1. Motivation

2. Orientation

Definition 2.1. V finite-dimensional real vector space and $\beta = \{v_1, ..., v_k\}$, $\beta' = \{w_1, ..., x_k\}$ are two ordered basis. Then there exists a unique linear isomorphism $A : V \to V$ such that $\beta' = A\beta$. Then β and β' are equivalently oriented if det $A > 0$.

Definition 2.2. An orientation of V is an arbitrary decision to affix a positive sign to the elements of one equivalence class and a negative sign to the others.

Definition 2.3. An orientation for a manifold with boundary X, is a smooth assignment of orientations of the tangent spaces $T_x(X)$.

Definition 2.4. A manifold with boundary X is orientable if we can give an orientation on X.

Proposition 2.5. *A connected, orientable manifold with boundary admits exactly two orientations.*

Definition 2.6. (Product Orientation) If X and Y are oriented and one of them is boundaryless, then $X \times Y$ has a product orientation as follows. Given any $(x, y) \in X \times Y$,

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y). \tag{2.7}$$

And given α, β ordered bases for $T_x(X)$ and $T_y(Y)$, $(\alpha \times 0, 0 \times \beta)$ is an ordered basis for $T_{(x,y)}(X \times Y)$. Define the orientation by

$$sign(\alpha \times 0, 0 \times \beta) = sign(\alpha)sign(\beta). \tag{2.8}$$

Definition 2.9. (Boundary Orientation) An orientation on X induces a boundary orientation on ∂X. For each point $x \in \partial X$, dim $T_x(X)-$ dim $T_x(\partial X) = 1$. And thus there are exactly two unit vectors in $T_x(X)$ perpendicular to $T_x(\partial X)$. One inward and one outward. Let the outward vector by n_x, then define the orientation for $\{v_1, ...v_{k-1}\}$ an ordered basis for $T_x(\partial X)$ by sign$(n_x, v_1, ...v_{k-1})$ in $T_x(X)$.

Observation 2.10. *The sum of the orientation numbers at the boundary points of any compact oriented one-dimensional manifold with boundary is zero.*

Definition 2.11. (Preimage Orientation) Let $f : X \to Y$ be a smooth map and $f \pitchfork Z$ and $\partial f \pitchfork Z$, where X, Y, Z are all oriented. Then we can define the preimage orientation on the manifold with boundary $S = f^{-1}(Z)$. Let $N_x(S; X)$ be the orthogonal complement to $T_x(S)$ in $T_x(X)$. Then we have

$$N_x(S; X) \oplus T_x(S) = T_x(X), \tag{2.12}$$

so we need an orientation on $N_x(S; X)$ to get a direct sum orientation on $T_x(S)$. Also using the transversality condition we have

$$df_x T_x(X) + T_z(Z) = T_z(Y). \tag{2.13}$$

Since the $T_x(S)$ is the entire preimage of $T_z(Z)$, we have

$$df_x N_x(S; X) \oplus T_z(Z) = T_z(Y). \tag{2.14}$$

In fact we can replace $N_x(S;X)$ by any other subspace of $T_x(X)$ complementary to $T_x(S)$, and we have

$$df_xH \oplus T_z(Z) = T_z(Y) \tag{2.15}$$

$$H \oplus T_x(S) = T_x(X). \tag{2.16}$$

Proposition 2.17. $\partial[f^{-1}(Z)] = (-1)^{codimZ}(\partial f)^{-1}(Z)$.

3. Applications

Proposition 3.1. *The relation of being "equivalently oriented" is an equivalence relation on ordered bases.*

Proposition 3.2. *Suppose that V is the direct sum of V_1 and V_2. Then the direct sum orientation from $V_1 \oplus V_2$ equals $(-1)^{(dimV_1)(dimV_2)}$ times the orientation from $V_2 \oplus V_1$.*

Proof. Let $\alpha = \{v_1, ..., v_k\}$ and $\beta = \{w_1, ..., w_l\}$ be ordered bases for V_1 and V_2. Then the ordered basis in $V_1 \oplus V_2$ is $\{v_1, ..., v_k, w_1, ...w_l\}$ and the ordered basis in $V_2 \oplus V_1$ is $\{w_1, ..., w_l, v_1, ...v_k\}$. And it takes kl permutations to switch these two ordered bases. Then determinant is $(-1)^{kl}$. □

Proposition 3.3. *∂H^k can be both oriented by a boundary orientation of H^k and by the standard orientation of $\mathbb{R}^{k-1}$. Then the boundary orientation agrees with the standard orientation if and only if k is even.*

Proof. Let $\alpha = \{a_1, ..., a_{k-1}\}$ be an ordered basis for ∂H^k as $\mathbb{R}^{k-1}$. And let $A = (a_1, ..., a_{k-1})$ be $k-1$ by $k-1$ matrix. Then $\text{sign}(\alpha) = \det A$ in $\mathbb{R}^{k-1}$. On the other hand, the outward normal vector is $n = (-1, 0, ..., 0)$ and when consider in the boundary orientation $\text{sign}(\alpha) = \text{sign}(\{n, v1, .., v_{k-1}\}) = (-1)^k \det A$. □

Proposition 3.4. *Let X and Z be transversal submanifolds of Y, all three being oriented. Let $X \cap Z$ denote the intersection manifold with the orientation prescribed by the inclusion map $i : X \to Y$. Now suppose*

$$dimX + dimZ = dimY, \tag{3.5}$$

so $X \cap Z$ is zero dimensional. Then at any point $y \in X \cap Z$,

$$T_y(Y) \oplus T_y(Z) = T_y(Y). \tag{3.6}$$

The orientation number of y in $X \cap Z$ is 1 if the orientations of X and Z add up to the orientation of Y.

Proposition 3.7. *If dim $X+$ dim Z = dim Y and X, Z intersect transversally, then*

$$X \cap Z = (-1)^{(dimX)(dimZ)} Z \cap X. \tag{3.8}$$

Proof. For $y \in X \cap Z$, the orientation is prescribed by the inclusion map $i_1 : X \to Y$, and is determined by

$$T_y(X) \oplus T_y(Z) = T_y(Y). \tag{3.9}$$

And for $y' \in Z \cap X$, the orientation is prescribed by the inclusion map $i_2 : X \to Y$, and is determined by

$$T_y(Z) \oplus T_y(X = T_y(Y). \tag{3.10}$$

And by previous proposition, the orientation is differed by $(-1)^{(dimT_y(X))(dimT_y(Z))}$. $\square$

Proposition 3.11. *The definition of boundary orientation uses the outward unit normal n_x to ∂X at x. But the perpendicularity is unnecessary.*

Proof. Given any $h_x \in T_x(X)$ vector pointing outward, we have $h_x = cn_x + v$, where c is a positive real number and $v \in T_x(\partial X)$, then we have

$$sign(h_x, v_1, ..., v_k) = c^{k+1} sign(n_x, v_1, ..., v_k), \tag{3.12}$$

where $\{v_1, .., v_k\}$ is an ordered basis of $T_x(\partial X)$. Then the signs are the same since c is positive. $\square$

Proposition 3.13. *The orthogonality is not needed in defining preimage orientations. Specifically, if*

$$H \oplus T_x(S) = T_x(X), \tag{3.14}$$

then this equation, together with the condition

$$df_x H \oplus T_z(Z) = T_z(Y), \tag{3.15}$$

defines the same preimage orientation as the one defined by the orthogonal complement.

Proof. H can be decomposed into sum of some orthogonal N and some elements in $T_x(S)$. Then following the similar argument as previous exercise. $\square$