THE LEFSCHETZ FIXED POINT THEOREM

ISABEL VOGT

1. Introduction

The Lefschetz Fixed Point Theorem generalizes a collection of fixed point theorems for different topological spaces, including maps on the n-sphere and the n-disk. Although the theorem is easily written in terms of compact manifolds, in this paper we will work entirely with topological spaces that are simplicial complexes or retracts of simplicial complexes. After developing the fundamentals of simplicial approximation, we will present a proof of the Lefschetz fixed point theorem and apply the theorem to maps on several topological spaces.

2. Simplicial Approximation

The Classical Lefschetz fixed point theorem is formulated for spaces X which are retracts of finite simplicial complexes K. The machinery of simplicial complexes forms the basis of the proof of the Lefschetz Fixed Point Theorem. As such, we develop here the key theorem that will function in the proof of the Lefschetz Fixed Point Theorem: the Simplicial Approximation Theorem. The Simplicial Approximation Theorem illustrates one of the "nice" features of simplicial complexes that makes their manipulation easier. In particular, any continuous map between simplicial complexes is homotopic to a map which is linear on the simplices of a barycentric subdivision of the domain complex. The concept of a map of simplicial complexes that is linear on the domain complex is summarized in the definition of a simplicial map [1].

Definition If K and L are simplicial complexes, then a map $f : K \to L$ is said to be *simplicial* if it sends the simplices of K to simplices in L by a linear map that takes the vertices of K to vertices in L.

There are two important points in this definition. First, a simplicial map takes every simplex in K into the span of a simplex in L. Note that this implies that the map takes vertices into vertices. The second point is that this map is linear when extended to the whole of each simplex of K. Using the realization of an n-simplex as the convex hull in $\mathbb{R}^{n+1}$ of the $n+1$ standard basis vectors, any point x on the simplex σ can be described as

$$x = \sum_{i}^{n+1} t_i e_i$$

with the requirement that $\sum t_i = 1$. It is then easy to see the generalization that, using barycentric coordinate, any point x in a simplex σ of K can be written as the sum

$$x = \sum_{i}^{n+1} t_i v_i$$

Thus, the image of $f : K \to L$ on the vertices $\{v_i\}$ uniquely determines the image of f on any point x in simplex, as we merely extend by linearity:

$$f : \sum_{i}^{n+1} t_i v_i \longmapsto \sum_{i}^{n+1} t_i f(v_i)$$

As alluded to above, the simplicial approximation theorem is the main tool which will be used in the proof of the Lefschetz Fixed Point Theorem.

Theorem 2.1 (Simplicial Approximation Theorem [1])**.** *If K is a finite simplicial complex and L is an arbitrary simplicial complex, then any map $f : K \to L$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K.*

In essence, this theorem states that any continuous map between the underlying topological spaces of a finite simplicial complex K and a simplicial complex L can be approximated by a homotopic map that is simplicial on some level of the barycentric subdivision of K.

In the further discussion we will use the definition of the star of a simplex σ ($\mathrm{St}\,\sigma$).

Definition The *star* of a simplex σ in a simplicial complex X is defined to be the subcomplex of X which is the union of all simplicies in X that contain σ.

Now, we will use the following direct corollary of the simplicial approximation theorem.

Corollary 2.2. *For $g : K' \to L$ the simplicial approximation homotopic to $f : K \to L$, for all $\sigma \in K'$, $f(\sigma) \subseteq St(g(\sigma))$.*

Note that a simplicial approximation g to f is often cited [2] as a continuous simplicial map such that for all vertices v in K,

$$f(\mathrm{St}\,(v)) \subset \mathrm{St}\,(g(v))$$

from which this corollary follows immediately.

Using this theorem we can proceed to the formulation and proof of the Lefschetz Fixed Point Theorem, which relies heavily on this result.

3. The Lefschetz Fixed Point Theorem

The Lefschetz fixed point theorem determines when there exist fixed points of a map on a finite simplicial complex using a characteristic of the map known as the Lefschetz number. For a map $f : X \to X$, where X is a space whose homology groups are finitely generated and vanish above some dimension, the Lefschetz number $\tau(f)$ is defined as follows:

$$\tau(f) = \sum_{n} (-1)^n \mathrm{tr}(f_* : H_n(X) \to H_n(X))$$

Here, as $H_n(X)$ need not be free abelian, the definition of the trace comes from the induced map on the torsion-free part of $H_n(X)$. To be explicit, the group homomorphism $f_* : H_n(X) \to H_n(X)$ induces a homomorphism $\bar{f}_* : H_n(X)/\text{torsion} \to H_n(X)/\text{torsion}$; we define $\mathrm{tr}(f_*) = \mathrm{tr}(\bar{f}_*)$.

The proof of the Lefschetz fixed point theorem will require the following purely algebraic lemma.

Lemma 3.1. *For A, B, and C finitely generated abelian groups in a short exact sequence, and $\alpha : A \to A$, $\beta : B \to B$, and $\gamma : C \to C$ endomorphisms that make the following diagram commute for the rows exact,*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

the trace is additive, ie. $tr(\beta) = tr(\alpha) + tr(\gamma)$.

Proof. We begin by reducing to the case that A, B, and C are free abelian. Using the factorization of finitely generated abelian groups, we write

$$0 \to A_{\text{free}} \oplus A_{\text{tor}} \to B_{\text{free}} \oplus B_{\text{tor}} \to C_{\text{free}} \oplus C_{\text{tor}} \to 0$$

where A_{free} maps to B_{free} and similarly with the torsion by exactness. As the trace of a finitely generated abelian group is defined to be the trace of the transformation on the free part, we may begin by quotienting B and A by their torsion subgroups. By the above comment that A_{free} maps to B_{free} and A_{tor} maps to B_{tor}, this preserves exactness at A. Similarly, we can remove the torsion from C by quotienting by the torsion subgroup, which preserves exactness at C by a similar reasoning. However, at this point, the sequence may not be exact at $B' = B/B_{\text{tor}}$, ie. there may exist elements of B/B_{tor} mapped to 0 that are not in the image of A/A_{tor}. To resolve this we merely replace A/A_{tor} by the larger group $A' \subset B'$ with $A/A_{\text{tor}} \subset A'$ of finite index. The altered endomorphisms α', β', and γ' have the same trace as the maps from which they derive by the definition of the trace of a finitely generated abelian group and the fact that α is related to α' by a change of basis, which preserves the trace.

Theorem 3.2 (Lefschetz Fixed Point Theorem)**.** *If X is a finite simplicial complex, or the retract of some finite simplicial complex, and $f : X \to X$ is a map with $\tau(f) \neq 0$, then f has a fixed point.*

We will prove this theorem by contraposition; that is, if $f : X \to X$ has no fixed points, then $\tau(f) = 0$. In brief, the proof goes as follows: as X is compact and f has no fixed points, it is possible to barycentrically subdivide X iteratively until every simplex is mapped to a simplex disjoint from the original. At this point we will be able to use the above lemma to conclude that the Lefschetz number of the map must be 0.

Proof. First, we can reduce to the case of a X a finite simplicial complex as follows. Assume that X is the retract of some finite simplicial complex K, ie. there exists a map $r : K \to X$ which is a retraction of K onto X such that $r|_X = \mathbb{1}_X$. Now, for a map $f : X \to X$, we can form the composition $fr : K \to X \subset K$. The group $H_n(K)$ splits into a direct sum as

$$H_n(K) = H_n(X) \oplus H_n(K, X)$$

Thus the map induced by the retraction on homology $r_* : H_n(K) \to H_n(X)$ is projection onto the first factor of the direct sum. Thus $\text{tr}(f_*) = \text{tr}(f_* r_*)$, and $\tau(f) = \tau(fr)$. In this way, K has fixed points if and only X has fixed points, and we may work simply with simplicial complexes.

Assume that for X a finite simplicial complex, $f : X \to X$ has no fixed points. Now, we will show that there exists a barycentric subdivision L of X and a further subdivision K of L such that there exists a simplicial map $g : K \to L$ homotopic to f with $g(\sigma) \cap \sigma = \emptyset$ for all $\sigma \in K$. This follows directly from the simplicial approximation theorem. Given a metric d on X, as f has no fixed points, $d(x, f(x)) > 0$ for all $x \in X$. By the compactness of X, we can choose a uniform $\epsilon > 0$ such that for all $x \in X$, $d(x, f(x)) > \epsilon$. Now, through iterative barycentric subdivision of X we form the complex L such that for all σ in L, $\mathrm{diam}(\sigma) < \epsilon/2$.

The corollary 2.2 to the simplicial approximation theorem says that there exists a barycentric subdivision K of the complex L, and a simplicial map $g : K \to L$ such that for all $\sigma \in K$, $f(\sigma) \subseteq \mathrm{St}(g(\sigma))$. For any $x \in \sigma$, and for any other point $y \in \sigma$, $d(x, y) < \epsilon/2$ as they lie in the same simplex $\sigma \in K$ of diameter at most $\epsilon/2$ as K is a subdivision of L. However, by the corollary above, $f(x) \in \mathrm{St}(g(\sigma))$, thus $f(x)$ is at most one simplex removed from the simplex $g(\sigma)$, and we can conclude that $d(f(x), g(\sigma)) < \epsilon/2$. Now we note that for any $x \in K$, the lack of fixed points of f translates into the requirement $d(x, f(x)) > \epsilon$. By an ϵ comparison, we can thus conclude that $g(\sigma) \cap \sigma = \emptyset$.

As f and g are homotopic maps, they are equivalent on homology, and thus $\tau(f) = \tau(g)$. By the simplicial property of g, it takes simplices of K into simplices of L. Thus it also takes the n-skeleton K^n of K into the n-skeleton L^n of L. Furthermore, as K is a subdivision of L, $L^n \subset K^n$ and, invoking g, $g(K^n) \subset K^n$. Thus, as g takes the n-skeleton of K into itself for all n, it induces a chain map from the cellular chain complex $\{H_n(K^n, K^{n-1})\}$ to itself as shown below.

$$
\begin{tikzcd}
\cdots \arrow[r] & H_{n-1}(K^{n-1}, K^{n-2}) \arrow[r] \arrow[d, "g_*"] & H_n(K^n, K^{n-1}) \arrow[r] \arrow[d, "g_*"] & H_{n+1}(K^{n+1}, K^n) \arrow[r] \arrow[d, "g_*"] & \cdots \\
\cdots \arrow[r] & H_{n-1}(K^{n-1}, K^{n-2}) \arrow[r, "d_{n-1}"] & H_n(K^n, K^{n-1}) \arrow[r, "d_n"] & H_{n+1}(K^{n+1}, K^n) \arrow[r] & \cdots
\end{tikzcd}
$$

Let $C_n = H_n(K^n, K^{n-1})$, and $\ker(d_n) = Z_n$ and $\mathrm{Im}(d_{n+1}) = B_n$. Then we have two different short exact sequences for each n:

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n \longrightarrow 0 \tag{2}$$

The map g_* induces endomorphisms on each of the groups in this short exact sequence. Let g_{B_n}, g_{Z_n}, and g_{C_n} be the endomorphisms on B_n, Z_n, and C_n respectively. Invoking lemma 3.1 for both (1) and (2):

$$\mathrm{tr}(g_{C_n}) = \mathrm{tr}(g_{Z_n}) + \mathrm{tr}(g_{B_{n-1}})$$
$$\mathrm{tr}(g_{Z_n}) = \mathrm{tr}(g_{B_n}) + \mathrm{tr}(g_{H_n})$$

We may substitute the second equation into the first to arrive at

$$\mathrm{tr}(g_{C_n}) = \mathrm{tr}(g_{B_n}) + \mathrm{tr}(g_{H_n}) + \mathrm{tr}(g_{B_{n-1}})$$

Multiplying by $(-1)^n$ and summing over all n, the terms involving B cancel in pairs as for $n < 0$ and $n > \dim(X)$, $tr(g_{B_n}) = 0$, so the non-overlapping end terms are trvial. What remains is the sum

$$\sum_n (-1)^n g_{C_n} = \sum_n (-1)^n g_{H_n}$$

Which, of course, implies that we can compute $\tau(g)$ using the following sum

$$\tau(g) = \sum_n (-1)^n \mathrm{tr}\left(g_* : H_n(K^n, K^{n-1}) \to H_n(K^n, K^{n-1})\right)$$

Finally, considering the map $g_* : H_n(K^n, K^{n-1}) \to H_n(K^n, K^{n-1})$, as $g(\sigma) \cap \sigma = \emptyset$, the vertices of each n-simplex are not brought into themselves, and thus the matrix representation of g_* has zeros down the diagonal. The $\mathrm{tr}(g_* : H_n(K^n, K^{n-1}) \to H_n(K^n, K^{n-1})) = 0$ for all n. Plugging this into the formula for the Lefschetz number, we derive the result that $\tau(g) = \tau(f) = 0$. ∎

Note that by the universal coefficient theorem, this entire analysis could have been expressed in terms of cohomology instead of homology. The standard formulation of the Lefschetz number for a map $f : X \to X$ is

$$\tau(f; F) = \sum_n \mathrm{tr}(f^* : H^n(X; F) \to H^n(X; F))$$

where F is any field, usually taken to be $\mathbb{Q}$. This is often a preferable formulation as the added structure of the cup product that makes the cohomology groups into a graded ring elucidates many calculations.

4. Some Examples

4.1. **$\mathbb{Q}$-acyclic spaces.** A connected simplicial complex is a $\mathbb{Q}$-acyclic space if $H_k(Y; \mathbb{Q}) =$ for all $k \neq 0$ [3]. Then for any map $f : Y \to Y$, the Lefschetz number $\tau(f)$ is given by the trace of the induced map on homology

$$f^* : H_0(Y; \mathbb{Q}) = \mathbb{Q} \to H_0(Y; \mathbb{Q}) = \mathbb{Q}$$

which is simply the identity function and thus has nonzero trace. Thus every map on a $\mathbb{Q}$-acyclic space has at least one fixed point.

Hence, any topological space Y, satisfying the conditions of the simplicial approximation theorem, with the same homology groups as a point, modulo torsion, will always have a fixed point. That is $H_0(Y) = \mathbb{Z}$, and $H_n(Y)/\text{torsion} = 0$ for all $n \neq 0$. This holds for $\mathbb{RP}^{2k}$, for $k \in \mathbb{Z}$. Thus every map on $\mathbb{RP}^n$ has a fixed point for n even.

An even clearer application is the proof that the Lefschetz fixed point theorem implies the Brouwer fixed point theorem. The Lefschetz fixed point theorem is a considerable generalization of the Brouwer fixed point theorem, which is only concerned with maps on the n-disk.

Theorem 4.1 (Brouwer Fixed Point Theorem)**.** *For every map $f : D^n \to D^n$, there exists a point $x \in D^n$ such that $f(x) = x$.*

We can prove this theorem through a rather simple application of the Lefschetz fixed point theorem. As examined above, we could simply note that D^n has the same homology groups as a point, and thus a fixed point always exists. More explicitly, $H_n(D^n) = 0$ for all $n \neq 0$. As D^n is a single path component, $H_0(D^n) = \mathbb{Z}$. Thus

$$\tau(f) = \mathrm{tr}(f_* : H_0(D^n) \to H_0(D^n))$$

For every map f, the induced map on $H_0(D^n)$ is nonzero, and thus has nonzero trace. Thus the Lefschetz number is nonzero and there exists a fixed point.

4.2. **Projective Spaces.** As noted above, every map on $\mathbb{R}\mathrm{P}^n$, for n even, has a fixed point. This follows directly from the fact that $\mathbb{R}\mathrm{P}^n$ has the same homology groups as a point. The connection to linear algebra is rather nice, as linear transformations $f : \mathbb{R}^n \to \mathbb{R}^n$ take lines through the origin into lines through the origin, and therefore induce maps $\tilde{f} : \mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$. Each eigenvector of f spans a 1-dimensional subspace that is brought into itself under f. Thus eigenvectors of f correspond to fixed points of $\tilde{f}$. For n odd (and thus $n-1$ even), the characteristic polynomial $\rho_f(t) = \det(tI - [f])$ is of odd degree n, and as such, has at least one real root. Hence we also reach the conclusion that $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ for n even has at least one fixed point. Conversely, it is alway possible to construct a map $f : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ which is a rotation of all of $\mathbb{R}^{2k}$ and has no eigenvectors. Namely, consider the function

$$f(x_1, x_2, ..., x_{2k}) = (x_2, -x_1, x_4, -x_3, ..., x_{2k}, -x_{2k-1})$$

This has no eigenvectors as a transformation in $\mathbb{R}^{2k}$, as it the direct sum of rotations through $\pi/2$ of the space spanned by each pair of coordinates. Thus the projectivization $\tilde{f} : \mathbb{R}\mathrm{P}^{2k-1} \to \mathbb{R}\mathrm{P}^{2k-1}$ has no fixed point.

These results are mirrored in the case of $\mathbb{C}\mathrm{P}^n$. Using only maps induced from linear transformations of $\mathbb{C}^n$, as every linear transformation $f : \mathbb{C}^n \to \mathbb{C}^n$ has at least one eigenvalue, the induced map $\tilde{f} : \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ has a fixed point. However, this does not cover all maps on $\mathbb{C}\mathrm{P}^n$; in particular, a map which is 'conjugate-linear' still takes lines through the origin in $\mathbb{C}^n$ into themselves. Using this fact, it is possible to construct a conjugate linear map $f : \mathbb{C}^{2k} \to \mathbb{C}^{2k}$ given by

$$f(z_1, z_2, ..., z_{2k}) = (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3, ..., \bar{z}_{2k}, -\bar{z}_{2k-1})$$

the projectivization of which, $\tilde{f} : \mathbb{C}\mathrm{P}^{2k-1} \to \mathbb{C}\mathrm{P}^{2k-1}$, fails to have fixed points by the same reasoning as in the real case.

For the case of maps $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ for n even, we can invoke the definition of the Lefschetz number on cohomology to use the structure of the cohomology ring of $\mathbb{C}\mathrm{P}^n$. Let α be the generator of $H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$. Then $\{1, \alpha, \alpha^2 ... \alpha^n\}$ are generators of the cohomology groups $H^0(\mathbb{C}\mathrm{P}^n; \mathbb{Z}), H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z}), H^4(\mathbb{C}\mathrm{P}^n; \mathbb{Z}), ..., H^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$, where $\alpha^2 := \alpha \smile \alpha$. All other cohomology groups are 0.

For $f^* : \alpha \mapsto \lambda\alpha, \lambda \in \mathbb{Z}$, let $\mathrm{tr}(f^* : H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \to H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})) = \lambda$. Then $f^* : \alpha^k \mapsto \lambda^k \alpha^k$, and $\mathrm{tr}(f^* : H^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \to H^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})) = \lambda^k$. As all nonzero homology groups are in even dimensions, the Lefschetz number is given by the sum

$$\tau(f) = \sum_{i=0}^{n} \lambda^i$$

This polynomial p in λ has no integer roots for i even by the rational root theorem, as trivially $p(1) \neq 0$, and $p(-1) \neq 0$ by the fact that, for n is even, there is one more term of $+1$ than -1. Thus for all maps $f : \mathbb{C}\mathrm{P}^{2k} \to \mathbb{C}\mathrm{P}^{2k}$, f fixes some point in $\mathbb{C}\mathrm{P}^{2k}$.

Note that this argument does not use the fact that the map f was on $\mathbb{C}\mathrm{P}^n$ for n even until the final step. Thus, in general, for $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$

$$\tau(f) = \sum_{i=0}^{n} \lambda^i$$

Thus for all $\lambda \neq -1$, f has a fixed point irrespective of the parity of n. However, for n odd, if $\lambda = -1$, then $p(-1) = 0$, and f is not guaranteed to have a fixed point. In other words, f is guaranteed to have a fixed point except for $f^*(\alpha) = -\alpha$. An immediate corollary of this result is that every map $f : S^2 \to S^2$ has a fixed point unless it is the antipodal map, or homotopic to the antipodal map. This is a special case of the theorem that for $h : S^n \to S^n$ with degree not equal to $(-1)^{n+1}$, then h has a fixed point. This result follows naturally from the Lefschetz fixed point theorem as the homology groups of S^n are given by $H_0(S^N) = H_n(S^n) = \mathbb{Z}$ and for all other $k \neq 0, n, H_k(S^n) = 0$. For $f : S^n \to S^n$, $f_* : H_0(S^n) \to H_0(S^n)$ is degree, and thus trace, 1. Thus

$$\tau(f) = 1 + (-1)^n \mathrm{tr}(f_* : H_n(S^n) \to H_n(S^n))$$

Hence $\tau(f) \neq 0$ except for $\mathrm{tr}(f_* : H_n(S^n) \to H_n(S^n)) = -1$ for n even, and $+1$ for n odd - ie for $\mathrm{tr}(f_* : H_n(S^n) \to H_n(S^n)) = (-1)^{n+1}$.

4.3. **Maps on the n-Torus.** As $T^n = \mathbb{R}^n/\mathbb{Z}^n$, an $n \times n$ matrix A with integer coefficients brings the lattice of $\mathbb{Z}^n \subset \mathbb{R}^n$ into itself, and thus induces a map $\tilde{A}$ on the $\mathbb{R}^n/\mathbb{Z}^n = T^n$. To determine the Lefschetz number $\tau(\tilde{A})$, we will examine the maps $\tilde{A}^*$ on cohomology. In general, $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$, as the obvious cell-structure of T^n has $\binom{n}{k}$ k-cells with the boundary map $d_k = 0$ for all k. The same is true for the dual chain complex and thus the cohomology groups are the same.

The action of $\tilde{A}^*$ on $H^1(T^n)$ is the same as the action of A on $\mathbb{Z}^n$. That is, the first cohomology group is generated by n homomorphism maps that are the identity on one 1-cell in T^n. Each of these n 1-cells is the image of a standard basis vector e_i in $\mathbb{R}^n$ after quotienting by $\mathbb{Z}^n$. Thus the action of A on $\mathbb{Z}^n$ exactly gives the action of $\tilde{A}^*$ on $H^1(T^n)$. In particular, $\mathrm{tr}(\tilde{A}^* : H^1(T^n) \to H^1(T^n)) = \mathrm{tr}(A)$.

We generalize to all $H^k(T^n)$ using the cup-product structure on the n-torus derived in [1]. In brief, let S_k be the set of the $\binom{n}{k}$ possible maps of the numbers $\{1, ..., k\}$ to a subset of k numbers from the set $\{1, ...n\}$; call the maps $\sigma \in S_k$. Then for $\alpha_1, ..., \alpha_n$ the generators of $H^1(T^n)$, $H^k(T^n)$ is generated by the set

$$\bigcup_{\sigma_i \in S_k} \beta_i = \alpha_{\sigma_i(1)} \smile \alpha_{\sigma_i(2)} \smile ... \smile \alpha_{\sigma_i(k)}$$

Note that order does not matter in the generators as the terms in the cup product anti-commute, so we define the convention that $\sigma(1) < ... < \sigma(k)$. Now, to calculate the action of the map $\tilde{A}^*$, we simply extend by the action already determined on the α_i for each of the β_i, using the fact that $\alpha_i \smile \alpha_i = 0$ and $\alpha_i \smile \alpha_j = -\alpha_j \smile \alpha_i$ for $i \neq j$. This calculation is made simpler by noting that the cohomology ring $H^*(T^n)$ is the exterior algebra $\wedge_{\mathbb{Z}}[\alpha_1, ..., \alpha_n]$, with the empty product representing the generator of $H^0(T^n)$, as shown in [1]. Thus the action of $\tilde{A}^*$ on $H^k(T^n)$ is given by the action of $\wedge^k A$ on $\wedge^k \mathbb{Z}^n \to \wedge^k \mathbb{Z}^n$, and subsequently $\mathrm{tr}(\tilde{A}^*) = \mathrm{tr}(\wedge^k A)$. We then find a formula for the Lefschetz number as

$$\tau(\tilde{A}) = \sum_n (-1)^n tr(\wedge^n A)$$

This can be simplified by a rather nice trick. The terms $\mathrm{tr}(\wedge^n A)$ form the coefficients of the characteristic polynomial

$$\rho_A(t) = \det(tI - A) = \sum_{k=0}^{n} (-1)^k \mathrm{tr}(\wedge^k A) t^{(n-k)}$$

where I is the $n \times n$ identity matrix. We have written it in this suggestive summation form to indicate that $\tau(\tilde{A})$ is given by $\rho_A(t = 1)$, or more more suggestively $\tau(\tilde{A}) = \det(I - A)$. Thus the map $\tilde{A}$ that is induced on the n-torus is guaranteed to have fixed points if $I - A$ is invertible.

References

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