1. Lefschetz fixed point theorem

Definition 1.1. The global Lefschetz number of f is the intersection number $I(\Delta, graph(f))$, denoted $L(f)$.

Theorem 1.2. *(Smooth Lefschetz Fixed-Point Theorem) Let $f : X \to X$ be a smooth map on a compact orientable manifold. If $L(f) \neq 0$, then f has a fixed point.*

Proposition 1.3. *$L(f)$ is a homotopy invariant.*

Proposition 1.4. *If f is homotopic to the identity, then $L(f)$ equals the Euler characteristic of X. In particular, if X admits a smooth map $f : X \to X$ that is homotopic to the identity and has no fixed points, then $\chi(X) = 0$.*

Definition 1.5. $f : X \to X$ is a Lefschetz map if $graph(f) \pitchfork \Delta$

Proposition 1.6. *Every map $f : X \to X$ is homotopic to a Lefschetz map.*

Given any x a fixed point of a Lefschetz map, we have $graph(f) \pitchfork \Delta$ if and only if

$$graph(df_x) + \Delta_x = T_x(X) \times T_x(X). \tag{1.7}$$

And this implies that df_x has no nonzero fixed point.

Definition 1.8. A fixed point x is a Lefshetz fixed point of f if df_x has no nonzero fixed point.

So f is a Lefschetz map if and only if all its fixed points are Lefschetz. If x is a Lefschetz fixed point, we denote the orientation number of (x, x) in the intersection $\Delta \cap graph(f)$ by $L_x(f)$, called the local Lefschetz number of f at x. Thus for f Lefschetz map,

$$L(f) = \sum_{f(x)=x} L_x(f). \tag{1.9}$$

x is a Lefschetz fixed point if and only if$df_x - I$ is an isomorphism of $T_x(X)$.

Proposition 1.10. *The local Lefschetz number $L_x(f)$ at a Lefschetz fixed point is 1 if the isomorphism $df_x - I$ preserves orientation on $T_x(X)$, and -1 if the isomorphism reverses orientation. That is the sign of $L_x(f)$ equals the sign of the determinant of $df_x - I$.*

Proposition 1.11. *(Splitting Proposition) Let U be a neighborhood of the fixed point x that contains no other fixed points of f. Then there exists a homotopy f_t of f such that f_t has only Lefschetz fixed points in U, and each f_t equals f outside some compact subset of U.*

Definition 1.12. Suppose that x is an isolated fixed point of f in $\mathbb{R}^k$. If B is a small closed ball centered at x that contains no other fixed point, then the degree of map

$$z \to \frac{f(z) - z}{|f(z) - z|} \tag{1.13}$$

is called the local Lefschetz number of f at x, denoted $L_x(f)$.

Proposition 1.14. *At Lefschetz fixed points, the two definitions of $L_x(f)$ agree.*

Proposition 1.15. *Suppose that the map f in $\mathbb{R}^k$ has an isolated fixed point at x, and let B be a closed ball around x containing no other fixed point of f. Choose any map f_1 that equals f outside some compact subset of $Int(B)$ but has only Lefschetz fixed points in B. Then*

$$L_x(f) = \sum_{f_1(z)=z} L_z(f_1), \tag{1.16}$$

for any $z \in B$.

Theorem 1.17. *(Local Computation of the Lefschetz Number). Let $f : X \to Y$ be any smooth map on a compact manifold, with only finitely many fixed points. Then the global Lefschetz number equals the sum of the local Lefschetz numbers:*

$$L(f) = \sum_{f_x(x)} L_x(f). \tag{1.18}$$

2. Examples

Example 2.1. The Euler characteristic of S^2 is 2.

Proof. Let $\pi : \mathbb{R}^3 \to S^2$ be the projection $\pi(x) = x/|x|$. Then we define the map $f : S^2 \to S^2$:

$$f(x) = \pi(x + (0, 0, -1/2)). \tag{2.2}$$

This map has a source at $(0, 0, 1)$, a sink at $(0, 0, -1)$ and no fixed point elsewhere, which gives $L(f) = 2$. Also the map $F : S^2 \times I \to S^2$ defined by

$$F(x, t) = \pi(x + (0, 0, -t/2)) \tag{2.3}$$

is a homotopy between f and the identity map. By Proposition 1.4, we know that

$$\chi(S^2) = 2. \tag{2.4}$$

□

Corollary 2.5. *Every map of S^2 that is homotopic to the identity must possess a fixed point. In particular, the antipodal map is not homotopic to the identity.*

Example 2.6. The surface of genus k admits a Lefschetz map homotopic to the identity, with one source, one sink, and $2k$ saddles. Consequently, its Euler characteristic is $2 - 2k$.

Example 2.7. The Euler characteristic of a compact connected Lie group is zero.

Proof. Let the compact connected Lie group be G and let $g \in G$ such that $g \neq 1$. Define $f : G \to G$ by $f(x) = g \cdot x$. This smooth map is homotopic to id_G but has no fixed point. Then we know

$$\chi(G) = L(f) = 0. \tag{2.8}$$

□

3. Homology version

Theorem 3.1. *(The Lefschetz fixed point theorem) Let X be a closed smooth manifold and let $f : X \to X$ be a smooth map with all fixed points nondegenerate. Then*

$$L(f) = \sum_i (-1)^i Tr(f_* : H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q})). \tag{3.2}$$

Example 3.3. (Brouwer Fixed Point Theorem) Every smooth $f : D^n \to D^n$ has a fixed point.

Proof. Since D^n is contractible, we know that the only nontrivial homology group of D^n is $H_0(D^n) = \mathbb{Z}$. Then we know that

$$L(f) = L(f) = \sum_i (-1)^i Tr(f_* : H_i(X) \to H_i(X)) = 1. \tag{3.4}$$

Then f must have a fixed point. $\square$