1. A Kunneth Formula

Definition 1.1. The cross product is the map

$$H^*(X;R) \times H^*(Y;R) \to H^*(X \times Y;R) \tag{1.2}$$

given by $a \times b = p_1^*(a) \smile p_2^*(b)$ where p_1 and p_2 are the projections of $X \times Y$ onto X and Y.

Since cup product is distributive, the cross product is bilinear.

Theorem 1.3. *The cross product $H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y;R)$ is a finitely generated R-module for all k.*

Proposition 1.4. *If a natural transformation between unreduced cohomology theories on the category of CW pairs is an isomorphism when the CW pair is (point,∅), then it is an isomorphism for all CW pairs.*

Theorem 1.5. *For CW pairs (X, A) and (Y, B) the cross product homomorphism $H^*(X, A;R) \otimes_R H^*(Y, B;R) \to H^*(X \times Y, A \times B;R)$ is an isomorphism of rings if $H^k(Y, B;R)$ is a finitely generated free R-module for each k.*

2. Exercise

Proposition 2.1. $S^2 \vee S^1 \vee S^1 = X$.

Proof. By Mayer-Vietoris, $H^n(S^2 \vee S^1 \vee S^1) = H^n(S^2) \oplus H^n(S^1) \oplus H^n(S^1)$, for $n > 0$. Then we have $H^0(X) = \mathbb{Z}$, $H^1(X) = \mathbb{Z}^2$ and $H^2(X) = \mathbb{Z}$. And $H^1(X) \simeq Hom(\mathbb{Z}^2, \mathbb{Z})$ with basis $\{\alpha, \beta\}$. We have α represented by cocycle ϕ and β represented by cocycle ψ. Where *phi* and *psi* take 1 on each simplex represented by two S^1 and 0 else where. Then we look at their cup product $\phi \smile \psi$. Given any 2-simplex c, we have $(\phi \smile \psi)(c) = 0$ and thus $\phi \smile \psi = 0$ is not a generator of $H^2(X)$. □