Proposition 0.1. *Let $k, i \in \mathbb{N}$ and $1 \leq i \leq k/2$, there exists k graph property on v vertices that has sensitivity $O(v^{k-i})$ and block sensitivity $\Omega(v^k)$.*

We prove the proposition by first proving two lemmas.

Lemma 0.2. *In a k-uniform hypergraph on v vertices, there are $\Omega(v^k)$ disjoint $K_{k+1}^{(k)}$.*

Proof. there are $\binom{v}{k+1}$ such cliques and for every clique chosen we eliminate any other cliques with more than $k-1$ points in common. In this way we get at least

$$\frac{\binom{v}{k+1}}{\binom{k+1}{k}(v-k)} = c * v^k \tag{0.3}$$

many cliques and any two of them have no more than $k-1$ points in common, which guarantees that they are disjoint cliques. □

We also need a special case of the Ray-Chaudhuri-Wilson's Theomrem.

Lemma 0.4. *Given $\mathcal{F}$ a family of subsets of $[n]$ such that any element of $\mathcal{F}$ has size $s \geq i$ and the intersection of any two elements has size at most $i-1$. Then we have*

$$|\mathcal{F}| \leq \binom{n}{i}. \tag{0.5}$$

Proof. Since the intersection of any element of $mathcalF$ has size at most $i-1$, any subset of $[n]$ with size i is contained in at most one element of $\mathcal{F}$. And any element of $\mathcal{F}$ contains at least one i subset of $[n]$, we have

$$|\mathcal{F}| \leq \binom{n}{i}. \tag{0.6}$$

□

With these two lemmas we can define the desired k graph property and analyze its block sensitivity and sensitivity.

Proof. Define f be the graph property that there exists a $K_{k+1}^{(k)}$ inside the graph such that $|K_{k+1}^{(k)} \cap E| \leq i-1$ for any edge E lies not entirely inside the clique. We claim that this is the desired k-graph property.

First we calculate the block sensitivity. Consider the empty graph, by the first lemma, there are $\Omega(v^k)$ disjoint cliques and each of them is a sensitive block. Hence we have

$$bs(f) = \Omega(v^k). \tag{0.7}$$

Then we want to show that the sensitivity of f is $O(v^{k-i})$. We calculate the sensitivity by looking at $s^0(f)$ and $s^1(f)$ separately

When $f = 1$, there is a desired $K_{k+1}^{(k)}$ clique inside the graph. To change the value of f, we need to either remove an edge from $K_{k+1}^{(k)}$ or add an edge with more than $i-1$ common vertices with the clique. We have

$$s^1(f) \leq \binom{k+1}{2} + \binom{k+1}{i}\binom{v-i}{k-i} \leq C * v^{k-i}, \tag{0.8}$$

for some constant C.

When $f = 0$, there doesn't exist such $K_{k+1}^{(k)}$ in the graph. We call a $k+1$-tuple

$\{v_1, ..., v_{k+1}\}$ sensitive if adding or removing an edge from the graph will make $\{v_1, ..., v_{k+1}\}$ the vertices of a desired clique. Any sensitive edge is associated with a sensitive tuple by this definition. Also, we can show that there is precisely 1 sensitive edge associated with each sensitive tuple since if there are two sensitive edges associated with a $k+1$-tuple, we can't construct a desired clique by just removing or adding just one edge. Thus we have a injection from the set of sensitive edges into the set of sensitive tuples, and let $\mathcal{F}$ denote the set of all sensitive tuples, we have

$$s^0(f) \leq |\mathcal{F}|. \tag{0.9}$$

Given two sensitive tuples, if they have more than $i-1$ vertices in common, without loss of generality, assume that $\{v_1, ..., v_i\}$ are vertices in common. If both of them can form a desired clique by adding an edge to the graph, there are no less than 1 edge through these i points and adding any edge will not remove these edges, which contradicts the fact that these two tuples can form a desired clique by adding an edge. If there exists one tuple can form a desired clique by removing an edge, there exists no less than 2 edges through $\{v_1, ..., v_i\}$ in one of the tuples, and thus another tuple can't form a desired clique by either removing or adding exactly 1 edge, which contradicts the fact that the tuple is sensitive. Then given any two sensitive tuples, they have at most $k-i-1$ common vertices.

Then $\mathcal{F}$ is a family of subsets of $[v]$ such that any element has size $k+1$ and any two elements have at most $i-1$ intersections. By our second lemma,

$$|\mathcal{F}| \leq \binom{v}{i} \leq C' * v^i, \tag{0.10}$$

for some constant C'.

From above $s^0(f) = O(v^{k-i})$ and $s^1(f) = O(v^i)$ we conclude that $s(f) = O(v^{k-i})$ since $i \leq k/2$. Hence, f is a k-graph property with sensitivity v^{k-i} and block sensitivity v^k. □

Corollary 0.11. *When k is even, there exists k graph property on v vertices that has sensitivity $O(v^{k/2})$ and block sensitivity $\Omega(v^k)$. When k is odd, there exists k graph property on v vertices that has sensitivity $O(v^{(k+1)/2})$ and block sensitivity $\Omega(v^k)$.*

Proof. When k is even, choose $i = k/2$ and when k is odd, choose $i = (k-1)/2$ in above proposition. □

Remark 0.12. Note that we don't actually require the "isolated" graph to be a clique, i.e. $K_{k+1}^{(k)}$ in our analysis of the block sensitivity and sensitivity. So we can replace $K_{k+1}^{(k)}$ by any k hypergraph $\mathcal{H}$ on $k+1$ vertices, such that given any i vertices of $\mathcal{H}$, there are at least two edges through these i vertices. Then we can construct many k graph properties with the desired lower bound on block sensitivity and upper bound on sensitivity in a similar way.