1. Lefschetz Fixed-Point Theory

Definition 1.1. The global Lefschetz number of f is the intersection number $I(\Delta, graph(f))$, denoted $L(f)$.

Theorem 1.2. *(Smooth Lefschetz Fixed-Point Theorem) Let $f : X \to X$ be a smooth map on a compact orientable manifold. If $L(f) \neq 0$, then f has a fixed point.*

Proposition 1.3. *$L(f)$ is a homotopy invariant.*

Proposition 1.4. *If f is homotopic to the identity, then $L(f)$ equals the Euler characteristic of X. In particular, if X admits a smooth map $f : X \to X$ that is homotopic to the identity and has no fixed points, then $\chi(X) = 0$.*

Definition 1.5. $f : X \to X$ is a Lefschetz map if $graph(f) \pitchfork \Delta$

Proposition 1.6. *Every map $f : X \to X$ is homotopic to a Lefschetz map.*

Given any x a fixed point of a Lefschetz map, we have $graph(f) \pitchfork \Delta$ if and only if

$$graph(df_x) + \Delta_x = T_x(X) \times T_x(X). \tag{1.7}$$

And this implies that df_x has no nonzero fixed point.

Definition 1.8. A fixed point x is a Lefshetz fixed point of f if df_x has no nonzero fixed point.

So f is a Lefschetz map if and only if all its fixed points are Lefschetz. If x is a Lefschetz fixed point, we denote the orientation number of (x, x) in the intersection $\Delta \cap graph(f)$ by $L_x(f)$, called the local Lefschetz number of f at x. Thus for f Lefschetz map,

$$L(f) = \sum_{f(x)=x} L_x(f). \tag{1.9}$$

x is a Lefschetz fixed point if and only if$df_x - I$ is an isomorphism of $T_x(X)$.

Proposition 1.10. *The local Lefschetz number $L_x(f)$ at a Lefschetz fixed point is 1 if the isomorphism $df_x - I$ preserves orientation on $T_x(X)$, and -1 if the isomorphism reverses orientation. That is the sign of $L_x(f)$ equals the sign of the determinant of $df_x - I$.*

Proposition 1.11. *The Euler characteristic of S^2 is 2.*

Corollary 1.12. *Every map of S^2 that is homotopic to the identity must possess a fixed point. In particular, the antipodal map is not homotopic to the identity.*

Proposition 1.13. *The surface of genus k admits a Lefschetz map homotopic to the identity, with one source, one sink, and $2k$ saddles. Consequently, its Euler characteristic is $2 - 2k$.*

Proposition 1.14. *(Splitting Proposition) Let U be a neighborhood of the fixed point x that contains no other fixed points of f. Then there exists a homotopy f_t of f such that f_t has only Lefschetz fixed points in U, and each f_t equals f outside some compact subset of U.*

Definition 1.15. Suppose that x is an isolated fixed point of f in $\mathbb{R}^k$. If B is a small closed ball centered at x that contains no other fixed point, then the degree of map

$$z \to \frac{f(z)-z}{|f(z)-z|} \tag{1.16}$$

is called the local Lefschetz number of f at x, denoted $L_x(f)$.

Proposition 1.17. *At Lefschetz fixed points, the two definitions of $L_x(f)$ agree.*

Proposition 1.18. *Suppose that the map f in $\mathbb{R}^k$ has an isolated fixed point at x, and let B be a closed ball around x containing no other fixed point of f. Choose any map f_1 that equals f outside some compact subset of $Int(B)$ but has only Lefschetz fixed points in B. Then*

$$L_x(f) = \sum_{f_1(z)=z} L_z(f_1), \tag{1.19}$$

for any $z \in B$.

Theorem 1.20. *(Local Computation of the Lefschetz Number). Let $f : X \to Y$ be any smooth map on a compact manifold, with only finitely many fixed points. Then the global Lefschetz number equals the sum of the local Lefschetz numbers:*

$$L(f) = \sum_{f_x(x)} L_x(f). \tag{1.21}$$

2. Exercises

Proposition 2.1. *Let $A : V \to V$ be a linear map. Then the following statements are equivalent:*

(1) 0 is an isolated fixed point of A.
(2) $A - I : V \to V$ is an isomorphism.
(3) 0 is a Lefschetz fixed point of A.
(4) A is a Lefschetz map.

Proposition 2.2. *The following are equivalent*

(1) x is a Lefschetz fixed point of $f : X \to X$.
(2) 0 is a Lefschetz fixed point of $df_x : T_x(X) \to T_x(X)$.
(3) df_x is a Lefschetz map.

Proof. If x Lefschetz fixed point of f, we have $df_x - I$ isomorphism and then by previous proposition, this is equivalent to 0 is a Lefschetz fixed point of df_x. And also by 3,4 in previous proposition, we know that 2,3 are equivalent. $\square$

Proposition 2.3. *The map $f(x) = 2x$ on $\mathbb{R}^k$ has $L_0(f) = 1$ and $f(x) = 0.5x$ has $L_0(f) = (-1)^k$.*

Proof. Only fixed point of f is 0 and 0 is a Lefschetz fixed point since $df_0 - I = I or - 0.5I$ isomorphism. Then we have $L_0(f) = det(df_0 - I)$. $\square$

Proposition 2.4. $\chi(X \times Y) = \chi(X)\chi(Y)$.

Proof. Since any map f is homotopic to some Lefschetz maps, we can pick f, g homotpic to id_X and id_Y. Thus $\chi(X) = L(f)$ and $\chi(Y) = L(g)$. Also $f \times g$ is homotopic to $id_X \times id_Y = id_{X \times Y}$. Then we have $\chi(X \times Y) = L(f \times g)$. To compute $L(f \times g)$, we know

(2.5)
$$L(f \times g) = \sum_{x,y: f(x)=x, g(y)=y} L_{(x,y)}(f \times g) = \sum_{x,y: f(x)=x, g(y)=y} L_x(f) L_y(g) = L(f)L(g).$$

Then we have

$$\chi(X \times Y) = \chi(X)\chi(Y). \tag{2.6}$$

□

Proposition 2.7. *Summing local Lefschetz numbers does not define a homotopy invariant without the compactness assumption.*

Proof. Pick any A, B $n \times n$ matrices such that $det(A - I) > 0$ and $det(B - I) < 0$. $\mathbb{R}^n$ contractible so we have A homotopic to B but $L(A) = 1$ while $L(B) = -1$. □

Proposition 2.8. *The Euler characteristic of a compact connected Lie group is zero.*

Proof. Let the compact connected Lie group be G and let $g \in G$ such that $g \neq 1$. Define $f : G \to G$ by $f(x) = g \cdot x$. This smooth map is homotopic to id_G but has no fixed point. Then we know

$$\chi(G) = L(f) = 0. \tag{2.9}$$

□

3. Exterior Algebra

Theorem 3.1. *Let $\{\phi_1, ..., \phi_k\}$ be a basis for V^*. Then the p-tensors $\{\phi_{i_1}, ..., \phi_{i_p}\}$ form a basis for $\mathcal{T}^p(V^*)$. Consequently, the dimension is k^p.*

Lemma 3.2. *If $Alt(T) = 0$, then $T \wedge S = S \wedge T = 0$.*

Theorem 3.3. *If $\{\phi_1, ..., \phi_k\}$ is a basis for V^*, then $\phi_I = \phi_{i_1} \wedge ... \wedge \phi_{i_p}$ such that $1 \leq i_1, ..., i_p \leq k$ is a basis for $\Lambda^p(V^*)$. Consequently, dimension is $\binom{k}{p}$.*

Corollary 3.4. *The wedge product satisfies the following anticommutativity relation:*

$$T \wedge S = (-1)^{pq} S \wedge T, \tag{3.5}$$

when $T \in \Lambda^p(V^*)$ and $S \in \Lambda^q(V^*)$.

Theorem 3.6. *If $A : V \to V$ is a linear map, then $A^*T = (detA)T$ for every $T \in \Lambda^k(V^*)$, where $k = dimV$. In particular, if $\phi_1, ..., \phi_k \in \Lambda^1(V^*)$, then*

$$A^*\phi_1 \wedge ... \wedge A^*\phi_k = (detA)\phi_1 \wedge ... \wedge \phi_k. \tag{3.7}$$