The Lefschetz Fixed Point Theorem

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Overview

- Homology and cohomology
- Intersection of cycles
- The Lefschetz Fixed Point Theorem
- A good p-adic cohomology for the affine line
- Monsky-Washnitzer cohomology

Homology

- Chain complex K is a sequence $\{C_n, \partial_n\}_{n\in\mathbb{Z}}$ of Abelian groups

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

  and boundary maps (homomorphisms) such that $\partial_n\partial_{n+1} = 0$.

- Since $\partial_n\partial_{n+1} = 0$ one has $\mathrm{Im}\,\partial_{n+1} \subset \mathrm{Ker}\,\partial_n$ and

$$H_n(K) := \mathrm{Ker}\,\partial_n/\mathrm{Im}\,\partial_{n+1}$$

  is the n-th homology group of K.

- Example: singular homology.

Singular Homology

- n-simplex: convex hull of $n+1$ points $x_0, \ldots, x_n$ not in $n-1$-dimensional subspace.
- Standard n-simplex σ_n: $x_0 = (1, 0, \ldots, 0), \ldots, x_n = (0, 0, \ldots, 1)$.
- A singular n-simplex of a topological space X is continuous function $\phi : \sigma_n \to X$.
- For each $0 \le i \le n$ we obtain a singular $n-1$-simplex

$$(\partial^{(i)}\phi)(t_0, \ldots, t_{n-1}) = \phi(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

- Boundary operator ∂ is given by

$$\partial_n = \partial^{(0)} - \partial^{(1)} + \cdots + (-1)^n \partial^{(n)}$$

SINGULAR HOMOLOGY

- Let $S_n(X)$ be free abelian group with basis singular n-simplices

$$S_n(X) = \{\sum_{\phi} n_\phi \cdot \phi \mid n_\phi \neq 0 \text{ finitely many } \}$$

- By linearity $\partial_n : S_n(X) \leftarrow S_{n-1}(X)$ and $\partial_n \circ \partial_{n+1} = 0$.
- Element $c \in S_n(X)$ is n-cycle if $\partial_n(c) = 0$.
- Element $d \in S_n(X)$ is n-boundary if $d = \partial(e)$ for $e \in S_{n+1}(X)$.
- n-th singular homology group

$$H_n(K) := \text{Ker } \partial_n / \text{Im } \partial_{n+1}$$

Singular Homology

Cohomology

- Cochain complex is a sequence $\{C^n, d_n\}_{n\in\mathbb{Z}}$ of Abelian groups

$$\cdots \xrightarrow{d_{n-2}} C_{n-1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \xrightarrow{d_{n+1}} \cdots$$

  and coboundary maps or differentials such that $d_n d_{n-1} = 0$.

- Since $d_n d_{n-1} = 0$ one has $\mathrm{Im}\ d_{n-1} \subset \mathrm{Ker}\, d_n$ and

$$H^n(K) := \mathrm{Ker}\, d_n / \mathrm{Im}\ d_{n-1}$$

  is the n-th cohomology group of K.

- Example: algebraic de Rham cohomology.

Algebraic de Rham Cohomology

- X smooth, affine variety over K of char 0 with coordinate ring

$$A := K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

- Module of Kähler differentials $\Omega^1_{A/K}$ generated by dg with $g \in A$

$$\Omega^1_{A/K} = (A\ dx_1 + \cdots + A\ dx_n)/(\sum_{i=1}^{m} A(\frac{\partial f_i}{\partial x_1}\ dx_1 + \cdots + \frac{\partial f_i}{\partial x_n}\ dx_n))\,.$$

- $\Omega^i_{A/K} = \bigwedge^i \Omega^1_{A/K}$ and $d_i : \Omega^i_{A/K} \to \Omega^{i+1}_{A/K}$ exterior diff.
- Since $d_{i+1} \circ d_i = 0$ we get the de Rham complex $\Omega_{A/K}$

$$0 \longrightarrow A \xrightarrow{d_0} \Omega^1_{A/K} \xrightarrow{d_1} \Omega^2_{A/K} \xrightarrow{d_2} \Omega^3_{A/K} \cdots$$

- i-th de Rham cohomology group of is defined as

$$H^i_{DR}(A/K) := \mathrm{Ker}\ d_i / \mathrm{Im}\ d_{i-1}$$

Intersection of Cycles

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Intersection of Cycles

- Let A and B two cycles that intersect tranversely at point p.
- The intersection number of A and B is

$$\#(A \cdot B) = \sum_{p \in A \cap B} \imath_p(A \cdot B)$$

- Intersection index $\imath_p(A \cdot B) \in \{-1, +1\}$ depends on orientation.
- $\#(A \cdot B)$ only depends on homology classes of A and B!
- General: intersection number defines pairing

$$H_k(M, \mathbb{Z}) \times H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}$$

- Poincaré: for any k-cycle A on M there is closed $(n-k)$-form φ_A

$$\#(A \cdot B) = \int_B \varphi_A$$

The Lefschetz Fixed Point Theorem

- Let M be compact oriented manifold of dimension n and $f : M \to M$ an endomorphism.

- The Lefschetz number of f is defined as

$$L(f) = \sum_{i=0}^{n} (-1)^i \mathrm{Trace}(f_* | H^i_{DR}(M)) \, .$$

- A point $p \in M$ is called a fixed point of f is

$$f(p) = p$$

- Question: what is $\#\{p \in M \mid f(p) = p\}$?

The Lefschetz Fixed Point Theorem

- Diagonal $\Delta \subset M \times M$ and graph $\Gamma_f = \{(p, f(p)) | p \in M\}$ of f.

$$\boxed{\text{fixed point} = \text{intersection of } \Delta \text{ and } \Gamma_f}$$

The Lefschetz Fixed Point Theorem

- If f has only nondegenerate fixed points then

$$\#(\Delta \cdot \Gamma_f)_{M\times M} = \sum_{f(p)=p} \imath_f(p)$$

- The Lefschetz Fixed Point Formula

$$\boxed{\sum_{f(p)=p} \imath_f(p) = L(f) = \sum_i (-1)^i \mathrm{Trace}(f_*|H^i_{DR}(M))}$$

- Proof:

$$\#(\Delta \cdot \Gamma_f)_{M\times M} = \int_{\Gamma_f} \varphi_\Delta$$

- φ_Δ Poincaré dual of homology class of diagonal.

The Lefschetz Fixed Point Theorem

- Corollary 1: $\#\{p \in M : f(p) = p\} \geq |L(f)|$.
- Corollary 2: If $L(f) \neq 0$, then f has a fixed point.
- Theorem: for analytic cycles V and W of compact complex manifold meeting transversally $\imath_p(V \cdot W) = +1$.
- Lefschetz Fixed Point Theorem: Let M be a compact complex analytic manifold and $f : M \to M$ an analytic map. Assume that f only has isolated nondegenerate fixed points then

$$\boxed{\#\{p \in M \mid f(p) = p\} = L(f) = \sum_i (-1)^i \mathrm{Trace}(f_* | H^i_{DR}(M))}$$

A p-adic Cohomology of the Affine Line

- Frobenius $\overline{F} : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p : x \mapsto x^p$ then $x \in \mathbb{F}_p$ iff $\overline{F}(x) = x$.
- Consider $\overline{C} : xy - 1 = 0$ with coordinate ring $\overline{A} = \mathbb{F}_p[x, 1/x]$, then

$$N_r = \#\overline{C}(\mathbb{F}_{p^r}) = \#\text{ fixed points of }\overline{F}^r = p^r - 1$$

- Construct de Rham cohomology in characteristic p?
  - Only possible to compute $N_r \pmod{p}$.
  - $\Omega^1(\overline{A}) := \overline{A}\,dx/(d\,\overline{A})$ is infinite dimensional.
  - $x^k\,dx$ with $k \equiv -1 \pmod{p}$ cannot be integrated.

p-ADIC NUMBERS

- p-adic norm $|\cdot|_p$ of $r \neq 0 \in \mathbb{Q}$ is

$$|r|_p = p^{-\rho}, \quad r = p^\rho u/v, \quad \rho, u, v \in \mathbb{Z}, \quad p \not| \, u, p \not| \, v.$$

- Field of p-adic numbers $\mathbb{Q}_p$ is completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$,

$$\sum_m^\infty a_i p^i, \quad a_i \in \{0, 1, \ldots, p-1\}, \quad m \in \mathbb{Z}.$$

- p-adic integers $\mathbb{Z}_p$ is the ring with $|\cdot|_p \leq 1$ or $m \geq 0$.

- Unique maximal ideal $M = \{x \in \mathbb{Q}_p \mid |x|_p < 1\} = p\mathbb{Z}_p$ and $\mathbb{Z}_p/M \cong \mathbb{F}_p$.

A p-adic Cohomology of the Affine Line

First attempt: lift situation to $\mathbb{Z}_p$ and try again?

- Consider two lifts to $\mathbb{Z}_p$

$$A_1 = \mathbb{Z}_p[x, 1/x] \quad \text{and} \quad A_2 = \mathbb{Z}_p[x, 1/(x(1+px))]$$

- A_1 and A_2 are not isomorphic; both x and $1+px$ invertible in A_2.
- $H^1_{DR}(A_1/\mathbb{Q}_p) = \langle \frac{dx}{x} \rangle$ and $H^1_{DR}(A_2/\mathbb{Q}_p) = \langle \frac{dx}{x}, \frac{dx}{1+px} \rangle$.
- Frobenius does not always lift:
  - Example: $\overline{A} = \mathbb{F}_3[x]/(x^2-2)$ and $A = \mathbb{Z}_3[x]/(x^2-2)$

A p-adic Cohomology of the Affine Line

Second attempt: use p-adic completion.

$$A_1^{\infty} \cong A_2^{\infty} \cong \{\sum_{i\in\mathbb{Z}} \alpha_i x^i \in \mathbb{Z}_p[[x, 1/x]] \mid \lim_{|i|\to+\infty} \alpha_i = 0\}$$

- However: $H^1_{DR}(A^{\infty}/\mathbb{Q}_p)$ is again infinite dimensional!
- $\sum_i p^i x^{p^{i-1}}$ is in A^{∞} but integral $\sum_i x^{p^i}$ is not.
- Convergence property lost in integration.

A p-adic Cohomology of the Affine Line

Third attempt: consider the dagger ring or weak completion

$$A^\dagger = \{\sum_{i\in\mathbb{Z}} \alpha_i x^i \in \mathbb{Z}_p[[x, 1/x]] \mid \exists \epsilon \in \mathbb{R}_{>0}, \delta \in \mathbb{R} : v_p(\alpha_i) \geq \epsilon|i| + \delta\}$$

- Note: $A_1^\dagger$ is isomorphic to $A_2^\dagger$, since $1 + px$ invertible in $A_1^\dagger$.

$$\frac{1}{1+px} = \sum_{i=0}^{\infty} (-1)^i p^i x^i$$

A p-adic Cohomology of the Affine Line

- Monsky-Washnitzer := de Rham cohomology of $A^\dagger \otimes \mathbb{Q}_p$
- $H^1(\overline{A}/\mathbb{Q}_p) = (A^\dagger \otimes \mathbb{Q}_p)dx/(d(A^\dagger \otimes \mathbb{Q}_p))$ and clearly for $k \neq -1$

$$x^k dx = d(\frac{x^{k+1}}{k+1})$$

- Conclusion: $H^1(\overline{A}/\mathbb{Q}_p)$ has basis $\frac{dx}{x}$
- Lifting Frobenius F to $A^\dagger$: infinitely many possibilities

$$F(x) \in x^p + pA^\dagger$$

- Examples: $F_1(x) = x^p$ or $F_2(x) = x^p + p$

A p-adic Cohomology of the Affine Line

- Action of F_1 on basis $\frac{dx}{x}$ is given by

$$F_{1*}\left(\frac{dx}{x}\right) = \frac{d(F_1(x))}{F_1(x)} = \frac{d(x^p)}{x^p} = p\frac{dx}{x}$$

- Action of F_2 on basis $\frac{dx}{x}$ is given by

$$F_{2*}\left(\frac{dx}{x}\right) = \frac{d(F_2(x))}{F_2(x)} = \frac{d(x^p + p)}{x^p + p} = \frac{px^{p-1}}{x^p + p}dx = \frac{p}{1 + px^{-p}}\frac{dx}{x}$$

- Power series expansion: $(1 + px^{-p})^{-1} = \sum_{i=0}^{\infty}(-1)^i p^i x^{-ip} \in A^\dagger$

$$F_{2*}\left(\frac{dx}{x}\right) = p\frac{dx}{x} + d\left(\sum_{i=1}^{\infty}\frac{(-1)^{i+1}p^{i-1}}{i}x^{-ip}\right)$$

A p-adic Cohomology of the Affine Line

- Action of F_1 and F_2 are equal on $H^1(\overline{A}/\mathbb{Q}_p)$!

$$F_*(\frac{dx}{x}) = p\frac{dx}{x} \Rightarrow F_*^{-1}\left(\frac{dx}{x}\right) = \frac{1}{p}\frac{dx}{x}$$

- Lefschetz Trace formula applied to $\overline{C}$ gives

$$\#\overline{C}(\mathbb{F}_{p^r}) = \text{Trace}\left((pF_*^{-1})^r|H^0(\overline{C}/\mathbb{Q}_p)\right) - \text{Trace}\left((pF_*^{-1})^r|H^1(\overline{C}/\mathbb{Q}_p)\right)$$

- Conclusion:

$$\boxed{\#\overline{C}(\mathbb{F}_{p^r}) = p^r - 1}$$

Monsky-Washnitzer cohomology

- $\overline{X}$ smooth affine variety over $\mathbb{F}_q$ with coordinate ring $\overline{A}$.
- Exists $A := \mathbb{Z}_q[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ with $A \otimes_{\mathbb{Z}_q} \mathbb{F}_q \cong \overline{A}$
- Dagger ring or weak completion $A^\dagger$ is defined

$$A^\dagger := \mathbb{Z}_q\langle x_1, \ldots, x_n\rangle^\dagger/(f_1, \ldots, f_m)$$

  with $\mathbb{Z}_q\langle x_1, \ldots, x_n\rangle^\dagger$ overconvergent power series

$$\left\{ \sum_I a_I x^I \in \mathbb{Z}_q[[x_1, \ldots, x_n]] \mid \liminf_{|I| \to \infty} \frac{v_p(\alpha_I)}{|I|} > 0 \right\}$$

- M-W cohomology is the de Rham cohomology of $A^\dagger \otimes \mathbb{Q}_q$.

Monsky-Washnitzer cohomology

- Definition only depends on $\overline{A}$ and not on choices made!
- Every morphism $\overline{G} : \overline{A} \rightarrow \overline{B}$ lifts to $G : A^{\dagger} \rightarrow B^{\dagger}$.
- Induced map on $H^i(\overline{A}/\mathbb{Q}_q) \rightarrow H^i(\overline{B}/\mathbb{Q}_q)$ only depends on $\overline{G}$.
- Cohomology groups $H^i(\overline{A}/\mathbb{Q}_q)$ are finite dimensional.
- Lefschetz trace formula: for $\overline{X}$ of dimension d

$$N_r = \sum_{i=0}^{d} (-1)^i \mathrm{Tr}\left((q^d F_*^{-1})^r | H^i(\overline{X}/\mathbb{Q}_q) \right)$$

- Let $\overline{C}$ be a projective, smooth curve of genus g over $\mathbb{F}_q$
  - S a set of m $\mathbb{F}_q$-points and $\overline{A}$ coordinate ring of $\overline{C} \setminus S$

$$\dim H^1(\overline{A}/\mathbb{Q}_q) = 2g + m - 1$$