TDA - fall 2025

Homology
(finally!)

Recap

Last time: Chain complexes

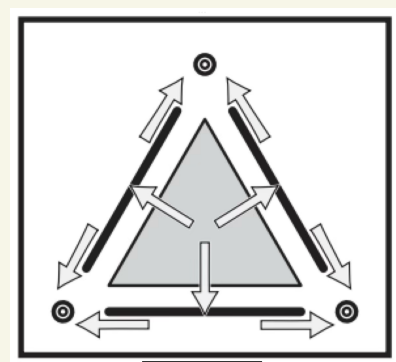
A p-chain: formal sum of p-simplices in K:

$$\alpha = \sum a_i \sigma_i$$

$$\cdots \to C_{p+1}(K) \xrightarrow{\partial_{p+1}} \overbrace{C_p(K)}^{p^{\text{th}} \text{ chain group}} \xrightarrow{\partial_p} C_{p-1}(K) \to \cdots$$

Boundary maps ∂_p : for any simplex $\sigma \in C_p$,

$$\partial_p(\sigma) =$$

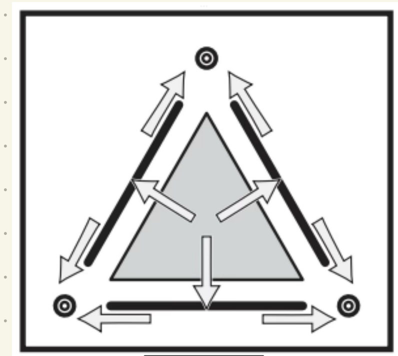


Cycles $Z_p \subseteq C_p$:

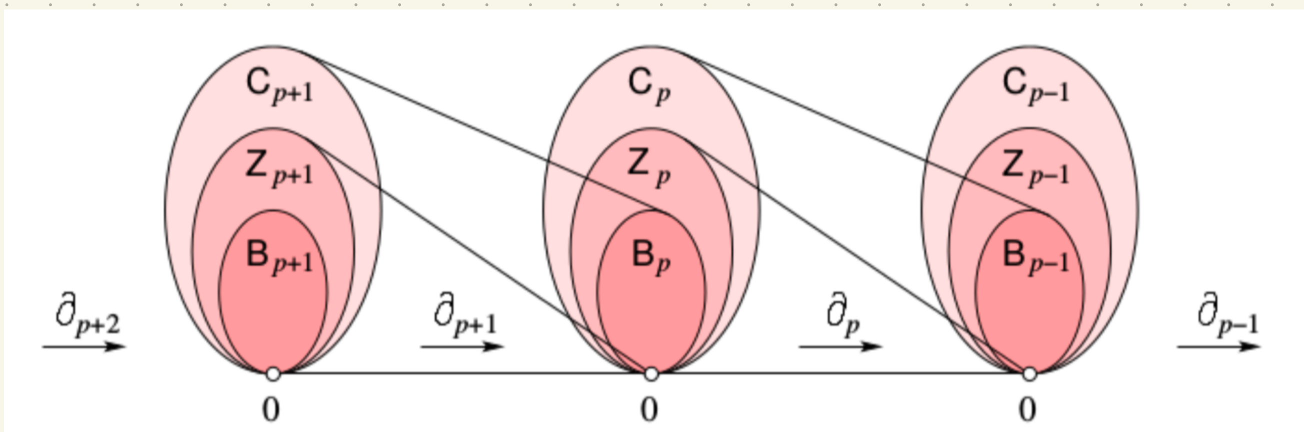
Boundaries $B_p \subseteq C_p$:

Note: Since $\partial_p \partial_{p+1}(\alpha) = 0 \; \forall \alpha \in C_{p+1}(K)$

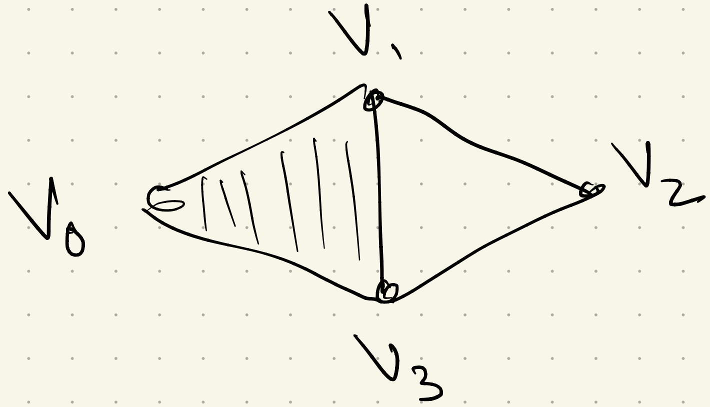
$\Rightarrow$ every p-boundary is also a p-cycle



So we get:



Example:

$K =$ 

Generators of $B_1(K)$?

Generators of $Z_1(K)$?

Quotient space

Take a vector space V over field F, and $W \subset V$ a subspace.

Define $\sim$ on V by $x \sim y$ iff $x - y \in W$.

Equivalence class of x:

$$[x] = x + W = \{x + w \mid w \in W\}$$

$$y \in [x] \Rightarrow x - y \in W$$

Then, quotient space V/W is $\{[x] \mid x \in V\}$.

Fact: V/W is a vector space with

- Scalar multiplication

$$a[x] =$$

- Addition:

$$[x] + [y] =$$

Homology

The p^{th} homology group is the quotient space:

$$H_p(K) := Z_p(K) / B_p(K)$$

Recall: $$C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1}$$

$$[\alpha] \in H_p(K) \quad \alpha \in Z_p$$

α is a cycle

$$\Rightarrow \{ \alpha + \beta \mid \beta \in B_p \}$$

$$= \{ \alpha + \partial_{p+1} \gamma \mid \gamma \in C_{p+1} \}$$

We say $\alpha, \beta \in C_p(K)$ are homologous

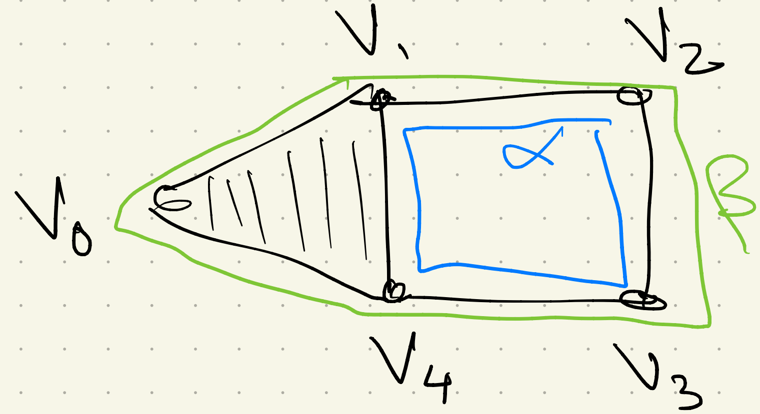
if $[\alpha] = [\beta]$ in $H_p(K)$

so: $\alpha = \beta + \partial\gamma$ for $\gamma \in C_{p+1}(K)$

cycle cycle boundary of higher dim chain

Time for an example ...

Can we find homologous 1-cycles?

$K =$ 

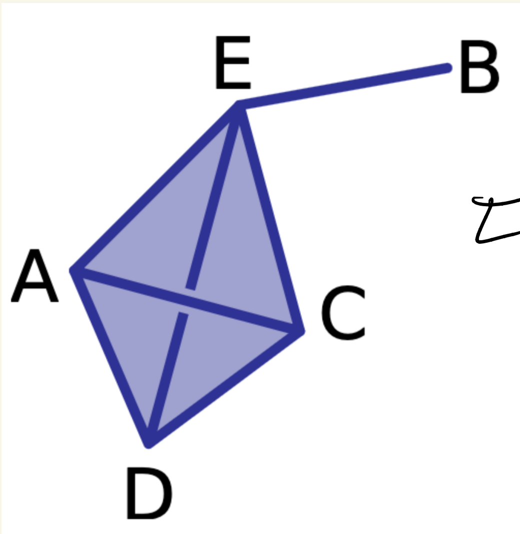
Consider: $\alpha =$

$\beta =$

If homologous, need a 2-chain γ s.t.
$\alpha = \beta + \partial_2 \gamma$.

Here, $H_1(K) = \langle \quad \rangle$

Another: What is $H_2(K)$?



no tetrahedron inside this time!

Well: $C_3(K) \xrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K)$

& $H_2(K) = \ker(\partial_2) / \operatorname{im}(\partial_3)$

$= Z_2 / B_2$

What is in $\operatorname{im}(\partial_3)$?

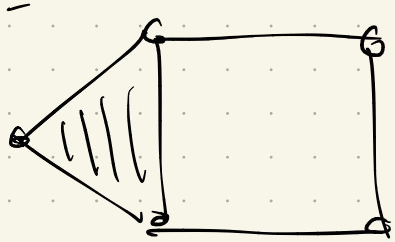
What about $\ker(\partial_2)$?

So: $H_1(K) =$

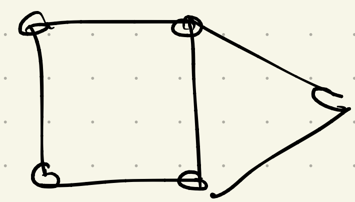
Betti numbers

The p^{th} Betti number is the rank of the p-dim homology: $\beta_p = \text{rank}(H_p)$

$K_1 =$



$K_2 =$



$\beta_1(K_1) =$

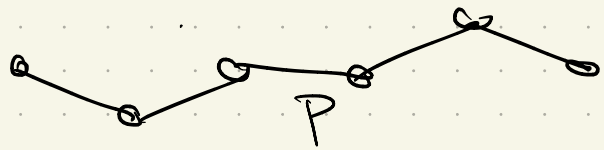
$\beta_1(K_2) =$

Some common spaces

① **Graphs**: 1d simplicial spaces

$$C_2(G)=0 \xrightarrow{\partial_2} C_1(G) \xrightarrow{\partial_1} C_0(G) \xrightarrow{\partial_2} \phi$$

- $\partial_0 = 0$, so every vertex is a 0-cycle
- B_0: boundaries of 1-chains (=paths)



$\partial(p) =$

- So $H_0(G) =$
- For H_1: no 2-cells! $\Rightarrow$ $B_1 =$

  What is Z_1?

  basis for H_1:

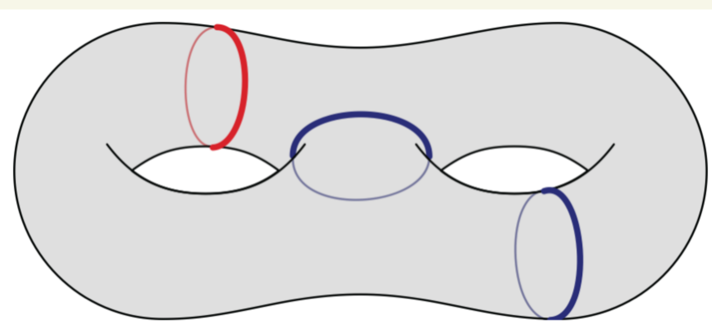
2 Surfaces:

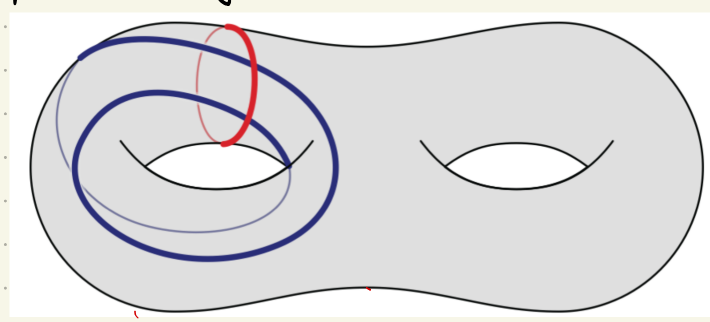
$$C_3=\emptyset \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \emptyset$$

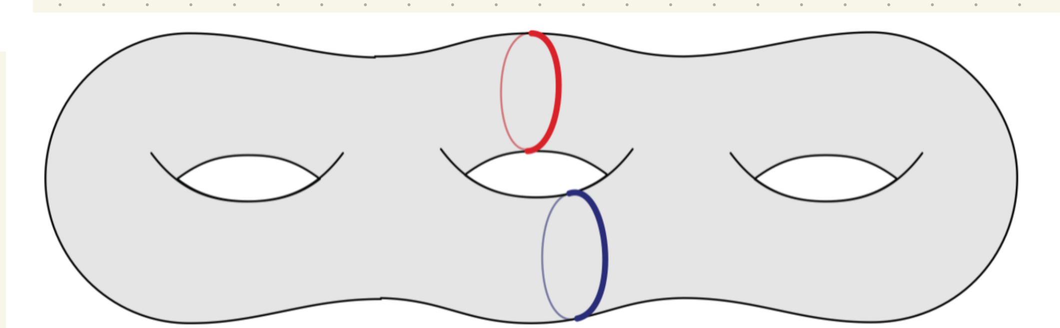
- H_0: same as graphs
- $H_1 = Z_1/B_1 = \ker(\partial_1)/\mathrm{im}(\partial_2)$

  Z_1: Still unions of cycles

  B_1: differ by ∂(some $\cup$ of Δ's)



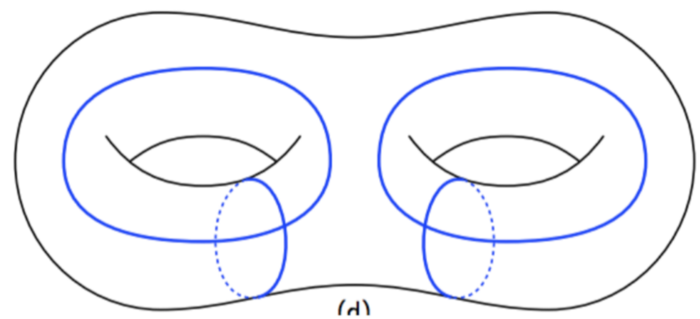




Surfaces (cont):

In the end: non-zero for H_0, H_1, & H_2:

$$\dim H_k(S_g) = \begin{cases} 1 & : \quad k = 0 \\ 2g & : \quad k = 1 \\ 1 & : \quad k = 2 \\ 0 & : \quad k > 2 \end{cases} .$$



Erickson-Whittlesey 2005

H_2: the only 2-cycle is the union of all Δ's

H_1: 2g cycles per handle

Computing homology groups

To compute Betti number:

$$\beta_p = \dim(H_p(K))$$

Well, for any linear transformation $f: U \to V$,

$$\dim(U) = \dim(\ker f) + \dim(\operatorname{im} f)$$

set $f = \partial_p$:

$$\dim(C_p) =$$

Also, for a quotient space V/W,

$$\dim(V/W) = \dim(V) - \dim(W)$$

$$\implies \beta_p =$$

So: computing!

Back to boundary matrices:

$$\partial_p \circ \underbrace{\alpha}_{\text{p-simplex}} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n_p} \\ b_{2,1} & & & b_{2,n_p} \\ \vdots & & & \\ b_{n_{(p-1)},1} & \cdots & & b_{n_{p-1},n_p} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{n_p} \end{bmatrix}$$

Rows are a basis for C_{p-1}

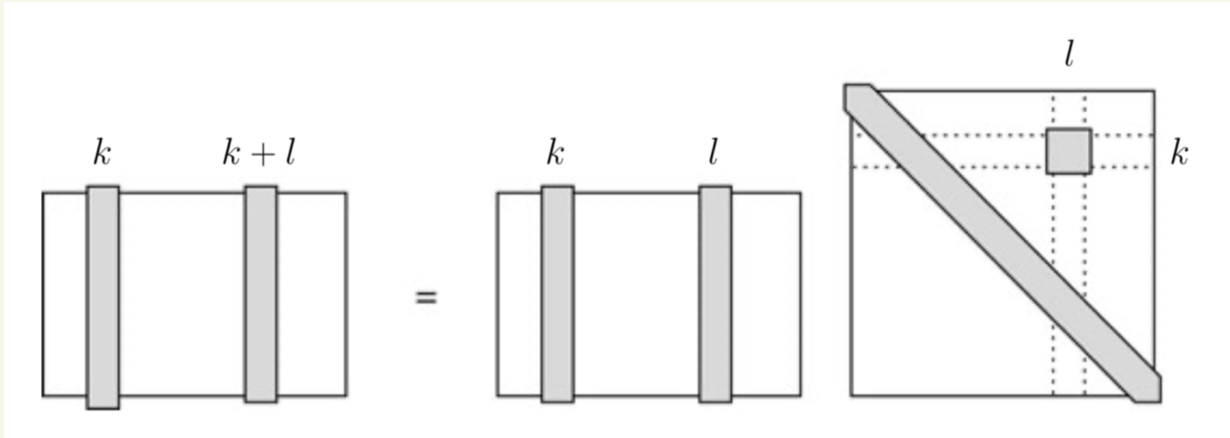
Columns are a basis for C_p

How to find rank?

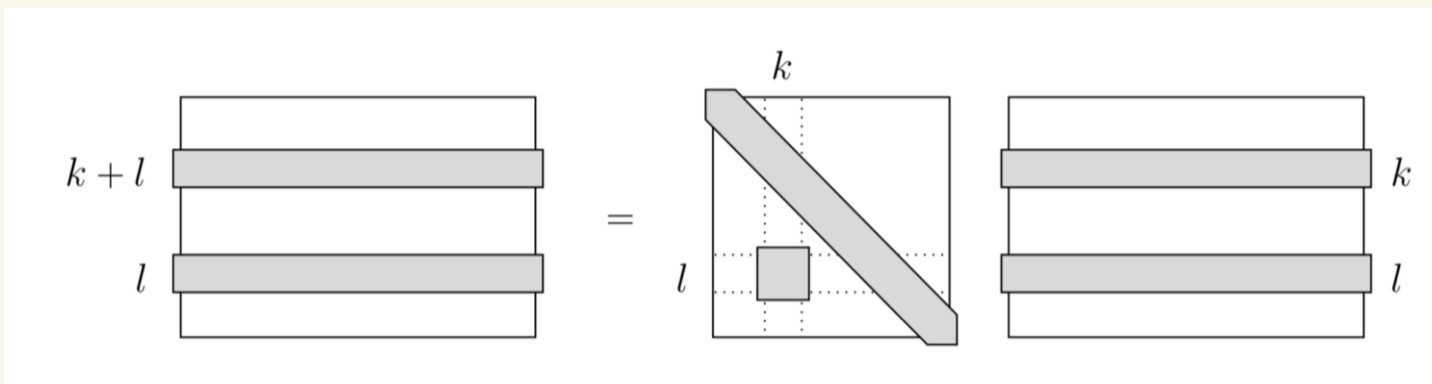
Operations on matrices

Simplify to Smith-Normal form. How?

Add columns



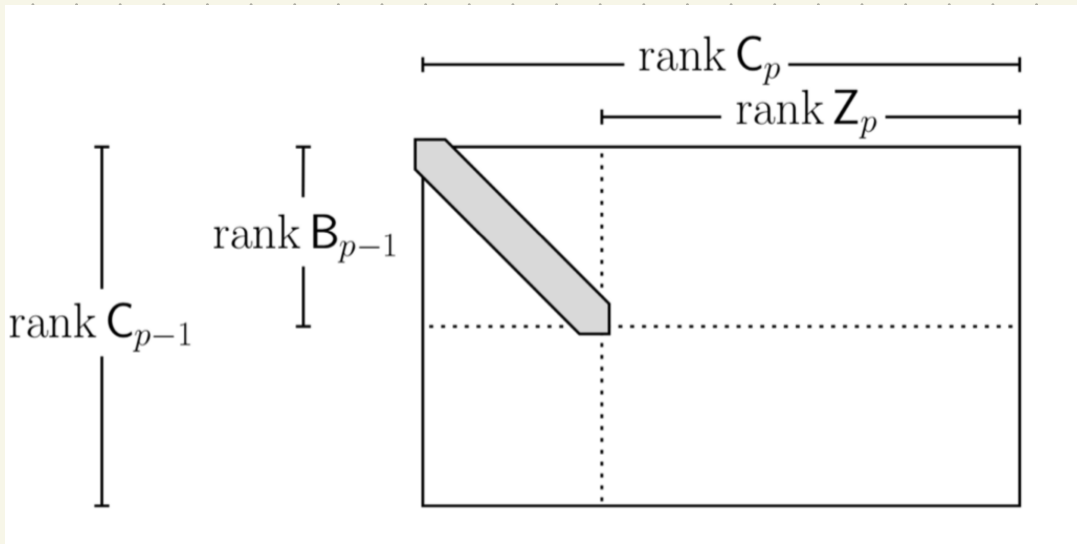
Add rows



(+ exchange rows/columns with 0 on diagonal)

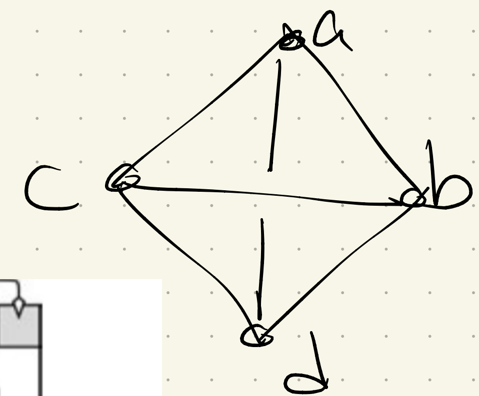
Goal: Move 1's to diagonal

Smith-Normal form: $N_p = U_{p-1}\, \partial_p\, V_p$

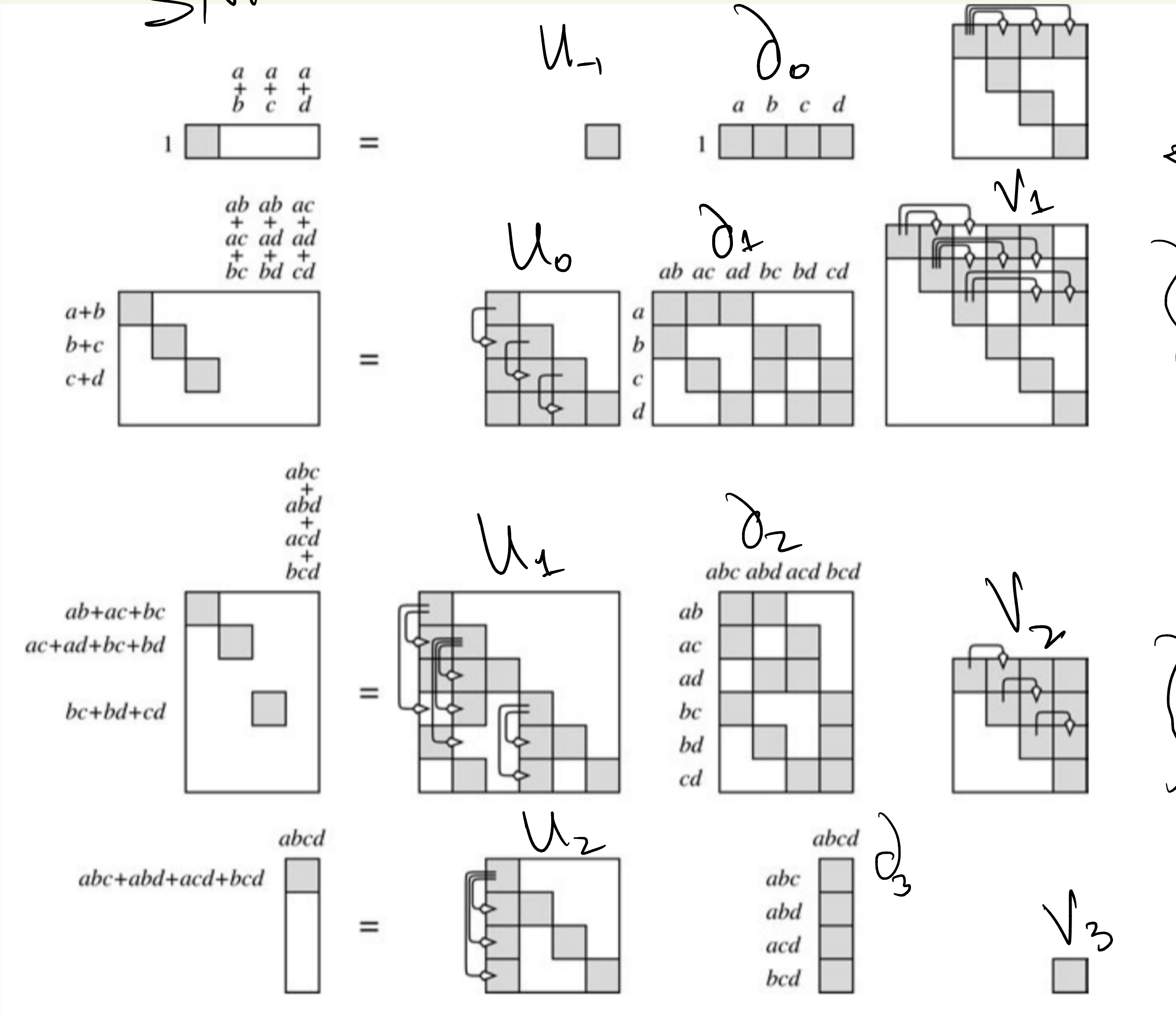
$N_p =$ 

then $\beta_p = \text{rank}(Z_p) - \underbrace{\text{rank}(B_p)}_{?}$

An example: solid tetrahedron



SNF



rank Z_0 =

rank B_0 =

rank Z_1 =

rank B_1 =

rank Z_2 =

rank B_2 =

Next time:

Simplicial Maps & induced homology

Diagram Chasing

& hopefully persistent homology