Robotic Navigation and Exploration

Week 8: Multi-view Geometry & Transformation Optimization

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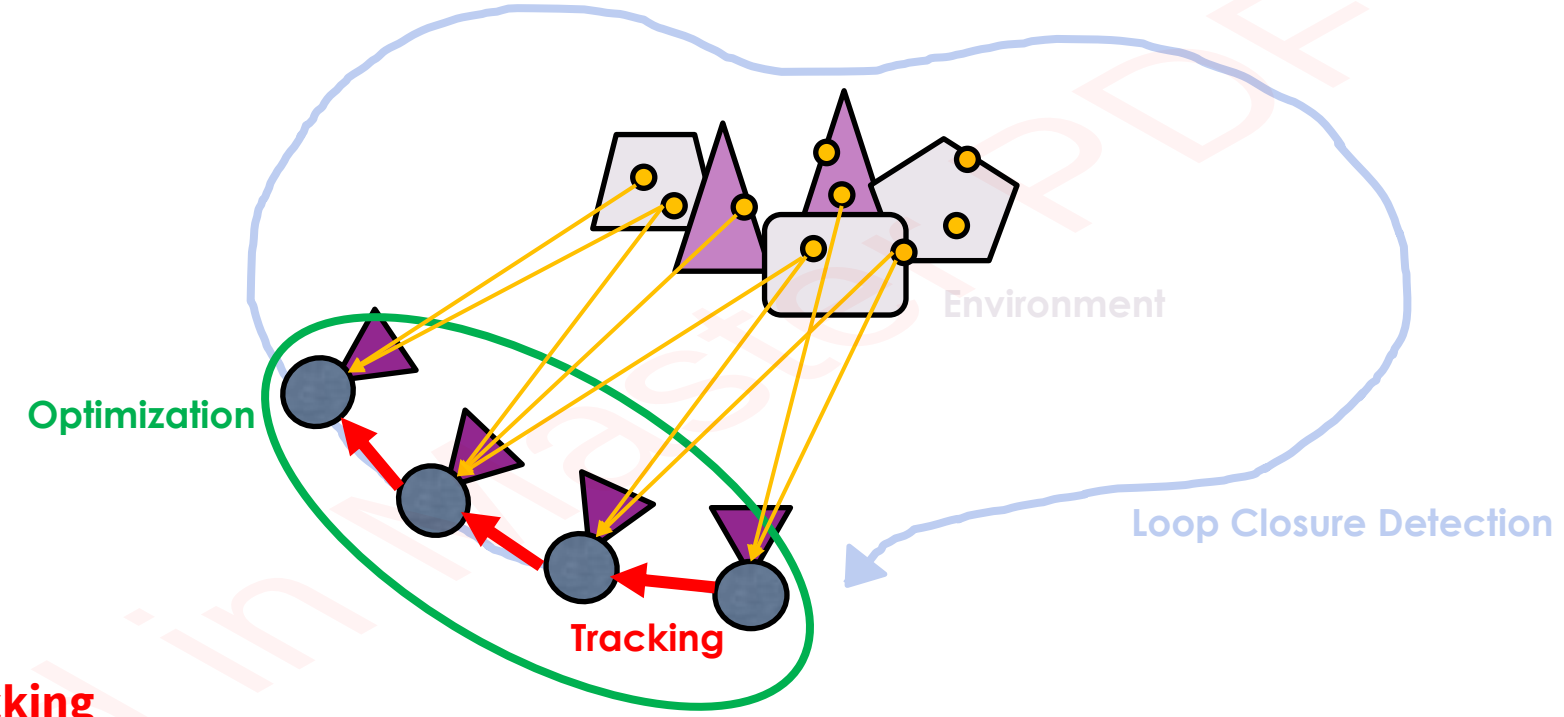
SLAM Overview



Pose Tracking

Using continuous measurement to estimate the movement

Local Optimization

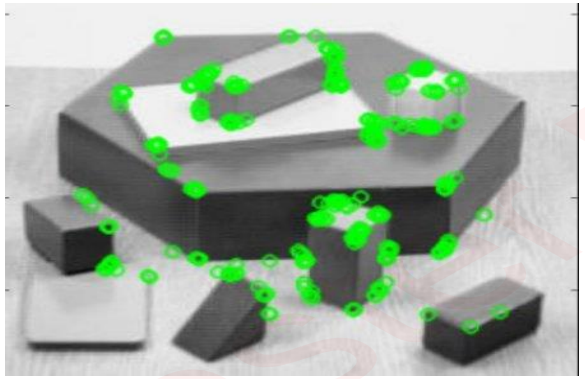
Using several measurement to optimize the error of the map

Loop Closure Detection

Detecting the loop to stabilize the global structure

Information from Image Data

Sparse



Sparse Feature Points

Dense

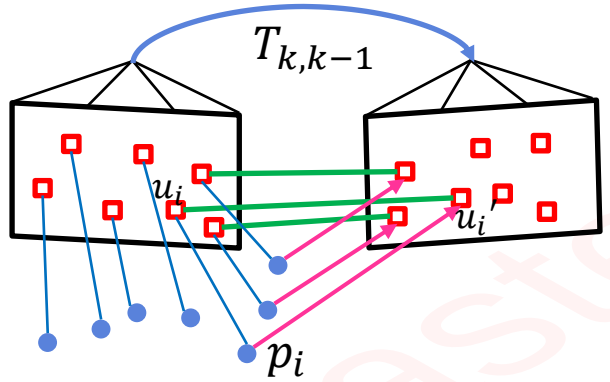
All Points

Semi-Dense

Important Points

Objective Function

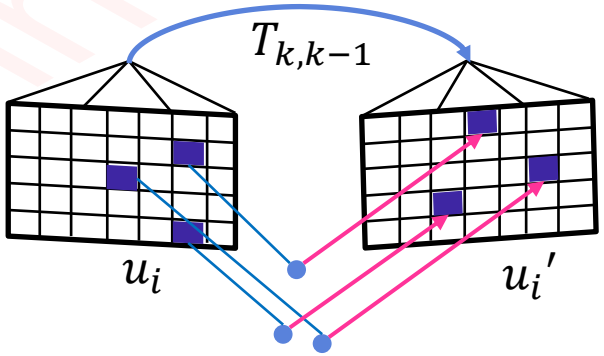
Indirect Method



$$T_{k,k-1} = \underset{T}{argmin} \sum_{i}^{N} ||u_i' - \pi p_i||^2$$

Minimize Geometric Error (Reprojection)

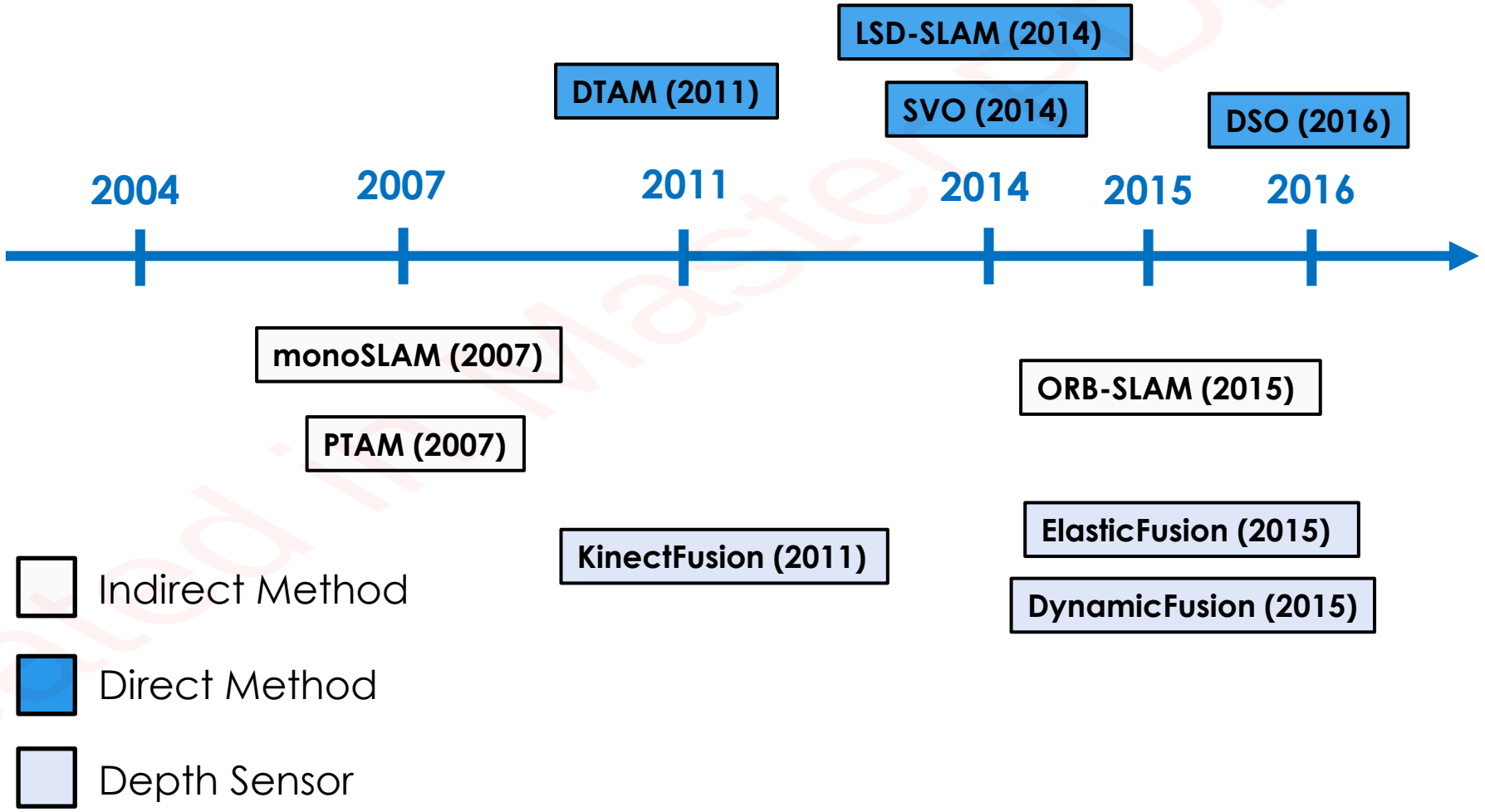
Direct Method



$$T_{k,k-1} = \underset{T}{argmin} \sum_{i}^{N} ||I_k(u_i') - I_{k-1}(u_i)||^2$$

Minimize Photometric Error (Pixel Grayscale)

History of Visual SLAM



History of Visual SLAM

First dense monocular SLAM algorithm.
Using GPU to accelerate the computation and build dense point cloud.

DTAM (2011)

Improve the speed of DTAM by only building the **semi-dense map** of whole image.

LSD-SLAM (2014)

PTAM (2007)

First real-time monocular SLAM algorithm.
Separate the system into two thread: tracking and mapping. The pipeline is the basis of modern SLAM system.

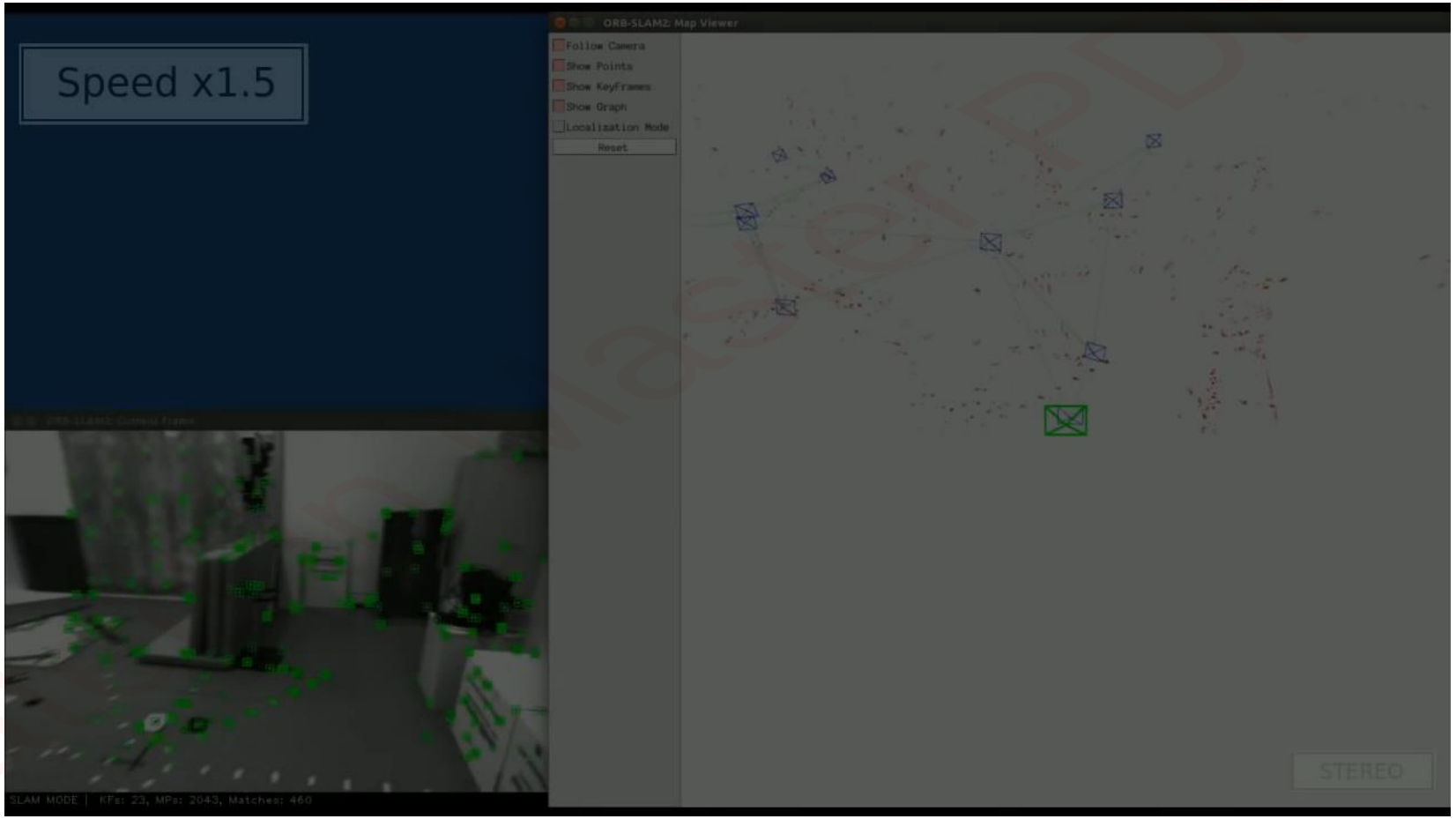
ORB-SLAM (2015)

Assembles recent researches of **feature-based SLAM**. Use similar pipeline as PTAM.
A stable and reliable monocular SLAM system.

KinectFusion (2011)

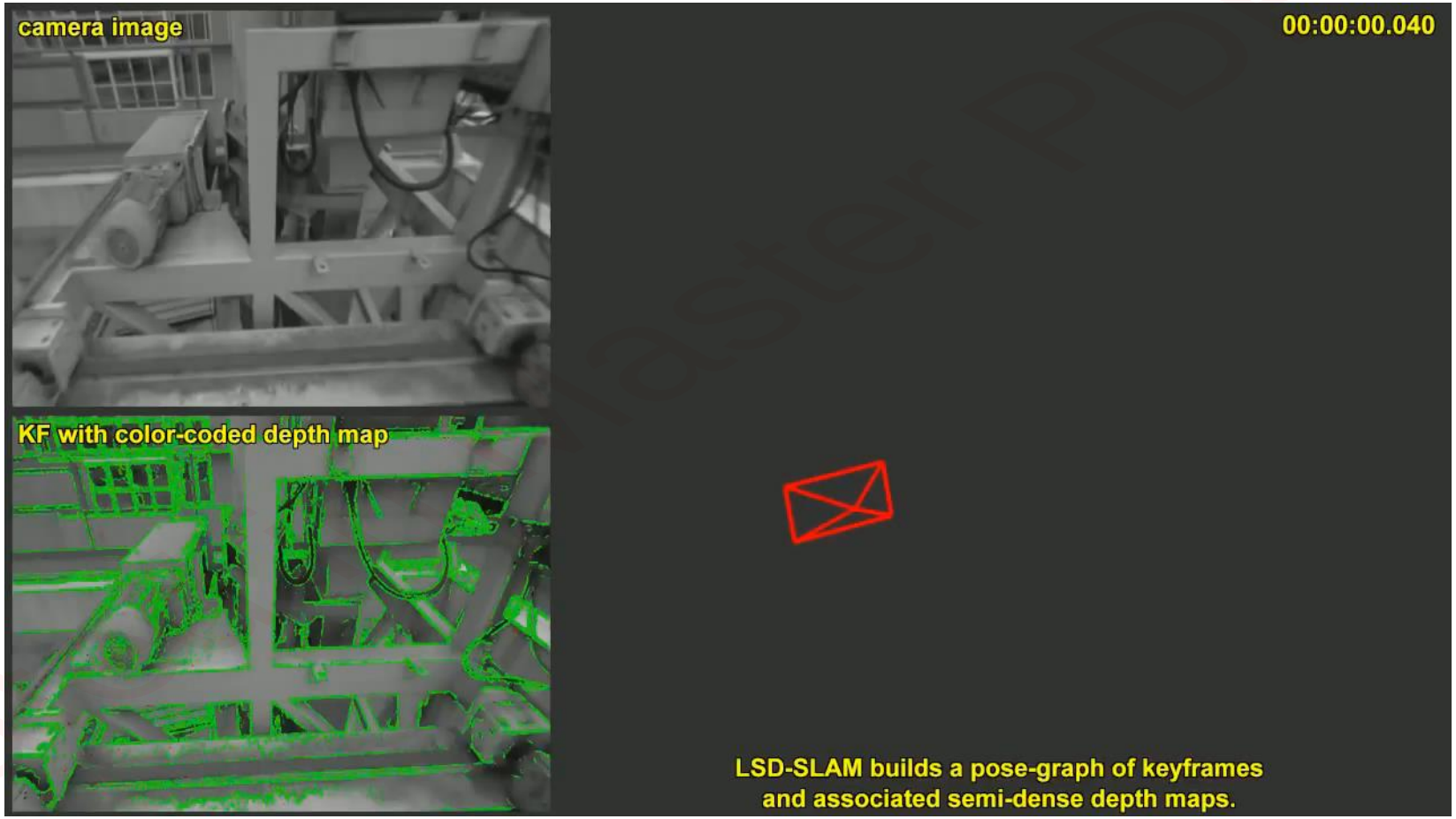
First depth SLAM algorithm.
Using the volumetric fusion map to construct complete and beautiful dense 3D point cloud.

ORB-SLAM



https://www.youtube.com/watch?v=IuBGKxgaxS0

LSD-SLAM



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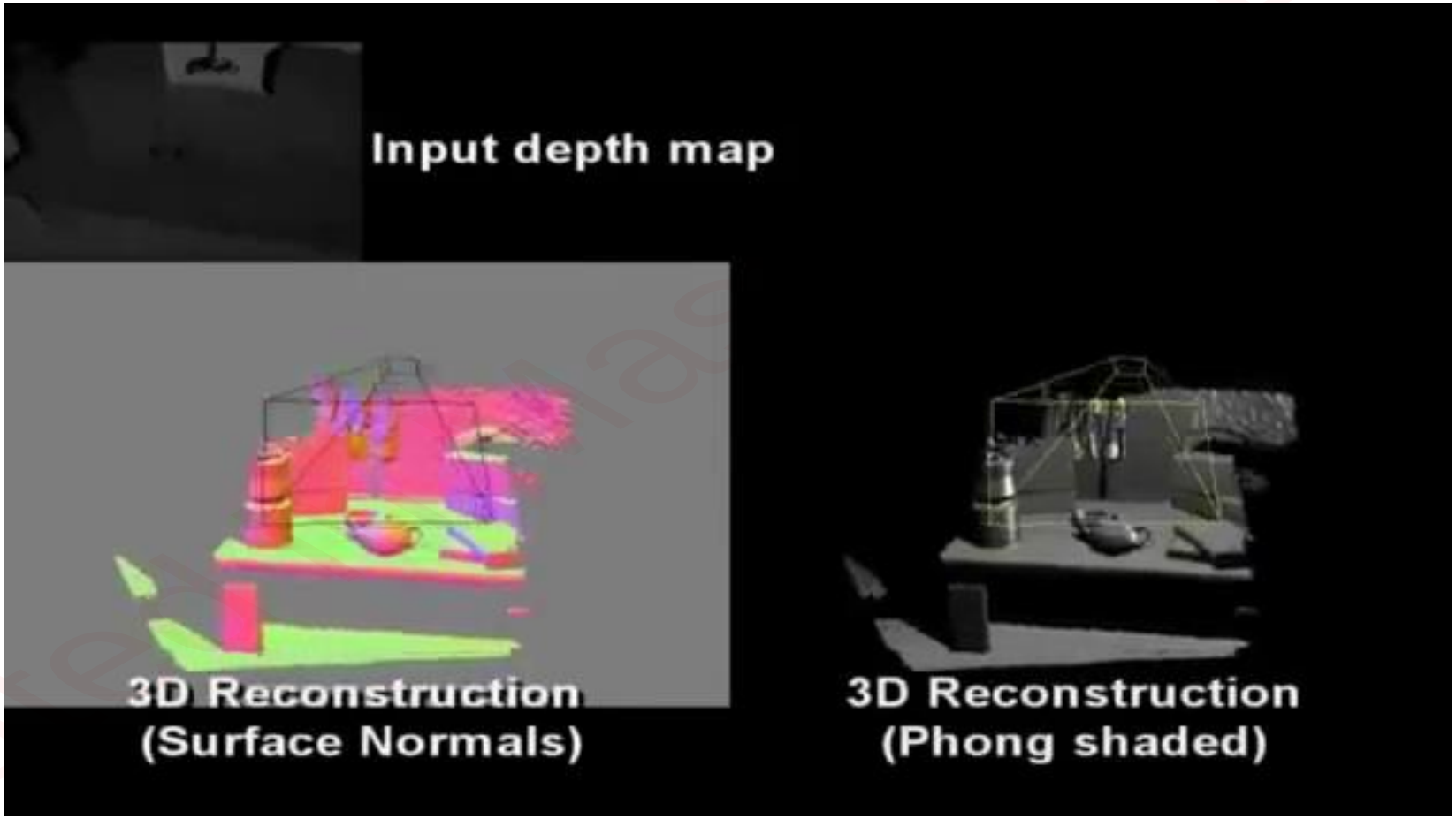
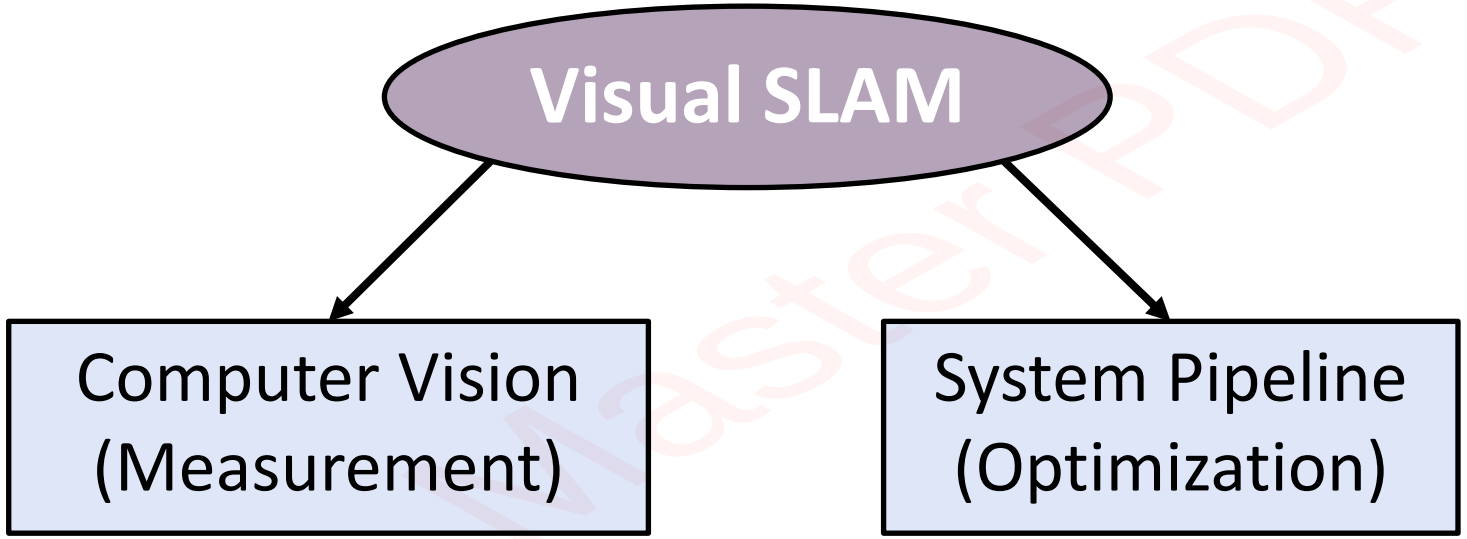
Kinect Fusion



https://www.youtube.com/watch?v=KOUSSIKUJ-A

Feature-based Visual SLAM



Feature Points Matching

Perspective-n-Points

Bundle Adjustment

Epipolar Geometry

Map storage

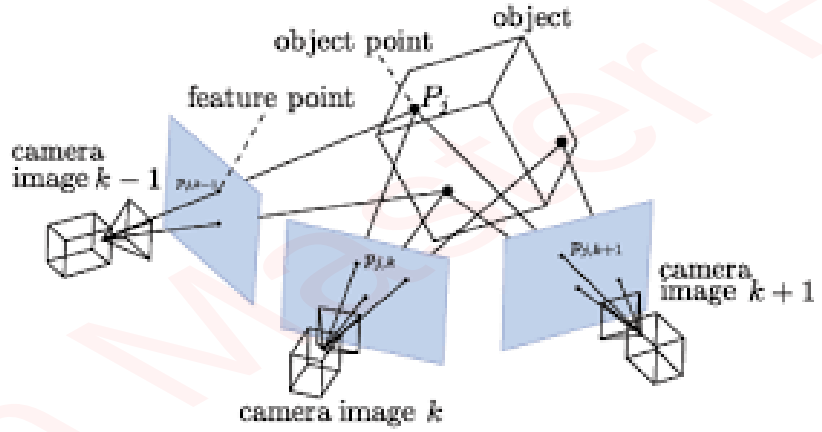
Graph Optimization

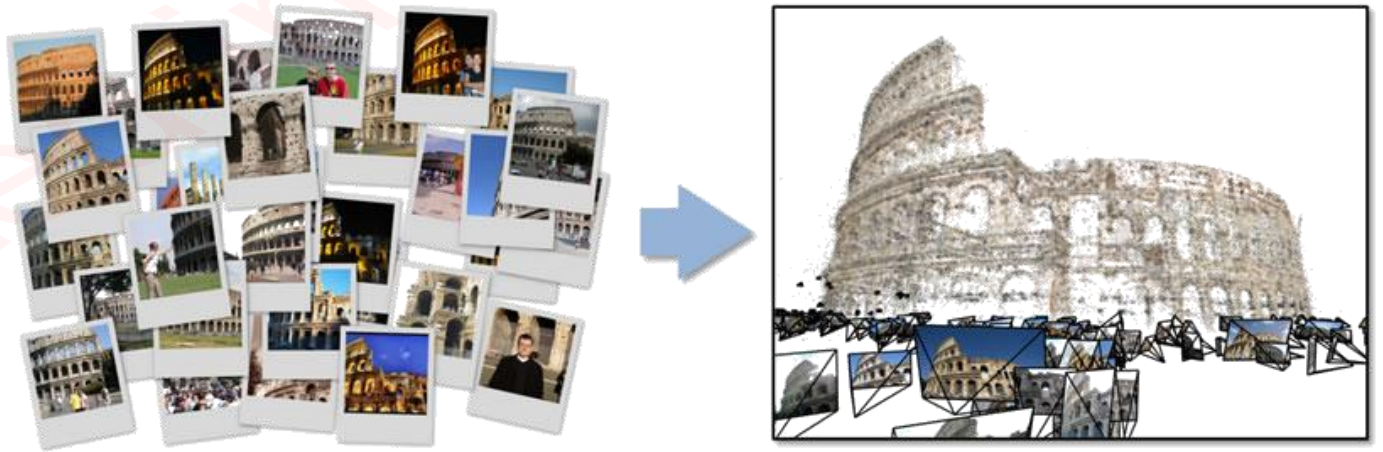
Loop Closure Detection

Computer Vision / Multi-view Geometry

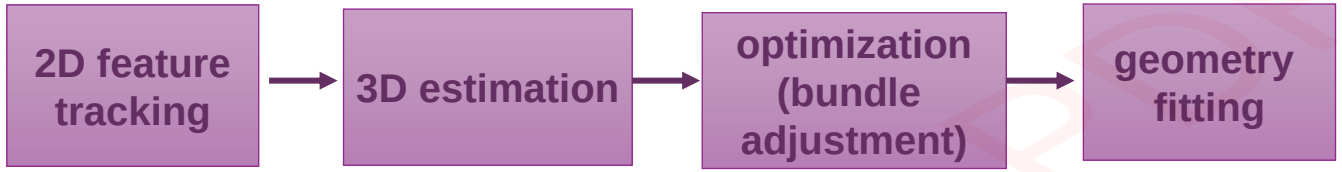
Structure from Motion (SfM)

- Structure from motion: automatic recovery of **camera motion** and **scene structure** from two or more images.

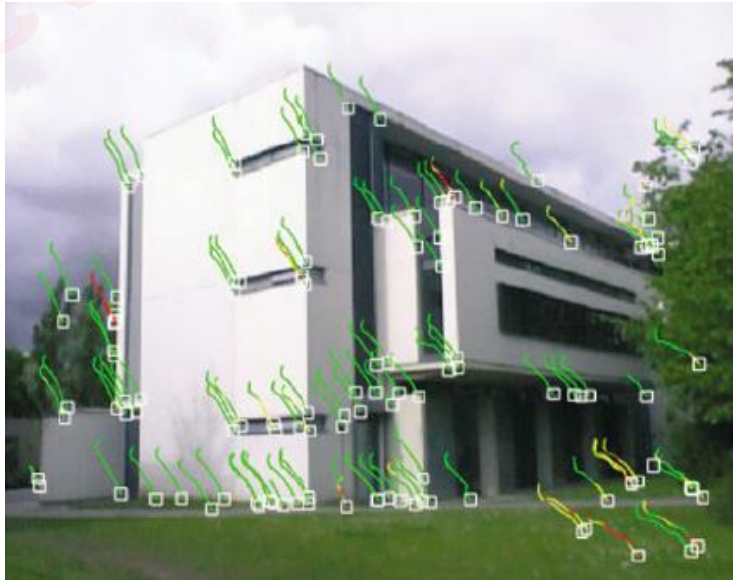




SfM Pipeline

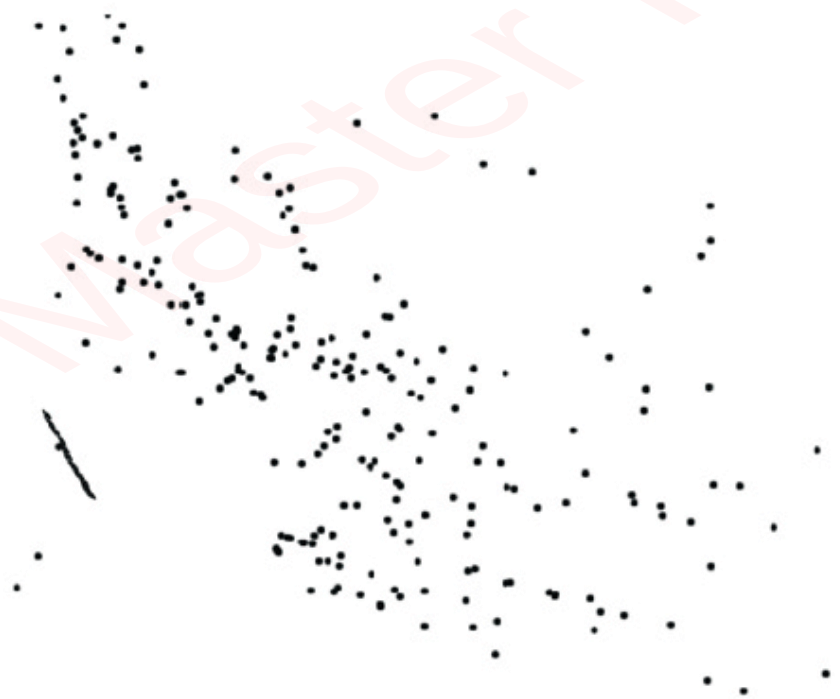


- Step 1: Track Features
  - Detect good features (SIFT)
  - Find correspondences between frames
    - Lucas & Kanade-style motion estimation
    - window-based correlation
    - SIFT matching



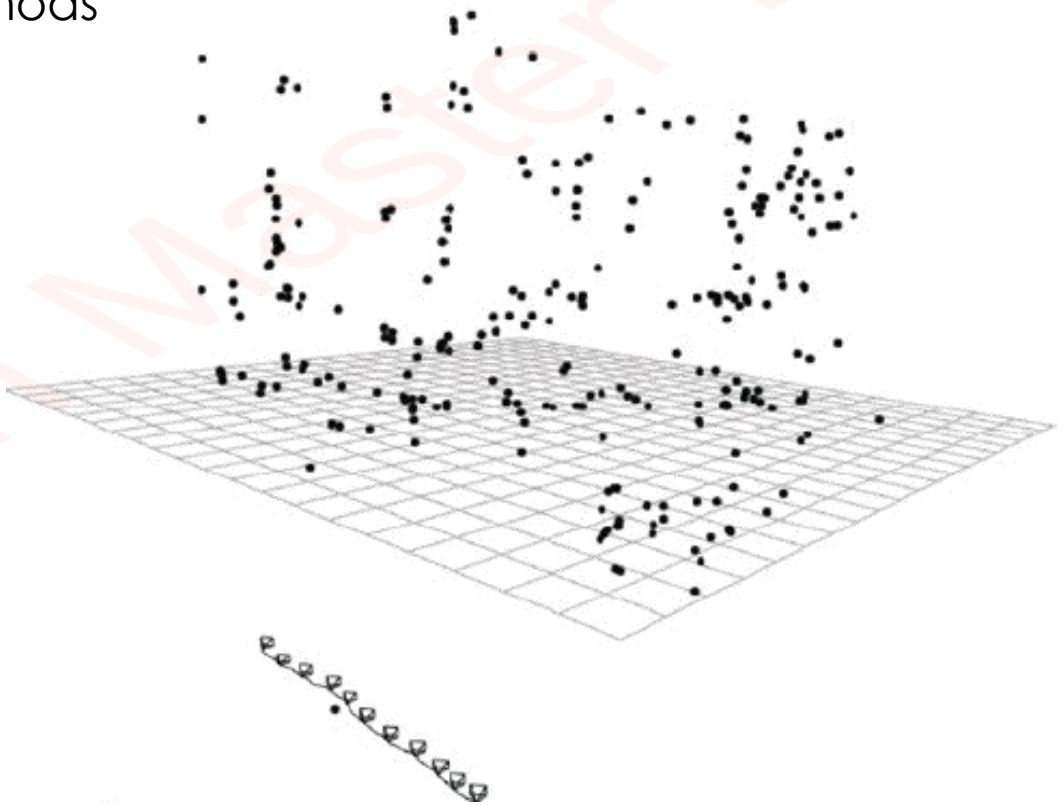
SfM Pipeline

- Step 2: Estimate Motion and Structure
  - Simplified projection model
  - 2 or 3 views at a time



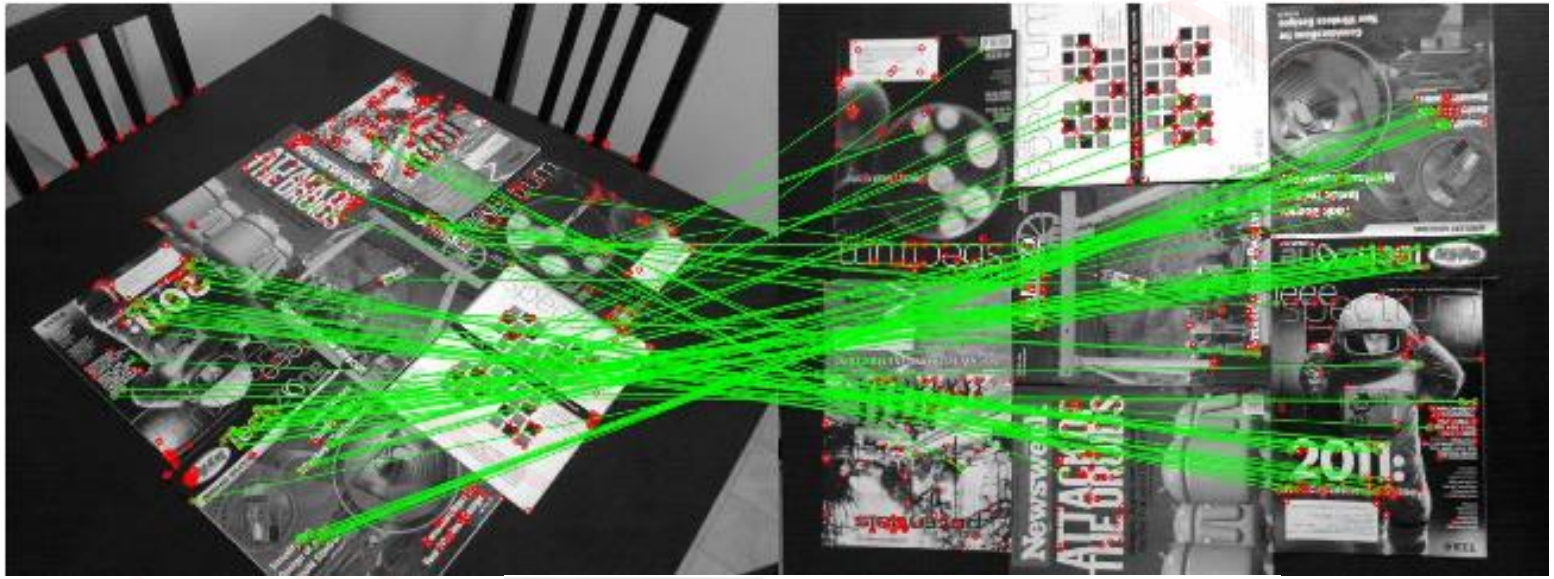
SfM Pipeline

- Step 3: Optimization to refine estimation
  - "Bundle adjustment" in photogrammetry
  - Other iterative methods

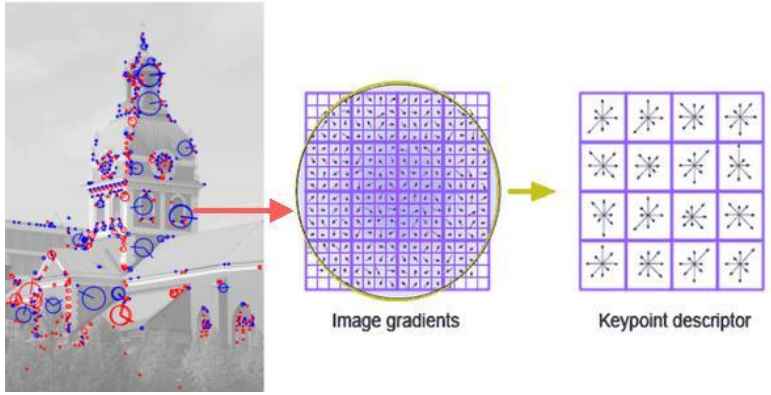


Feature Points Matching

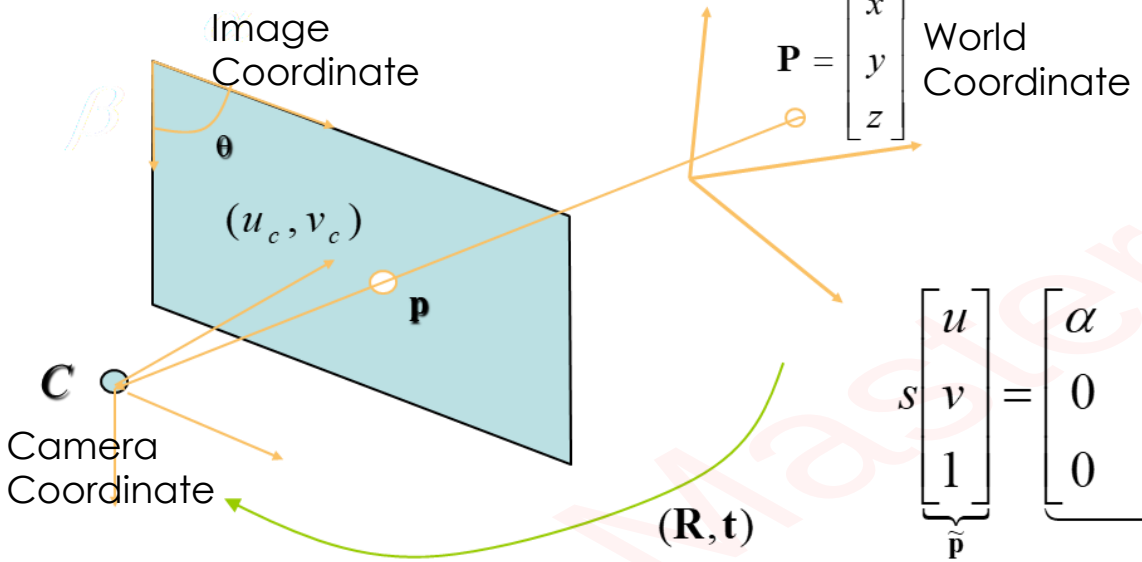
Feature Points Detection/Description



SIFT, SURF, ORB



Camera Calibration



$$\mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

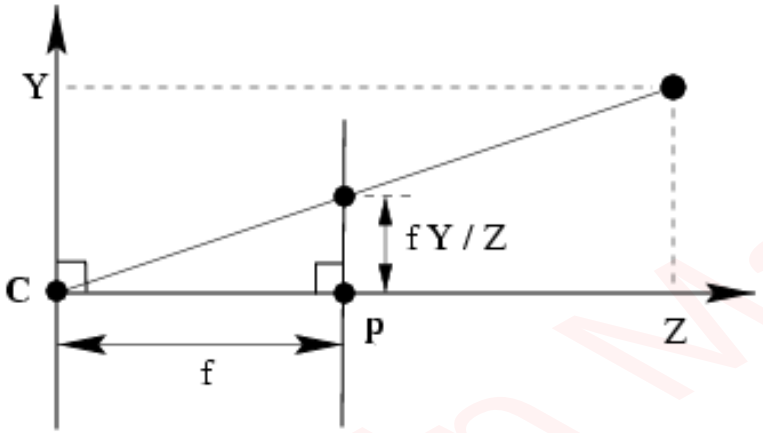
World Coordinate

$$s\underbrace{\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}}_{\tilde{\mathbf{p}}} = \underbrace{\begin{bmatrix} \alpha & \gamma & u_c \\ 0 & \beta & v_c \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \end{bmatrix}}_{[\mathbf{R}\ \mathbf{t}]} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\tilde{\mathbf{P}}}$$

- Intrinsics:
  - scale factor
  - focal length
  - aspect ratio
  - principle point
  - radial distortion
- Extrinsics
  - optical center
  - camera orientation

A camera is calibrated when intrinsics/extrinsics are known.

Pinhole Camera Projection Model



$$x = \frac{fX}{Z} \qquad y = \frac{fY}{Z}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

相机内参 相机外参

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} \sim \mathbf{K}[\mathbf{I}|0]\mathbf{X}$$

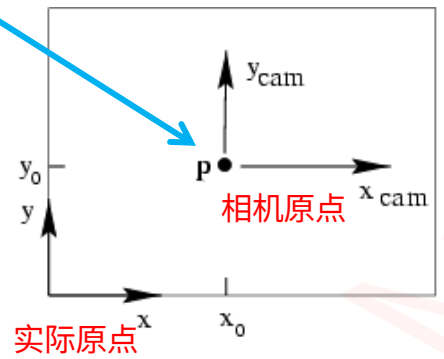
intrinsic matrix

extrinsic matrix (Camera Coordinate =World Coordinate)

旋转矩阵

Principal Point Offset

principal point

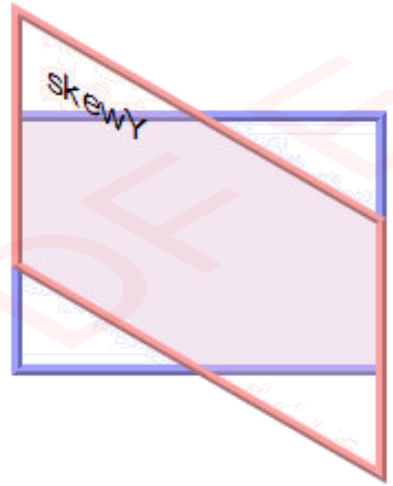


$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} \sim \mathbf{K}[\mathbf{I}|0]\mathbf{X}$$

Intrinsic Matrix

$$\mathbf{K} = \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{K} = \begin{bmatrix} fa & s_y & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$



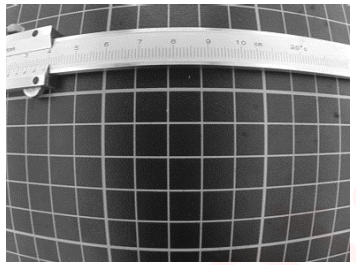
Good enough for modeling the camera projection?

a : aspect ratio (for non-square pixels) 长宽比(对于非正方形像素)

s_y : skew (for non-rectangular pixels) 倾斜(对于非矩形像素)

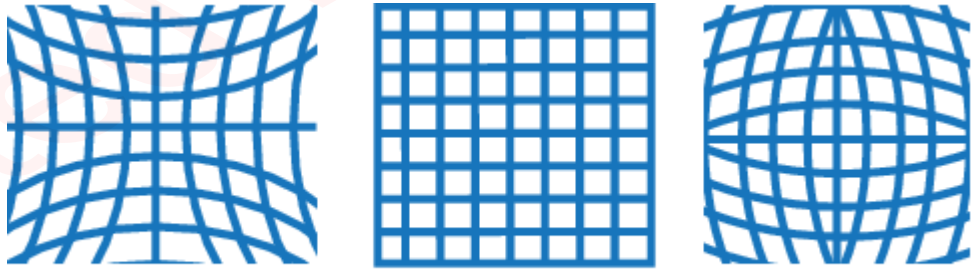
$$x_{distorted} = x + [2 * p_1 * x * y + p_2 * (r^2 + 2 * x^2)]$$

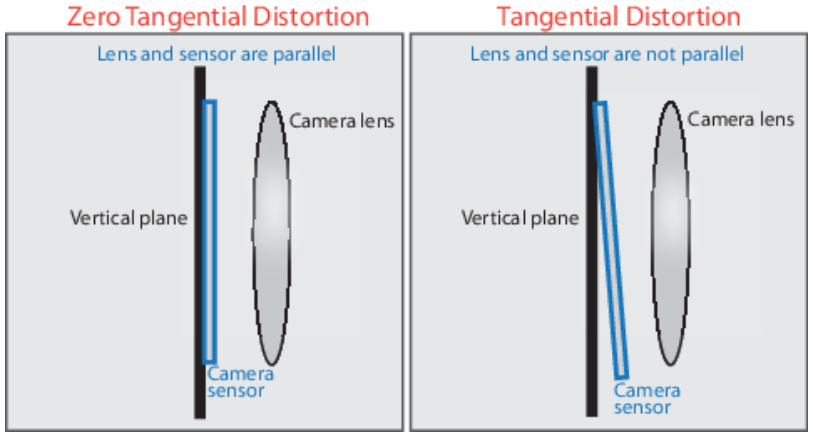
$$y_{distorted} = y + [p_1 * (r^2 + 2 * y^2) + 2 * p_2 * x * y]$$



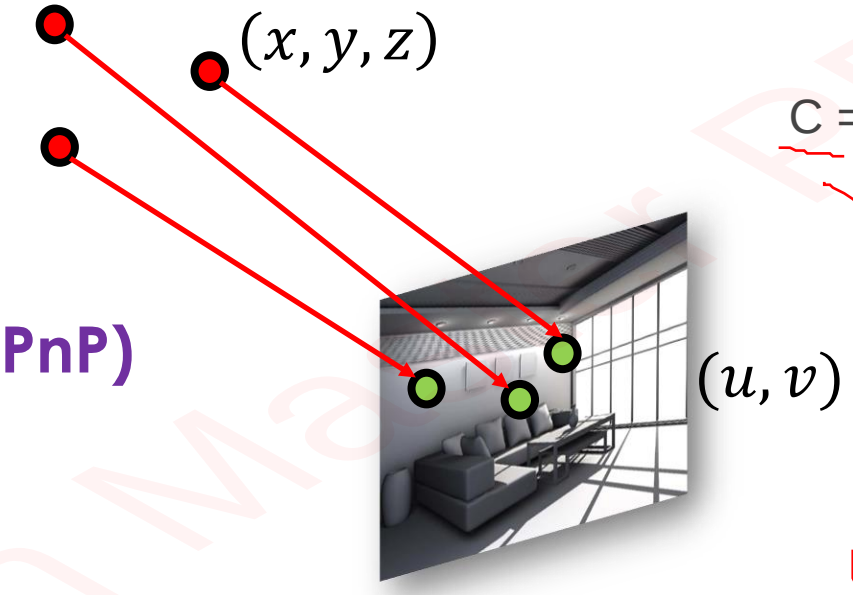
$$x_{distorted} = x(1 + k_1*r^2 + k_2*r^4 + k_3*r^6)$$

$$y_{distorted} = y(1 + k_1*r^2 + k_2*r^4 + k_3*r^6)$$





Transformation Matrix Estimation by Reprojection



C = K [R | t] = [KR | KRT] = [A | AT]

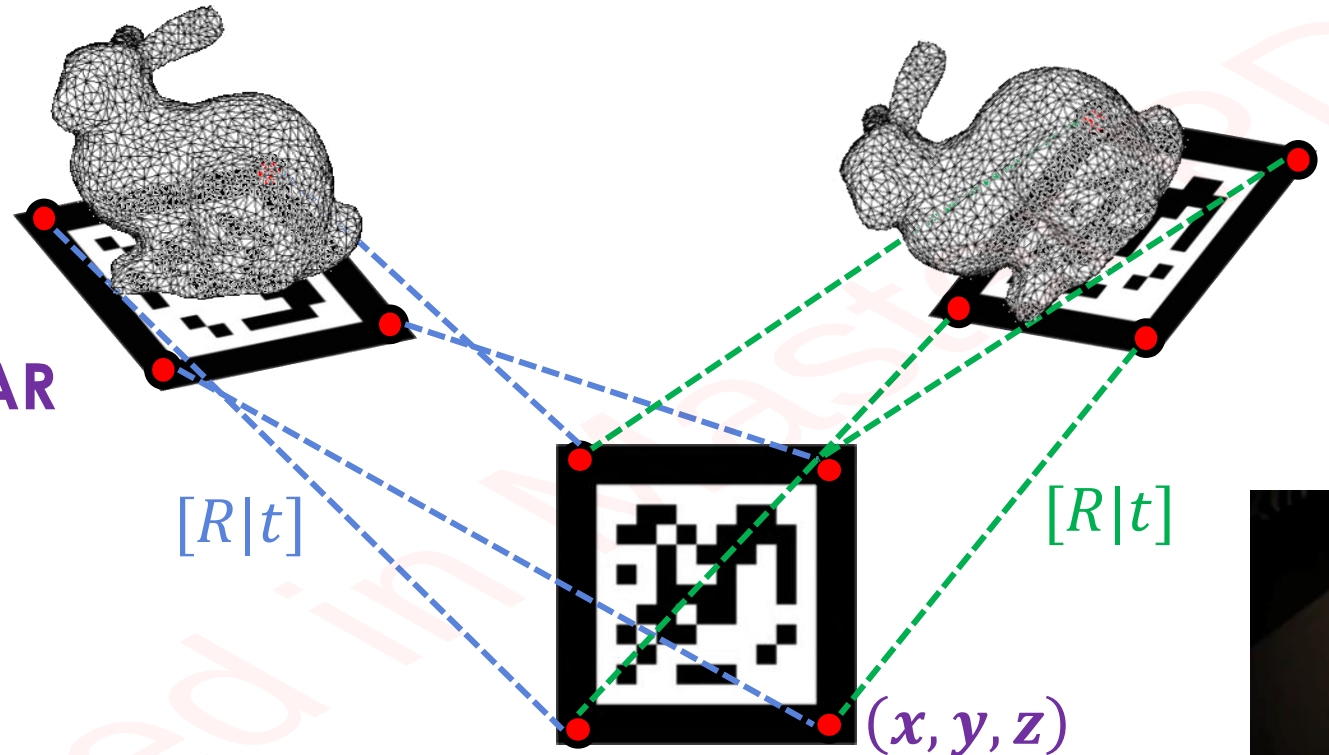
Perspective-n-Point (PnP)

Intrinsic Matrix *Extrinsic Matrix*

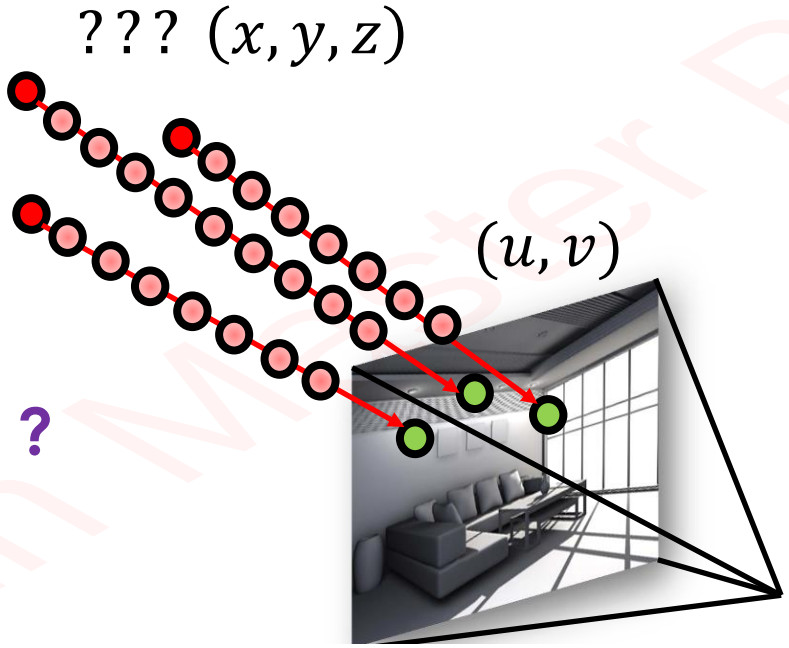
$$s\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & \gamma & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Unknown Need 3 Points to Solve (P3P)

Transformation Matrix Estimation by Reprojection



Transformation Estimation by Reprojection

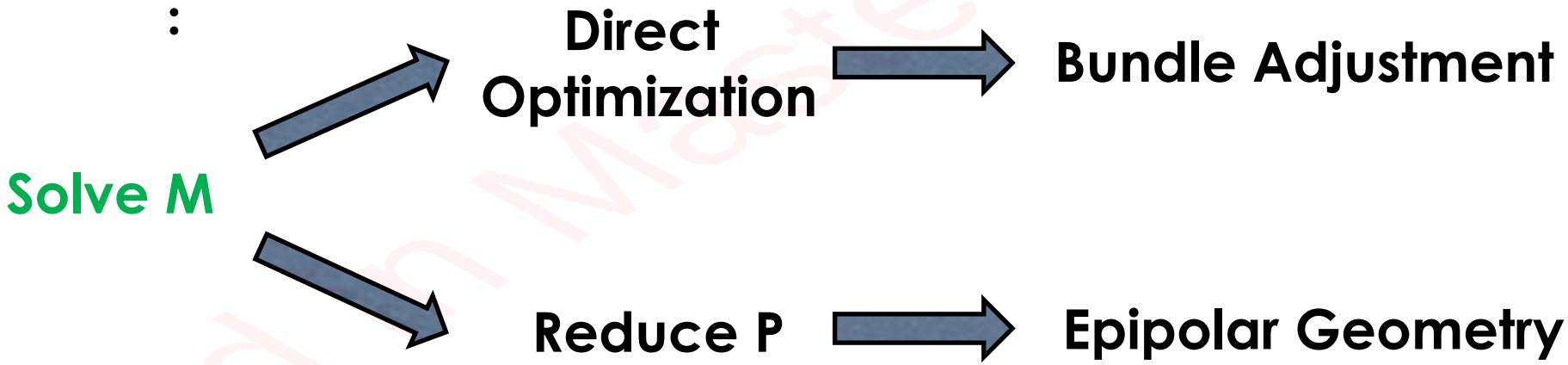


Unknown Structure ?

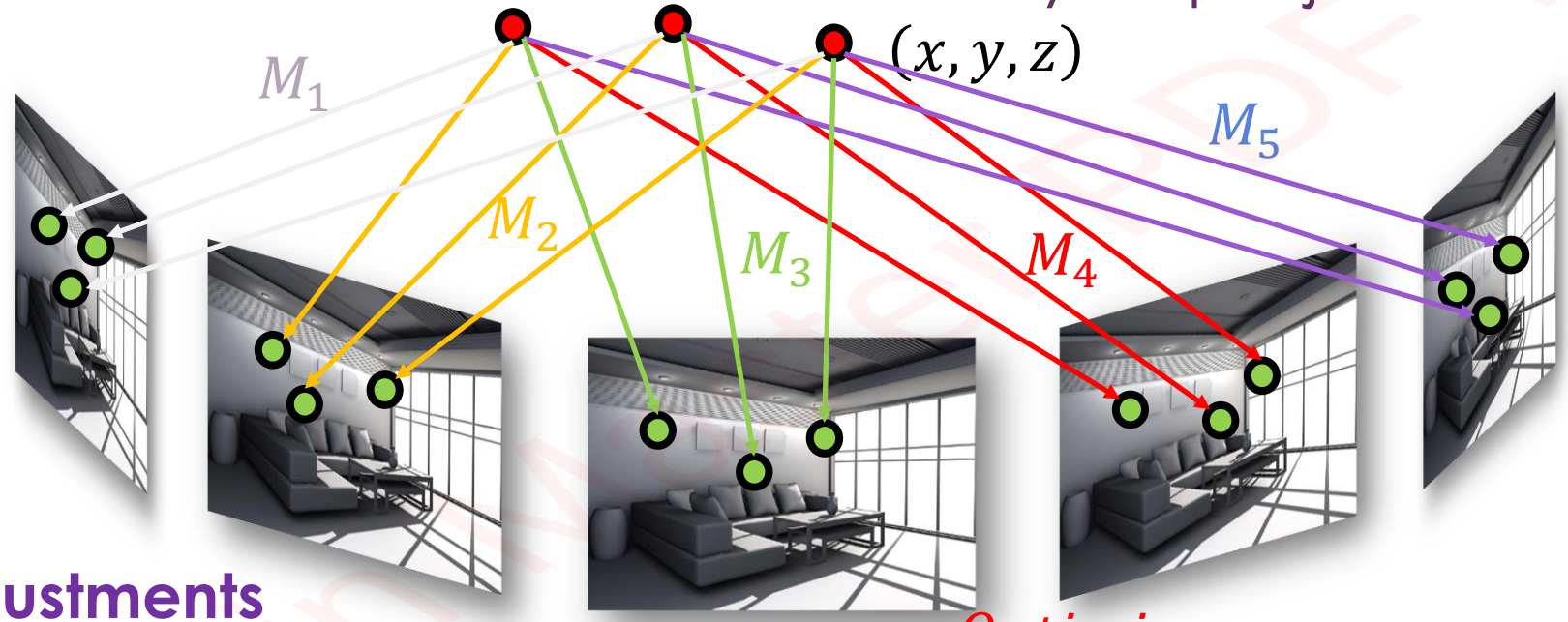
$$s\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & \gamma & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Transformation Matrix Estimation by Reprojection

$$sx_1 = KM_1P$$
$$sx_2 = KM_2P$$
$$\vdots$$



Transformation Matrix Estimation by Reprojection



Bundle Adjustments

Camera Matrix

Optimize

$$s\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & \gamma & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

K 假设已知

M

Unknown Structure Initialization

$$\overrightarrow{C_0p_0} \cdot \left(\overrightarrow{C_0C_1} \times \overrightarrow{C_1p_1}\right) = 0$$

法向量

$$\mathbf{p}_0 \cdot (\mathbf{t} \times \mathbf{R}\mathbf{p}_1) = 0$$

$$\mathbf{p}_0^T [\mathbf{t}]_\times \mathbf{R}\mathbf{p}_1 = 0$$

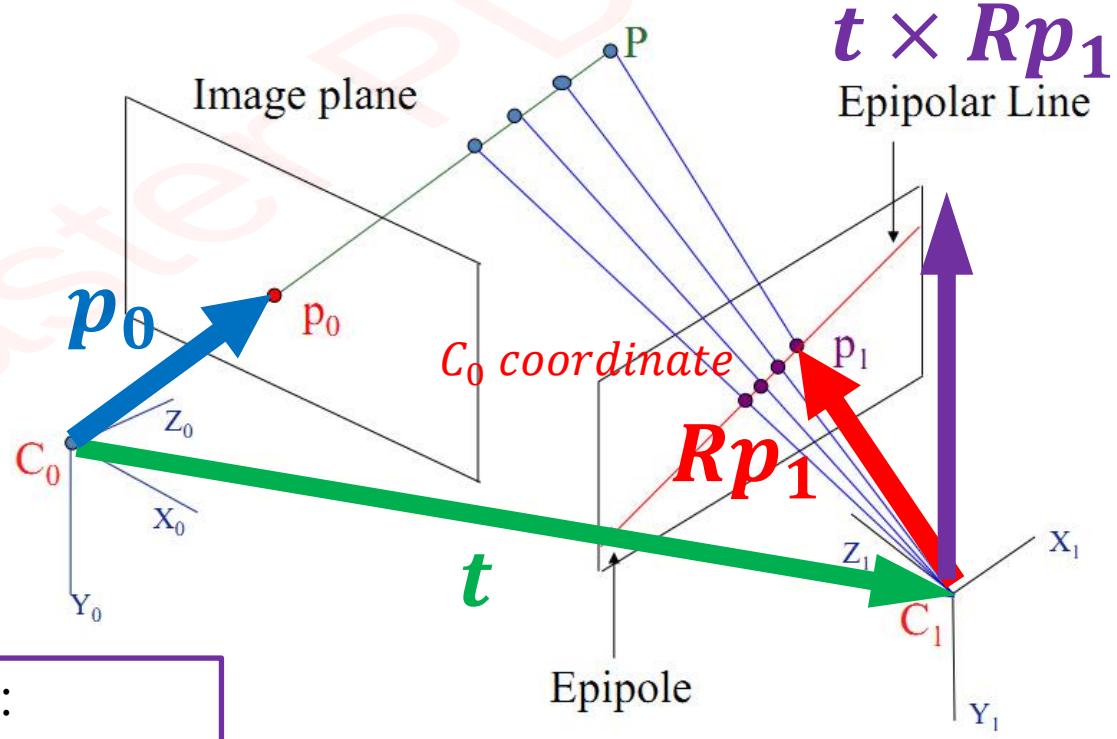
$$\mathbf{p}_0^T \mathbf{E}\mathbf{p}_1 = 0$$

Essential Matrix

Cross Matrix :

$$[t]_\times = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}$$

Epipolar Geometry

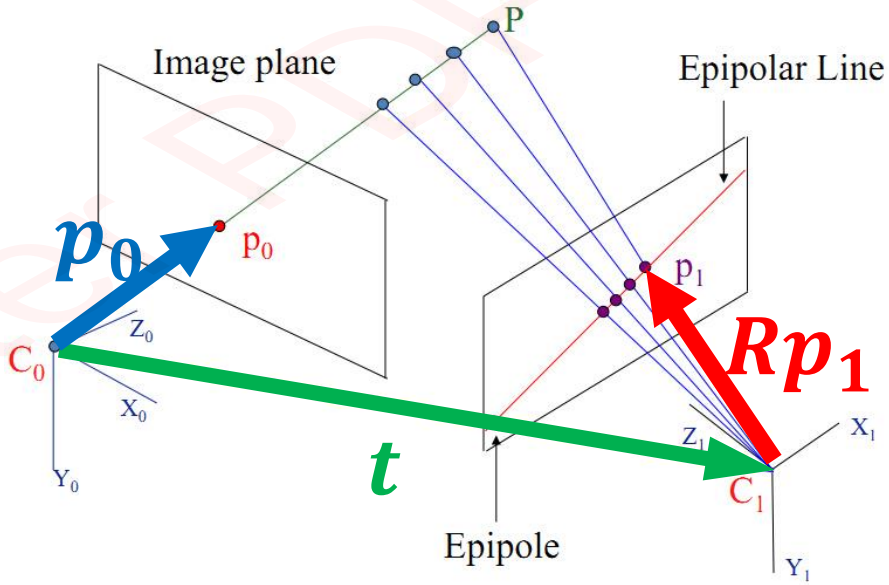


Unknown Structure Initialization

$$\mathbf{p}_0^T \, \mathbf{E} \, \mathbf{p}_1 = 0$$

$$\begin{pmatrix} x_0 & y_0 & 1 \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = 0$$

Epipolar Geometry



Write as **A x = 0**, where **x** = (E11, E12, E13, … , E33)

$$\begin{pmatrix} x_0x_1 & x_0y_1 & x_0 & y_0x_1 & y_0y_1 & y_0 & x_1 & y_1 & 1 \end{pmatrix} \begin{pmatrix} E_{11} \\ E_{12} \\ E_{13} \\ \vdots \\ E_{33} \end{pmatrix} = 0$$

Unknown Structure Initialization

Essential Matrix Decomposition

$$E = [t]_\times R = U\Sigma V^T = U \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

Scale

Loss of Rank

$$Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, ZW = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

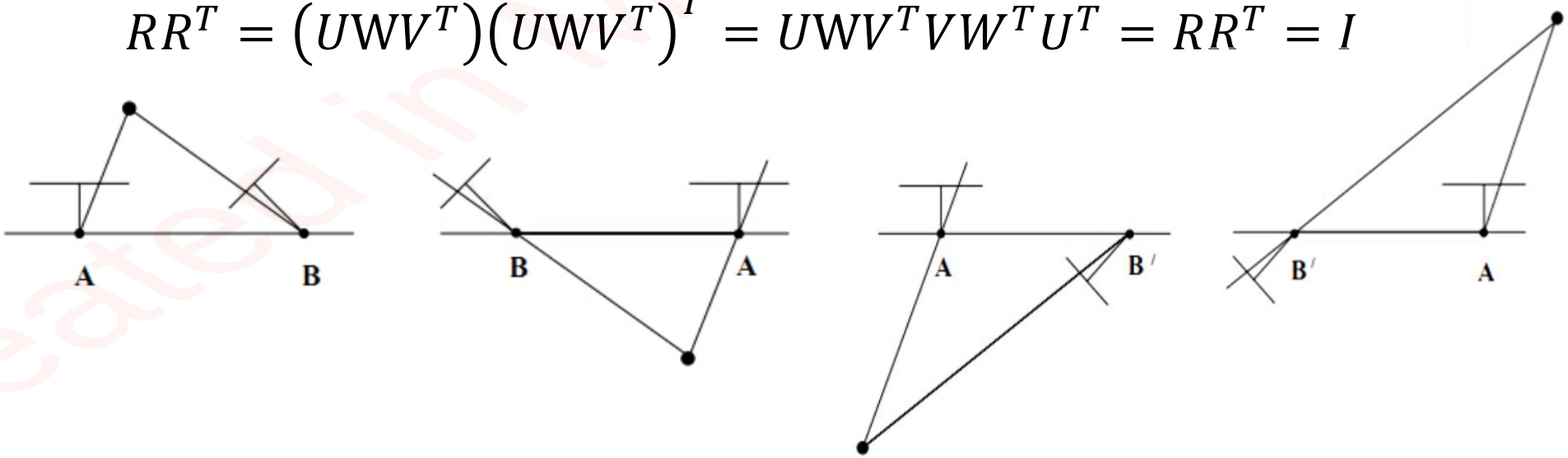
$$E = UZWV^T = \underbrace{(UZU^T)}_{[t]_\times}\underbrace{(UWV^T)}_{R} = [\boldsymbol{t}]_\times R$$

Unknown Structure Initialization

Essential Matrix Decomposition

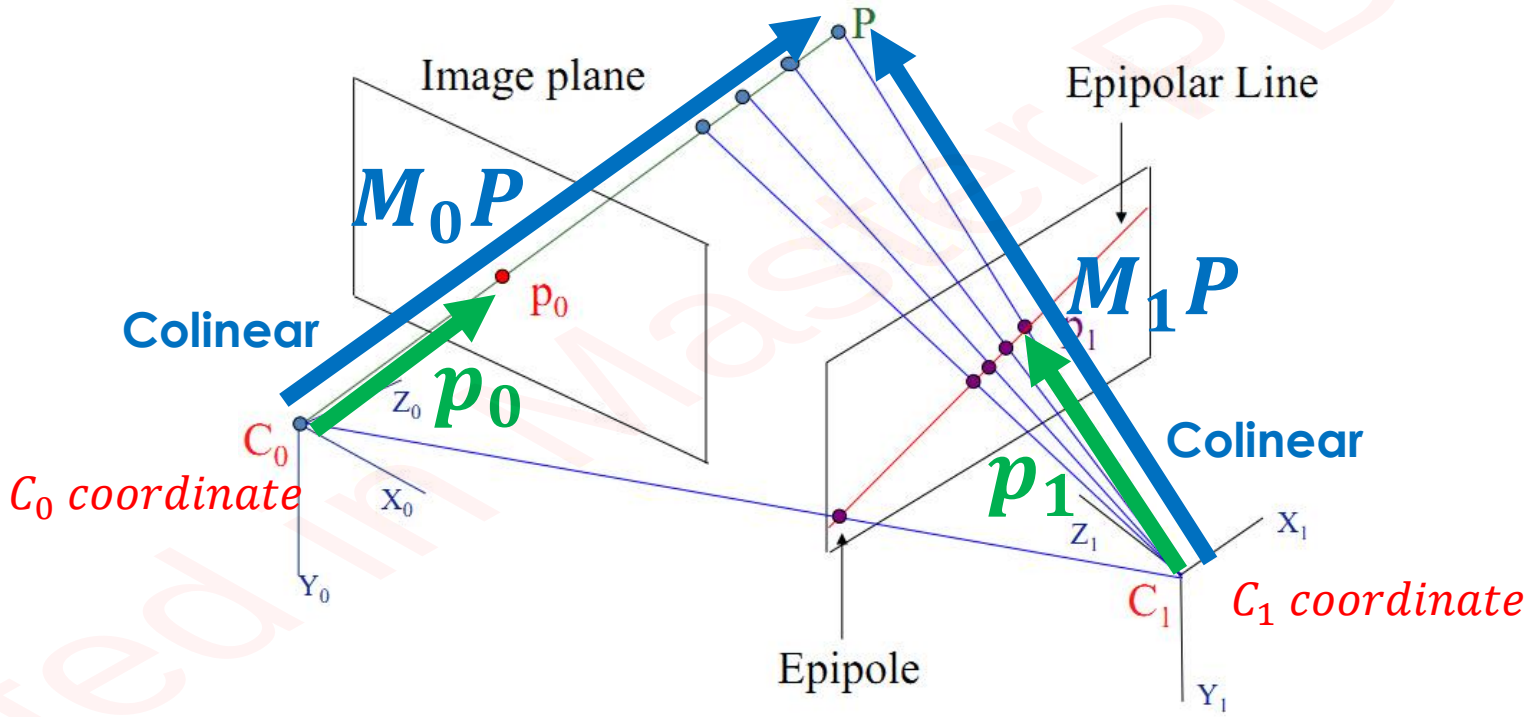
$$\begin{bmatrix} 0 & u_{33} & -u_{23} \\ -u_{33} & 0 & u_{13} \\ u_{23} & -u_{13} & 0 \end{bmatrix} \longrightarrow [\boldsymbol{t}]_{\times}$$

$$RR^T = \left(UWV^T\right)\left(UWV^T\right)^T = UWV^TVW^TU^T = RR^T = I$$



Unknown Structure Initialization

3D Structure Recovering (Triangulation)

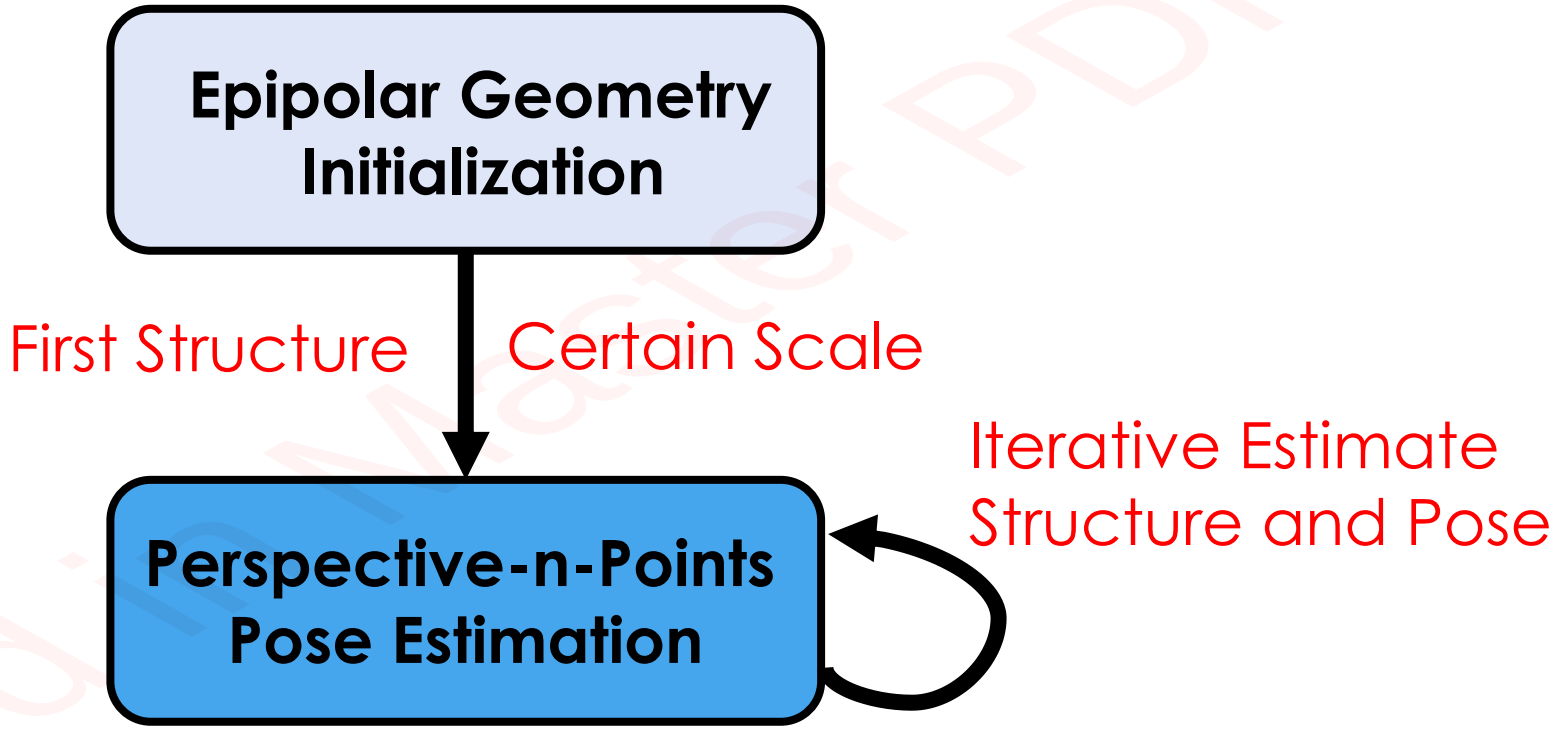


$$M_0 = [R_0|t_0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, M_1 = [R_1|t_1]$$

$$\begin{aligned} p_0 \times M_0 P &= 0 \\ p_1 \times M_1 P &= 0 \end{aligned} \Rightarrow \begin{pmatrix} [(p_0)_\times] M_0 \\ [(p_1)_\times] M_1 \end{pmatrix} P = 0$$

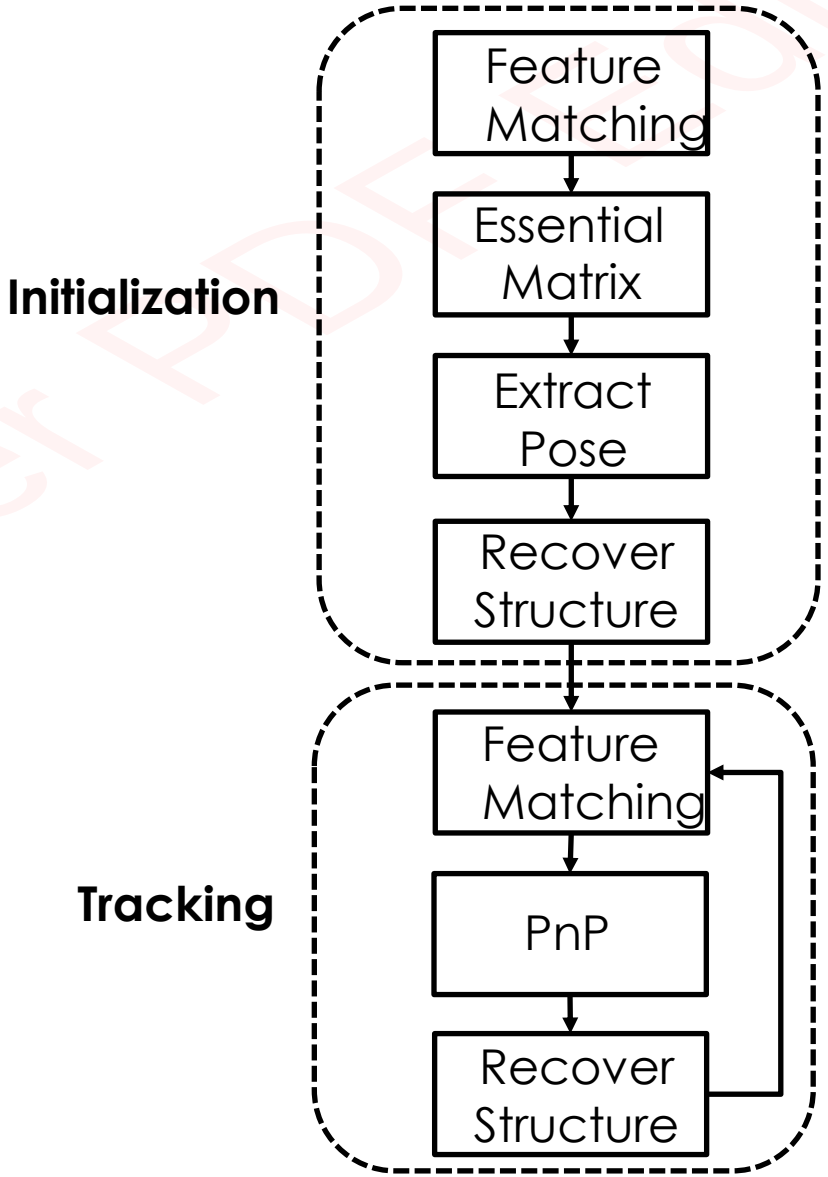
Basic Initialization and Tracking

Concept



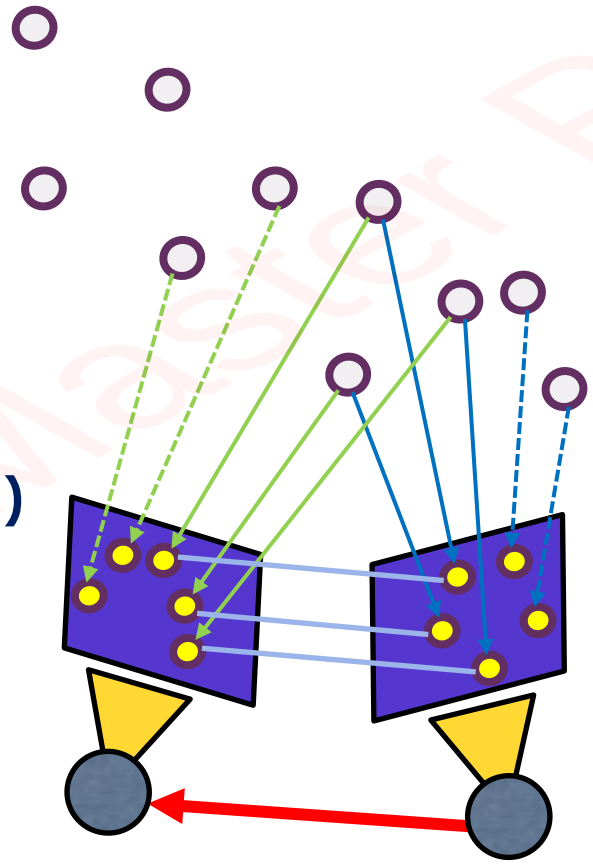
Use known structure points to estimate the pose then compute new points by triangulation.

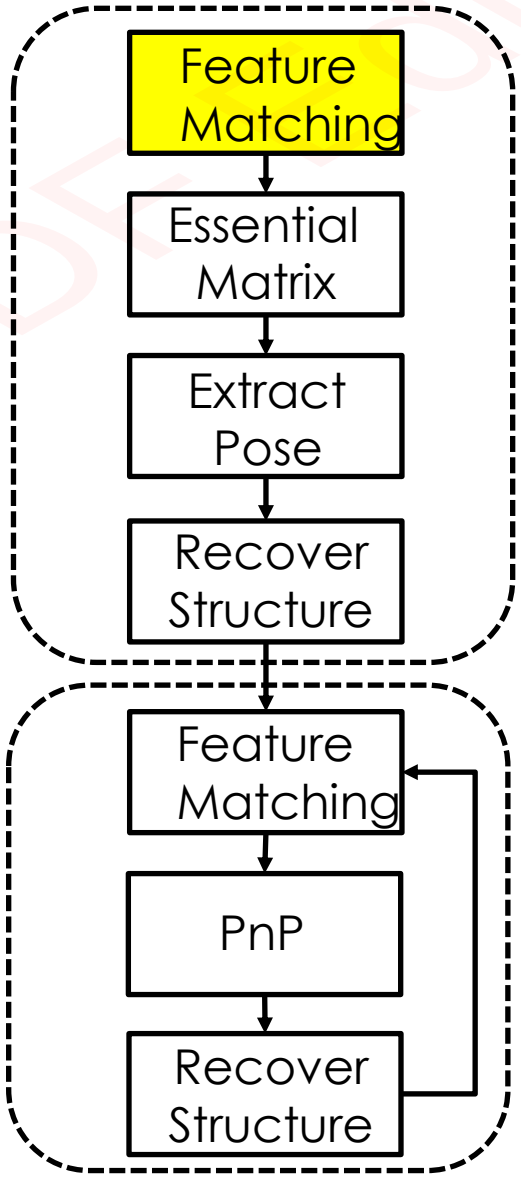
Basic Initialization and Tracking



Basic Initialization and Tracking

Initialization (Feature Matching)

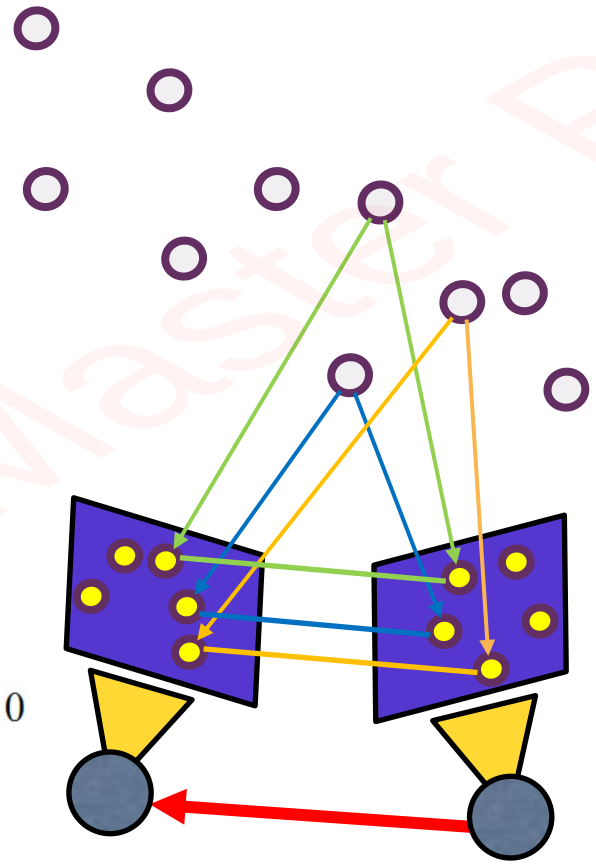


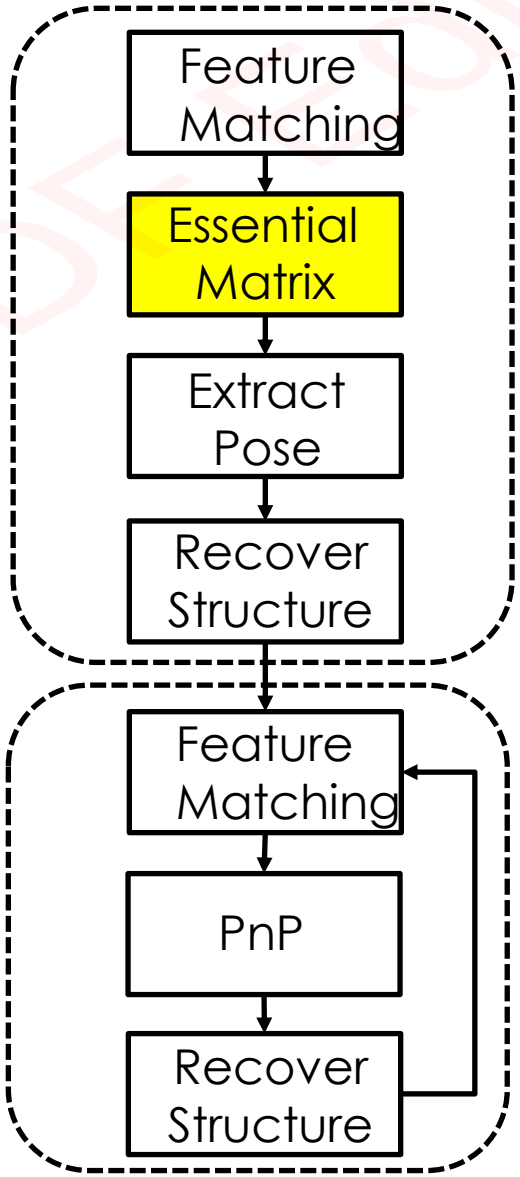


Basic Initialization and Tracking

Initialization (Essential Matrix)

$$\begin{pmatrix} x_0 & y_0 & 1 \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = 0$$

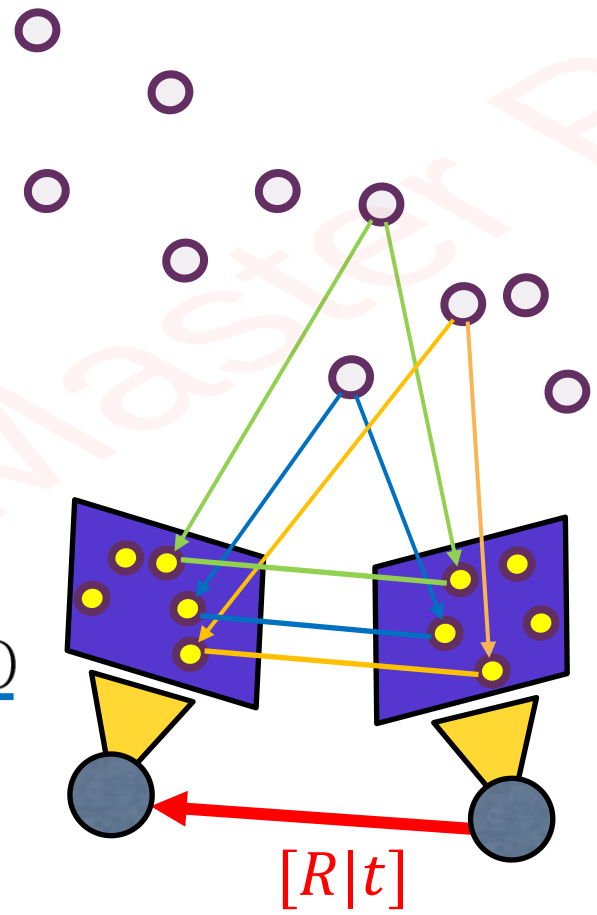


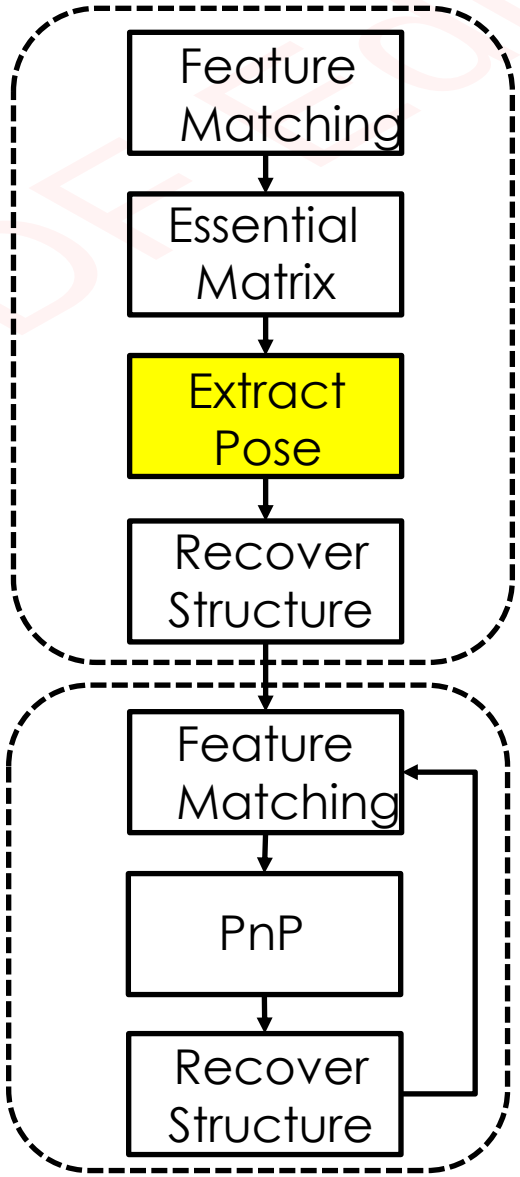


Basic Initialization and Tracking

Initialization (Extract Pose)

$$E = UZWV^T = \underbrace{(UZU^T)}_{S}\underbrace{(UWV^T)}_{R}$$

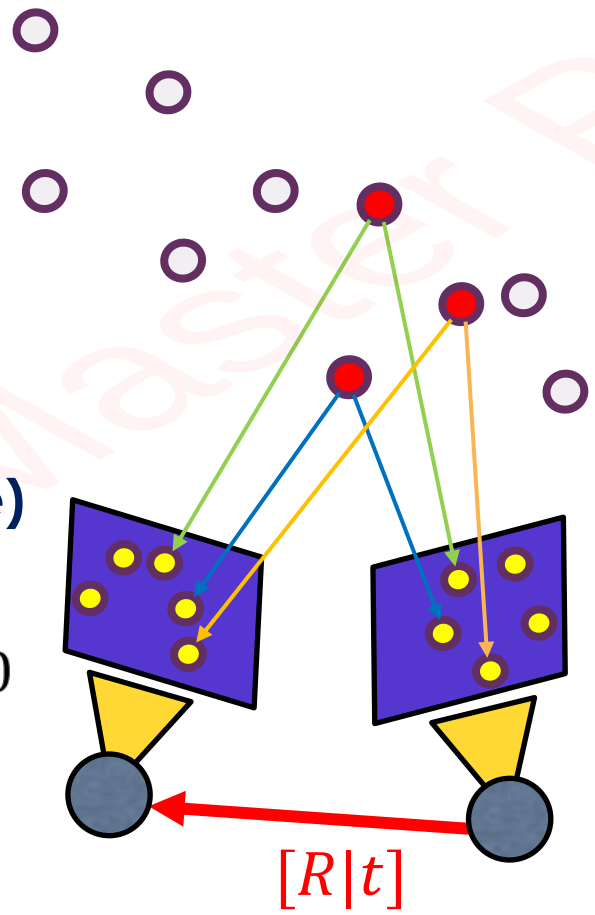


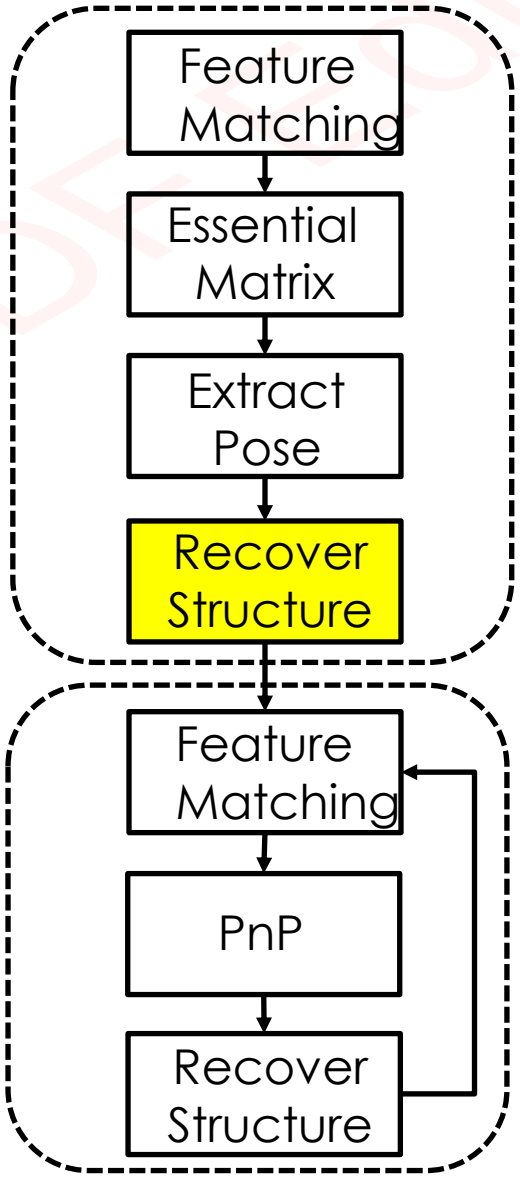


Basic Initialization and Tracking

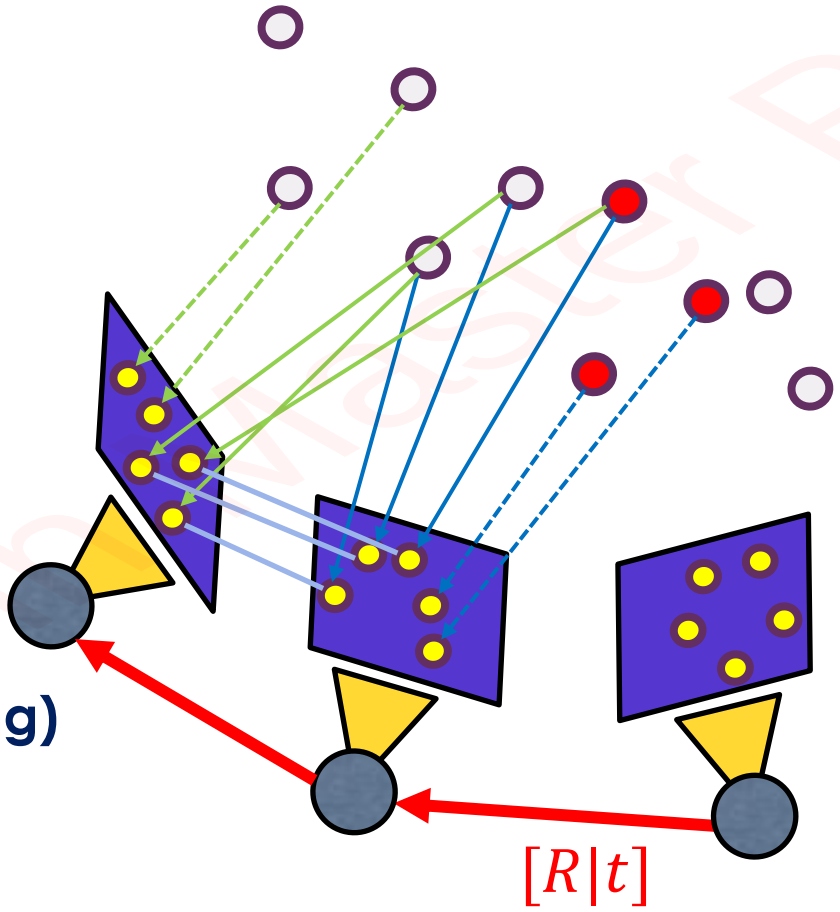
Initialization
(Recover Structure)

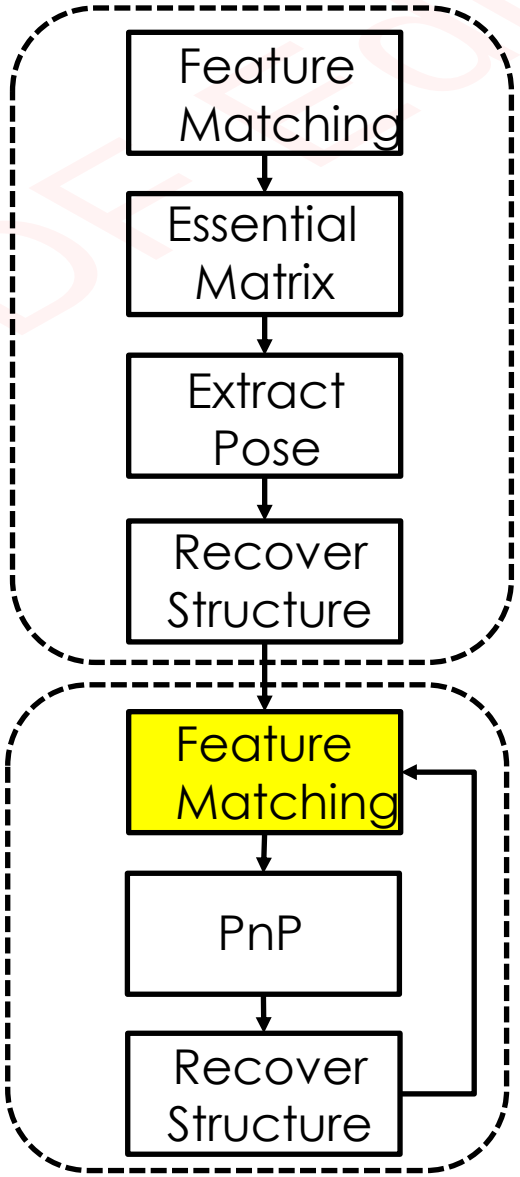
$$\begin{pmatrix} [(P_0)_\times]M_0 \\ [(P_1)_\times]M_1 \end{pmatrix} P = 0$$



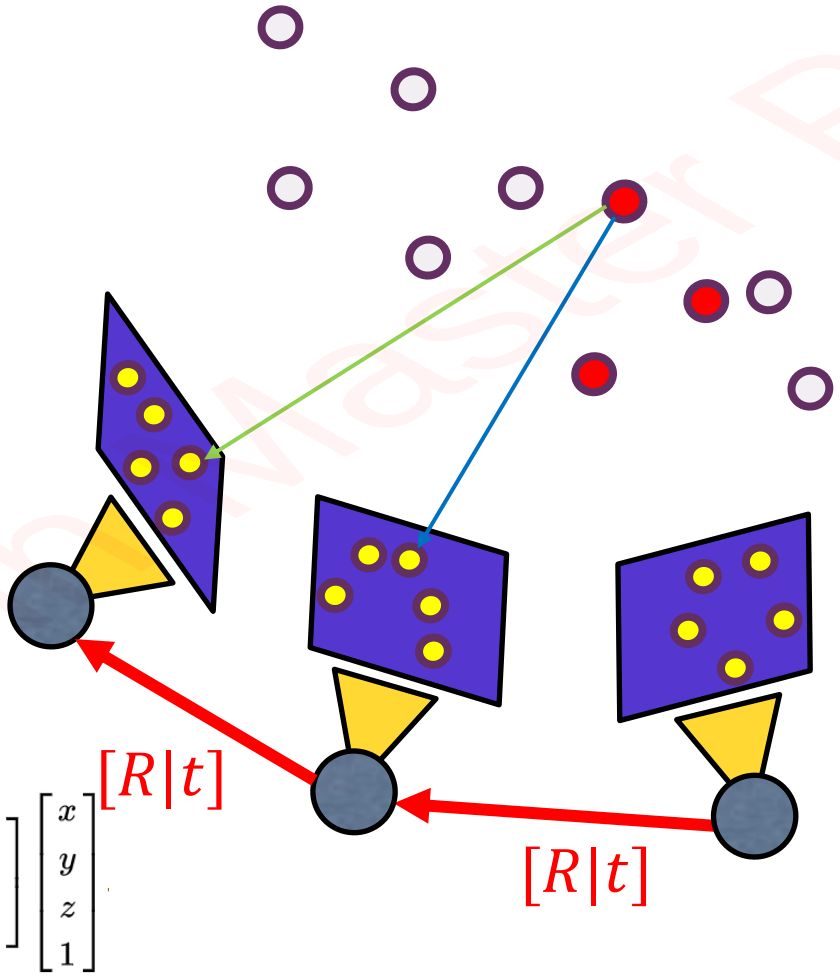


Basic Initialization and Tracking



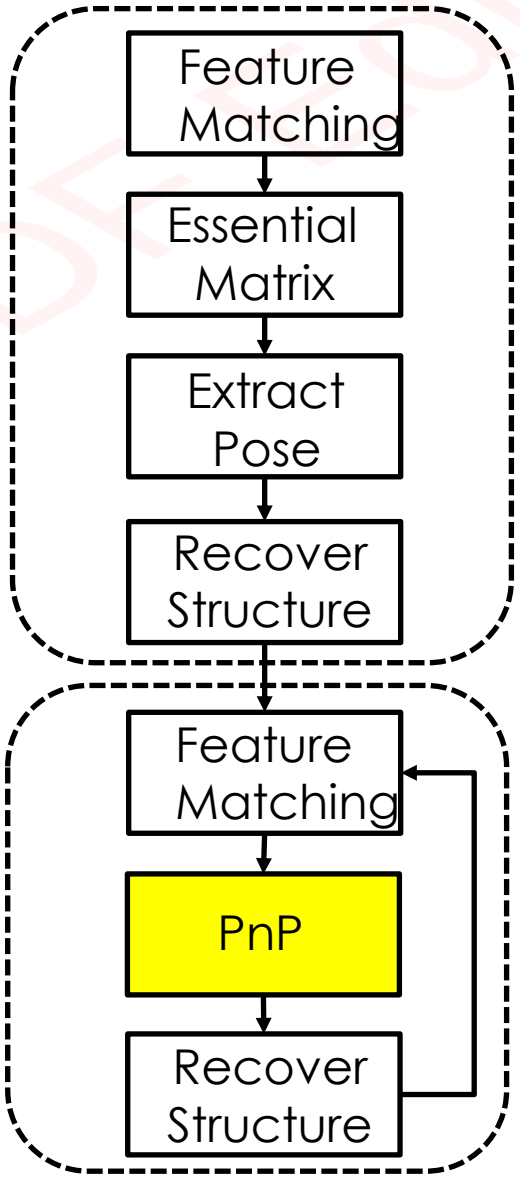


Basic Initialization and Tracking

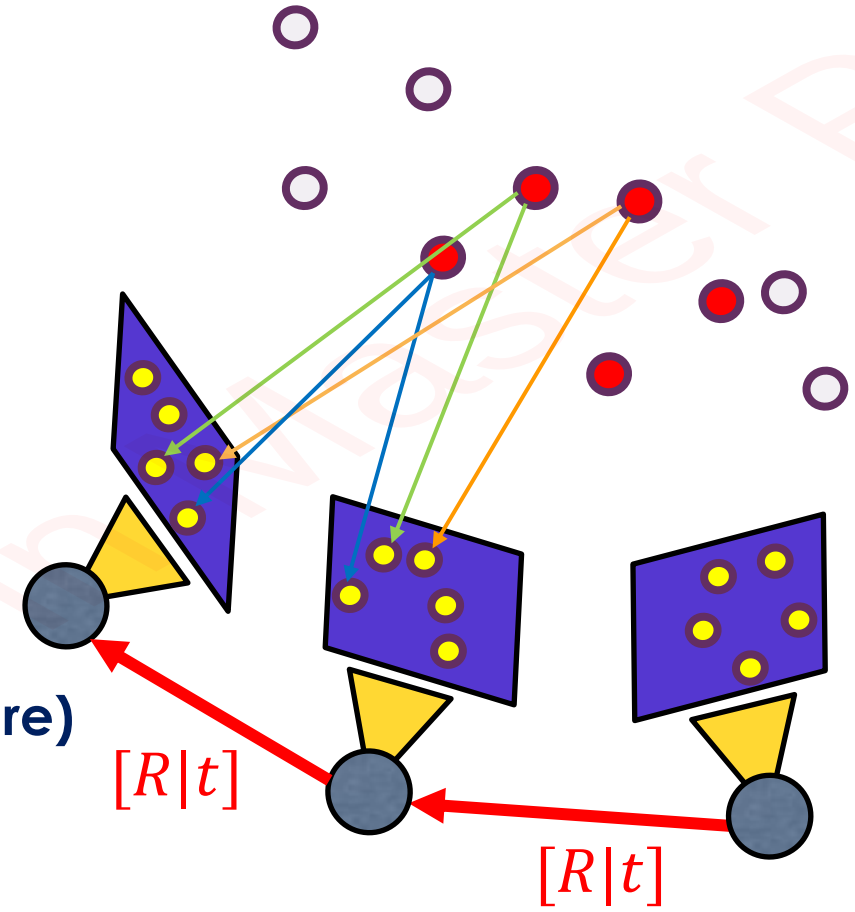


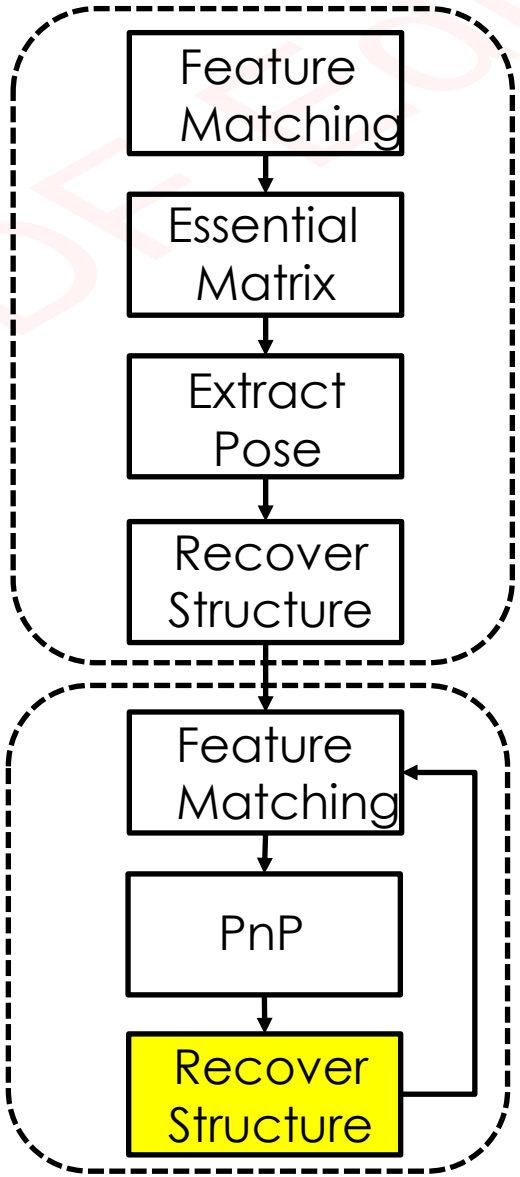
Start Tracking 1 (PnP)

$$s\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & \gamma & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

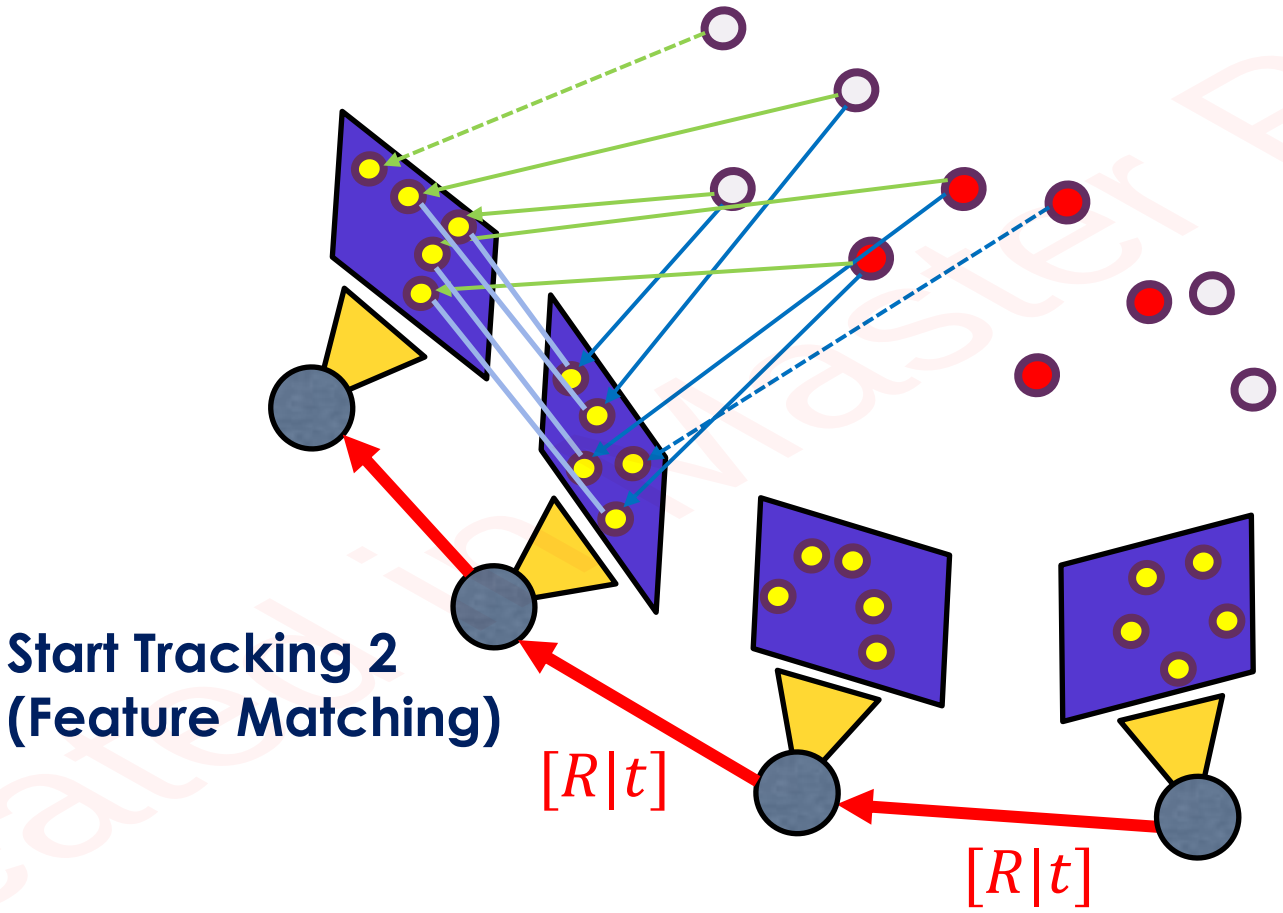


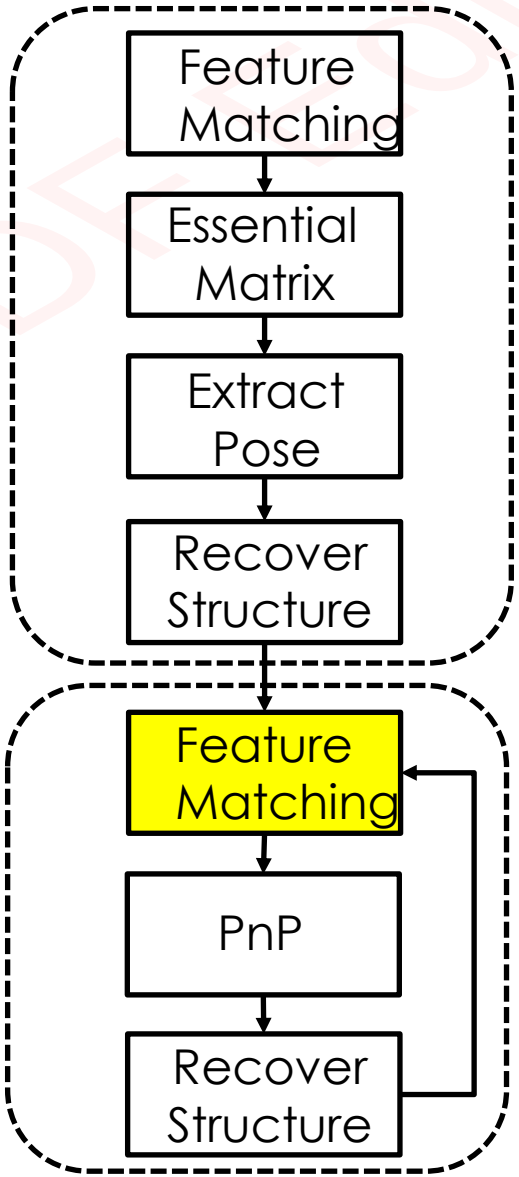
Basic Initialization and Tracking



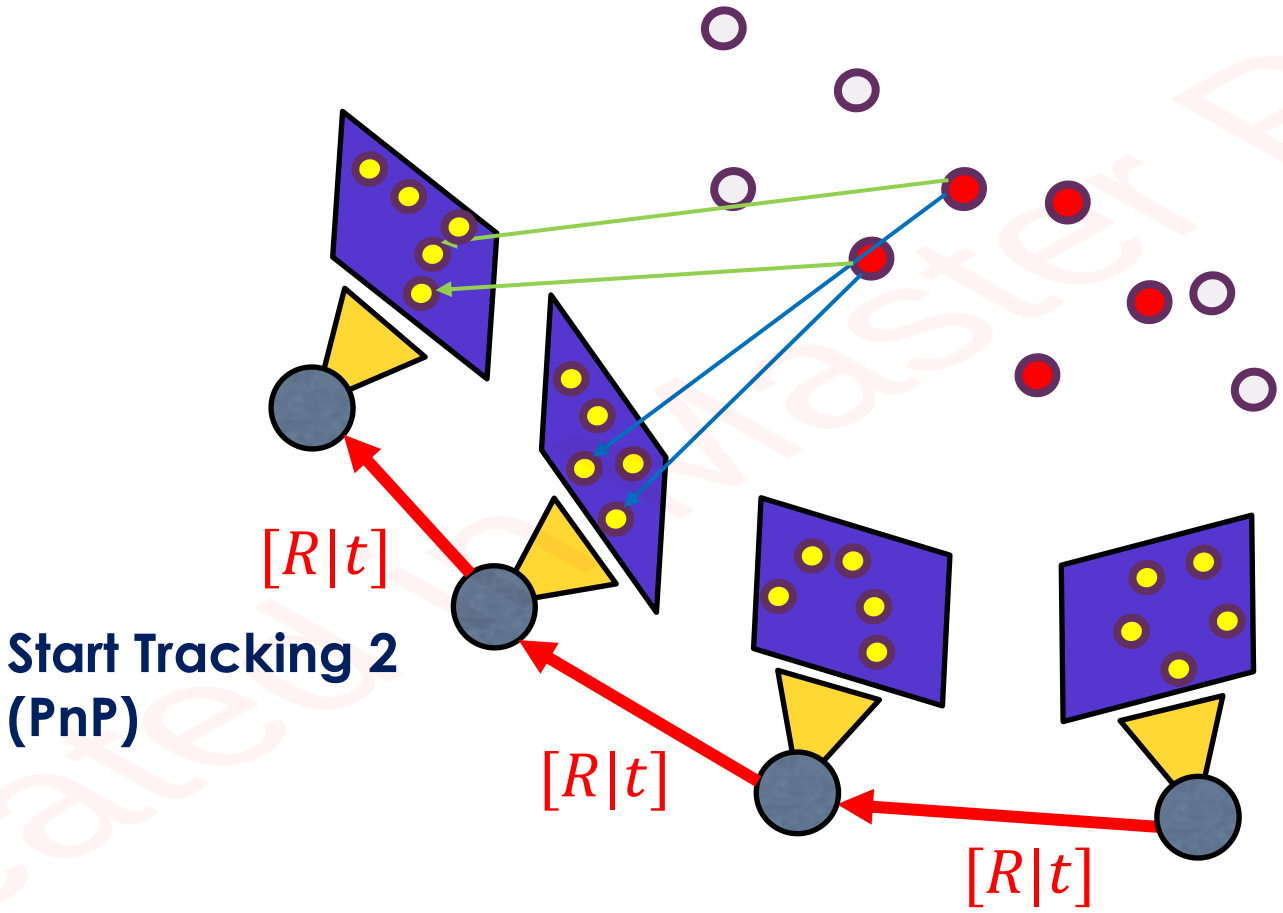


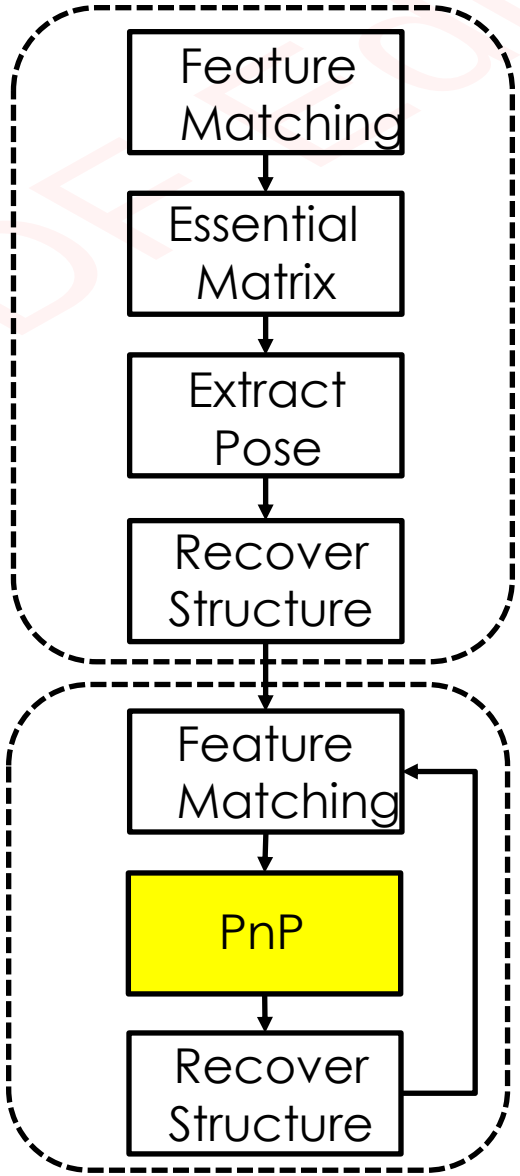
Basic Initialization and Tracking



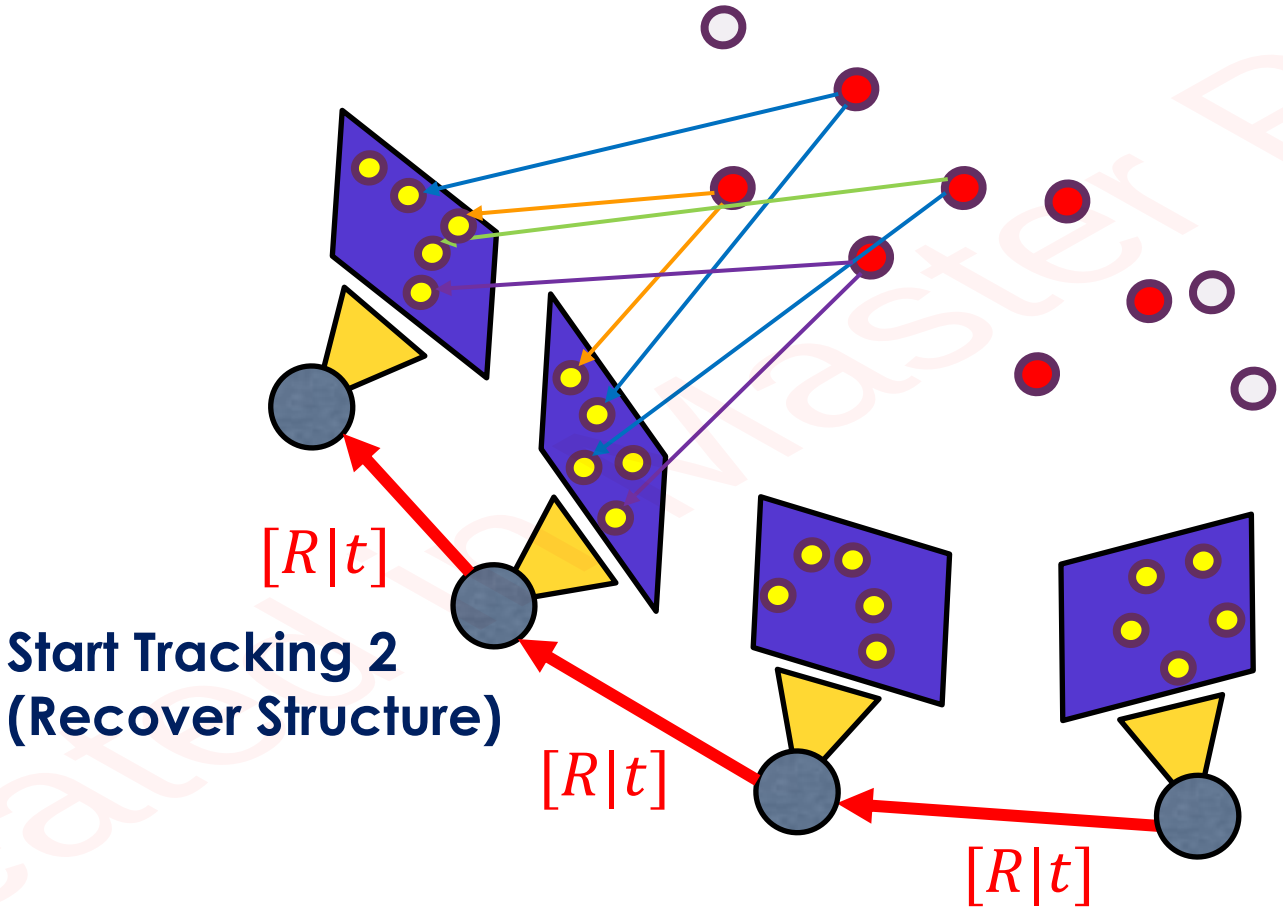


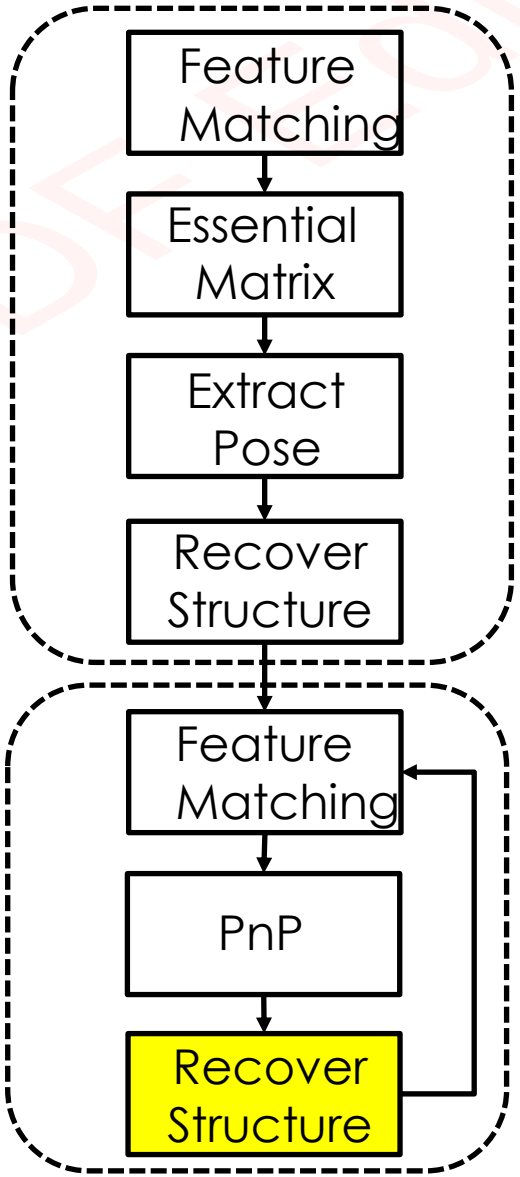
Basic Initialization and Tracking



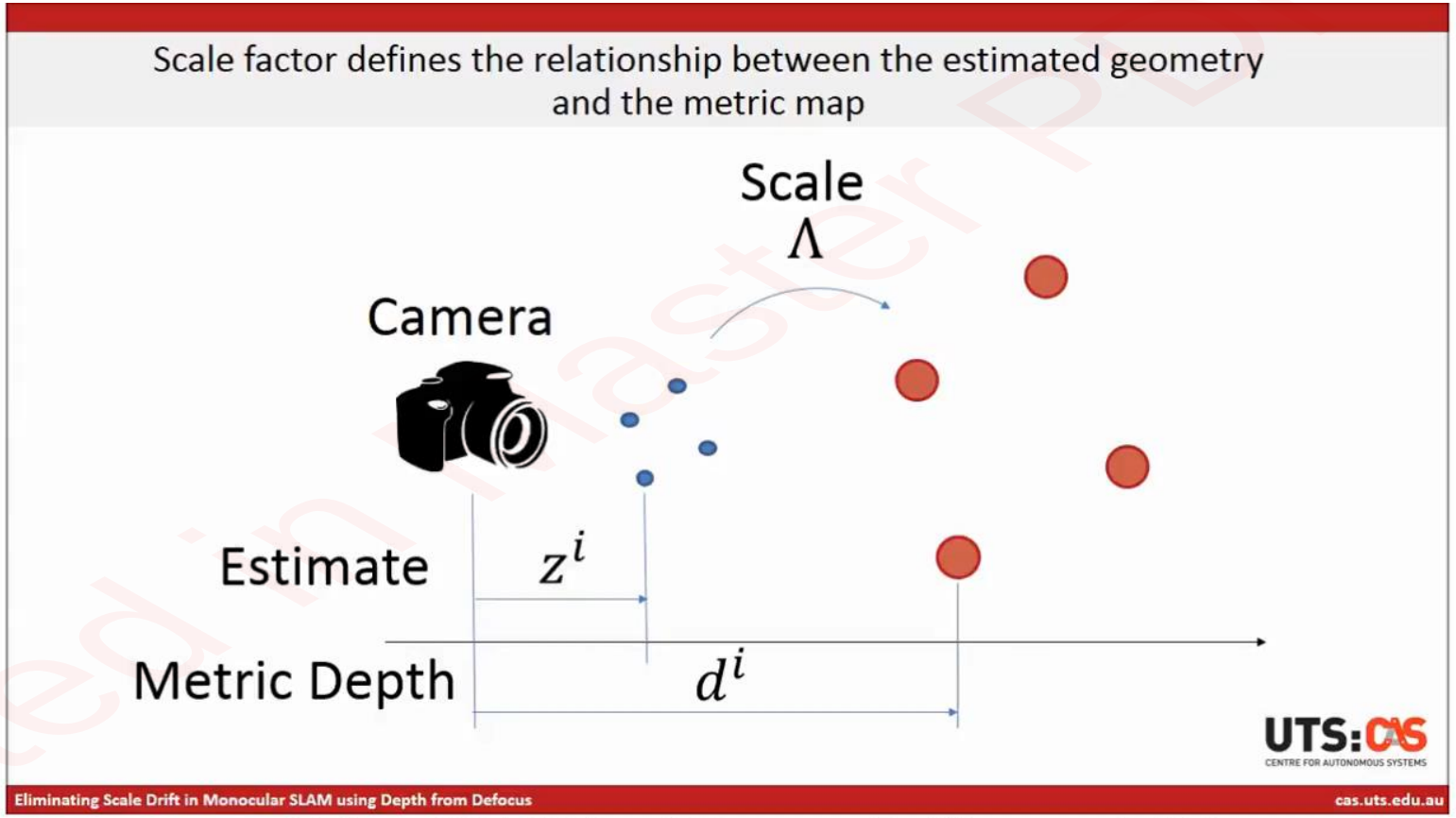


Basic Initialization and Tracking





Scaling Drift Problem



 Tue AM Pod U.2

https://www.youtube.com/watch?v=-oWxwL1KSfA

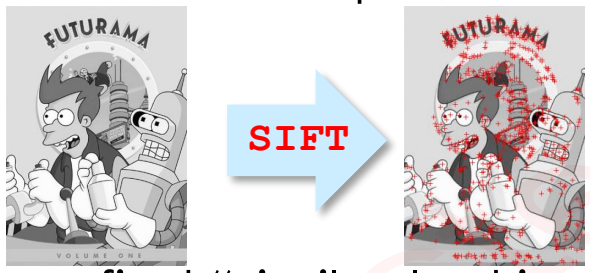
Image Matching

- How do we detect an object in an image?

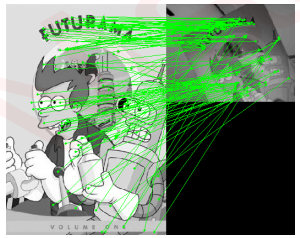
- Combines ideas from **image transformations**, **least squares**, and **robustness**

Object Matching in Three Steps

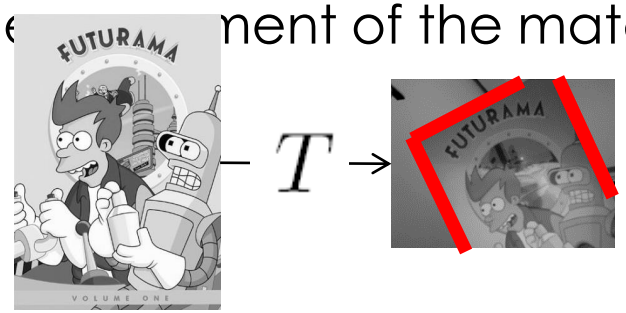
1. Detect features in the template and search images



2. Match features: find "similar-looking" features in the two images



3. Find a transformation $\boldsymbol{T}$ that explains the ment of the matched features



Affine Transformations

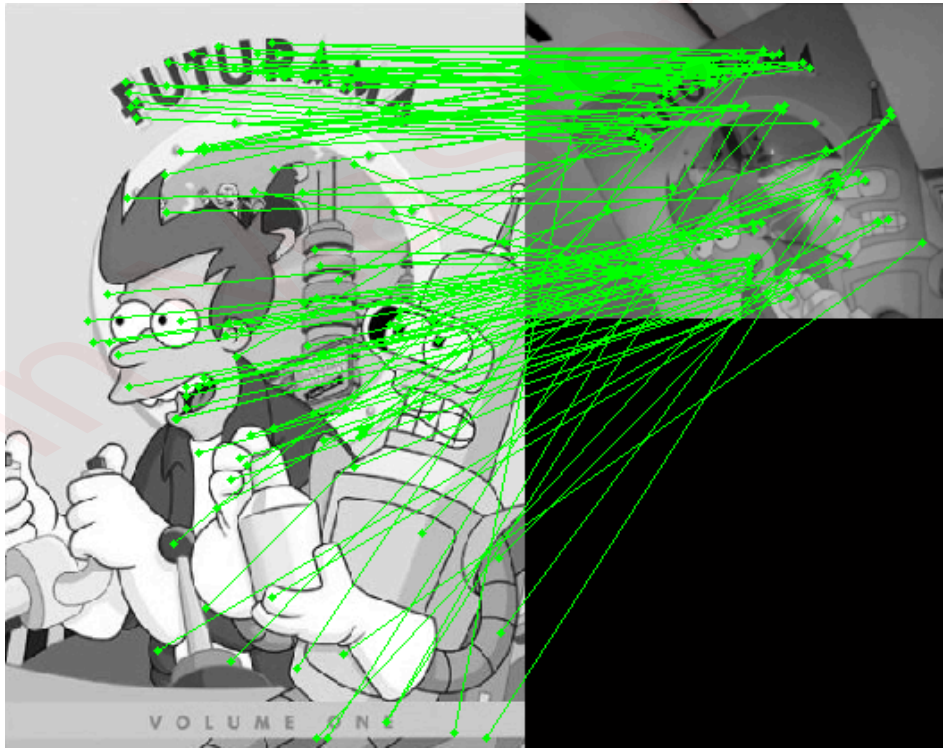
- A *2D affine transformation* has the form:

$$T = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

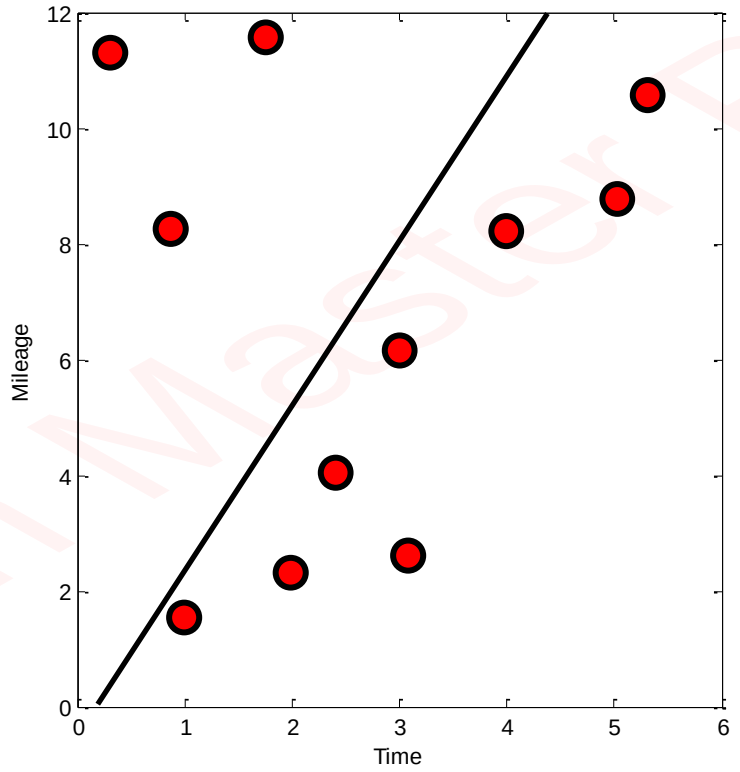
$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

Fitting affine transformations

- We will fit an affine transformation to a set of feature matches
  - Problem: there are many incorrect matches

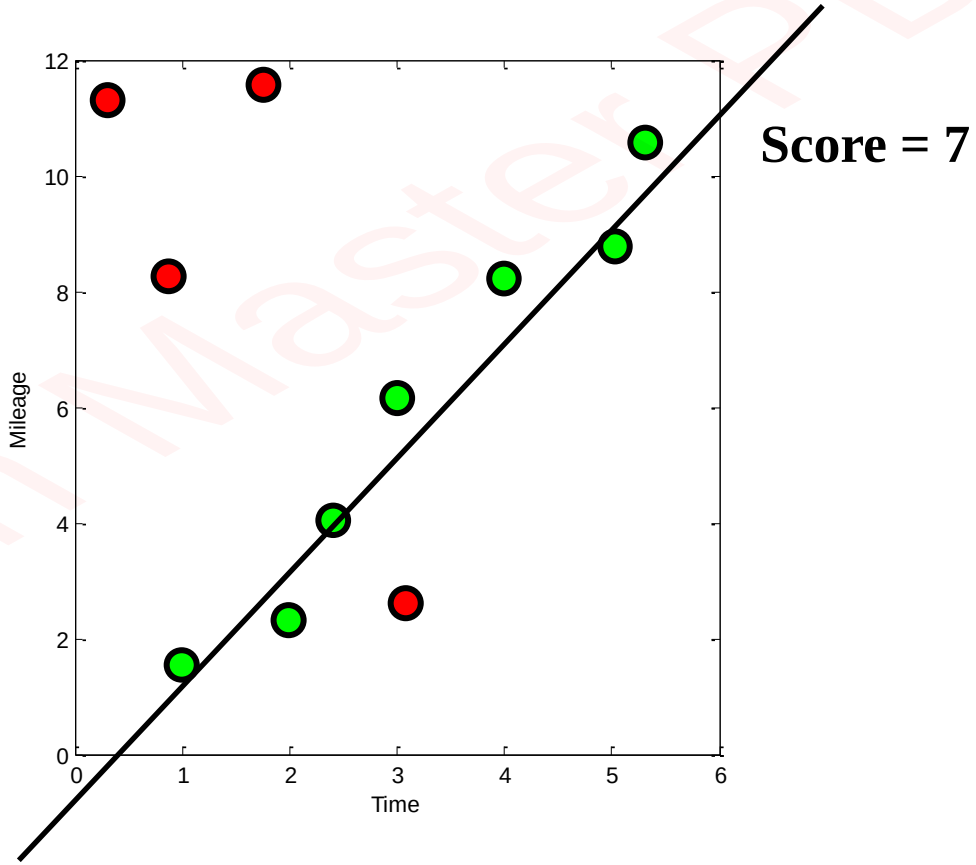


Linear Regression



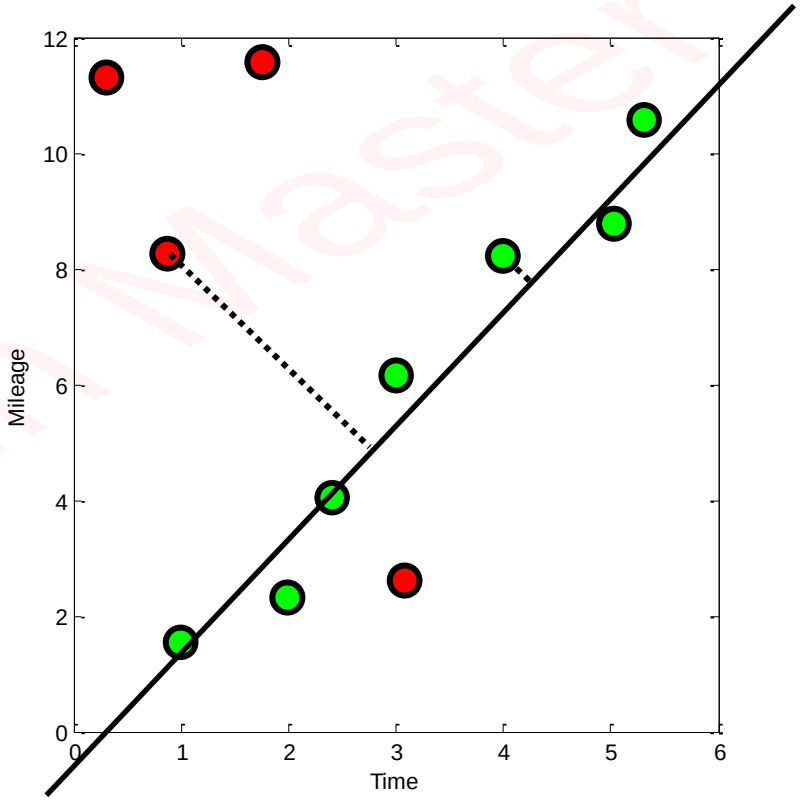
Testing Goodness

- Idea: *count* the number of points that are "close" to the line



Testing Goodness

- How can we tell if a point agrees with a line?
  - Compute the distance the point and the line, and threshold

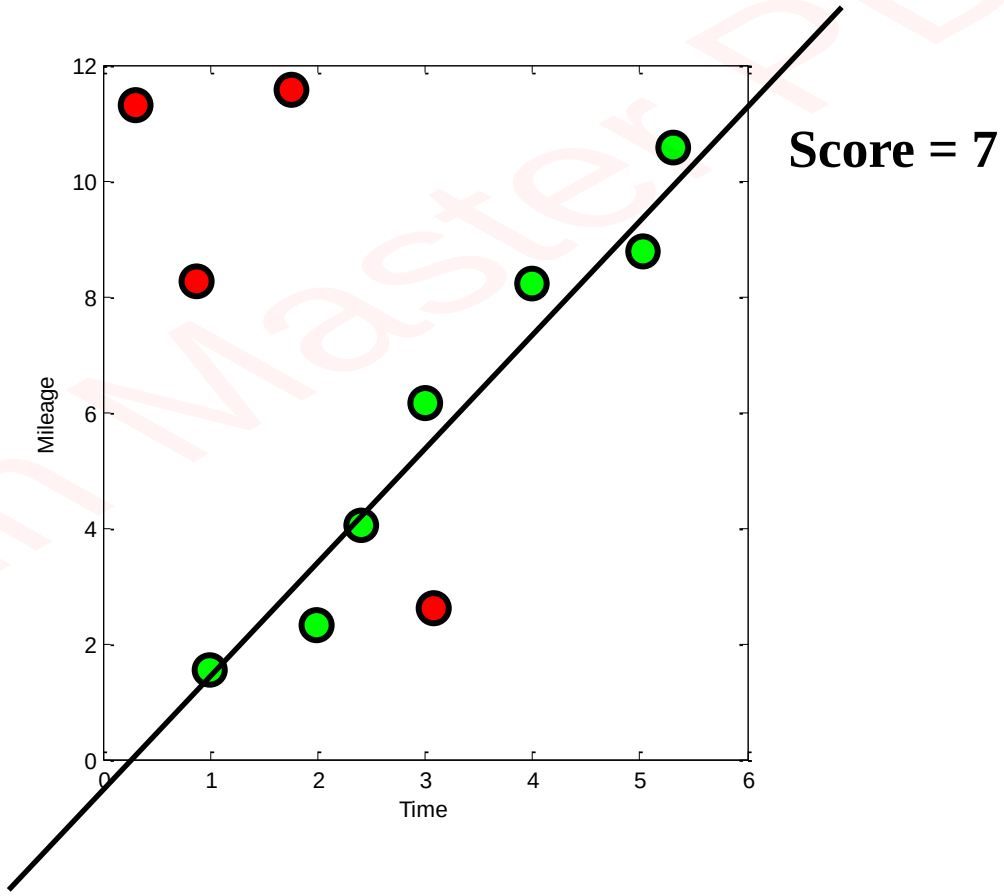


Testing Goodness

- If the distance is small, we call this point an *inlier* to the line
  - For an inlier point and a good line, this distance will be close to (but not exactly) zero
- If the distance is large, it's an *outlier* to the line
  - For an outlier point or bad line, this distance will probably be large
- Objective function: find the line with the most inliers (or the fewest outliers)

Optimizing for Inlier Count

- How do we find the best possible line?



Algorithm (RANSAC)

1. Select two points at random
2. Solve for the line (L) between these two points
3. Count the number of inliers to the line *L*
4. If *L* has the highest number of inliers so far, save it
5. Repeat for N rounds, return the best *L*

Testing goodness

(随机样本一致)

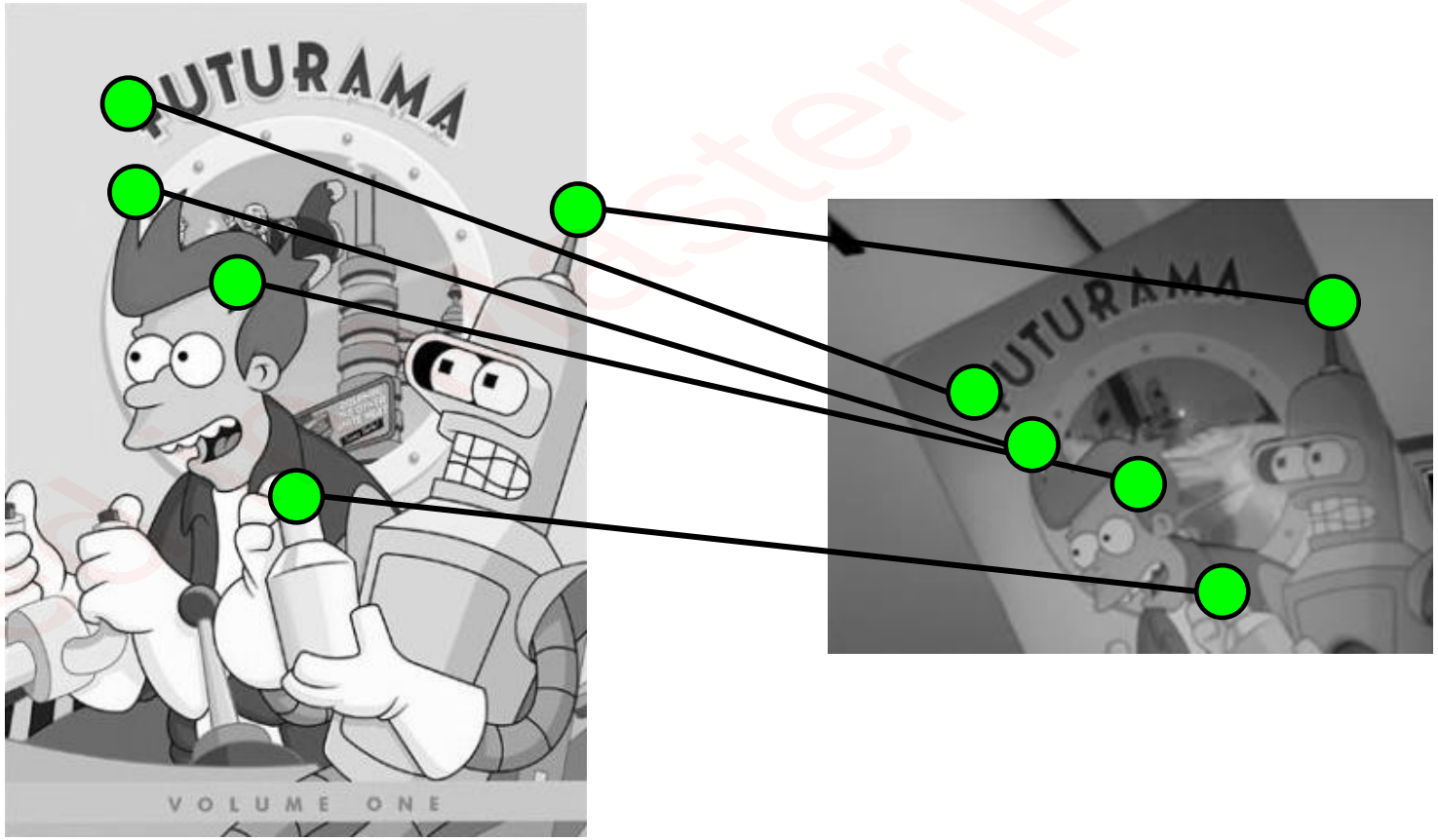
- This algorithm is called RANSAC (RANdom SAmple Consensus) – example of a randomized algorithm
- Used in an amazing number of computer vision algorithms
- Requires two parameters:
  - The agreement threshold (how close does an inlier have to be?)
  - The number of rounds (how many do we need?)

Randomized algorithms

- Very common in computer science
  - In this case, we avoid testing an infinite set of possible lines, or all $O(n^2)$ lines generated by pairs of points
- These algorithms find the right answer with some probability
- Often work very well in practice

Very Similar Idea

- Given two images with a set of feature matches, how do we compute an affine transform between the two images?



Multi-variable Fitting

- Let's consider a 2D affine transformation that maps a 2D point to another 2D point

$$T = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

- We have a set of n matches

$[x_1 \; y_1] \rightarrow [x_1' \; y_1']$
$[x_2 \; y_2] \rightarrow [x_2' \; y_2']$
$[x_3 \; y_3] \rightarrow [x_3' \; y_3']$
…
$[x_n \; y_n] \rightarrow [x_n' \; y_n']$

Fitting an Affine Transformation

- Consider just one match $[x_1 \; y_1] \rightarrow [x_1' \; y_1']$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1' \\ y_1' \\ 1 \end{bmatrix}$$

$$ax_1 + by_1 + c = x_1'$$

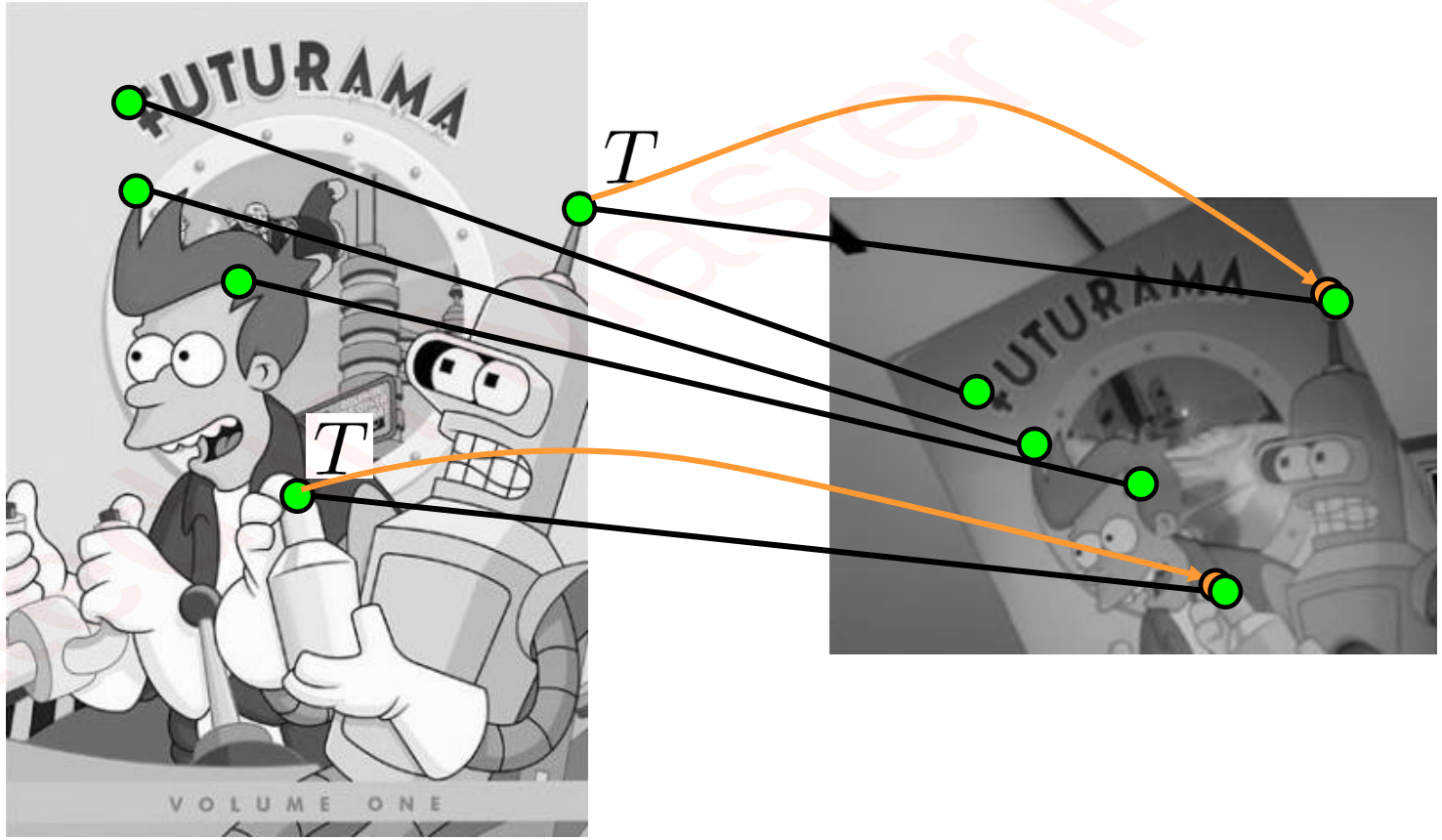
$$dx_1 + ey_1 + f = y_1'$$

- 2 equations, 6 unknowns

→ we need at least 3 matches, but can fit n using least squares

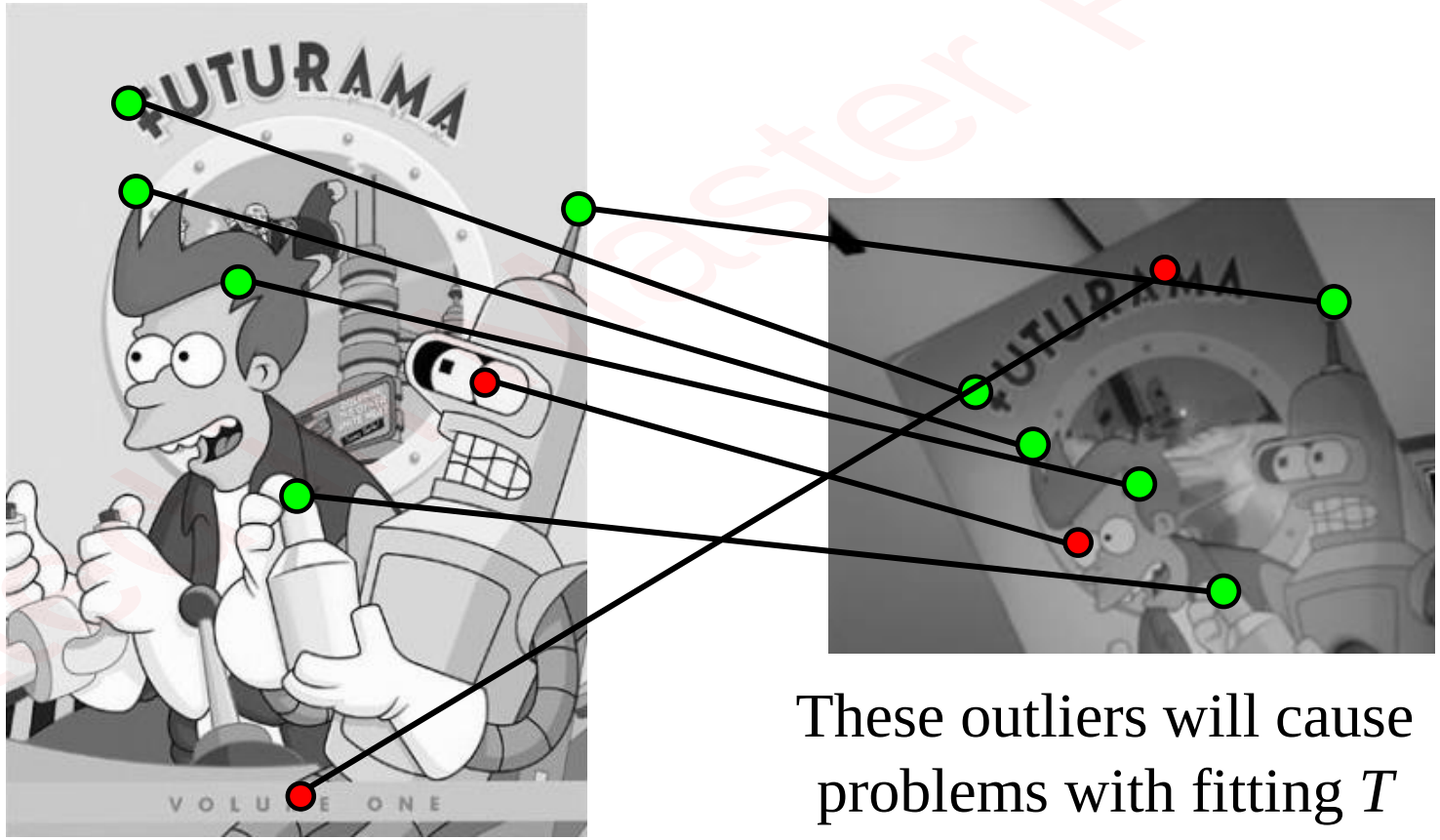
Fitting an Affine Transformation

- Find 2D affine transformation T that maps points in image 1 as close as possible to their matches in image 2



Back to Fitting

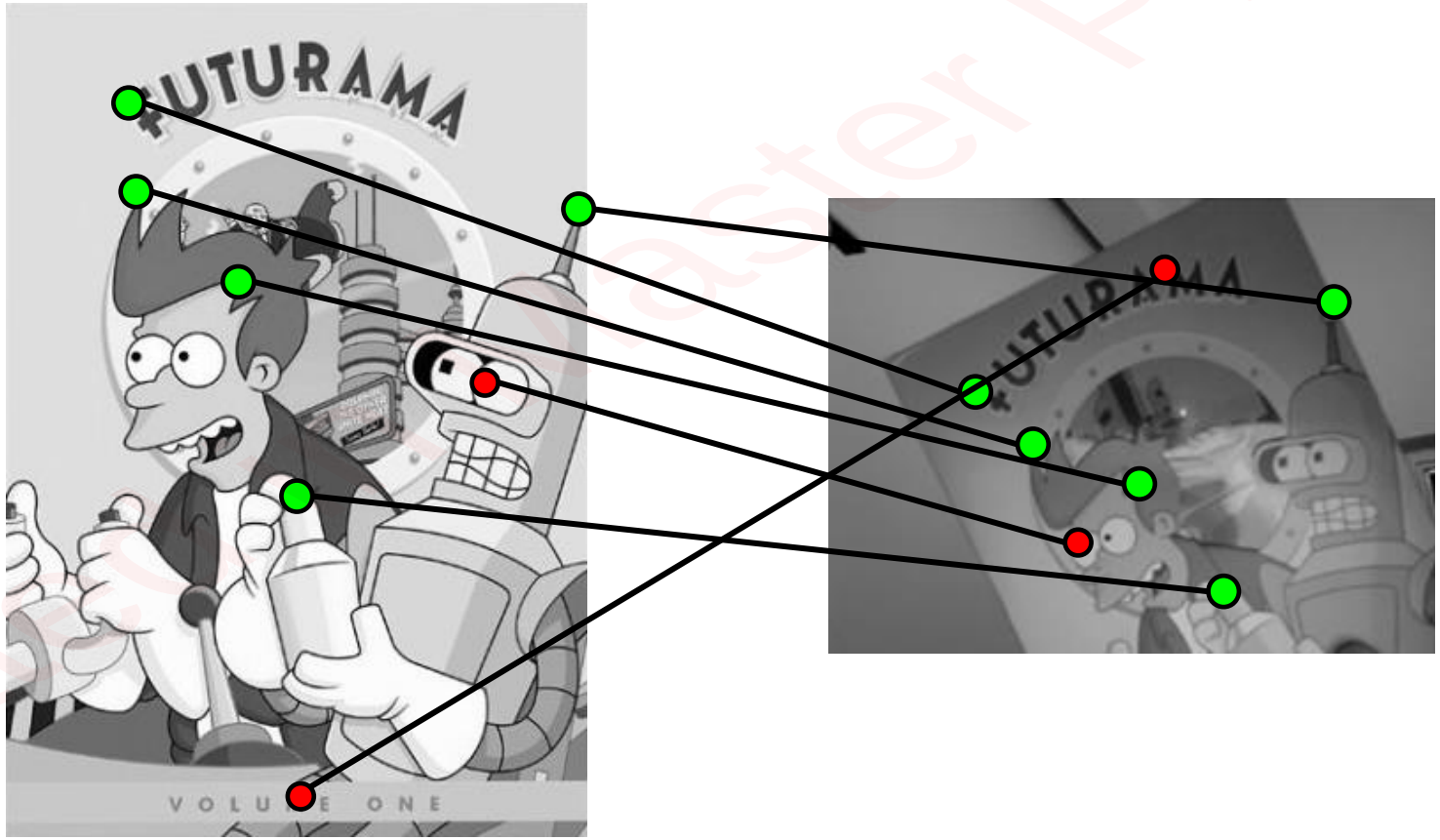
- Just like in the case of fitting a line, we have some bad data (incorrect matches)



These outliers will cause problems with fitting T

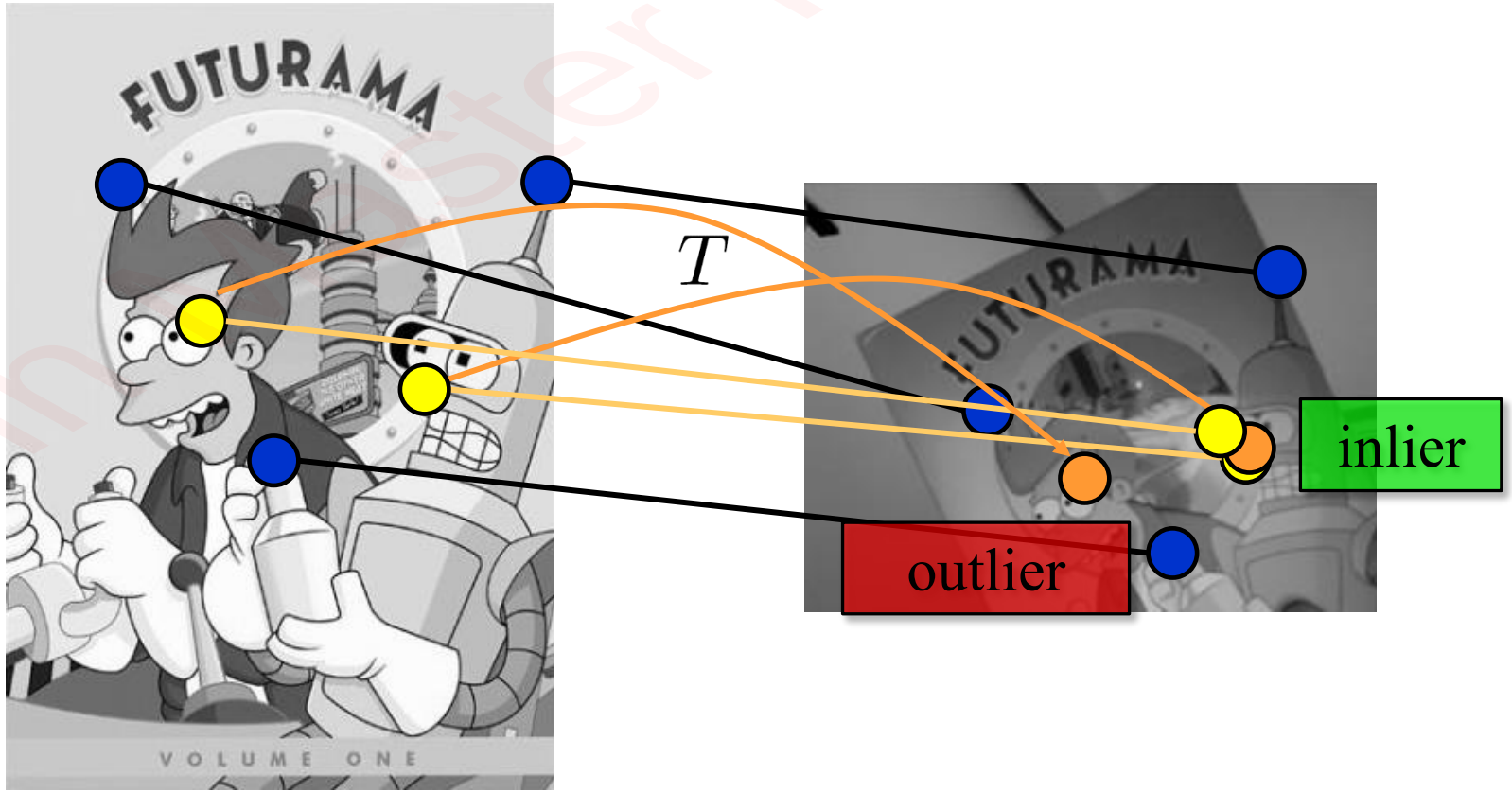
How Do We Fix This?

- RANSAC to the rescue!



Generate and Test an Affine Transformation

- From set of matches, select 3 at random
- Fit a transformation T to the selected matches
- Count inliers



Transform Fitting Algorithm (RANSAC)

1. Select three matches at random
2. Solve for the affine transformation T
3. Count the number of matches that are inliers to *T*
4. If *T* has the highest number of inliers so far, save it
5. Repeat for N rounds, return the best *T*

How do we solve for T given 3 matches?

- Three matches give a linear system with six equations:

$[\ x_1\ y_1\ ] \rightarrow [\ x_1'\ y_1'\ ]$

$$ax_1 + by_1 + c = x_1'$$
$$dx_1 + ey_1 + f = y_1'$$

$[\ x_2\ y_2\ ] \rightarrow [\ x_2'\ y_2'\ ]$

$$ax_2 + by_2 + c = x_2'$$
$$dx_2 + ey_2 + f = y_2'$$

$[\ x_3\ y_3\ ] \rightarrow [\ x_3'\ y_3'\ ]$

$$ax_3 + by_3 + c = x_3'$$
$$dx_3 + ey_3 + f = y_3'$$

Two 3x3 linear systems

$$ax_1 + by_1 + c = x_1'$$
$$ax_2 + by_2 + c = x_2'$$
$$ax_3 + by_3 + c = x_3'$$

$$dx_1 + ey_1 + f = y_1'$$
$$dx_2 + ey_2 + f = y_2'$$
$$dx_3 + ey_3 + f = y_3'$$

Solving a 3x3 system

$$ax_1 + by_1 + c = x_1'$$

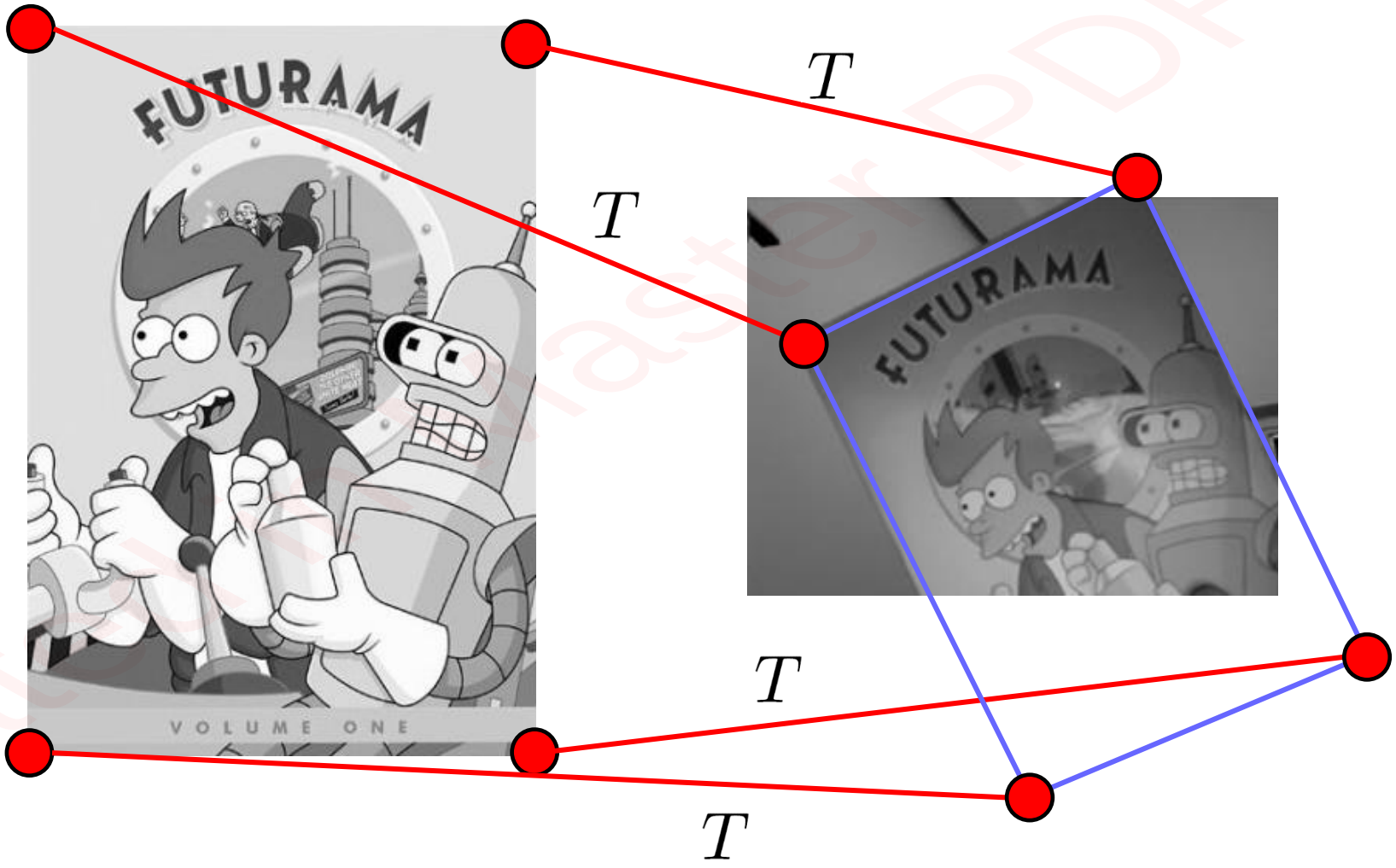
$$ax_2 + by_2 + c = x_2'$$

$$ax_3 + by_3 + c = x_3'$$

- We can write this in matrix form:

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

Finding the Object Boundary



RANSAC

RANdom SAmple Consensus

repeat

- select minimal sample (8 matches)
- compute solution(s) for F
- determine inliers

until $\Gamma(\#inliers, \#samples)$>95% or too many times

compute F based on all inliers

Lee Group & Lee Algebra

李群和李代数

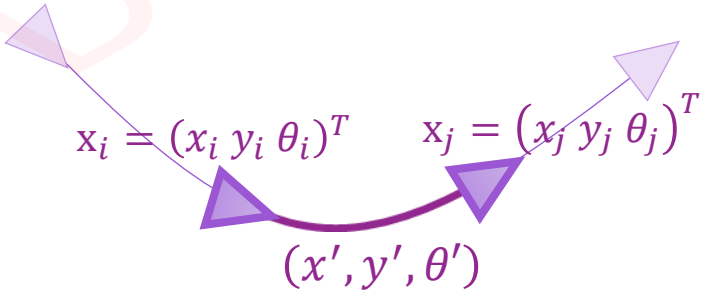
Graph Optimization for 2D Pose

- Consider the relation between two poses:

面向 $\begin{bmatrix} x_j \\ y_j \\ \theta_j \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} + \begin{bmatrix} R_i * \begin{bmatrix} x' \\ y' \end{bmatrix} \\ \theta' \end{bmatrix}$, in which $R_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$

前一个时间点　变化量

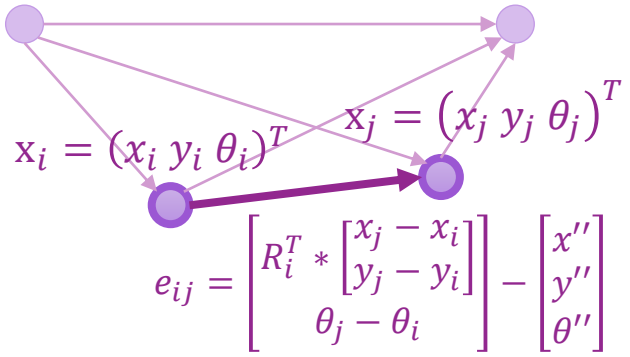
And get $\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} R_i^T * \begin{bmatrix} x_j - x_i \\ y_j - y_i \end{bmatrix} \\ \theta_j - \theta_i \end{bmatrix}$



- After measuring the transform (x'', y'', θ'') between two nodes, we can write down the error term:

$$e_{ij} = \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} - \begin{bmatrix} x'' \\ y'' \\ \theta'' \end{bmatrix} = \begin{bmatrix} R_i^T * \begin{bmatrix} x_j - x_i \\ y_j - y_i \end{bmatrix} \\ \theta_j - \theta_i \end{bmatrix} - \begin{bmatrix} x'' \\ y'' \\ \theta'' \end{bmatrix}$$

差异越小越好



Optimization of Transformation

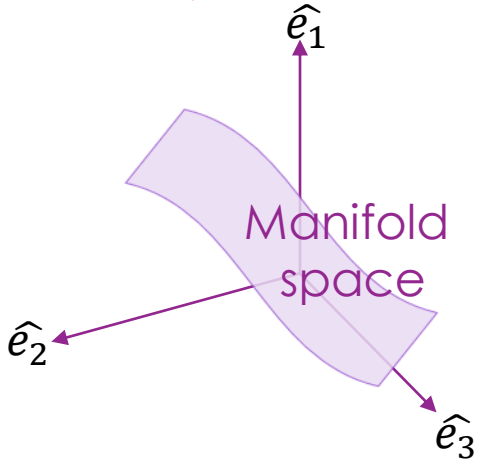
- For optimization problem, the most important thing we care about is to compute the **derivative of the control parameters**.
- In 2D case, the rotation can be represented by one parameter θ. However, the derivative of rotation becomes more complicated in 3D space.
- The key difficulties is that the **rotation matrix does not satisfy the "Closure" property on addition operation**. In other words, we can only optimize the rotation matrix along the manifold of the space instead of the full space.

关键难点是旋转矩阵在加法运算中不满足“闭包”性质。换句话说，我们只能优化沿着空间流形的旋转矩阵，而不是整个空间。

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\frac{\partial f(R)}{\partial R} = \lim_{\Delta R \to 0} \frac{f(R + \Delta R) - f(R)}{\Delta R}$$

??



Rotation Representations

- Rotation matrix
- Euler Angles
- Axis-angle
- Quaternion
- and many more…

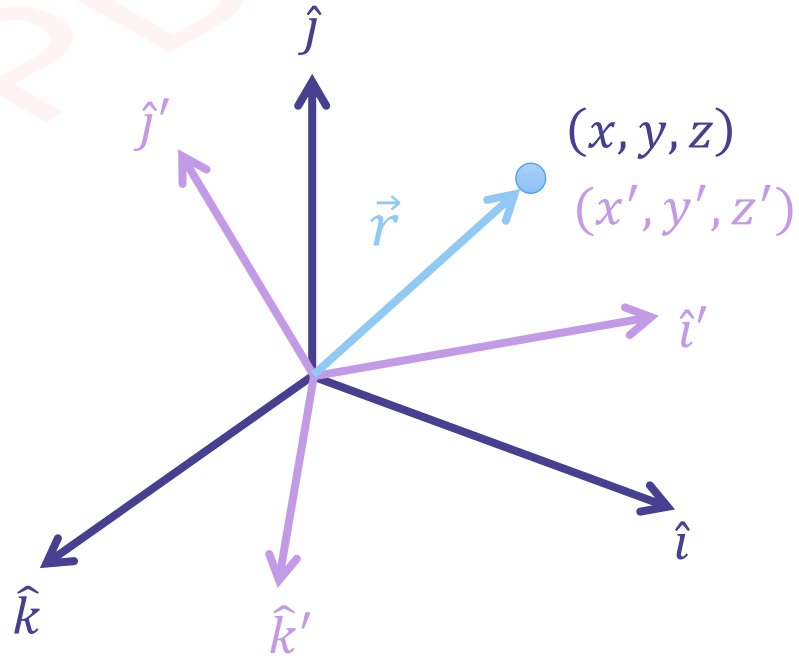
Rotations

A project by @BERKELEY.EDU dynamics

http://rotations.berkeley.edu

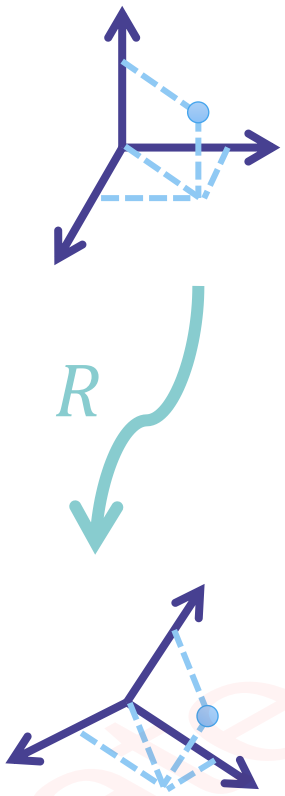
Change Axes in Cartesian Coordinate

- ***Geometric information = coordinates + unit basis***
  - Coordinates are meaningless without unit basis
- $\vec{r}$: displacement vector
- $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'$
- Two types of transformations
  - Coordinate-system transformations
    - Transform basis vector
    - Vector is the same, but components change
  - Transform vector in *the same* coordinate
    - Vector is different from original one



$\vec{r}$ is fixed!
But its components change!!

Rotation Matrix



$$\mathbf{a} = [\mathbf{e_1} \quad \mathbf{e_2} \quad \mathbf{e_3}] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$$

orthogonal matrix

$$[\mathbf{e_1} \quad \mathbf{e_2} \quad \mathbf{e_3}] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [\mathbf{e_1}' \quad \mathbf{e_2}' \quad \mathbf{e_3}'] \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \mathbf{e_1}^T\mathbf{e_1}' & \mathbf{e_1}^T\mathbf{e_2}' & \mathbf{e_1}^T\mathbf{e_3}' \\ \mathbf{e_2}^T\mathbf{e_1}' & \mathbf{e_2}^T\mathbf{e_2}' & \mathbf{e_2}^T\mathbf{e_3}' \\ \mathbf{e_3}^T\mathbf{e_1}' & \mathbf{e_3}^T\mathbf{e_2}' & \mathbf{e_3}^T\mathbf{e_3}' \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \mathbf{R}\mathbf{a}'$$

R: 3D rotations centered at the origin

$$\mathbf{a}' = \mathbf{R}^{-1}\mathbf{a} = \mathbf{R}^{\mathrm{T}}\mathbf{a}$$

SO: Special Orthogonal Group $\mathrm{SO}(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} \mid \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}, \det(\mathbf{R}) = 1\}$

Transformation Matrix

$$\mathbf{a}' = \mathbf{R}\mathbf{a} + \mathbf{t}$$

$$\mathbf{a}_1 = \mathbf{R}_{12}\mathbf{a}_2 + \mathbf{t}_{12}$$

新坐标系　　旧坐标系

$$\mathbf{b} = \mathbf{R}_1\mathbf{a} + \mathbf{t}_1$$

$$\mathbf{c} = \mathbf{R}_2\mathbf{b} + \mathbf{t}_2$$

$$\mathbf{c} = \mathbf{R}_2(\mathbf{R}_1\mathbf{a} + \mathbf{t}_1) + \mathbf{t}_2$$

Too tedious !

$$\begin{bmatrix}\mathbf{a}' \\ 1\end{bmatrix} = \begin{bmatrix}\mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\mathrm{T}} & 1\end{bmatrix}\begin{bmatrix}\mathbf{a} \\ 1\end{bmatrix} \overset{\text{def}}{=} \mathbf{T}\begin{bmatrix}\mathbf{a} \\ 1\end{bmatrix}$$

homogeneous coordinate

T: 3D rotations with translations
(preserve *distance* and *orientation*)

$$\mathbf{T}^{-1} = \begin{bmatrix}\mathbf{R}^{\mathbf{T}} & -\mathbf{R}^{\mathbf{T}}\mathbf{t} \\ \mathbf{0}^{\mathrm{T}} & 1\end{bmatrix}$$

SE: Special Euclidean group $SE(3) = \left\{\mathbf{T} = \begin{bmatrix}\mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\mathrm{T}} & 1\end{bmatrix} \in \mathbb{R}^{4x4} | \mathbf{R} \in SO(3), \mathbf{t} \in \mathbb{R}^3\right\}$

Group

- A family of transformations forms a **group**
- A set $\mathbf{A}$ together with a binary operation $\circ$ defined on its elements is called a group **G**, if it satisfies the axioms of ***closure, identity, inverse and associativity***

Closure

$$\forall a_1, a_2 \in \mathbf{A}, \quad a_1 \circ a_2 \in \mathbf{A}$$

Inverse

$$\forall a \; \exists a^{-1} \in \mathbf{A}: \; a \circ a^{-1} = a^{-1} \circ a = \mathrm{e}$$

Identity

$$\exists \mathrm{e} \in \mathbf{A}: a \circ \mathrm{e} = \mathrm{e} \circ a = a$$

Associativity

$$\forall a_1, a_2, a_3 \in \mathbf{A}, \quad a_1 \circ (a_2 \circ a_3) = (a_1 \circ a_2) \circ a_3$$

Introduction to Higher Mathematics - Lecture 16: Group Theory
SIG'14: Mathematical Basics of Motion and Deformation in Computer Graphics

Interpolating Rotation Matrices

90°CW around z-axis *90°CCW around z-axis*

$$0.5\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.5\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Oops!! This is NOT a rotation matrix!!

Rotation matrix is a group with multiplication **NOT** addition

$$R_1 + R_2 \notin SO(3) \qquad R_1 R_2 \in \mathrm{SO}(3)$$

Lee Group

- **Lee group** is a group with the **smoothness** property.
  - SO(n) and SE(n) are both Lee groups
- To solve the optimization problem on manifold space, we need to figure out the concept of the "Lee Group" and the mapping to "Lee Algebra".

Property of Lee Group SO(3)

- An arbitrary rotation matrix satisfy the orthogonal property: $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$
- Assume $\mathbf{R}$ is the rotation of camera that changes along time, we can represent the rotation as a function of time: $\mathbf{R}(t)$
- Compute the derivative of the equation: $\mathbf{R}(t)\mathbf{R}(t)^{\mathrm{T}} = \mathbf{I}$

$$\dot{\mathbf{R}}(t)\mathbf{R}(t)^{\mathrm{T}} + \mathbf{R}(t)\dot{\mathbf{R}}(t)^{\mathrm{T}} = 0$$
$$\dot{\mathbf{R}}(t)\mathbf{R}(t)^{\mathrm{T}} = -\left(\dot{\mathbf{R}}(t)\mathbf{R}(t)^{\mathrm{T}}\right)^{\mathrm{T}}$$

- We can observe that $\dot{\mathbf{R}}(t)\mathbf{R}(t)^{\mathrm{T}}$ is an antisymmetric matrix constructed according to an unique vector. Define the operators "∧" and "∨" are the mapping between the antisymmetric matrix and the vector:

$$\mathbf{a}^{\wedge} = \mathbf{A} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \qquad \mathbf{A}^{\vee} = \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Property of Lee Group SO(3)

- Thus we can find a 3D vector $\boldsymbol{\phi}(t)$ corresponding to $\dot{\mathbf{R}}(t)\mathbf{R}(t)^{\mathrm{T}}$:

$$\dot{\mathbf{R}}(t)\mathbf{R}(t)^{\mathrm{T}} = \boldsymbol{\phi}(t)^{\wedge}$$

- The derivative of the rotation matrix can be obtained by multiplying $\mathbf{R}(t)$ on both sides :

$$\dot{\mathbf{R}}(t) = \boldsymbol{\phi}(t)^{\wedge}\mathbf{R}(t) = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \mathbf{R}(t)$$

- Assume $t_0 = 0, \mathbf{R}(0) = \mathbf{I}$, we can apply Taylor expansion near t_0 . The vector $\boldsymbol{\phi}$ is on the "tangent space" of the rotation matrix which reflect the derivative property of the rotation:

$$\mathbf{R}(t) \approx \mathbf{R}(t_0) + \dot{\mathbf{R}}(t_0)(t - t_0) = \mathbf{I} + \boldsymbol{\phi}(t_0)^{\wedge}(t)$$

Lee Algebra

- Assume the vector $\boldsymbol{\phi}$ is constant in small time interval around t_0:

$$\boldsymbol{\phi}(t_0) = \boldsymbol{\phi}_0 \qquad \dot{\mathbf{R}}(t) = \boldsymbol{\phi}(t)^{\wedge}\mathbf{R}(t) = \boldsymbol{\phi}_0^{\wedge}\mathbf{R}(t)$$

- Solve the differential equation of $\mathbf{R}$ with the initial condition $\mathbf{R}(0) = \mathbf{I}$, and we can find the rotation matrix can be computed by an exponential mapping of the asymmetric matrix:

$$\mathbf{R}(t) = \exp(\boldsymbol{\phi}_0^{\wedge} t)$$

- We then define the **Lee Algebra** as the set of $\boldsymbol{\phi}$ or $\boldsymbol{\Phi}$

$$\boldsymbol{\Phi} = \boldsymbol{\phi}^{\wedge} = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$\mathfrak{so}(3) = \{\boldsymbol{\phi} \in \mathbb{R}^3, \boldsymbol{\Phi} = \boldsymbol{\phi}^{\wedge} \in \mathbb{R}^{3\times 3}\}$$

Lee Algebra

- Every Lee Group has a corresponding Lee Algebra representing its **tangent space**
- A set $\mathbb{V}$ together with a field $\mathbb{F}$ and a binary operation [,] (Lee Bracket) defined on its elements is called a Lee Algebra $\mathfrak{g}$, if it satisfies the axioms of ***closure, bilinearity, alternativity, and Jacobi identity***

Closure

$$\forall \mathbf{X}, \mathbf{Y} \in \mathbb{V}, \qquad [\mathbf{X}, \mathbf{Y}] \in \mathbb{V}$$

Alternativity

$$\forall\, \mathbf{X} \in \mathbb{V}, \qquad [\mathbf{X}, \mathbf{X}] = 0$$

Bilinearity

$$\forall \mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{V} \quad \text{and} \quad a, b \in \mathbb{F}$$

$$[a\mathbf{X} + b\mathbf{Y}, \mathbf{Z}] = a[\mathbf{X}, \mathbf{Z}] + b[\mathbf{Y}, \mathbf{Z}],$$

$$[\mathbf{Z}, a\mathbf{X} + b\mathbf{Y}] = a[\mathbf{Z}, \mathbf{X}] + b[\mathbf{Z}, \mathbf{Y}]$$

Jacobi identity

$$\forall \mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{V}, \qquad \big[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]\big] + \big[\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]\big] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] = 0$$

Exponential Mapping of Lee Algebra

$$\phi = \theta a \qquad \|a\| = 1 \qquad \exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

$$a\hat{}a\hat{} = \begin{bmatrix} -a_2^2 - a_3^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & -a_1^2 - a_3^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & -a_1^2 - a_2^2 \end{bmatrix} = \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_1^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = aa^T - I$$

$$a\hat{}b = -b\hat{}a$$

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a_3 b_2 + a_2 b_3 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_2 b_3 + a_3 b_2 \\ a_1 b_3 - a_3 b_1 \\ -a_1 b_2 + a_2 b_1 \end{bmatrix}$$

$$a\hat{}a = -a\hat{}a = 0 \qquad a\hat{}a\hat{}a\hat{} = a\hat{}\left(aa^T - I\right) = -a\hat{}$$

Exponential Mapping of Lee Algebra

- The exponential of the matrix:

$$\exp(\phi^\wedge) = \exp(\theta a^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!}(\theta a^\wedge)^n$$

$$= I + \theta a^\wedge + \frac{1}{2!}\theta^2 a^\wedge a^\wedge + \frac{1}{3!}\theta^3 a^\wedge a^\wedge a^\wedge + \frac{1}{4!}\theta^4 (a^\wedge)^4 + \cdots$$

$$= aa^T - a^\wedge a^\wedge + \theta a^\wedge + \frac{1}{2!}\theta^2 a^\wedge a^\wedge - \frac{1}{3!}\theta^3 a^\wedge - \frac{1}{4!}\theta^4 (a^\wedge)^2 + \cdots$$

$$= aa^T + \left(\theta - \frac{1}{3!} + \frac{1}{5!} - \cdots\right) a^\wedge - \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right) a^\wedge a^\wedge$$

$$= aa^T + \sin\theta\, a^\wedge - \cos\theta\, (aa^T - I)$$

$$= \cos\theta\, I + (1 - \cos\theta) aa^T + \sin\theta\, a^\wedge$$ (Rodrigues Rotation)

Lee Group vs. Lee Algebra

Lee Group

$SO(3)$

$R \in \mathbb{R}^{3\times3}$

$RR^T = I$

$\det(R) = 1$

Mapping

Exponential Mapping

$$\exp(\phi^\wedge) = \cos\theta\, I + (1-\cos\theta)aa^T + \sin\theta\, a^\wedge$$

Logarithmic Mapping

$$\phi = (\ln R)^\vee = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(R-I)^{n+1}\right)^\vee$$

$$\theta = \arccos\frac{tr(R)-1}{2} \qquad Ra = a$$

Lee Algebra

$\mathfrak{so}(3)$

$\phi \in \mathbb{R}^3$

$$\phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}$$

BCH Equation and Approximation

- Can we apply addition operation on the Lee Algebra ?

$$\exp(\phi_1^\wedge)\exp(\phi_2^\wedge) = \exp\left((\phi_1+\phi_2)^\wedge\right) \; ?? \qquad \ln(\exp(A)\exp(B)) = A + B \; ??$$

- Baker-Campbell-Hausdorff (BCH) Equation

$$\ln(\exp(A)\exp(B)) = A + \sum_{n=0}^{\infty}(-1)^n\left(\frac{B_n}{n!}\right)[A,[A,\dots[A,B]\dots]] = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \cdots$$

- Linear Approximation

$$\ln(\exp(\phi_1^\wedge)\exp(\phi_2^\wedge)) \approx \begin{cases} J_l(\phi_2)^{-1}\phi_1 + \phi_2 & \text{Left Multiply Approximation} \\ J_r(\phi_1)^{-1}\phi_2 + \phi_1 & \text{Right Multiply Approximation} \end{cases}$$

$$J_l = \cos\theta\, I + (1-\cos\theta)aa^T + \sin\theta\, a^\wedge$$

Derivative of Lee Algebra

~~$\frac{\partial(Rp)}{\partial R}$~~ $\Rightarrow$

$$\frac{\partial(\exp(\phi^\wedge)\,p)}{\partial\phi} = \lim_{\delta\phi\to 0}\frac{\exp\big((\phi+\delta\phi)^\wedge\big)\,p - \exp(\phi^\wedge)\,p}{\delta\phi}$$

$$= \lim_{\delta\phi\to 0}\frac{\exp\big((J_l\delta\phi)^\wedge\big)\exp(\phi^\wedge)\,p - \exp(\phi^\wedge)\,p}{\delta\phi}$$

$$= \lim_{\delta\phi\to 0}\frac{(I+(J_l\delta\phi)^\wedge)\exp(\phi^\wedge)\,p - \exp(\phi^\wedge)\,p}{\delta\phi}$$

$$= \lim_{\delta\phi\to 0}\frac{(J_l\delta\phi)^\wedge\exp(\phi^\wedge)\,p}{\delta\phi}$$

$$= \lim_{\delta\phi\to 0}\frac{-(\exp(\phi^\wedge)\,p)^\wedge J_l\delta\phi}{\delta\phi}$$

$$= -(Rp)^\wedge J_l$$

$$\ln(\exp(\phi_1^\wedge)\exp(\phi_2^\wedge)) \approx \begin{cases} J_l(\phi_2)^{-1}\phi_1 + \phi_2 \\ J_r(\phi_1)^{-1}\phi_2 + \phi_1 \end{cases}$$

$$R_{init} = \exp(\phi_{init}^\wedge)$$

$$\phi_{new} = \phi_{init} - \alpha\left(\frac{\partial u}{\partial\phi}\right)^T \qquad \phi_{new} = \phi_{init} - \alpha J_l^T(Rp)\left.\frac{\partial u}{\partial(Rp)}\right|^T \qquad \frac{\partial u}{\partial\phi} = \frac{\partial u}{\partial(Rp)}\frac{\partial(Rp)}{\partial R} = \frac{\partial u}{\partial(Rp)}(Rp)^\wedge J_l$$

Perturbation Model

$$\boxed{\frac{\partial(\exp(\phi\hat{}\,)\,p)}{\partial\phi} = \lim_{\delta\phi\to 0}\frac{\exp\big((\phi+\delta\phi)\hat{}\,\big)\,p - \exp(\phi\hat{}\,)\,p}{\delta\phi}}$$

Left Perturbation

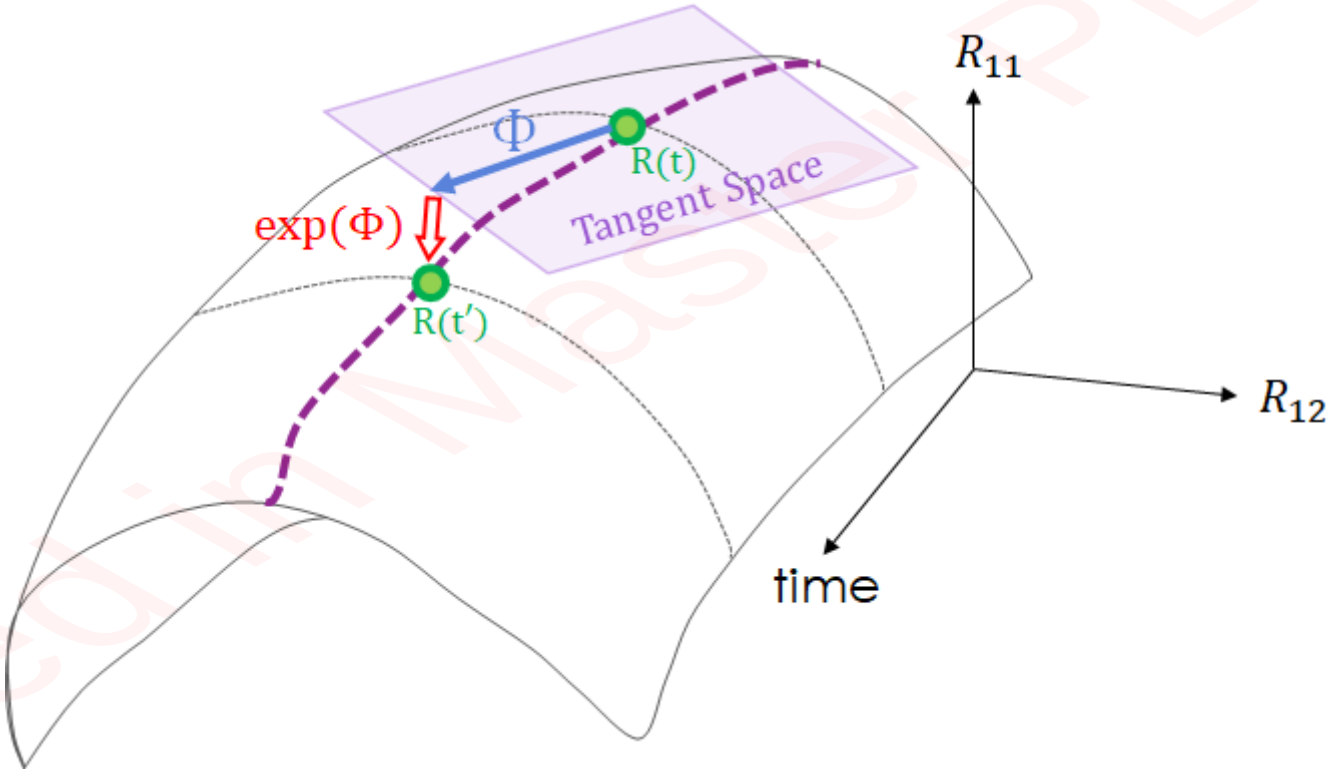
$$\frac{\partial(Rp)}{\partial\psi} = \lim_{\psi\to 0}\frac{\exp(\psi\hat{}\,)\exp(\phi\hat{}\,)\,p - \exp(\phi\hat{}\,)\,p}{\psi}$$

$$= \lim_{\psi\to 0}\frac{(I+\psi\hat{}\,)\exp(\phi\hat{}\,)\,p - \exp(\phi\hat{}\,)\,p}{\psi}$$

$$= \lim_{\psi\to 0}\frac{\psi\hat{}\,Rp}{\delta\psi} = \lim_{\psi\to 0}\frac{-(Rp)\hat{}\,\psi}{\psi} = -(Rp)\hat{}$$

$$R_{new} = \exp(\psi\hat{}\,)\,R_{init}$$

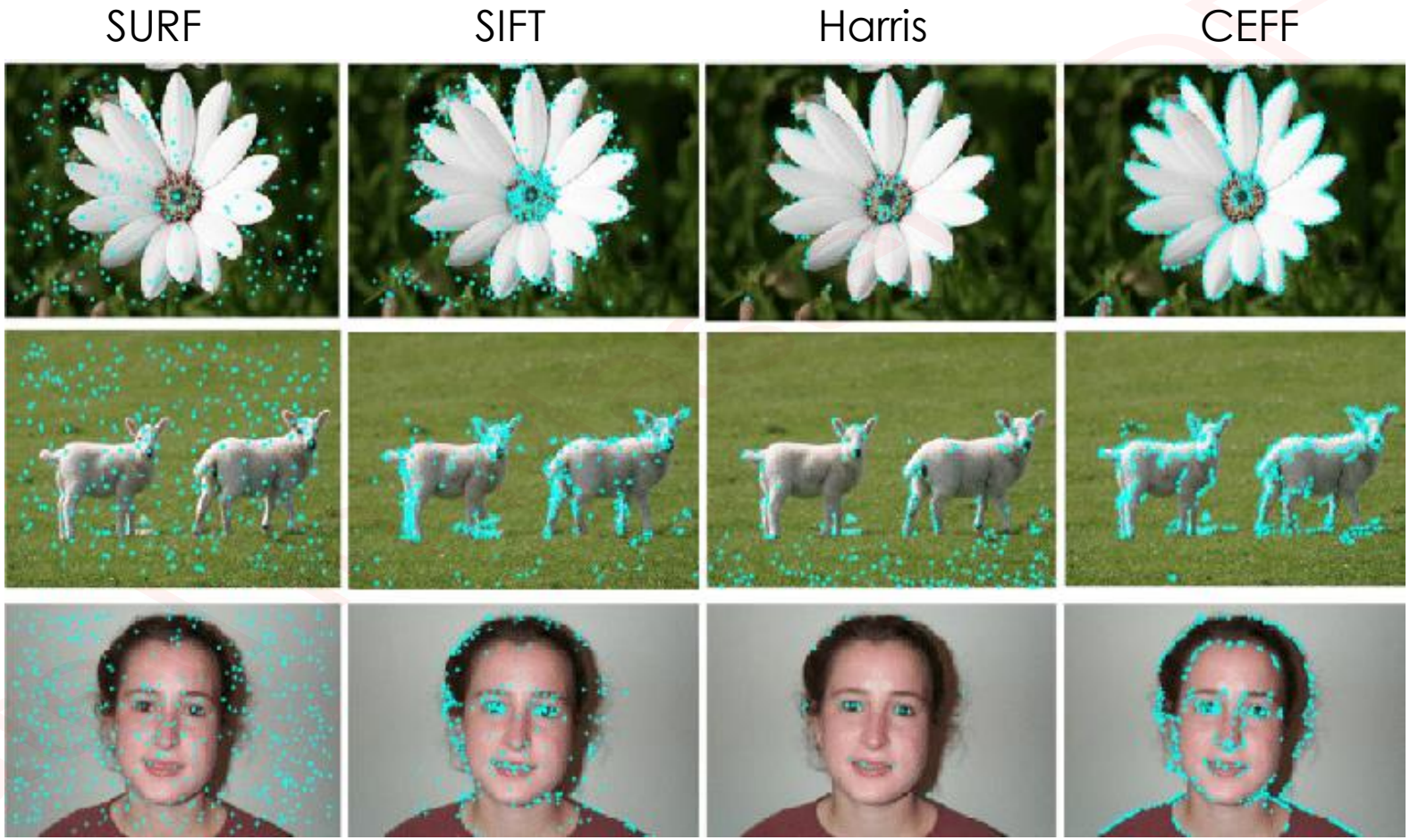
3D Transformation in Lee Group/Algebra



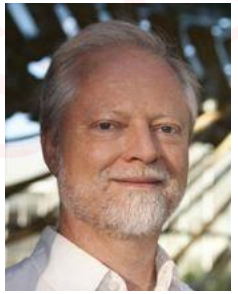
Q&A

Feature Point Extraction

Popular Feature Extractors

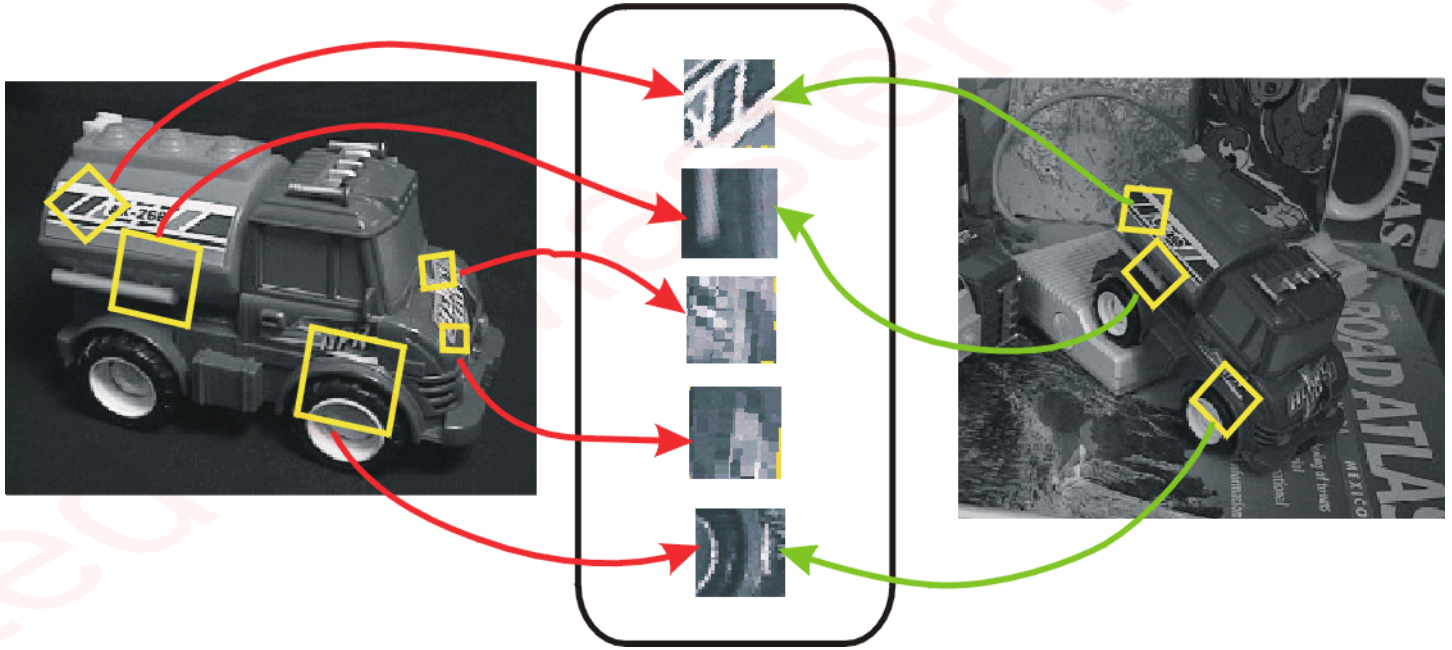


[Ref] Nawaz, Mehmood, et al. Clustering based one-to-one hypergraph matching with a large number of feature points. *Signal Processing: Image Communication*, 2019, 74: 289-298.



Idea of SIFT

- Image content is transformed into local feature coordinates that are invariant to translation, rotation, scale, and other imaging parameters

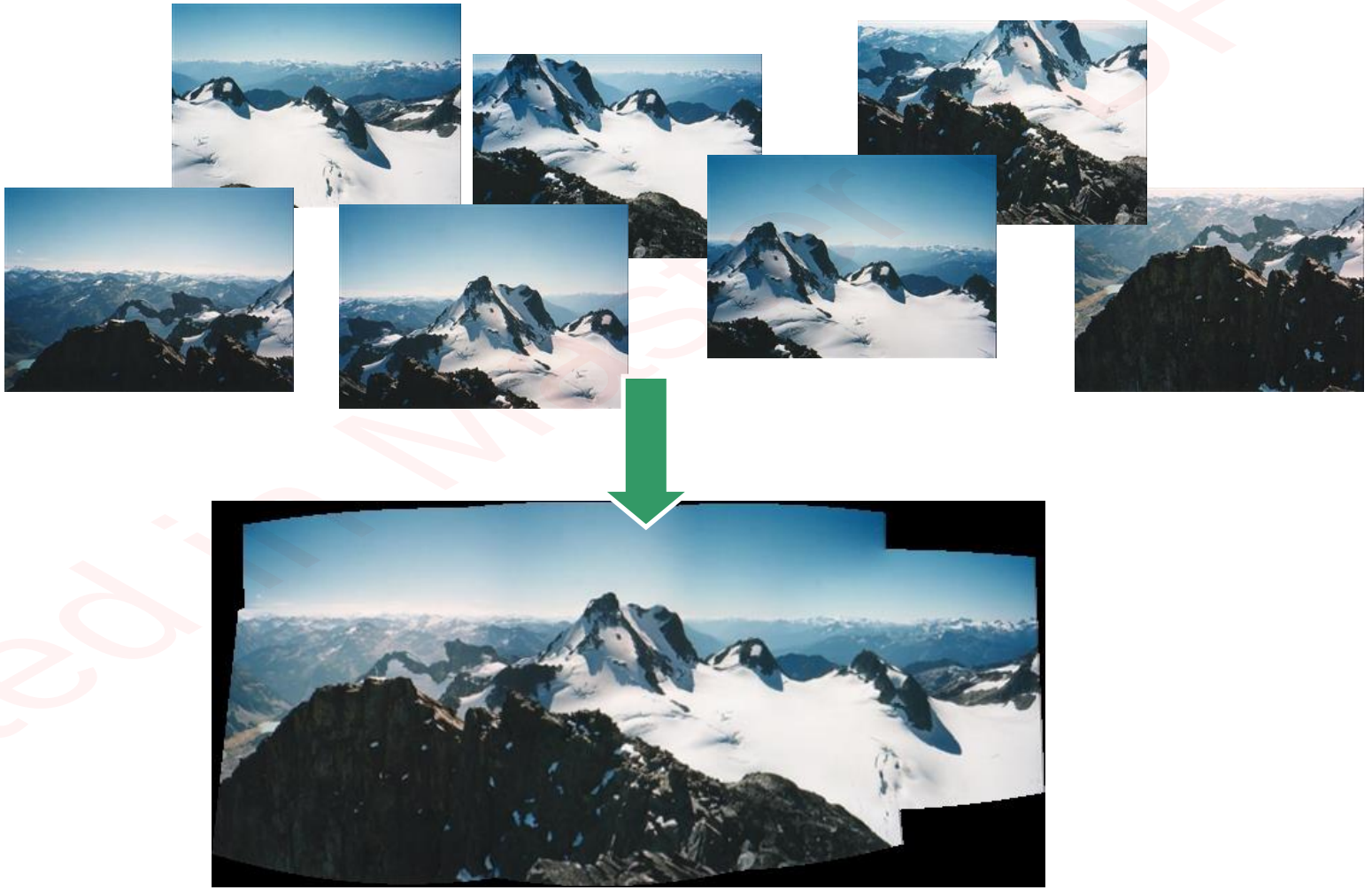


SIFT Features

[Ref] Lowe, David G. Distinctive image features from scale-invariant keypoints. *International journal of computer vision*, 2004, 60.2: 91-110.

Application: Object Recognition (Matching)

Application: Image Stitching



Application: Photosynth

Claimed Advantages of SIFT

- **Locality**
  - features are local, so robust to occlusion and clutter (no prior segmentation)
- **Distinctiveness**
  - individual features can be matched to a large database of objects
- **Quantity**
  - many features can be generated for even small objects
- **Efficiency**
  - close to real-time performance
- **Extensibility**
  - can easily be extended to wide range of other feature types, with each adding robustness

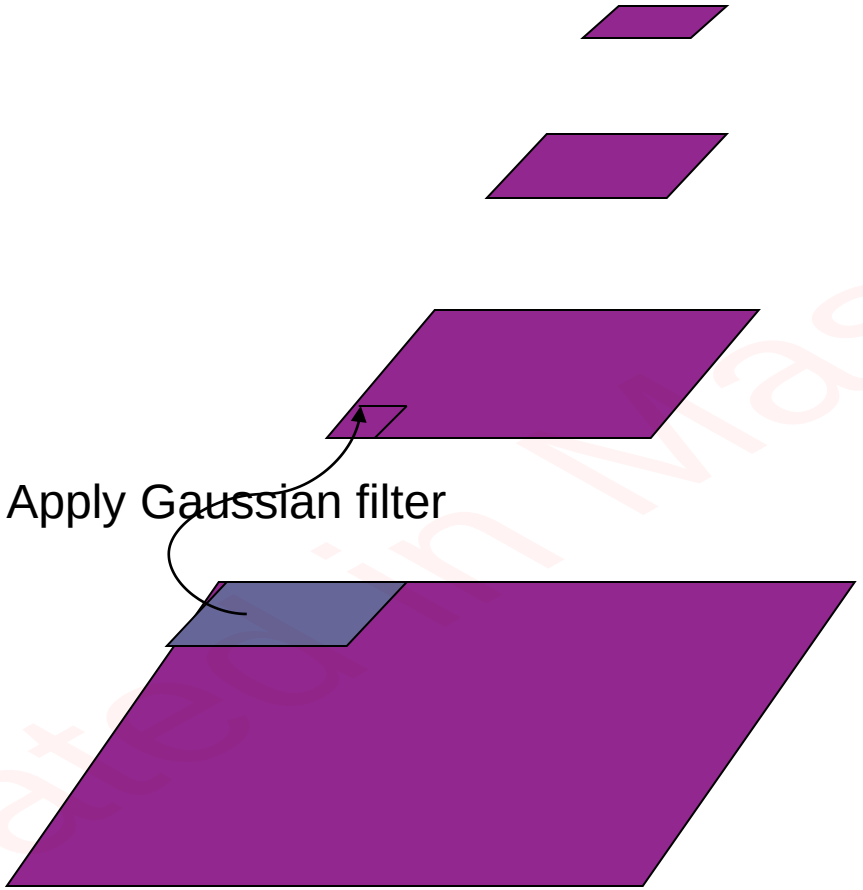
4 Steps of SIFT

- **Scale-space extrema detection**
  - Search over multiple scales and image locations
- **Keypoint localization**
  - Fit a model to determine location and scale
  - Select keypoints based on a measure of stability
- **Orientation assignment**
  - Compute best orientation(s) for each keypoint region
- **Keypoint descriptor**
  - Use local image gradients at selected scale and rotation to describe each keypoint region

1. Scale-space Extrema Detection

- Goal:
  - Identify locations and scales that can be repeatably assigned under different views of the same scene or object.
- Method:
  - Search for stable features across multiple scales using a continuous function of scale.
- Prior work has shown that under a variety of assumptions, the best function is a Gaussian function.
- The scale space of an image is a function $L(x,y,\sigma)$ that is produced from the convolution of a Gaussian kernel (at different scales) with the input image.

Gaussian Pyramid

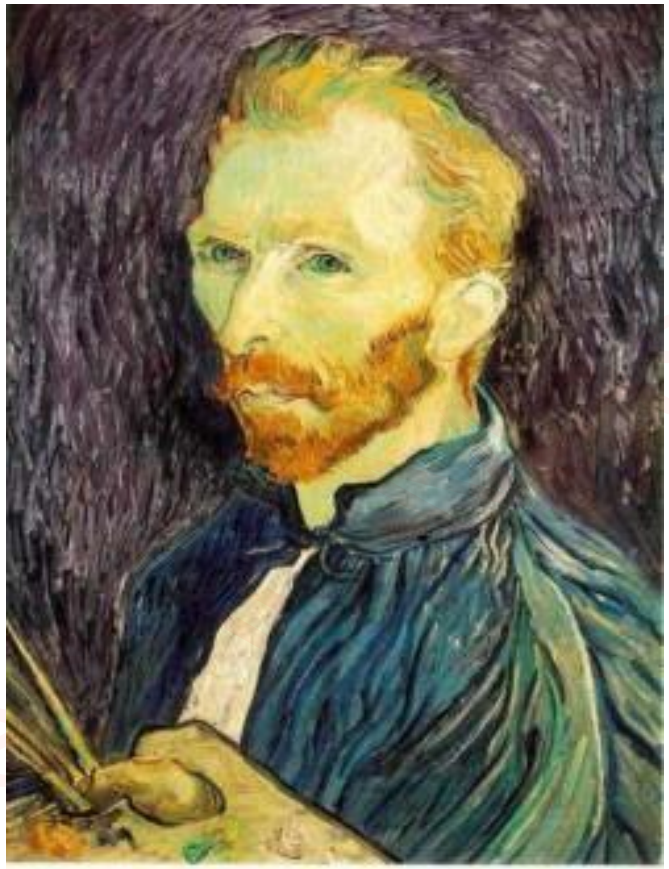


And so on.

At 2nd level, each pixel is the result of applying a Gaussian mask to the first level and then subsampling to reduce the size.

Bottom level is the original image.

Example

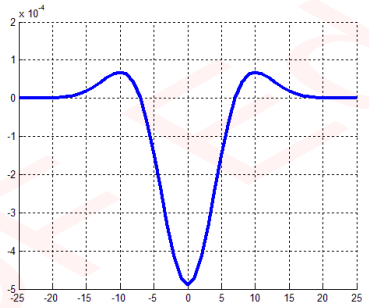


Gaussian 1/2

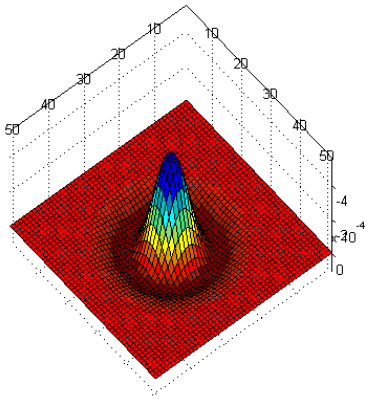
G 1/4

G 1/8

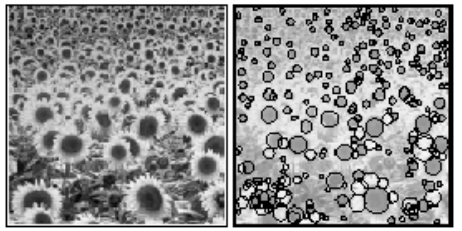
Lowe's Scale-space Interest Points

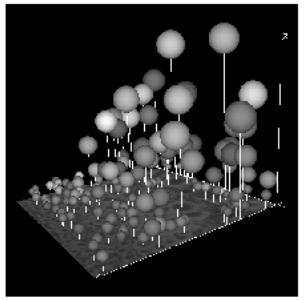


- **Laplacian of Gaussian** kernel
  - Scale normalized
  - Proposed by Lindeberg
- Scale-space detection
  - Find local maxima across scale/space
  - A good "blob" detector



$$G(x, y, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}}$$

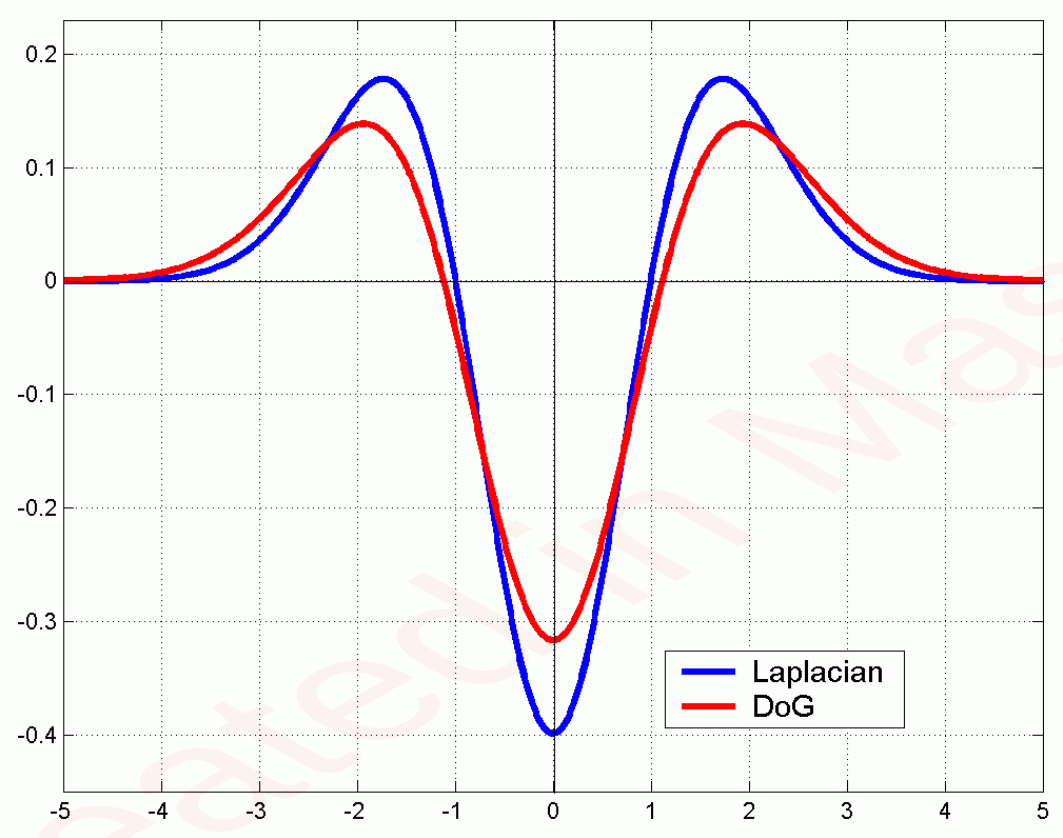




$$\Delta[G_\sigma(x, y) * f(x, y)] = [\Delta G_\sigma(x, y)] * f(x, y)$$

$$LoG = \Delta G_\sigma(x, y) = \frac{\partial^2 G_\sigma(x, y)}{\partial x^2} + \frac{\partial^2 G_\sigma(x, y)}{\partial y^2} = \frac{x^2 + y^2 - 2\sigma^2}{\sigma^4} e^{-(x^2+y^2)/2\sigma^2}$$

Lowe's Scale-space Interest Points: Difference of Gaussians



$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} exp(-\frac{x^2+y^2}{2\sigma^2})$$

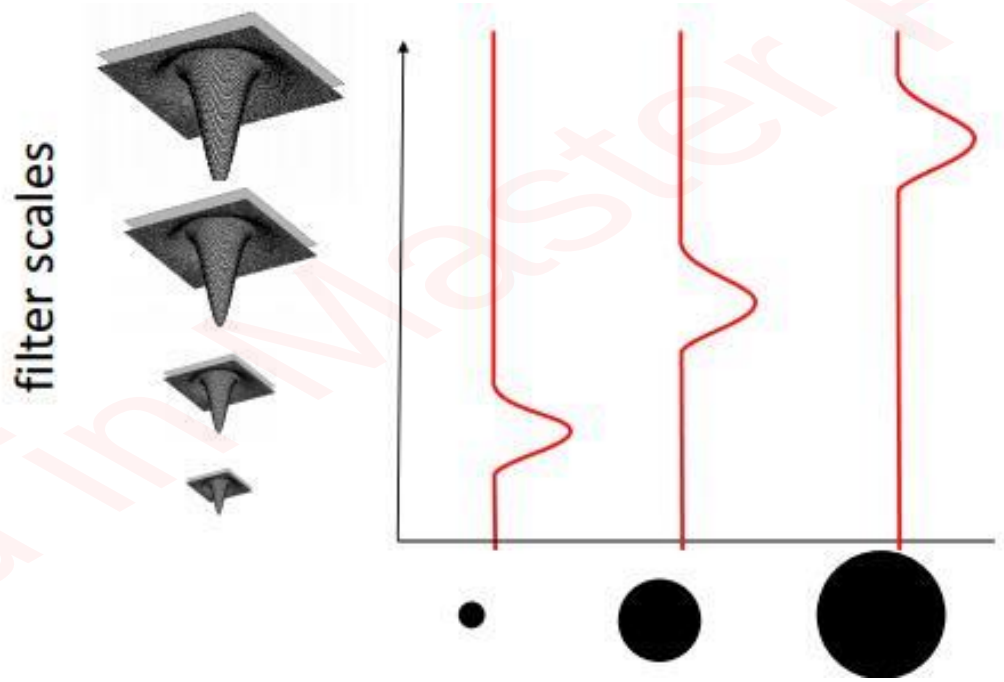
$$DoG = G_{\sigma 1} - G_{\sigma 2} = \frac{1}{\sqrt{2\pi}}[\frac{1}{\sigma_1} e^{-(x^2+y^2)/2\sigma_1^2} - \frac{1}{\sigma_2} e^{-(x^2+y^2)/2\sigma_2^2}]$$

$$\frac{\partial G}{\partial \sigma} = \sigma \nabla^2 G$$

$$\frac{\partial G}{\partial \sigma} \approx \frac{G(x,y,k\sigma) - G(x,y,\sigma)}{k\sigma - \sigma}$$

$$G(x,y,k\sigma) - G(x,y,\sigma) \approx (k-1)\sigma^2 \nabla^2 G$$

Lowe's Scale-space Interest Points: Difference of Gaussians



Lowe's Pyramid Scheme

$\sigma_{s+1}=2^{(s+1)/s}\sigma_0$

$\vdots$

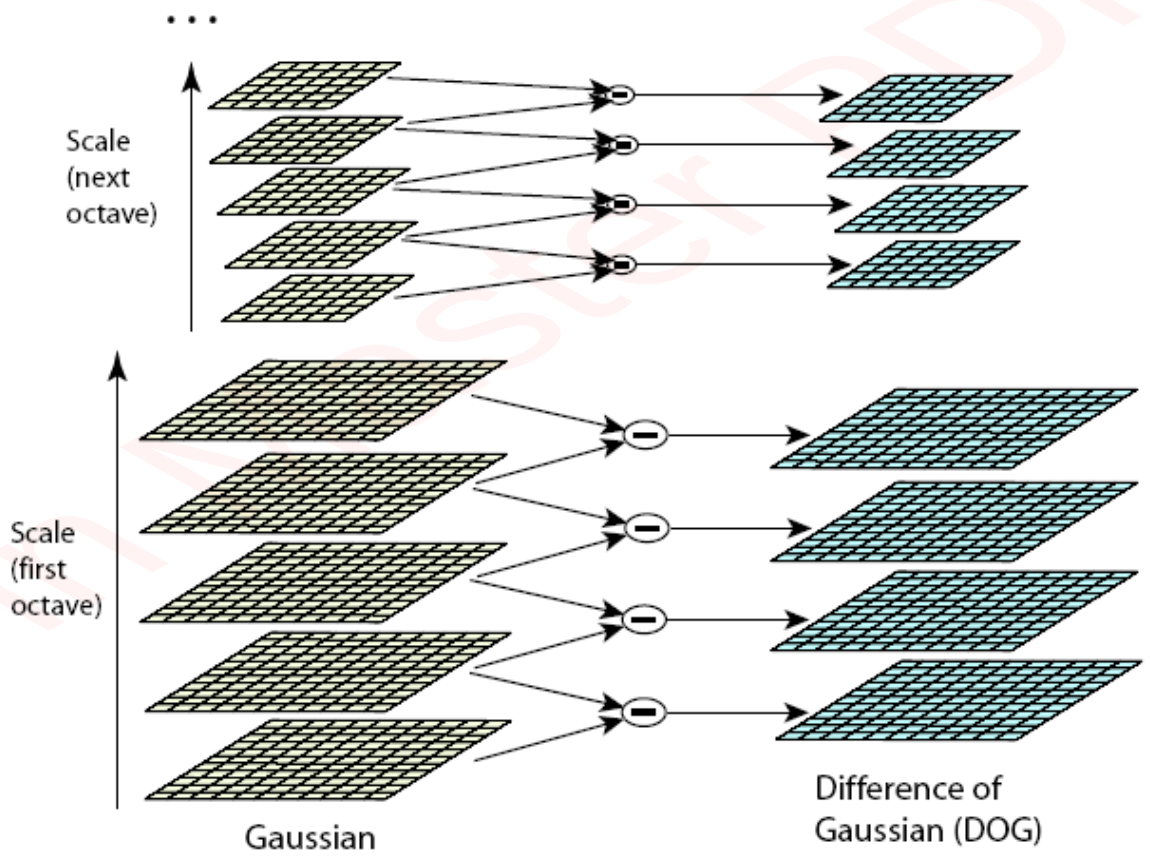
$\sigma_i=2^{i/s}\sigma_0$

$\vdots$

$\sigma_2=2^{2/s}\sigma_0$

$\sigma_1=2^{1/s}\sigma_0$

σ_0

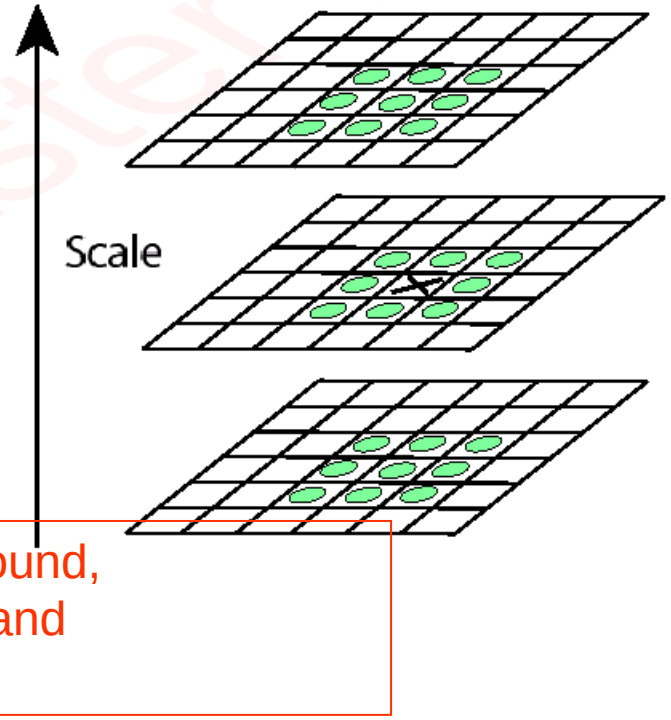


The parameter **s** determines the number of images per octave

2. Key point localization

- Detect maxima and minima of difference-of-Gaussian in scale space
- Each point is compared to its 8 neighbors in the current image and 9 neighbors each in the scales above and below

s+2 difference images.
Ignore top and bottom.
Search s planes.



For each max or min found, output is the **location** and the **scale**.

2. Keypoint Localization

- There are still a lot of points, some of them are not good enough
  - The locations of keypoints may be not accurate

Taylor series expansion

$$D(\mathbf{x}) = D + \frac{\partial D}{\partial \mathbf{x}}^T \mathbf{x} + \frac{1}{2}\mathbf{x}^\mathbf{T} \frac{\partial^2 D}{\partial \mathbf{x}^2}\mathbf{x}$$

$$\hat{\mathbf{x}} = -\frac{\partial^2 D}{\partial \mathbf{x}^2}^{-1} \frac{\partial D}{\partial \mathbf{x}}$$

$$D(\hat{\mathbf{x}}) = D + \frac{1}{2}\frac{\partial D}{\partial \mathbf{x}}^T \hat{\mathbf{x}}.$$

Eliminating the Edge Response

- Reject flats by a gradient threshold:
  - $|D(\hat{\mathbf{x}})| < 0.03$
- Reject edges by a ratio threshold: $\mathbf{H} = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{xy} & D_{yy} \end{bmatrix}$

$$\mathrm{Tr}(\mathbf{H}) = D_{xx} + D_{yy} = \alpha + \beta,$$
$$\mathrm{Det}(\mathbf{H}) = D_{xx}D_{yy} - (D_{xy})^2 = \alpha\beta.$$

Let α be the eigenvalue with larger magnitude and β the smaller.

$$\frac{\mathrm{Tr}(\mathbf{H})^2}{\mathrm{Det}(\mathbf{H})} = \frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{(r\beta+\beta)^2}{r\beta^2} = \frac{(r+1)^2}{r},$$

Let $r = \alpha/\beta$. So $\alpha = r\beta$

$(r+1)^2/r$ is at a min when the 2 eigenvalues are equal.

  - $r < 10$

Eliminating the Edge Response

233x189

input image

832

initial keypoints

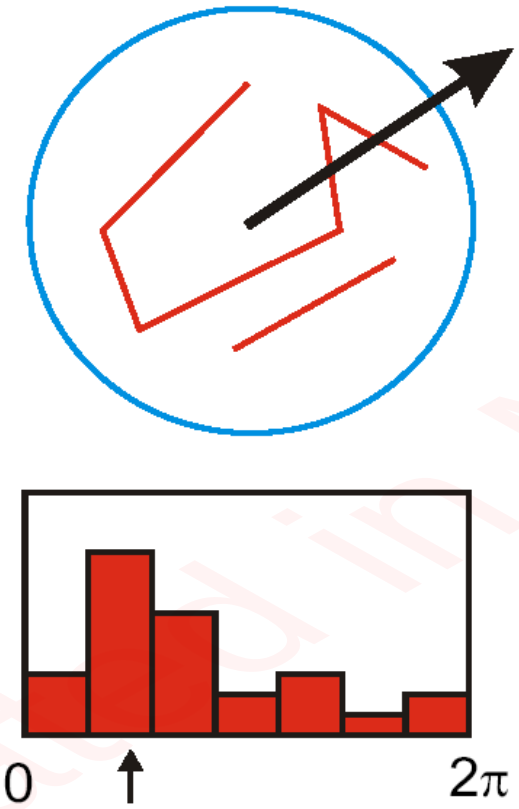
729

keypoints after gradient threshold

536

keypoints after ratio threshold

3. Orientation assignment



- Create histogram of local gradient directions at selected scale
- Assign canonical orientation at peak of smoothed histogram
- Each key specifies stable 2D coordinates

(x, y, scale, orientation)

If 2 major orientations, use both.

Orientation Assignment

- Assign an orientation to each keypoint, the keypoint descriptor can be represented relative to this orientation and therefore achieve invariance to image rotation
- Compute magnitude and orientation on the Gaussian smoothed images

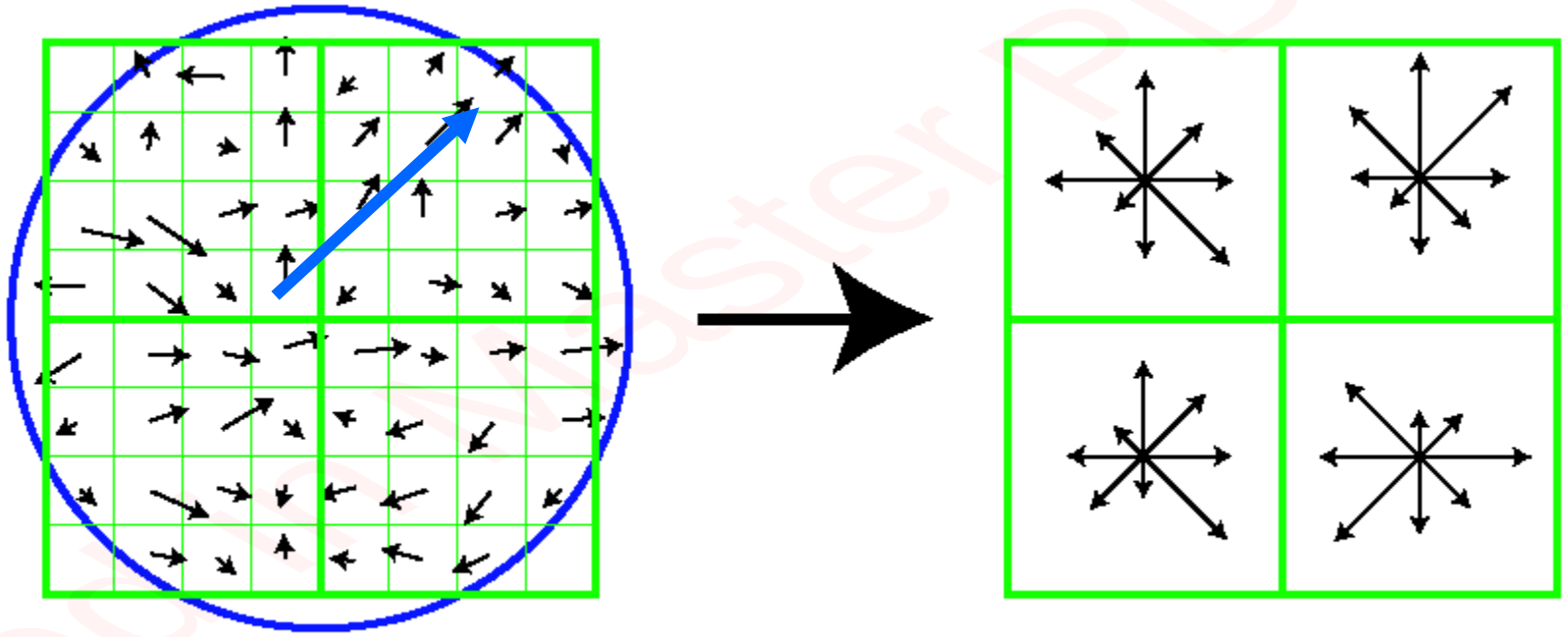
$$m(x,y) = \sqrt{(L(x+1,y) - L(x-1,y))^2 + (L(x,y+1) - L(x,y-1))^2}$$

$$\theta(x,y) = \tan^{-1}((L(x,y+1) - L(x,y-1))/(L(x+1,y) - L(x-1,y)))$$

4. Keypoint Descriptors

- At this point, each keypoint has
  - location
  - scale
  - orientation
- Next is to compute a descriptor for the local image region about each keypoint that is
  - highly distinctive
  - invariant as possible to variations such as changes in viewpoint and illumination

Lowe's Keypoint Descriptor (shown with 2 X 2 descriptors over 8 X 8)



In experiments, 4x4 arrays of 8 bin histogram is used,
a total of 128 features for one keypoint

Lowe's Keypoint Descriptor

- Use the normalized region about the keypoint
- Compute gradient magnitude and orientation at each point in the region
- Weight them by a Gaussian window overlaid on the circle
- Create an orientation histogram over the 4 X 4 subregions of the window
- 4 X 4 descriptors over 16 X 16 sample array were used in practice. 4 X 4 times 8 directions gives a vector of 128 values.

Application on Object Recognition

- The SIFT features of training images are extracted and stored
- For a query image
  1. Extract SIFT feature
  2. Efficient nearest neighbor indexing
  3. 3 keypoints, Geometry verification (RANSAC)

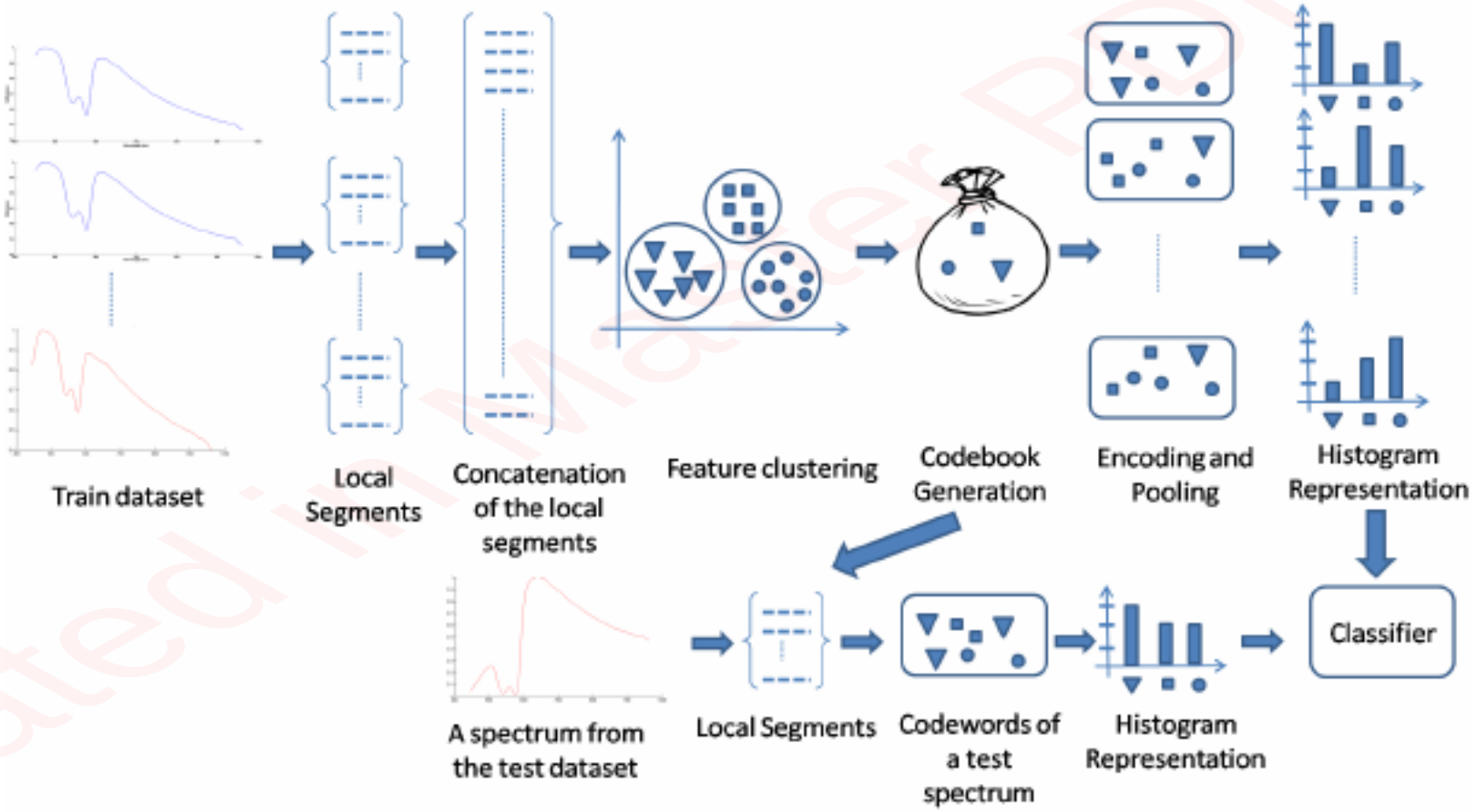
Extensions

- PCA-SIFT
  1. Work on patches wit size 41*41 pixels
  2. Compute vertical and horizontal gradient for all pizels (2*39*39 dimensions)
  3. Use PCA to project it to 20 dimensions

SURF

- Approximate SIFT
- Works almost equally well
- Very fast

Bag of Words



Image/ Document