1. Further Discussion on Possible Quadratic Gap and Open Problems

Though we have some k graph properties with sensitivity $O(v^{k/2})$ for k odd, a k graph property with a quadratic gap for k odd is still unknown since the lower bound on block sensitivity is strictly less than $\Theta(v^k)$. However, since we just have an upper bound for sensitivity, it's possible that for some hypergraph $\mathcal{H}$, integer i and $t \in [0,1]$, we get a graph property that gives quadratic gap.

First we can analyze for which i and t, it's impossible to have a quadratic gap for this k graph property by assuming the trivial upper bound on block sensitivity, $bs(f) = O(v^k)$.

It's easy to see that $s^1 = \Theta(v^{k-i(1-t)})$ but the best lower bound we get for s^0 is

$$s^0(f) = \Omega(\frac{\binom{v}{v^t}}{\binom{v^t}{i}\binom{v-i}{k-i}}) = \Omega(v^{i(1-2t)}), \tag{1.1}$$

by embedding $\mathcal{H}$ inside a clique of same size $\Theta(v^t)$ and choosing cliques with less than i vertices in common. Since $s(f) = max\{s^0(f), s^1(f)\}$, for $i \leq (k-1)/2$, $s(f) = \Omega(v^{(k+1)/2})$ and quadratic gap is impossible. For $i \geq (k+1)/2$, if quadratic gap is possible, i.e, $s^0 = O(v^{k/2})$ and $s^1 = O(v^{k/2})$, we have

$$i(1-2t) \leq k/2 \leq i(1-t), \tag{1.2}$$

which gives

$$\frac{1}{2}(1 - \frac{k}{2i}) \leq t \leq 1 - \frac{k}{2i}. \tag{1.3}$$

Hence for $t > 1 - \frac{k}{2i}$ or $t < \frac{1}{2}(1 - \frac{k}{2i})$ or $i \leq (k-1)/2$, there's no possible quadratic gap.

There are several ways we can improve this criterion. First of all, if we can improve the upper bound on block sensitivity for some $\mathcal{H}$, we have a larger interval on t for fixed i, in which a quadratic gap is impossible. Since there is no isolated vertex in $\mathcal{H}$, at $x = 0$, $bs(f, 0) = O(v^{k-t})$. However, 0 may not the the point where $bf(f, x)$ achieves maximum.

Another way we can improve this criterion is by improving the lower bound on sensitivity. It seems that for most "nice" Boolean functions f, we have $s^0(f)s^1(f) = \Omega(n)$. Specifically, for k graph properties, we have $s^0(f)s^1(f) = \Omega(v^k)$. If we assume this conjecture to be true, there's possibly a quadratic gap for $\mathcal{H}, i$ and k only if $s^1(f) = s^0(f) = O(v^{k/2})$.

Finally, if we combine these two ways, suppose for some hypergraph $\mathcal{H}$,i and t, the upper bound for block sensitivity is strictly less than $O(v^k)$. And also, we assume the conjecture to be true. We know that $s(f) = \Omega(v^{k/2})$, which means a quadratic gap is impossible.