3-UNIFORM GRAPH PROPERTIES WITH LOW SENSITIVITY

In this section, we show that there exists a class of 3-uniform graph properties with $o(v^2)$ sensitivity. Moreover, we extend this result to k-uniform hypergraphs.

Proposition 0.1. *Let $3 < t < v$. Then there exists a 3-uniform graph property with sensitivity* $\max\{\binom{t}{2}v, \binom{v}{2}/\binom{t}{2}\}$. *In particular, if $t = v^{1/4}$, we have $O(v^{3/2})$ sensitivity.*

Proof. Let H be a 3-uniform hypergraph. We say a clique K is "isolated" if $|V(K) \cap E| \leq 1$ for all $E \in H \setminus K$. Let f be the graph property that there exists an isolated K_t. We first calculate the $1 \to 0$ sensitivity.

To change from 1 to 0, we have to either delete an edge from an isolated K_t or add an edge that makes K_t not isolated. We have $\binom{t}{2}v$ ways of doing this since we have to choose 2 vertices from K_t and a last vertex from any of the v vertices. If we choose the last vertex to be one of the vertices of the isolated K_t, then we are deleting an edge from it. If we choose the last vertex outside of K_t, then we are adding an edge that makes K_t not isolated.

To change from 0 to 1, H must contain subgraphs that are 1 flip away from an isolated K_t. We shall refer to such subgraphs as sensitive K_t's. Notice that two sensitive K_t's share at most 1 vertex. If they shared 2 vertices, 1 flip is not enough to make one of them a sensitive K_t, contradicting their sensitiveness. Since sensitive K_t's share at most 1 vertex with each other, a pair of vertices in H corresponds to at most 1 sensitive K_t. There are $\binom{v}{2}$ pairs of vertices in H and each sensitive K_t corresponds to $\binom{t}{2}$ pairs of vertices. Hence, the maximum number of sensitive K_t is bounded above by $\binom{v}{2}/\binom{t}{2}$. Therefore, the sensitivity of f is $\max\{\binom{t}{2}v, \binom{v}{2}/\binom{t}{2}\}$.

If we set $t = v^\alpha$, then $s(f)$ is $O(\max\{v^{2\alpha+1}, v^{2(1-\alpha)}\})$ asymptotically since $\binom{t}{2} \sim t^2/2$ and $\binom{v}{2}/\binom{t}{2} \sim (v/t)^2$. Minimizing $s(f)$ with respect to α, we get $s(f) = O(v^{3/2})$ at $\alpha = 1/4$. □

We can generalize this result to k-uniform hypergraphs by extending the notion of "isolatedness".

Proposition 0.2. *Let $k < t < v$. Then there exists a k-uniform graph property with sensitivity* $\max\{\binom{t}{k-1}v, \binom{v}{k-1}/\binom{t}{k-1}\}$. *In particular, if $t = v^{\frac{k-2}{2(k-1)}}$, we have $O(v^{k/2})$ sensitivity.*

Proof. Let H be a k-uniform hypergraph. We say a clique K is "isolated" if $|V(K) \cap E| \leq k-2$ for all $E \in H \setminus K$. Let f be the graph property that there exists an isolated K_t.

First, we calculate the $1 \to 0$ sensitivity. As before, to change from 1 to 0, we either delete an edge from K_t or add an edge to K_t that makes it not isolated. There are $\binom{t}{k-1}v$ ways of doing this.

For the $0 \to 1$ sensitivity, H must contain subgraphs that are 1 flip away from an isolated K_t. Again, we refer to such subgraphs as sensitive K_t's. A pair of sensitive K_t's share at most $k-2$ vertices. Hence, $k-1$ vertices correspond to at most 1 sensitive K_t and the maximum number of K_t is bounded above by $\binom{v}{k-1}/\binom{t}{k-1}$. Therefore, the sensitivity of f is $\max\{\binom{t}{k-1}v, \binom{v}{k-1}/\binom{t}{k-1}\}$.

If we set $t = v^{\alpha}$, then $s(f)$ is $O(\max\{v^{(k-1)\alpha+1}, v^{(k-1)(1-\alpha)}\})$. Minimizing $s(f)$ with respect to α, we get $s(f) = O(v^{k/2})$ at $\alpha - \frac{k-2}{2(k-1)}$.

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