1. Oriented Intersection Number

The following will be assumed in this section: X, Y, Z are boundaryless manifolds, X is compact, Z is a closed submanifold of Y and dim $X+$ dim $Z =$ dim Y.

Definition 1.1. If $f : X \to Y$ is transversal to Z, then $f^{-1}(Z)$ is a finite number of points, each with orientation number 1 or -1 by the preimage orientation. Define the *intersection number* $I(f, Z)$ to be the sum of these orientation numbers.

Given any point x, such that $f(x) = z \in Z$, we have

$$df_x T_x(X) \oplus T_z(Z) = T_z(Z) \tag{1.2}$$

by our assumptions. Then the orientation number at x is 1 if the orientation on $df_x T_x(X) \oplus T_z(Z)$ is the same as the prescribed orientation on $T_z(Y)$, and -1 otherwise.

Proposition 1.3. *If $X = \partial W$ and $f : X \to Y$ extends to W, then $I(f, Z) = 0$.*

Proof. Suppose f extends to F, we may assume F transversal to Z by the Extension Theorem. And thus $f^{-1}(Z) = \partial F^{-1}(Z)$. Since $F^{-1}(Z)$ is an one-manifold with boundary, $I(f, Z) = 0$. □

Proposition 1.4. *In particular, homotopic maps always have the same intersection number.*

Then we can define the intersection number for any arbitrary function.

Definition 1.5. Given any $g : X \to Y$, pick f such that f homotopic to g and f transversal to Z. Define intersection number $I(g, Z) = I(f, Z)$.

By the previous proposition, the intersection number is well defined.

Definition 1.6. When Y is connected and has the same dimension as X, we define the degree of an arbitrary smooth map $f : X \to Y$ to be the intersection number $I(f, \{y\})$.

Proposition 1.7. *Suppose that $f : X \to Y$ is a smooth map of compact oriented manifolds having the same dimension and that $X = \partial W$. If f can be extended to all of W, then deg$(f) = 0$.*

Proposition 1.8. *Let W be a smooth compact region in $\mathbb{C}$ whose boundary contains no zeros of the polynomial p. Then the total number of zeros of p inside W counting multiplicities is the degree of the map $p/|p| : \partial W \to S^1$.*

Lemma 1.9. *Let U and W be subspaces of the vector space V. Then $U \oplus W = V$ if and only if $U \times W \oplus \Delta = V \times V$. Assume also, that U and W are oriented, and give V the direct sum orientation. Now assign Δ the orientation carried from V by the natural isomorphism $V \to \Delta$. Then the product orientation on $V \times V$ agrees with the direct sum orientation form $U \times W \oplus \Delta$ if and only if W is even dimension.*

Proposition 1.10. *$f \pitchfork g$ if and only if $f \times g \pitchfork \Delta$, and then*

$$I(f, g) = (-1)^{dim Z} I(f \times g, \Delta). \tag{1.11}$$

Definition 1.12. For arbitrary maps $f : X \to Y$, $g : Z \to Y$, we define $I(f,g) = (-1)^{dimZ} I(f \times g, \Delta)$.

Proposition 1.13. *If f_0 and g_0 are respectively homotopic to f_1 and g_1, then $I(f_0, g_0) = I(f_1, g_1)$.*

Corollary 1.14. *If Z is a submanifold of Y and $i : Z \to Y$ is its inclusion map, then $I(f,i) = I(f,Z)$ for any map $f : X \to Y$.*

Corollary 1.15. *If dim X = dim Y and Y is connected, then $I(f, \{y\})$ is the same for every $y \in Y$. Thus deg(f) is well defined.*

Proposition 1.16. $I(f,g) = (-1)^{(dimX)(dimZ)} I(f,g)$.

2. Exercise

Proposition 2.1. *Suppose that $f : X \to Y$ is a diffeomorphism of compact connected manifolds. Then deg(f) = 1 if f preserves orientation, and -1 otherwise.*

Proof. f diffeomorphism, then $\deg(f) = I(f, \{y\}) = sign(f^{-1}(y))$. If f orientation preserving, we have $sign(f^{-1}(y)) = 1$ and -1 otherwise. □

Proposition 2.2. *The antipodal map is homotopic to the identity if and only if k is odd.*

Proof. Degree of antipodal map is $(-1)^{k+1}$. If antipodal map is homotopic to the identity, then degree= 1, which implies that k is odd. When k is odd, we can find a homopoty between antipodal map and identity. □

Proposition 2.3. *Suppose that $f : X \to Y$ and $g : Y \to Z$, then we have $deg(g \circ f) = deg(f)deg(g)$.*

Proof. Given any $z \in Z$, we have $deg(g \circ f) = I(g \circ f, \{z\}) = \sum_{(g \circ f)(x) = z} sign(x) = \sum_{y:g(y)=z} \sum_{x:f(x)=y} sign(x) = \sum_y deg(g) sign(y) = deg(f)deg(g)$. □

Proposition 2.4. *Assume that $X \pitchfork Z$ both compact oriented and then*

$$I(X,Z) = (-1)^{(dimX)(dimZ)} I(Z,X). \tag{2.5}$$

Proof. Since $X \pitchfork Z$, we have

$$di_x T_x(X) \oplus T_z(Z) = T_z(Y), \tag{2.6}$$

where i is the inclusion map $X \to Z$. Similarly, we have

$$di'_z T_z(Z) \oplus T_x(X) = T_x(Y), \tag{2.7}$$

where i' is the inclusion map $Z \to X$. And it takes $(dimX)(dimZ)$ transposition to switch the basis. □

Proposition 2.8. *The map $S^1 \to S^1$ given by $z \to \bar{z}^m$ has degree $-m$.*

Proof. Define $f : S^1 \to S^1$ and $g : S^1 \to S^1$ such that $f(z) = \bar{z}$, $g(z) = z^m$. We have $deg(f) = -1$ and $deg(g) = m$. Then degree of the map is $deg(f)deg(g) = -m$. □