} \\ 1 & \text{o/w} \end{cases}$

$\chi_S(x) = (-1)^{\sum_{i\in S} x_i}$

$\sqrt{\langle \chi_S, \chi_S\rangle} = \sqrt{\mathbb{E}(\chi_S^2)} = 1$.

for $S \neq T$. $\langle \chi_S, \chi_T\rangle = \mathbb{E}(\chi_S \chi_T) = \mathbb{E}(\chi_{S \triangle T}) = 0$

$\forall f: \{0,1\}^n \longrightarrow \mathbb{R}$. $f = \sum_{S \subset [n]} \alpha_S \chi_S$

Def. $\hat{f}(S) = \alpha_S = \langle f, \chi_S\rangle = \mathbb{E}(f\chi_S)$.
$= \frac{1}{2^n} \sum_x f(x) \chi_S(x)$.

Ex flip a coin. head $\frac{1}{2}+\varepsilon$. $\quad 0$.
tail $\frac{1}{2}-\varepsilon$. $\quad 1$

$\Pr(\text{parity even}) = \frac{1}{2}(1 + (2\varepsilon)^n)$.

Thm. (Plancherel) $\langle f, g\rangle = \sum_S \hat{f}(S)\hat{g}(S)$.

Cor $\|f\|_2^2 = \sum (\hat{f}(S))^2$

$f: \{0,1\}^n \longrightarrow \{0,1\}$.
$f: \{0,1\}^n \longrightarrow \{\pm 1\}$

Prop. f boolean. then $\hat{f}(S) = 1 - 2\Pr(f(x) \neq \chi_S(x))$.

pf: $\hat{f}(S) = \frac{1}{2^n} \sum f(x)\chi_S(x)$

$= \frac{1}{2^n} \sum_{f(x) = \chi_S(x)} f(x)\chi_S(x) + \frac{1}{2^n} \sum_{f(x) \neq \chi_S(x)} f(x)\chi_S(x)$

$= \Pr(f(x) = \chi_S(x)) - \Pr(f(x) \neq \chi_S(x))$

$= 1 - 2\Pr(f(x) = \chi_S(x))$

$$= \Pr(f(x) = \chi_S(x)) - \Pr(f(x) \neq \chi_S(x))$$

$$= 1 - 2\Pr(f(x) = \chi_S(x)).$$

Prop. $\mathbb{E}(f) = \hat{f}(\phi)$.

Ex. $\deg(f)$ = what? st. ~~ch~~ can be read from the Fourier Coefficien.

$\max |S|$ st. $\hat{f}(S) \neq 0$.

Ex. Figure out what works. reformulate. for finite. $G = \mathbb{Z}_k$

Linearly testing

Test by looking at output on only a small # of inputs

- f linear.
- $\forall g$ linear. f, g disagree $\geq \varepsilon$ fraction of inputs.

Testing: $x, y \in \{0,1\}^n$ random. Check $f(x)f(y) = f(x+y)$

$\delta = \Pr(f(x) \cdot f(y) = f(x+y))$

Lemma $\delta \geq \varepsilon$.

Pf:
$$1 - \delta = \Pr(f(x)f(y) = f(x+y))$$
$$= \Pr(f(x)f(y)f(x+y) = 1)$$
$$= \mathbb{E}\left(\frac{1 + f(x)f(y)f(x+y)}{2}\right)$$
$$= \frac{1}{2} + \frac{1}{2}\mathbb{E}(f(x) \cdot f(y) \cdot f(x+y))$$
$$= \frac{1}{2} + \frac{1}{2}\left((\sum \hat{f}(S)\chi_S(x))(\sum \hat{f}(T)\chi_T(y))(\sum \hat{f}(U)\chi_U(x+y))\right)$$
$$= \frac{1}{2} + \frac{1}{2}\sum \hat{f}(S)\hat{f}(T)\hat{f}(U)\,\mathbb{E}(\chi_S(x)\chi_T(y)\chi_U(x+y))$$

$$\mathbb{E}(\chi_S(x)\chi_T(y)\chi_U(x+y)) = \mathbb{E}\left((-1)^{\sum x_i}(-1)^{\sum y_i}(-1)^{\sum x_i + y_i}\right)$$
$$= \mathbb{E}\left((-1)^{\sum_{S \Delta U} x_i}(-1)^{\sum_{T \Delta U} y_i}\right)$$
$$= \mathbb{E}(\chi_{S \Delta U}(x))\,\mathbb{E}(\chi_{T \Delta U}(y)) = \begin{cases} 1 & S = T = U \\ 0 & \text{o/w} \end{cases}$$

$$\Rightarrow 1 - \delta = \frac{1}{2} + \frac{1}{2}\left(\sum_S \hat{f}(S)^3\right)$$
$$\leq \frac{1}{2} + \frac{1}{2}\max \hat{f}(S) = \frac{1}{2} + \frac{1}{2}\hat{f}(T)$$
$$= \frac{1}{2} + \frac{1}{2}(1 - 2\Pr(f(x) = \chi_T(x)))$$
$$= \frac{1}{2} + \frac{1}{2} - \varepsilon = 1 - \varepsilon$$

Thm $f \in (d, S)$. $as(f) = O(\log S)^{d-1}$

$\rho \in R\left(\frac{1}{10}\left(\frac{1}{\log S}\right)^{d-1}, \frac{1}{2}\right)$ w.p. $\geq 1 - \frac{1}{S}$ f becomes same.

AND of ORs.

$$as(f) = \frac{1}{p}\mathop{\mathbb{E}}_{\rho \in R(p, \frac{1}{2})} as(f|_\rho)$$

$$= 10 \cdot (10 \log S)^{d-1}\left((1 - \tfrac{1}{S}) \cdot 1 + \tfrac{1}{S} n\right) \quad S > n \text{ done.}$$

$$= 10 \cdot (10 \log S)^{d-1} \underset{\rho \in R(p, \frac{1}{2})}{} \left((1 - \tfrac{1}{S}) \cdot 1 + \tfrac{1}{S} n \right)$$ $S > n$ done. if $S < n$.

$as(f \wedge g) < as(f) + as(g)$.
bound: n by S^d.

$as(f) = \mathbb{E}_{x \in \{0,1\}^n} S(f, x)$.

f is k intersection of half-planes $as(f) \leq O(\sqrt{n} \cdot \text{weight})$

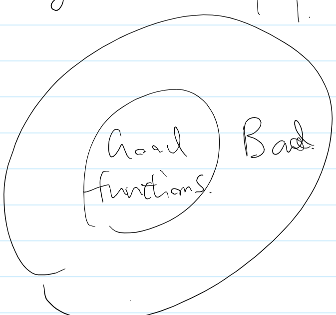
f t-CNF.
$\Pr\left(f|_\rho \text{ is } S\text{-DNF}\right) \leq \left(\frac{\sqrt{pt}}{q}\right)^S$. $\rho \in R(p, q)$.

Thm.
$as(f) \leq S$, then $\forall \varepsilon > 0$. $\exists\, C = C(\varepsilon)$ s.t. $\exists g$ depends only on C^S vars
$\Pr(f(x) \neq g(x)) < \varepsilon$

Simons. f depends on n vars. $S(f) \geq \Omega(\log n)$

$\Leftarrow$ if for $\forall g$ depends on C^S vars.

The Method of Approximation.
$f_1 = x_1$
$f_2 = x_2$
$\vdots$
$f_n = x_n$.
$f_i = \begin{cases} f_j \wedge f_k & j, k < i \\ f_j \vee f_k & j, k < i \end{cases}$
$C(g) = \min \{ i \mid \exists$ straight line program $\{f_n\}$ s.t. $f_i = g \}$.



all Boolean functions on n variables.
distance.
$\delta : B_n \times B_n \longrightarrow [0, 1]$.
- $\delta(f, f) = 0$
- $\delta(f, g) \leq \delta(f, h) + \delta(h, g)$.
- $\delta(f, g) = \delta(g, f)$

f, g good. $f \wedge g$ $(f \vee g)$ is ε-close to good function.

g really bad. $\Leftrightarrow$ $(\forall \text{ good } f)(\delta(f, g) = \Omega(1))$.
$\Rightarrow C(g) \geq \frac{\Omega(1)}{\varepsilon}$.

Razborov-S.
$AND(x_1, \ldots, x_m) = x_1 \cdots x_m$.
$MOD_p(x_1, \ldots, x_n) = 1 - (\sum x_i)^{p-1}$.
$OR(x_1, \ldots, x_m) = 1 - \prod(1 - x_i)$.

$OR(x_1, \dots x_m) = 1 - \prod(1-x_i)$.

$OR(x_1, \dots x_m) = OR(X_{S_1}) \vee OR(X_{S_2}) \vee \dots$

$S_i \subset [m]$.

$\widetilde{OR}(x_1, \dots x_m) = \bigvee_{\ell} \left(\sum_{j \in S_i} x_j \right)^{p-1}$

$S_1 \dots S_\ell \subset [m]$ random set.

$= 1 - \prod_{\ell} (1 - (\sum_{S_i} x_j)^{p-1})$ $\quad deg = \ell(p-1)$.

Claim $dist(OR, \widetilde{OR}_\ell) \le \frac{1}{2^\ell}$

$\exists S_1 \dots S_{\ell}$ st. for $\forall \ell$ $\quad dist = Pr(OR \ne \widetilde{OR}_\ell) \le \frac{1}{2^\ell}$

$\ell = O(\log^2 n)$

Thm. $f \in (d, S)$ Mod p gates. $(\forall \ell)(\exists p(x) \in \mathbb{F}_p[x_1 \dots x_n]$ st.
$deg\, p \le \ell^d$. $Pr(f(x) \ne p(x)) \le S \cdot 2^{-\ell})$.

$\Leftrightarrow S \ge 2^\ell Pr(f(x) \ne p(x)) = \Omega(1) \cdot 2^\ell$.

Find lower bound on S.

Negation ~~$f \in (d,S)$ st~~ $(\exists \ell)(\forall p(x) \in \mathbb{F}_p[x_1 \dots x_n]$ st. $deg\, p \le \ell^d$
$Pr(f(x) \ne p(x)) > S \cdot 2^{-\ell}) \Rightarrow f \notin (d, S)$.

Hilbert function.

$h_m(S)$

$\mathbb{F}_p[x_1, \dots x_m] \Big/ (x_1^2 = x_1, \dots x_m^2 = x_m) = R$

$S \subset \{0,1\}^n$

$h_m(S) = \dim \{ p(x)|_S : deg(p) \le m . p(x) \in R \}$

$\le \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{m} \le |S|$.

(Smolensky 87) For any $f: \{0,1\}^n \to \{0,1\}$. $\forall g: \{0,1\}^n \to \{0,1\}$.
of degree $\le d$,

$2^n Pr(f(x) \ne g(x)) \ge 2h_m(S) - |S|$

where $S = f^{-1}(0)$. $m \le \frac{n-1-d}{2}$

Pf: (graph).

$h_m(S) < \binom{n}{0} + \dots + \binom{n}{m}$

$\Rightarrow \exists\, p_1(x), p_2(x)$ of degree $\le m$. st.

$p_1(x)|_S \equiv p_2(x)|_S \Rightarrow p_1(x) - p_2(x)|_{S = f^{-1}(0)} \equiv 0$.

$h(x) = p_1(x) - p_2(x)$. of degree $\le m$. $\Rightarrow AI(f) \le m$.

Claim. $AI_{\mathbb{F}}(MAJ_n) = \lceil \frac{n}{2} \rceil$ for $\forall \mathbb{F}$

$\forall g(x).\ 2^n Pr_x(f(x) \ne g(x)) \ge 2h_m(S) - |S|$.

$\ge 2\left(\binom{n}{0} + \dots + \binom{n}{m} \right) - 2^{n-1}$.

$$\geq 2\left(\binom{n}{0}+\dots+\binom{n}{m}\right) - 2^{n-1}$$

$\left|\frac{n}{2} - m\right| \leq O(\sqrt{n})$ $\quad \geq 2\left(2^{n-1} - o(2^n)\right) - 2^{n-1}$

$\Rightarrow d = o(\sqrt{n})$ $\quad = 2^{n-1} - o(2^n)$

$\frac{\Omega(1)}{\varepsilon} = \Omega(2^l)$ as long as $l^d = o(\sqrt{n})$.
$= \Omega(2^{n^{1/(3d)}})$. $\quad l = n^{1/3d}$

Thm. $MAJ_n \notin (d, 2^{n^{\frac{1}{3d}}})$ with Mod p gates

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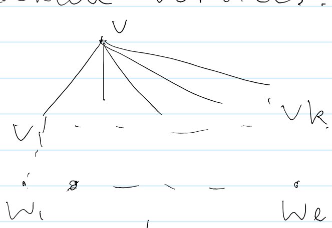
Saturday 7/5
Turan's paper.

Prop. For $\forall v \in \mathbb{N}$. $\exists$ graph property P. $s(f_P) = v-1$.
P = the graph has an isolated vertex.
Thm. (Turan) For any non trivial graph property P on v vertices. $s(f_P) \geq \lfloor \frac{1}{4} v \rfloor$.
Pf: Assume $s(f_P) < \lfloor \frac{1}{4} v \rfloor \Rightarrow P$ is trivial.
G - number of edges n, $n \leq \frac{1}{4} v$.
$s(f_P) = \max_G s(f_P, G)$.

Assume that $\forall G'$ w/ $|E(G')| < |E(G)|$ $P(G') = 0 \Rightarrow P(G) = 0$.
Claim: If $|E(G)| > v/4$, then the inductive step is trivial.
(If x 1-step minimal input w/ $f(x) = 1$ then $s(f) \geq |x|$.)

Then we can just assume $|E(G)| < v/4 \Rightarrow G$ has least $v/2$ isolated vertices. $l > v/2$.

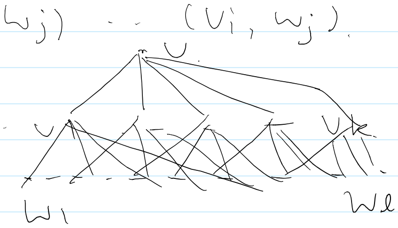


$G + (v_1, w_1) \cong G + (v_1, w_r)$
(wts if $G \notin P$, $G + (v_1, w_1) \notin P$).
$\Rightarrow$ if $G + (v_1, w_1) \in P$ then.
$s(f, G + (v_1, w_1)) \geq l > v/2$.
$\Rightarrow$ contradiction.

w_l: isolated vertices

if $i \leq v/4$. $G + \sum_{j=1}^{i} (v_i, w_j) \notin P$.

when remaining isolated pt. $< v/4$. and we delete edges instead if $G' - (v_1, w_j) \in P$. then $s(f, G') > v/4$.
$\Rightarrow$ we can add all (v_1, w_i) $i = 1, \dots, l$
And then we can do similar thing and add all edges $(v_2, w_j) \dots (v_i, w_j)$.

$G' =$ 

complete bipartite graph.
v is equivalent to w_j since its connected to $v_1 \dots v_k$.

$\Rightarrow$ we get a proper subgraph G'' st $G'' \notin P$

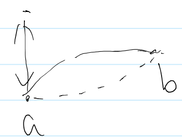
Thm. f invariant under transitive gp. then $bs(f) \geq n^{1/3}$.

$\Rightarrow$ we get a proper subgraph G' st. $G'' \notin P$

Thm. f invariant under transitive grp. then. $bs(f) \geq n^{1/3}$.

Lem. G, H graph. $V(G) = V(H)$. If $2\deg\max(G) \cdot \deg\max(H) < v$, then $\exists$ disjoint placement of G, H.

Pf: find z st. flip a, z will decrease # ~~inters~~ overlaps.



Assumption: $\forall$ a graph with $c \cdot v$ edges: $\exists$? $\Omega(v)$ copies of itself disj. placed?

Q: True or False? $\exists c > 0$. st $|E(G)| \leq c \cdot v$ edges, $\Rightarrow \exists \geq cv$ disj copies of G.?

$S_v^{(2)}$ $S_v^{(3)}$?

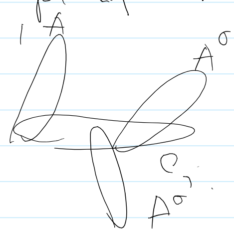
Lemma. (Ex).

$\mathcal{H}_1, \mathcal{H}_2$ k-unif, l-unif. both regular, $V(\mathcal{H}_1) = V(\mathcal{H}_2)$.
$\forall A \in \mathcal{H}_1, \forall B \in \mathcal{H}_2$ $|A \cap B| \geq t$, then $kl \geq t \cdot v$.

Ex. Infer from this, if f transitive nontrivial boolean functions, then certificate complexity of $f \geq \sqrt{n}$.

1-certificate. intersection. ~~not~~ empty

0-certificate. Supp 1 $\cap$ Supp 0. $\neq \phi$ and disagree at the pt of intersection.



look at translates of. A. transitive hypergraph $|A|$-~~regular~~ uniform and regular.

$|A \cap B| \geq 1$

$\Rightarrow k \cdot l \geq v$ $\Rightarrow |A||B| \geq v$

$\Rightarrow$ certificate complexity $\geq \sqrt{v}$.

Sourav.

$\exists f$ invariant under C_n. st. $s(f) = \Theta(n^{1/3})$ $f: \{0,1\}^n \to \{0,1\}$.

define $g: \{0,1\}^{k^2} \to \{0,1\}$ $g(x) = 1 \iff x = \underbrace{[1100\cdots0]}_{k}\underbrace{[\quad]}_{k}\cdots$

2nd block starts with 11111, rest doesn't matter. every other blocks starts with 11111 the same type of block

For the last block, should start with 11111 and ends with 111.

X looks like this $g(X) = 1$. $g(x) = 0$ o/w.

$f(x) = \begin{cases} 1 & \text{if } \exists i\ g(X_i X_{i+1} \cdots X_{i-1}) = 1 \\ 0 & \text{o/w} \end{cases}$

$S^0(f) = \Theta(\frac{n}{k^2})$. $S^1(f) = \Theta(k)$. Lets show this.

lower bound. Start with what's needed and everything else.

$1100\cdots0\,11111\,00\cdots0\,11111\,00\cdots0 \cdots 11111\,00\cdots0111$.

$k + (k-2)5 + 8. = 6k-2 = S(f, x)$.

Upper bound.

$\exists x \in \{0,1\}^n$, st. $\exists i \leq n$ st. $f(x^i) = 0$.

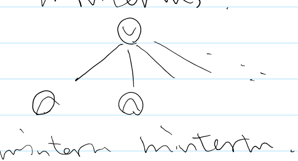
$f(x) = 1 \Rightarrow \exists$ strip look like what we want. $\Rightarrow$ we need to change value for some a which gives $6k-2$.

$\exists x' \in \{0,1\}^n$, s.t. $\exists i \le n$ s.t. $f(x') = 0$.
$f(x)=1 \Rightarrow \exists$ strip look like what we want. $\Rightarrow$ we need to change value for some g which gives $6k-2$.
$\Rightarrow \Theta(k) = s'(f)$.

Consider $s^0(f)$.
$m = \lfloor \frac{n}{k^2} \rfloor$ $r = n - k^2 m$.
lower bound $x = (100\cdots0|11110\cdots0|11110\cdots0|111100\cdots0111)^m 0^r$.
$s^0(f) \ge m = \lfloor \frac{n}{k^2} \rfloor$

upper bound. $\exists x \in \{0,1\}^n$. s.t. $\exists i \le n$. $f(x^i) = 1$.

weight $n^{1/3}$ no more than $n^{1/3}$ pairwise compatible copies
Minterm translate function.
several minterms.

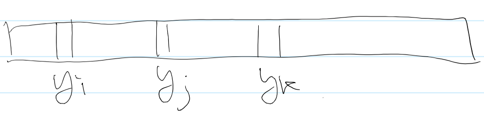


Sensitivity conjecture for DNF of several minterms.
no symm involved.
What if for graph properties?

subgraph partial graph. viewed as a pattern.
(all isomorphic copies.)
graph. compatible with all patterns. this is graph prop. but not monotone.

Reading gp.

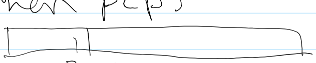
$\varphi(x_1, \cdots x_n)$ check whether this satisfy.
value $x_1 \cdots x_n$.
V choose three random bits.
And then check $(y_i \vee \overline{y_j} \vee y_k)$.



$(y_i \vee \overline{y_j} \vee y_k)$.
$(y_{i'} \vee \overline{y_{j'}} \vee y_{k'})$.

$\varphi \longrightarrow \varphi_V$. Such that
φ is SAT $\Rightarrow \varphi_V$ SAT.
φ is UNSAT $\Rightarrow \varphi_V$ is at most $c.f$ - SAT.
satisfied $\le c.f$ fraction $c.f$ fraction of the clauses. SAT.

V_1 (outer PCP)
accept. 1 or accept. w/ extremely low probability.
V_2 (inner PCP)



encoding symbol by symbol.
$\sim \Sigma$.
$c(\Sigma) \in \{0,1\}^k$.

M n

$c(\bar{\Sigma}) \in \{0,1\}^k$.

Monday.

Communication complexity.
$D^{cc}(f)$ = min #. bits over all protocol that computes f.

Fact $\log \operatorname{rank}(M_f) \le D^{cc}(f)$.

$[M_f]_{x,y} = f(x,y)$.
$\{0,1\}^{2^n \times 2^n}$ matrix.

Log-rank conjecture. lower bound is tight.
$D^{cc}(f) = O(\log^c \operatorname{rank}(M_f))$ for some constant c.

If conjecture true, then $\exists$ large rectangle in side $[M_f]$ that's constant.

Log-conjecture $\Leftrightarrow$. If $\operatorname{rank}(M) \le r$, then there exists mon-o chromatic rectangle of area $\ge 2^{2^n} / \log^c \operatorname{rank}(M)$.

For $f: \{0,1\}^n \times \{0,1\}^n \longrightarrow \{0,1\}$.
Upper bound:
Trivial: $D^{cc}(f) \le \operatorname{rank}(M_f)$.
Best before last year: $D^{cc}(f) \le \log(4/3) \cdot \operatorname{rank}(M_f)$.
Current: $D^{cc}(f) = O(\sqrt{\operatorname{rank}(M_f)}) \log \operatorname{rank}(M_f)$.
Lower bound:
$\exists f$ st. $D^{cc}(f) = \Omega(\log^\alpha \operatorname{rank}(M_f))$. where $\alpha = \log_3 6$.

XOR function.
Def. $F(x,y)$ is a XOR function if $F(x,y) = f(x \oplus y)$ for some $f: \{0,1\}^n \to \{0,1\}$
Notation: $f \circ \oplus$.
eigenvalues of $f \circ \oplus$ $\Longleftrightarrow$ Fourier coefficient of f
$\operatorname{rank}(f \circ \oplus)$ = # non-zero Fourier coefficient of f

Log-rank for XOR is true for
- Symmetric
- Monotone
- LTFs (linear Threshold functions)
- AC^0

In this work: Also true for
- function with small spectral norm.
- low degree polynomial

Fourier analysis:
$\forall f: \{0,1\}^n \longrightarrow \mathbb{R}$ $\quad f = \sum \hat{f}(S) \cdot \chi_S$. $\quad \chi_S(x) = (-1)^{\langle S, x \rangle}$
For $f: \{0,1\}^n \longrightarrow \{\pm 1\}$
Fact: (Parseval's Identity) $\quad \langle f, g \rangle = \mathbb{E}_x (f(x) g(x))$.
$\|f\|_2 = \sum_S \hat{f}^2(S) = 1$.

Fact: (Convolution Theorem)
$f \cdot g = \sum_S \left(\sum_T \hat{f}(T) \hat{g}(S+T) \right) \chi_S$.

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$\|\hat{f}\|_p = \left(\sum |\hat{f}(S)|^p \right)^{1/p}$ for $p > 0$.

If f has a small depth decision tree, then $f \circ \oplus$ has a low coset communication protocol.

$D^{cc}(f \circ \oplus) \le 2 \cdot D_{\oplus}(f)$

Construct a small depth PDT (parity DT).

View $f: \{0,1\}^n \to \{0,1\}$ as a polynomial over $GF(2)$.

e.g. $XOR(x_1, \dots x_n) = x_1 + \dots + x_n$.

$f(x_1, \dots x_n) = x_1 f_1(x_2, \dots x_n) + x_2 f_2(x_3, \dots x_n) + \dots +$
$x_{n-1} f(x_n) + x_n f_n$.

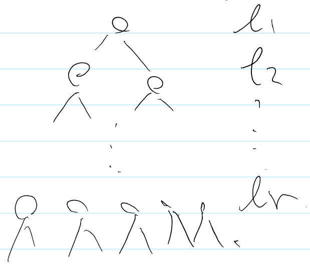
(*) $= l_1(x) f_1(x) + \dots + l_r(x) f_r(x) + f_0(x)$.

s.t. $\deg(l_i) = 1$ $\deg(f_i) \le d-1$. $d = \deg(f)$.

Def l-rank(f) = min r s.t. f can be expressed in the form (*).

Suppose $r = l$-rank(f). $f = l_1 f_1 + \dots + l_r f_r + f_0$.

the PDT constructed by the algorithm.



Fact $\deg(f) \le \log \|\hat{f}\|_0$

Fact l-rank$(f) \le C_{\oplus,\min}(f)$.

$C_{\oplus,\min}(f)$ = min codimension of all affine subspace of $\{0,1\}^n$ on which f is a constant.

$f|_H$ = constant H affine subspace

Lemma: $C_{\oplus,\min}(f) = O(\|\hat{f}\|_1)$

Lemma: $C_{\oplus,\min}(f) = O(2^{d/2} \log^{d-2} \|\hat{f}\|_1)$

$f(x_1, \dots x_n) = \sum_S \hat{f}(S) \cdot \chi_S(x_1, \dots x_n)$.

$f_0(x_1, \dots x_n) = f(0, x_2, \dots x_n)$.

$= \sum_{S: S_1 = 0} \hat{f}(S) \chi_S(0, x_2, \dots x_n)$

$+ \sum_{S: S_1 = 1} \hat{f}(S) \cdot \chi_S(0, x_2, \dots x_n)$

$= \sum_{S: S_1 = 0} \hat{f}(S) \cdot \chi_S(0, x_2, \dots x_n) = \sum_{S: S_1 = 0} (\hat{f}(S) + \hat{f}(S + e_1)) \chi_S(x_2, \dots x_n)$.

$f_1(x_1, \dots x_n) = f(1, x_2, \dots x_n)$

$= \sum_{S: S_1 = 1} \hat{f}(S) \chi_S(1, x_2 \dots x_n) = \sum (\hat{f}(S) - \hat{f}(S + e_1)) \chi_S(x_2, \dots x_n)$

the Fourier coefficient of $f|_{\{l(x) = 1\}}$ is of the form

$\hat{f}(\alpha) + (-1)^b \hat{f}(\alpha + \beta)$

$a_1 \ge a_2 \ge \dots \ge a_s$ to combine a_1 and a_2

$\shortparallel$ $\shortparallel$ to find a subspace such that

$a_1 \geq a_2 \geq \cdots \geq a_s$ — to combine a_1 and a_2

$|\hat{f}(\alpha_1)| \quad\quad |\hat{f}(\alpha_s)|$ — to find a sbspace H such that

$\max \{ |f|_H| \} = a_1 + a_2$

in $O(\|\hat{f}\|_1)$ steps, the resulting function has a Fourier coefficient $=1$

$\Rightarrow f = \pm \chi_S$ for some S

I. if $a_1 \leq 1/2$

Fact: $a_2 = \Omega\left(\frac{1}{\|\hat{f}\|_1}\right)$, $a_1 \geq \frac{1}{\|\hat{f}\|}$

Pf: use Parseval's identity

$\|f\|_2 = 1 \Rightarrow \sum \hat{f}(S)^2 = 1 \Rightarrow a_1 \cdot \sum |\hat{f}(S)| \geq \sum \hat{f}(S)^2 = 1$

$\Rightarrow a_1 \geq \frac{1}{\|\hat{f}\|}$

II. if $a_1 > 1/2$, then spectral norm drops by $\frac{2a_1 a_2}{a_2}$ in each step

$\geq 2a_1$

Low degree:

Lemma $C_{\oplus,\min}(f) = O(2^{\frac{d^2}{2}} \log^{d-2} \|\hat{f}\|_1)$

Pf sketch: Induction on $\deg f = d$.

True for $d=2$ by Dickson Theorem $C_{\oplus,\min}(f) = O(\log \|\hat{f}\|_1)$

$\Delta_e(f) = f(x) \cdot f(x+e)$

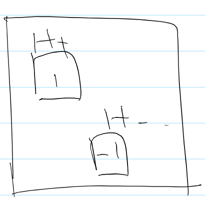
$\deg(\Delta_e f) < \deg f$

Fact: $\|\widehat{\Delta_e f}\|_1 \leq \|\hat{f}\|_1^2$

$D_\oplus(\Delta_e f) \leq \log \|\widehat{\Delta_e f}\|_1 \leq 2 \log \|\hat{f}\|_1$

$\exists H_+, H_-$ affine sbspaces of $\{0,1\}^n$ of codim $\leq 2\log \|\hat{f}\|_1$

$\Delta_e f|_{H_+} \equiv 1 \quad\quad \Delta_e f|_{H_-} \equiv -1$



$\Delta_e f$ — $f|_{H_+}$ or $f|_{H_-}$

WTS $\|\widehat{f|_{H_+}}\|_1 \leq \frac{\|\hat{f}\|_1}{2}$ or $\|\widehat{f|_{H_-}}\| \leq \frac{\|\hat{f}\|_1}{2}$

$\log \|\hat{f}\|_1$ steps reduced to 1.

Open problems:

1. $C_{\oplus,\min}(f) = O(\log^c \|\hat{f}\|_1)$ or even $O(\log^c \|\hat{f}\|_1)$

($\Leftrightarrow$ $D_\oplus(f) = O(\log^c \|\hat{f}\|_1)$)

2. whether there exists $t \neq 0$ st $\|\widehat{\Delta_t f}\|_1 = O(\|\hat{f}\|_1^{2-\varepsilon})$ for some $\varepsilon > 0$

Thm $\exists$ explicit family of f such that $C_{\oplus,\min}(f) = \Omega(\log^{\log_2 3} \|\hat{f}\|_1)$ (parity kill number)