1. Manifolds with Boundary

Theorem 1.1. *Let f be a smooth map of a manifold X with boundary onto a boundaryless manifold Y and suppose that both $f : X \to Y$ and $\partial f : \partial X \to Y$ are transversal with respect to a boundaryless submanifold Z. Then the preimage $f^{-1}(Z)$ is a manifold with boundary and the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.*

2. Transversality

Theorem 2.1. *(The Transversality Theorem) that $F : X \times S \to S$ is a smooth map of manifolds, where only X has boundary, and let Z be any boundaryless submanifold fo Y. If both F and ∂F are transversal to Z, then for almost every $sn \in S$, both f_s and ∂f_s are transversal to Z.*

Theorem 2.2. *(Transversality Homotopy Theorem)o For any smooth map $f : X \to Y$ and any boundaryless submainfold Z of the boundaryles manifold Y, there exists a smooth map $g : X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$.*

Theorem 2.3. *Suppose that Z is a closed submanifold of Y, both boundaryless, and C is a closed subset of X. Let $f : X \to Y$ be a smooth map with $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on C intersect ∂X. Then f can be extended to g which is transversal to Z.*

Corollary 2.4. *If, for $f : X \to Y$, the boundary map is transversal to Z, then there exists a map $g : X \to Y$ homotopic to f such that $\partial g = \partial f$ and $g \pitchfork Z$.*

3. Applications

Theorem 3.1. *(General Position Lemma) Let X and Y be submanifolds of $\mathbb{R}^N$. Then for almost every $a \in \mathbb{R}^N$ the translation $X + a$ intersects Y transversally.*

Proof. Define $F : X \times \mathbb{R}^N \to \mathbb{R}^N$ such that $F(x, a) = i(x) + a$, where i is the inclusion map of X. Then for any fixed $x \in X$, $F(x, a) = x + a$ just translate $\mathbb{R}^N$ by x and thus F and ∂F are submersions. Then F and ∂F are transversal to Z. By the Transversality Theorem, for almost every $a \in \mathbb{R}^N$, $f_a(x) = x + a$ is transversal to Z. □

Proposition 3.2. *Suppose that X is a submanifold of $\mathbb{R}^N$. Then almost every vector space V of any fixed dimension l in $\mathbb{R}^N$ intersects X transversally.*

Proof. Let $S \subset (\mathbb{R}^N)^l$ be the set of all linearly independent l-tuples of vectors in $\mathbb{R}^N$. S is open in $\mathbb{R}^N l$, which implies S is a manifold. Define a map $F : \mathbb{R}^l \times S \to \mathbb{R}^N$ such that

$$F[(t_1, ..., t_l), v_1, ..., v_l] = t_1 v_1 + ... + t_l v_l. \tag{3.3}$$

For fixed $(t_1, ...t_l)$, F is just a linear combination of $v_1, ..., v_l$ and thus F is a submersion. Then F and ∂F intersect X transversally. By the Transversality Theorem, for almost every $s = (v_1, ..., v_l) \in S$, $f_s(t_1, ...t_l) = F[(t_1, ...t_l), s]$ is transversal to X, which means

$$Image(df_s)_v + T_x(X) = T_x(\mathbb{R}^n), \tag{3.4}$$

where $f_s(t) = x$. By definition of f_s, $Image(df_s)_x$ is just $Splan\{v_1, ..., v_l\}$, which is $T_t(V)$, where V is the vector space spaned by $v_1, ..., v_l$. Hence for almost every vector space V of fixed dimension l, we have

$$T_x(V) + T_x(X) = T_x(\mathbb{R}^N). \tag{3.5}$$

V intersects X transversally. □

4. Intersection Theory Mod 2

Theorem 4.1. *If $f_0, f_1 : X \to Y$ are homotopic and both transversal to Z, then $I_2(f_0, Z) = I_2(f_1, Z)$.*

Corollary 4.2. *If $g_0, g_1 : X \to Y$ are arbitrary homotopic maps, then we have $I_2(g_0, Z) = I_2(g_1, Z)$.*

Theorem 4.3. *(Boundary Theorem) Suppose that X is the boundary of some compact manifold W and $g : X \to Y$ is a smooth map. If g may be extended to all of W, then $I_2(g, Z) = 0$ for any closed submanifold Z in Y of complementary dimension.*

Theorem 4.4. *If $f : X \to Y$ is a smooth map of a compact manifold X into a connected manifold Y and dim X = dim Y, then $I_2(f, \{y\})$ is the same for all points $y \in Y$. This common value is called the mod 2 degree of $deg_2(f)$.*

Theorem 4.5. *Homotopic maps have the same mod 2 degree.*

Theorem 4.6. *If $X = \partial W$ and $f : X \to Y$ may be extended to all of W, then $deg_2(f) = 0$.*

Proposition 4.7. *If the mod 2 degree of $p/|p| : \partial W \to S^1$ is nonzero, then the function has a zero inside W.*

5. Applications

Proposition 5.1. *Let $f : X \to Y$ and $g : Y \to Z$ be a sequence of smooth maps of manifolds , with X compact. Assume that g is transversal to a closed submanifold W of Z, then*

$$I_2(f, g^{-1}(W)) = I_2(g \circ f, W). \tag{5.2}$$

Proof. $I_2(f, g^{-1}(W)) = \text{card } f^{-1}(g^{-1}(W)) = \text{card } (g \circ f)^{-1}(W) = I_2(g \circ f, W)$. Furthermore, if $I_2(f, g^{-1}(W))$ is defined, we have $\dim X + \dim g^{-1}(W) = \dim Y$. Then $\dim X + \dim W = \dim Z$, i.e, $I_2(g \circ f, W)$ is defined. □

Proposition 5.3. *If $f : X \to Y$ is homotopic to a constant map, then $I_2(f, Z) = 0$ for all complementary dimensional closed Z in Y, except perhaps if $\dim X = 0$.*

Proof. If $\dim X > 0$, then $\dim Z < \dim Y$. Then f is homotopic to the constant map $g(x) = y$, where y is not i Z. Then we know

$$I_2(f, Z) = I_2(g, Z) = |g^{-1}(Z)| = 0. \tag{5.4}$$

□