TDA - fall 2025

- Zig-zag
  ↳ Persistence
- & extended
  persistence

Last time:

So MUCH Stats & ML!

Today:

Back to basics:

ties to Reeb graphs

- Extended persistence
- Zigzag persistence

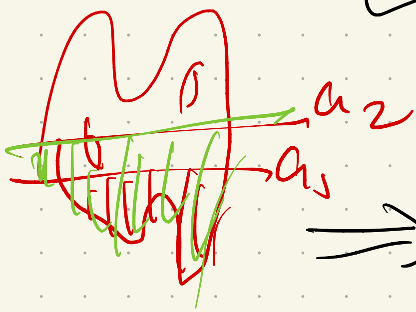
Zig-Zag Persistence Modules

"Normal" persistence: Given (X, f) + $a_0 \le \cdots \le a_n$

Morse critical

↳ filtration $X_{a_0} \hookrightarrow X_{a_1} \hookrightarrow \cdots \hookrightarrow X_{a_n}$

where $X_{a_i} = f^{-1}((-\infty, a_i])$



$$\Longrightarrow H_p(X): H_p(X_0) \longrightarrow H_p(X_1) \longrightarrow \cdots \longrightarrow H_p(X_n)$$

a b

dim ↳ barcodes

Viewing this abstractly, this is just a series of vector spaces & linear maps.

Can we generalize?

↳ quivers

↳ Steve Oudot book

Generalize: Consider n vector space with maps:

$$V_1 \overset{p_1}{\longleftrightarrow} V_2 \overset{p_2}{\longleftrightarrow} \cdots \overset{p_{n-1}}{\longleftrightarrow} V_n$$

$\rightarrow \quad \leftarrow \quad \rightarrow$

where p_i can be:

$$f_i : V_i \longrightarrow V_{i+1}$$

or

$$g_i : V_{i+1} \longrightarrow V_i$$

Simplest ex: $V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow \cdots \longrightarrow V_n$

In filtration terms:

not just adding simplices

↳ might remove or add

How can we get "backwards" maps?

Ex: Point clouds and subsamples

Take subsamples $X_1, \dots, X_n$ $\subseteq X$ of X, where we choose differently each time. $X_i \neq X_j$

No inclusion maps!

But, might be nice to understand which persistence points are correlated or are distinct.

So:

$$
\begin{tikzcd}[column sep=small]
X_1 \arrow[dr] & & X_2 \arrow[dl] \arrow[dr] & & X_3 \arrow[dl] & \cdots & X_n \arrow[dl] \\
& X_1 \cup X_2 & & X_2 \cup X_3 & \cdots & X_{n-1} \cup X_n &
\end{tikzcd}
$$

zigzag

Our matrix algorithm really only works if $X_a \subseteq X_b$ $\forall a \leq b$.

But! turns out there is a way to track some notion of "feature" across $\leftrightarrow$ maps.

Some heavy math (Gabriel's theorem + Krull-Remak-Schmidt)

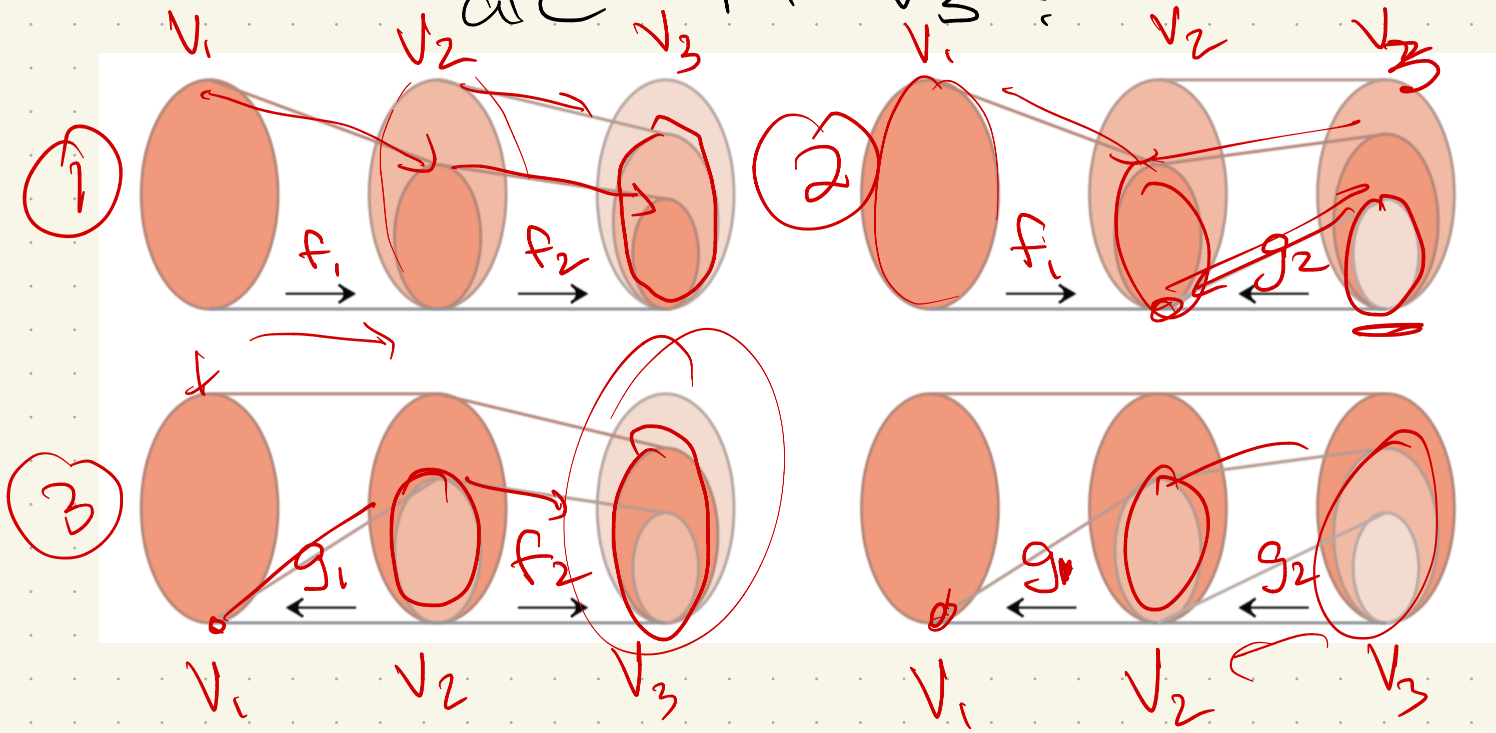
$\Rightarrow$ Still have barcodes!

Well, sort of – definitely lose some of the nice interpretation, but the algebra can still give insights $\rightsquigarrow$

Carlsson + de Silva 2010

Take $V_1 \overset{f_1}{\longleftrightarrow} V_2 \overset{f_2}{\longleftrightarrow} V_3$

4 cases: What interesting subspaces are in V_3?



① $f_2(V_2)$

$f_2(f_1(V_1))$

② $g_2^{-1}(0) \subseteq V_3$

$g_2^{-1}(f_1(V_1))$

③ $f_2(V_2)$

$f_2(g_1^{-1}(0))$

Example: Myers, Muñoz, Khasawneh & Munch 2023

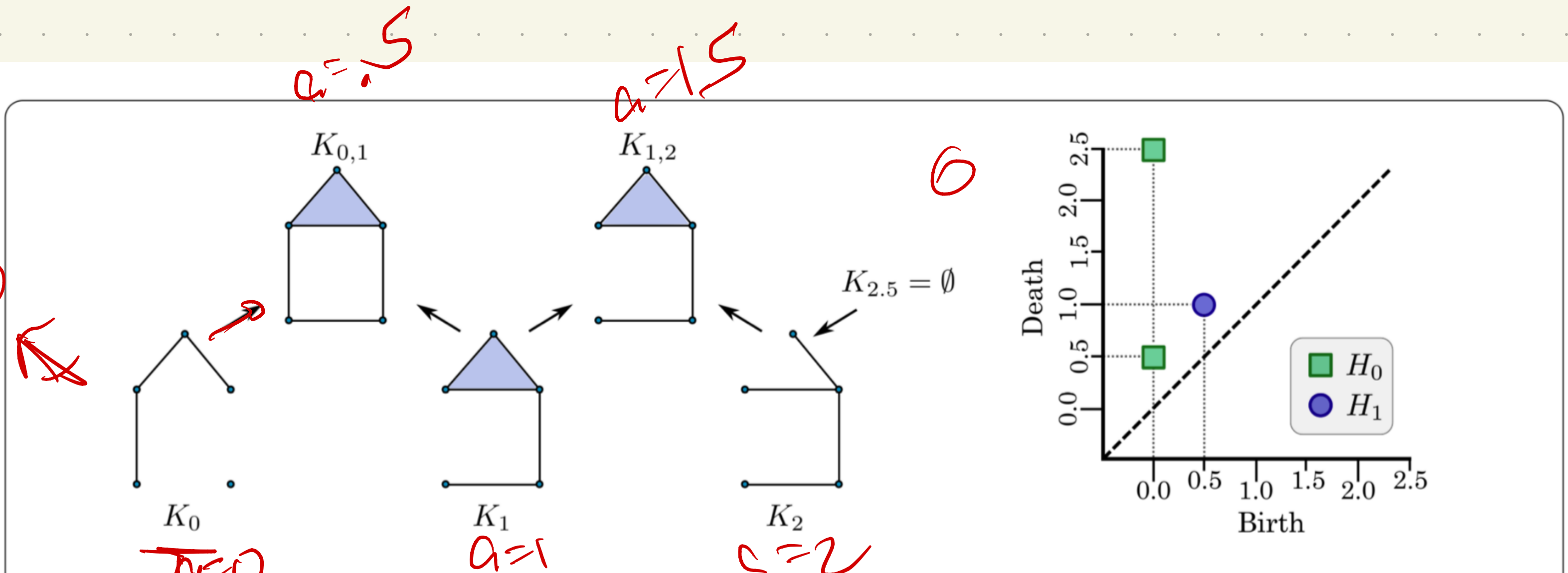
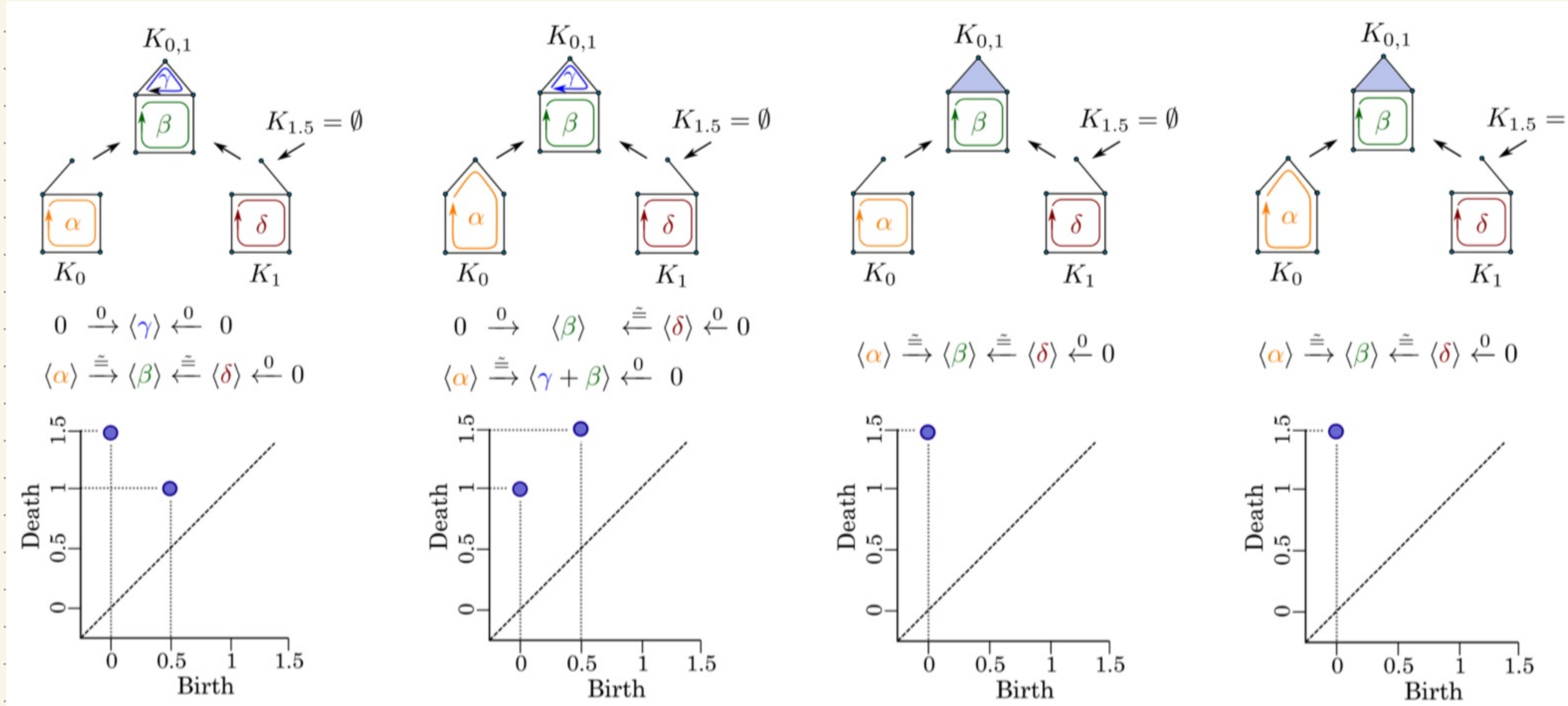


Figure 2 Example application of zigzag persistence to study changing topology of simplicial complex sequence. This example shows the sequence of simplicial complexes with intermediate unions and corresponding time stamps on the left and on the right the resulting zigzag persistence diagram on the right for dimension 0 and 1 as H_0 and H_1, respectively

(Here, K_{01} is time 0.5 & $K_{1,2}$ is time 1.5)

Some interesting subtleties!



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4

1 & 3 differ by in $K_{0,1}$
(same for 2 & 4)

What we still have: algorithms!

Carlsson, de Silva & Morozov 2009

With some care can adapt earlier matrix algorithm to handle removal, as well as additions.

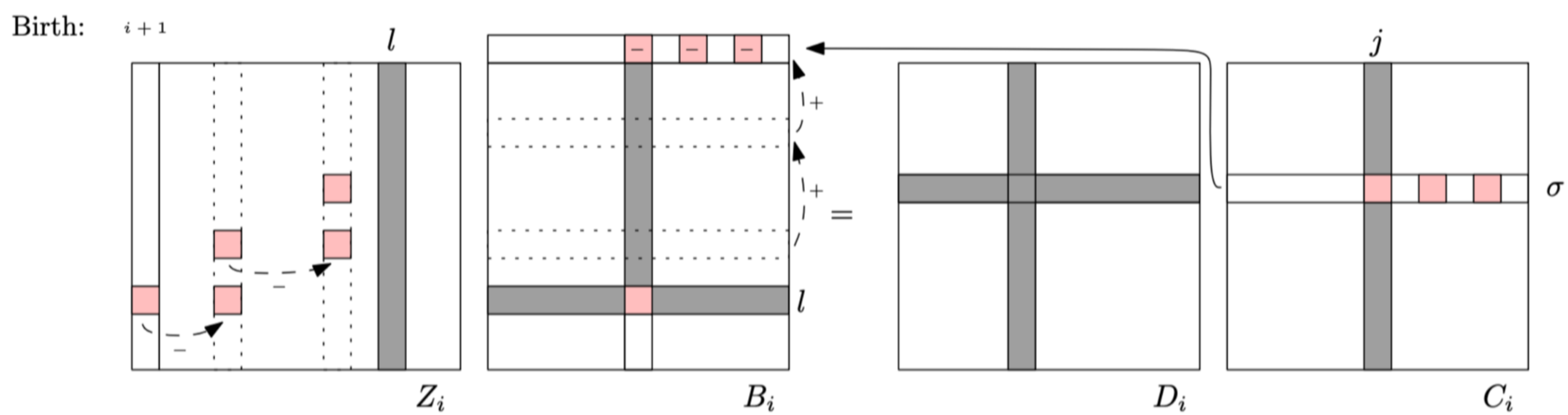


Figure 5: Adjustments made to matrices Z_i, B_i, D_i, and C_i in case of birth after the removal of simplex σ.

(Implemented in both GUDHI & Dionysus)

Applications

① Temporal graphs: edges appear & disappear

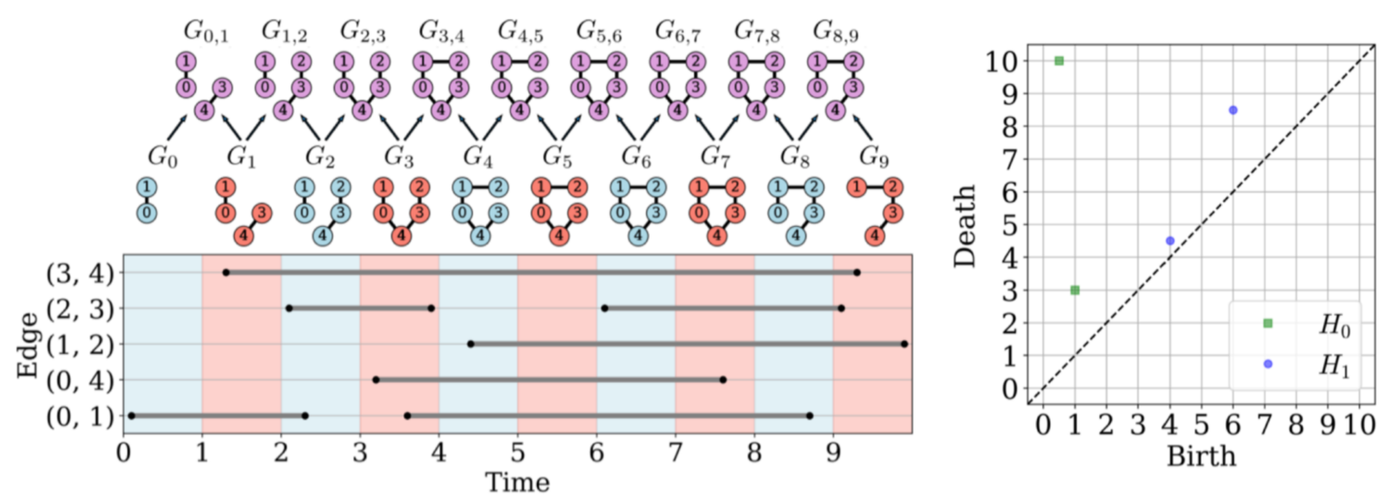
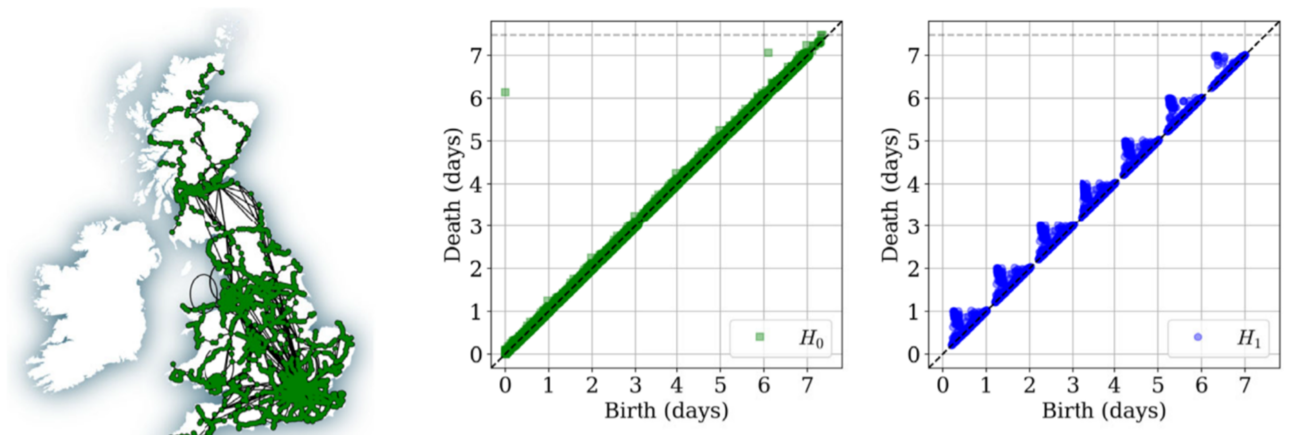


Figure 5 Example zigzag persistence applied to a simple temporal graph with temporal information stored for each edge as intervals. (Left) Edge intervals with sliding windows highlighted (alternating blue-red) with corresponding graphs and union graphs above. (Right) Zigzag persistence diagram for both H_0 and H_1. The birth and death of a feature is encoded as the midpoint of the interval where the event happened

Myers, Muñoz, Khasawneh & Munch

Examples: Transit networks



(a) Full Rail Travel Network.

(b) Zero-dimensional zigzag persistence.

(c) One-dimensional zigzag persistence.

Figure 6 Zigzag persistence diagrams of the rail transportation network of Great Britain

versus traditional methods:

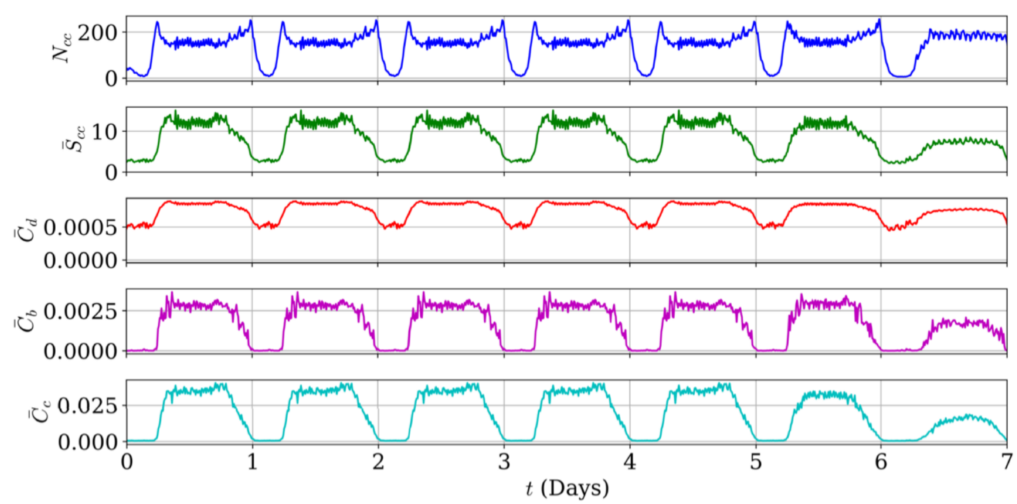


Figure 7 Connectivity and centrality analysis on temporal Great Britain rail network

Example: Ordinal partition networks

↳ graph representation of time series data based on permutation transitions

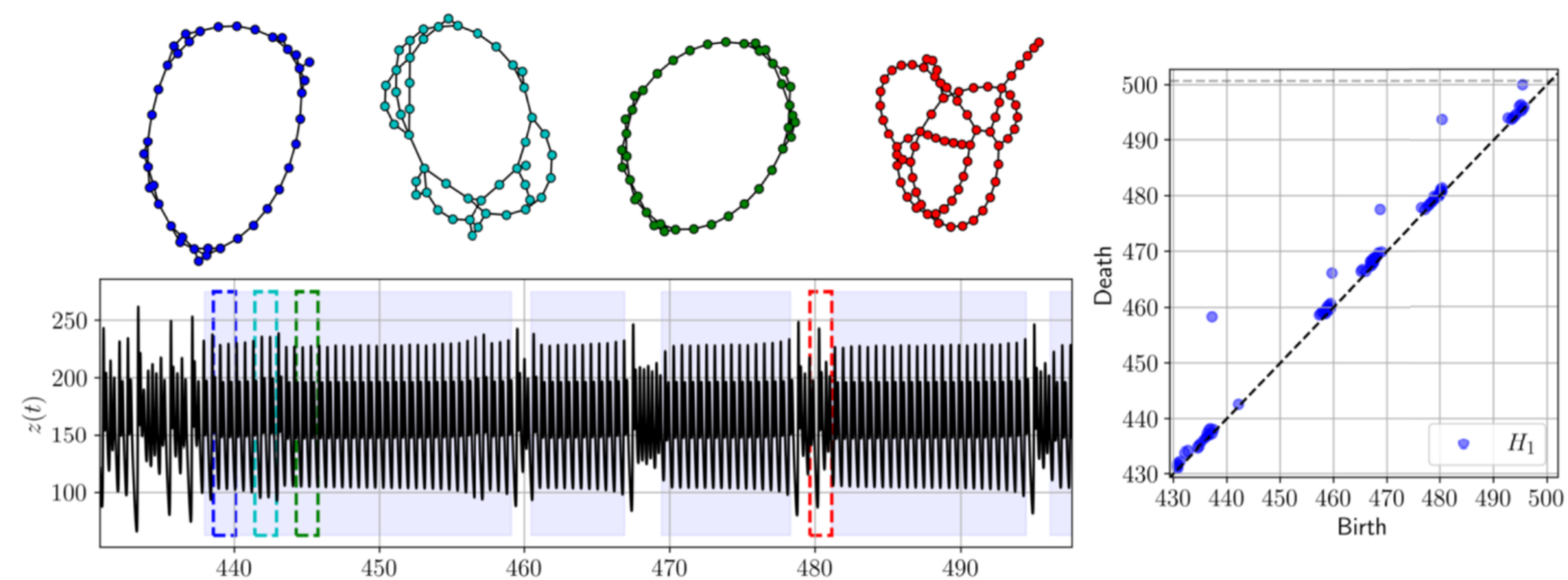
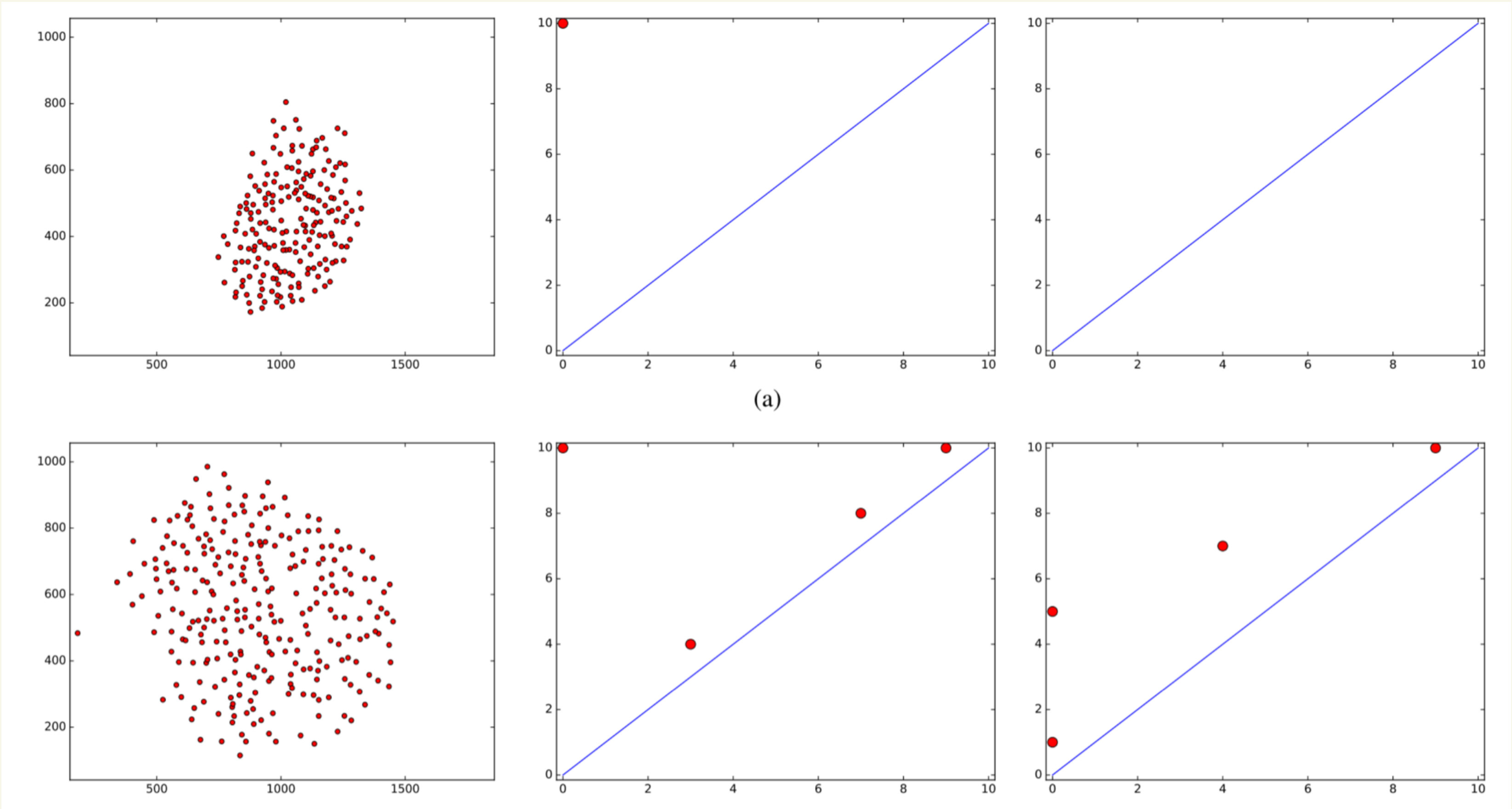


Figure 10 On the left, the z solution of the intermittent Lorenz system described in Eq. (5) is shown, along with four different graphs obtained from the corresponding ordinal partition networks in the windows of matching color. On the right, the one-dimensional zigzag persistence diagram

② Swarms of fish: Corcoran + Jones 2017

zig-zags + persistence lanscapes



↑ Fish location

H_0

H_1

Other applications

- Hopf bifurcations in dynamical systems

  Tymochko-Munch-Khasawneh 2020

- Stacks of neuron data

  Mata, Morales, Romero, Rubio 2015

Some thoughts:

- Slower to catch on

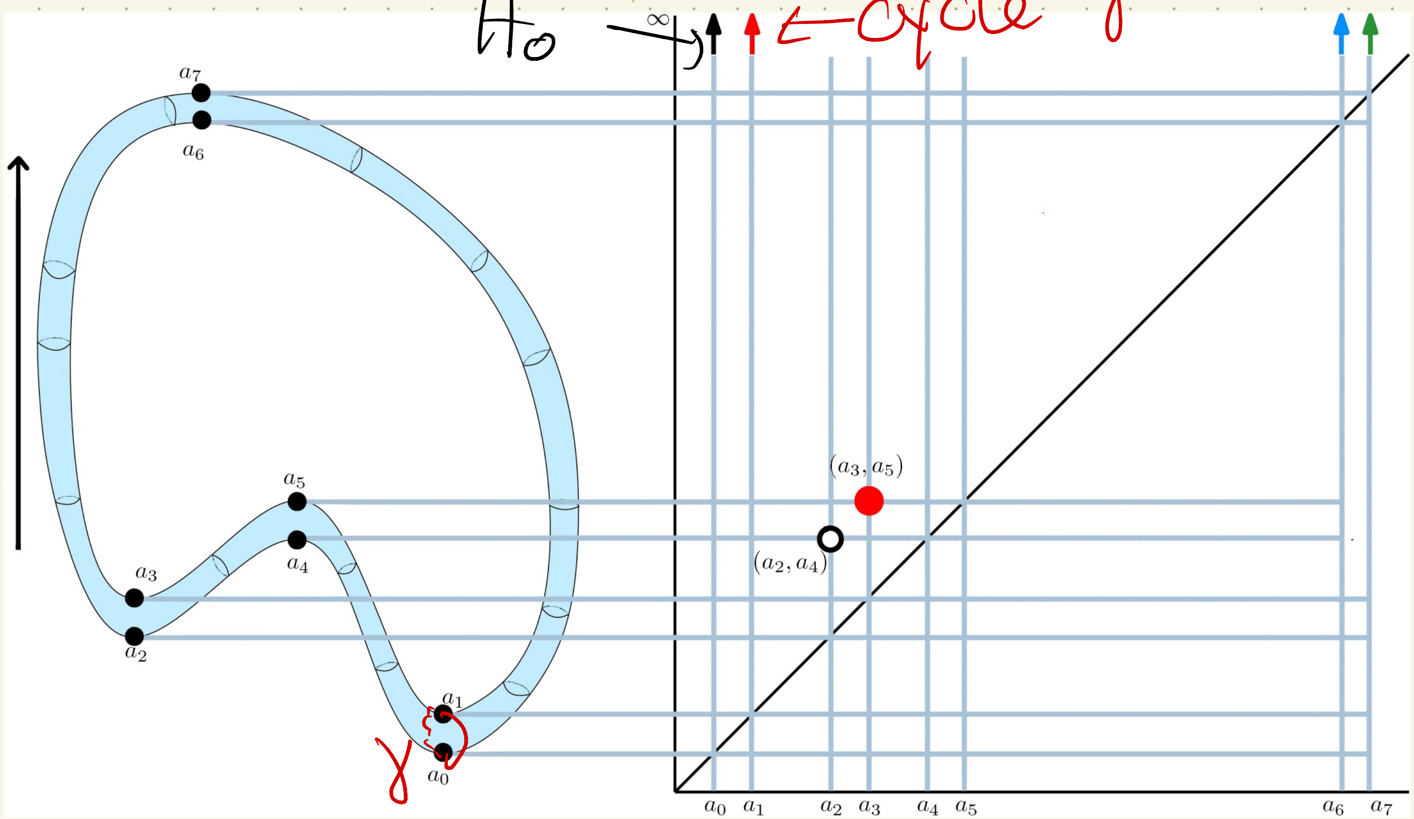
  ↳ but some strong potential!

Extended Persistence

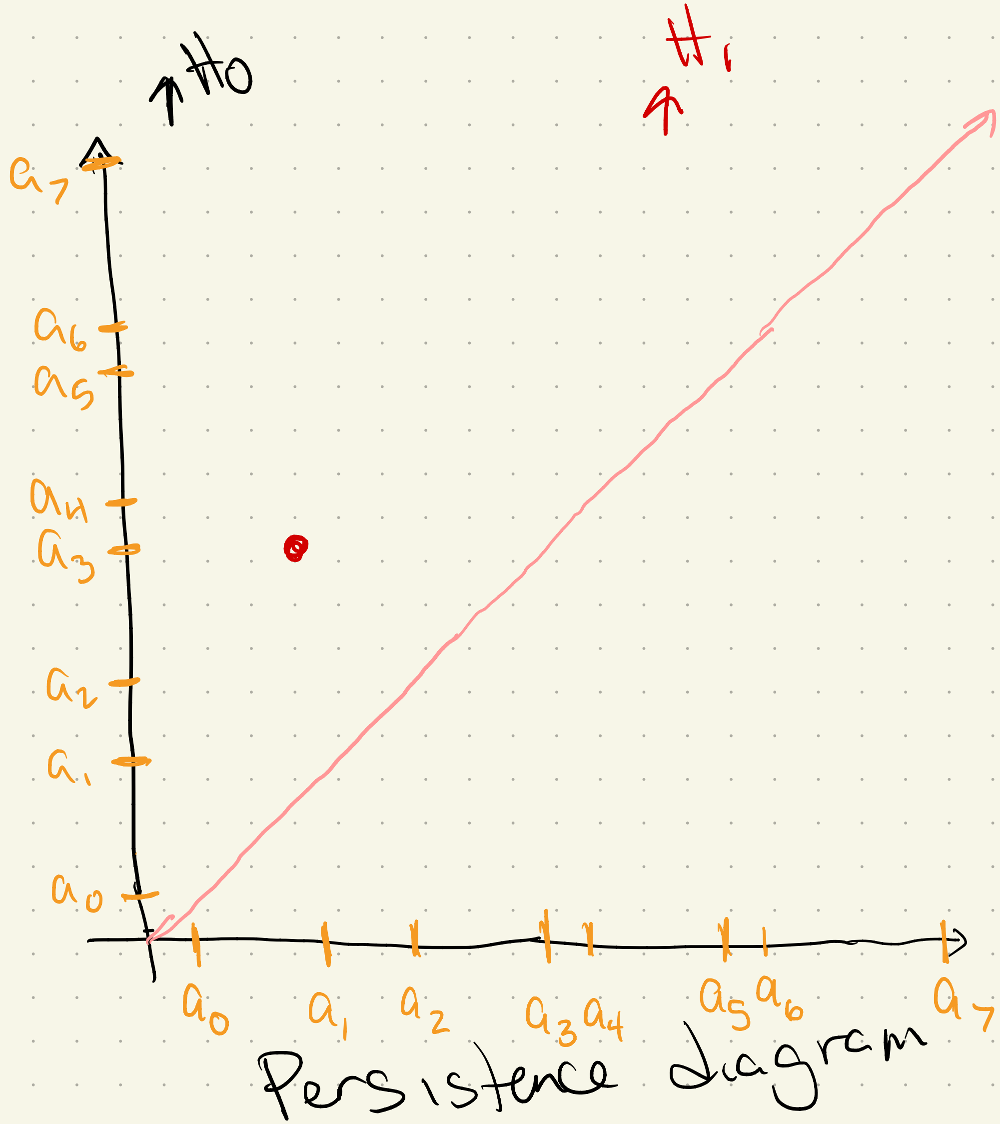
Odd parts of persistence:

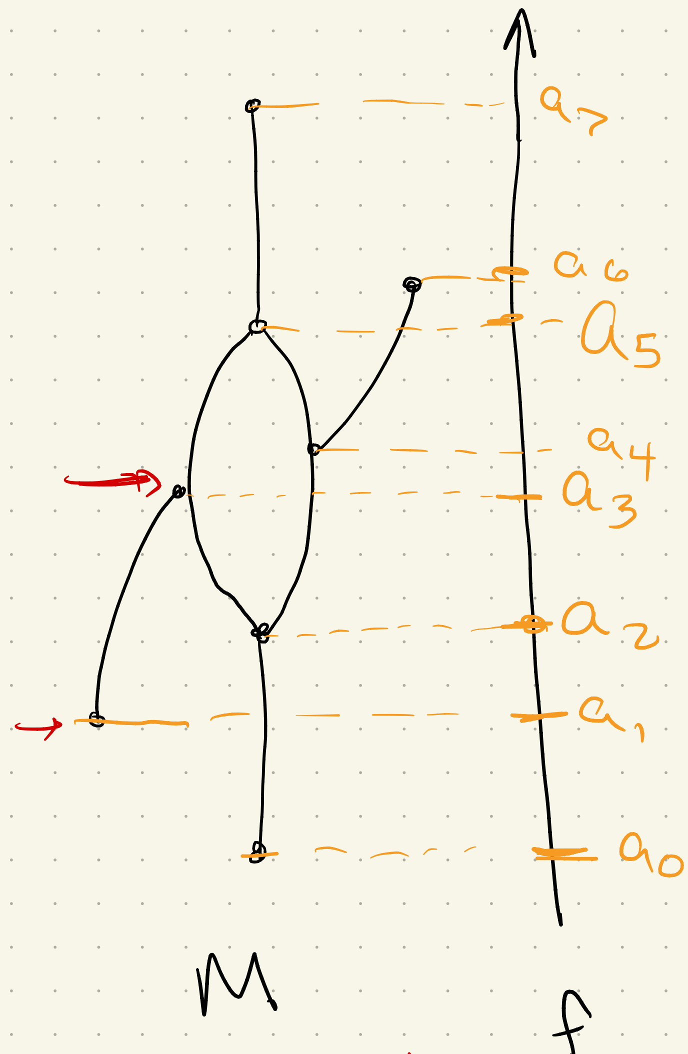
- points at infinity
- some Morse critical points don't seem to matter

$H_0 \to \infty$ ← cycle γ



Really becomes obvious on Reeb graphs
(more next week)





what about the rest!

$\boxed{\text{Agarwal-Edelsbrunner-Harer-Wang 2006}}$

$\Rightarrow$ $\boxed{\text{Cohen-Steiner, Edelsbrunner, Harer 2009}}$

Use relative homology to find other critical points, + get better pairings.

Relative homology Fix $L \subseteq K$

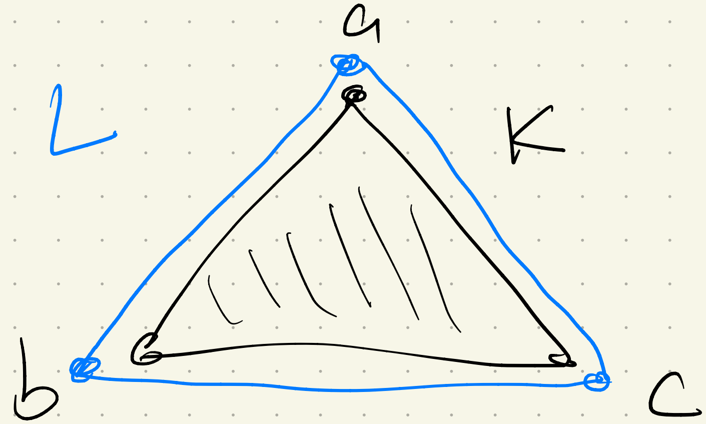
Define $C_p(K, L) := C_p(K) / C_p(L)$

$+ \ [\alpha] = \{ \gamma \in C_p(K) \mid \alpha + \gamma \in C_p(L) \}$

Then maps extend to homology.

Remember a month ago? $\longrightarrow$

Example:



$$C_2(K) = \langle 0, [a_0 a_1 a_2] \rangle$$

$$C_2(L) = 0$$

$$\Rightarrow C_2(K, L) = \langle 0, [a_0 a_1 a_2] \rangle$$

$$C_1(K) = \langle 0, [ab], [ac], [bc] \rangle$$

$$C_1(L) = \langle 0, [ab], [ac], [bc] \rangle$$

$$\Rightarrow C_1(K, L) = \langle 0 \rangle$$

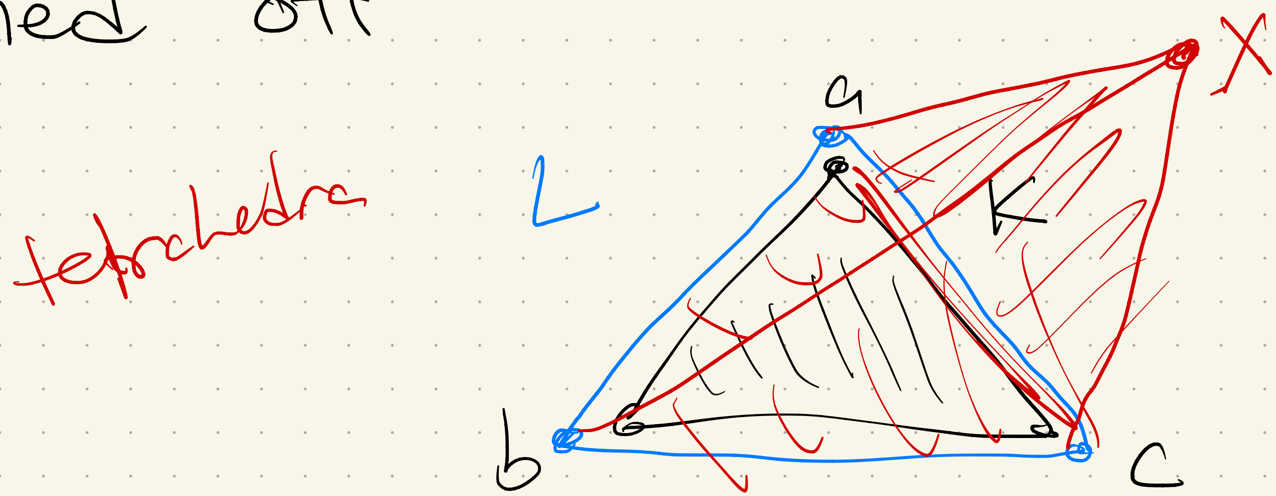
$$\& \quad C_0(K) = C_0(L) = \langle 0, [a], [b], [c] \rangle$$

$$\Rightarrow C_0(K, L) = 0$$

Fun fact

Let $K^* = K \cup \{x\} \cup \{\sigma \cup \{x\} \mid \sigma \in L\}$

"coned off"



Theorem:

$$H_p(K, L) = H_p(K^*) \text{ for } p > 0$$

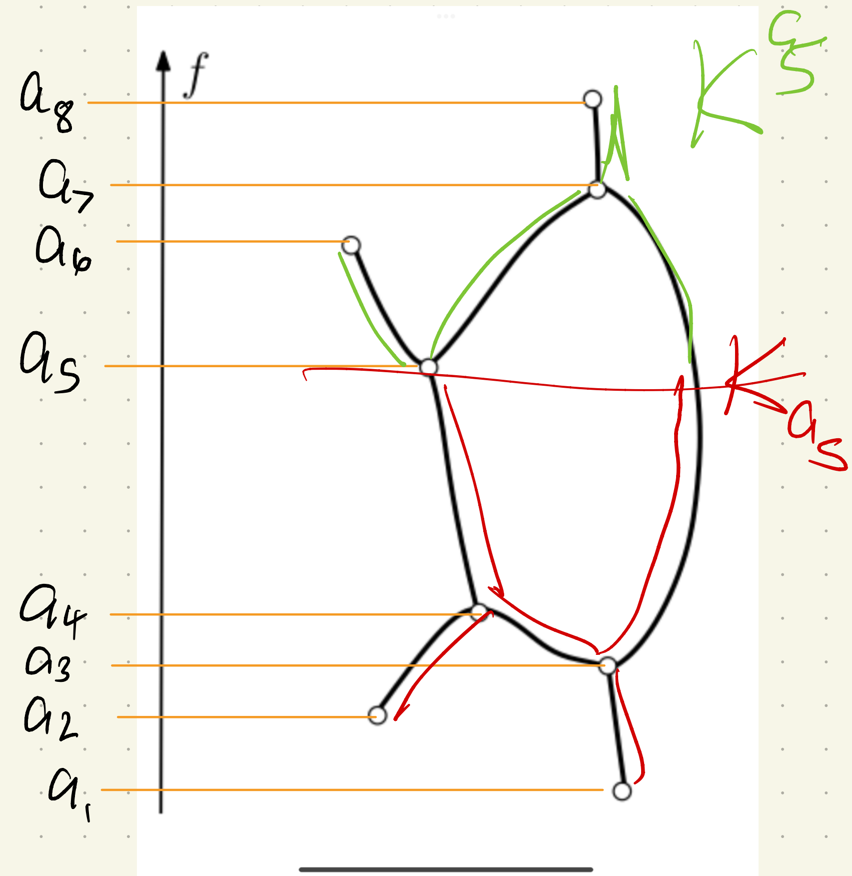
$$\& \quad \beta_0(K, L) = \beta_0(K^*) - 1$$

Here, we want to look at relative homology of the superlevel sets:

Given $f: K \to \mathbb{R}$

$$K_a = \{\sigma \in K \mid f(\sigma) \leq a\}$$

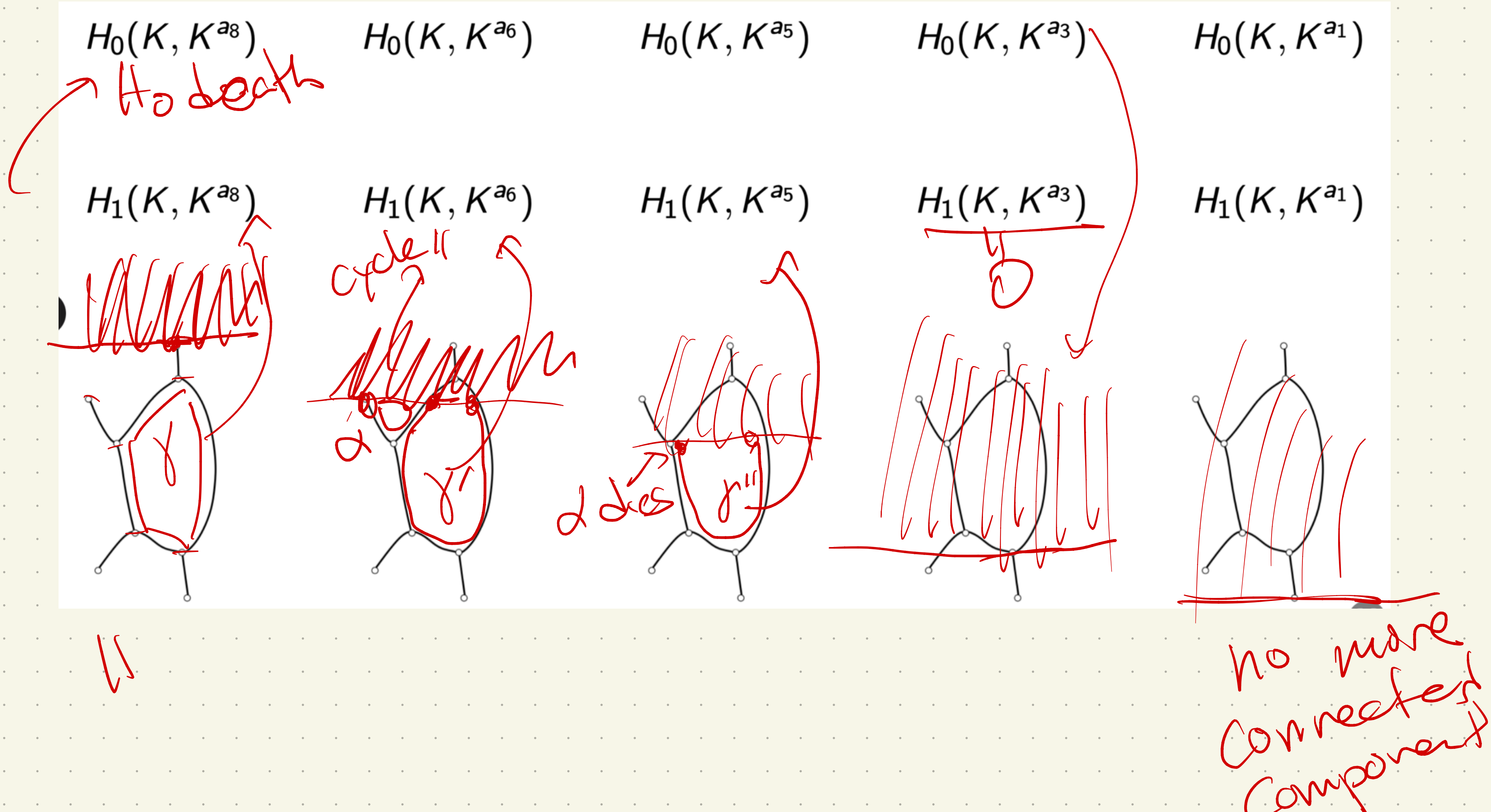
$$K^a = \{\sigma \in K \mid f(\sigma) \geq a\}$$



& study $H_p(K, K^a)$

(as well as $H_p(K_a)$)

What are important bits? ("cone off" K^{a_i})



Extended persistence Module

$$H_p(K_{a_1}) \to H_p(K_{a_2}) \to \cdots \to H_p(K_{a_n})$$

regular persistence

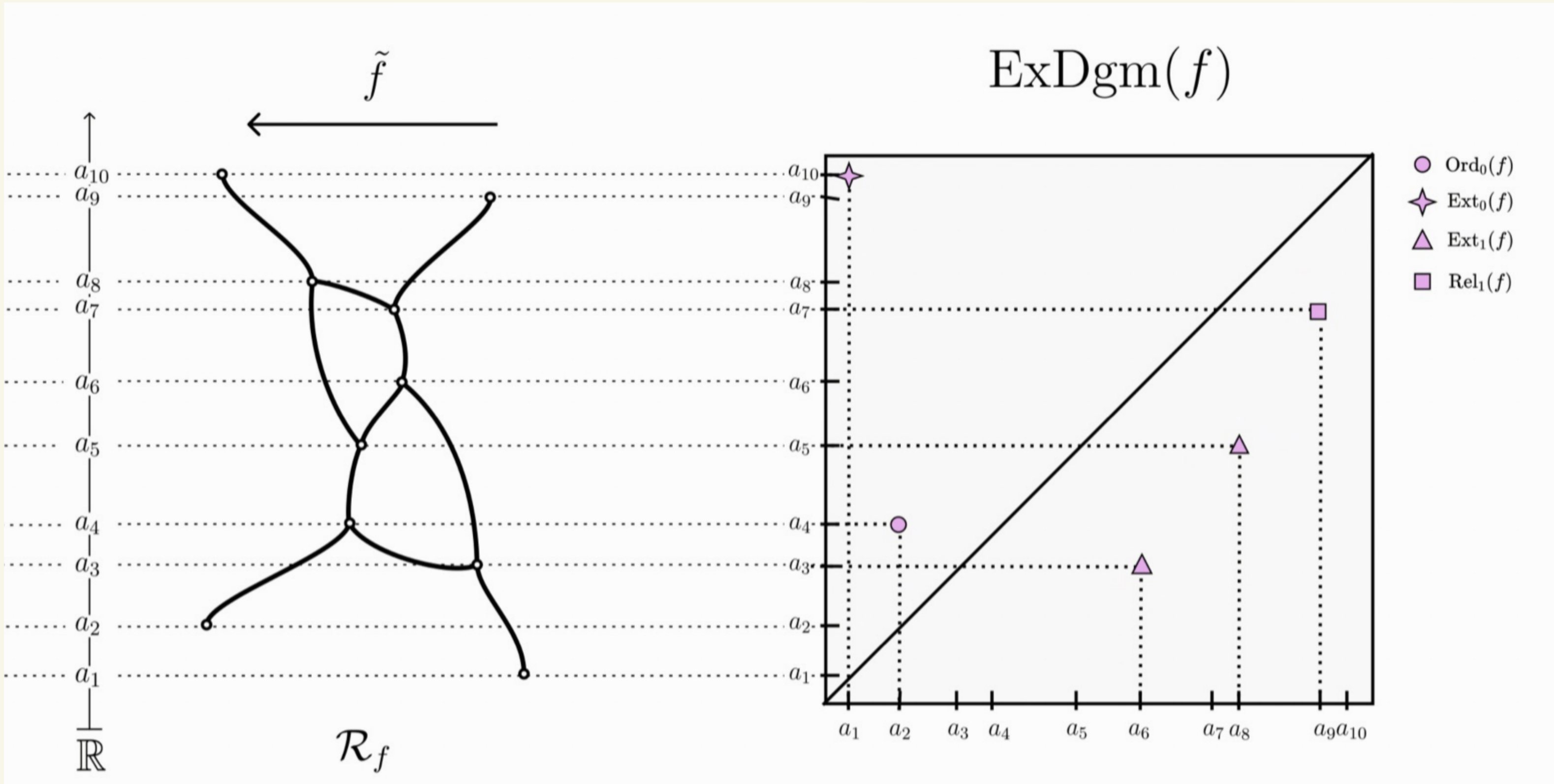
$$\to H_p(K, K^{a_n}) \to \cdots \to H_p(K, K^{a_1})$$

relative superlevel set homology

3 kinds of points:

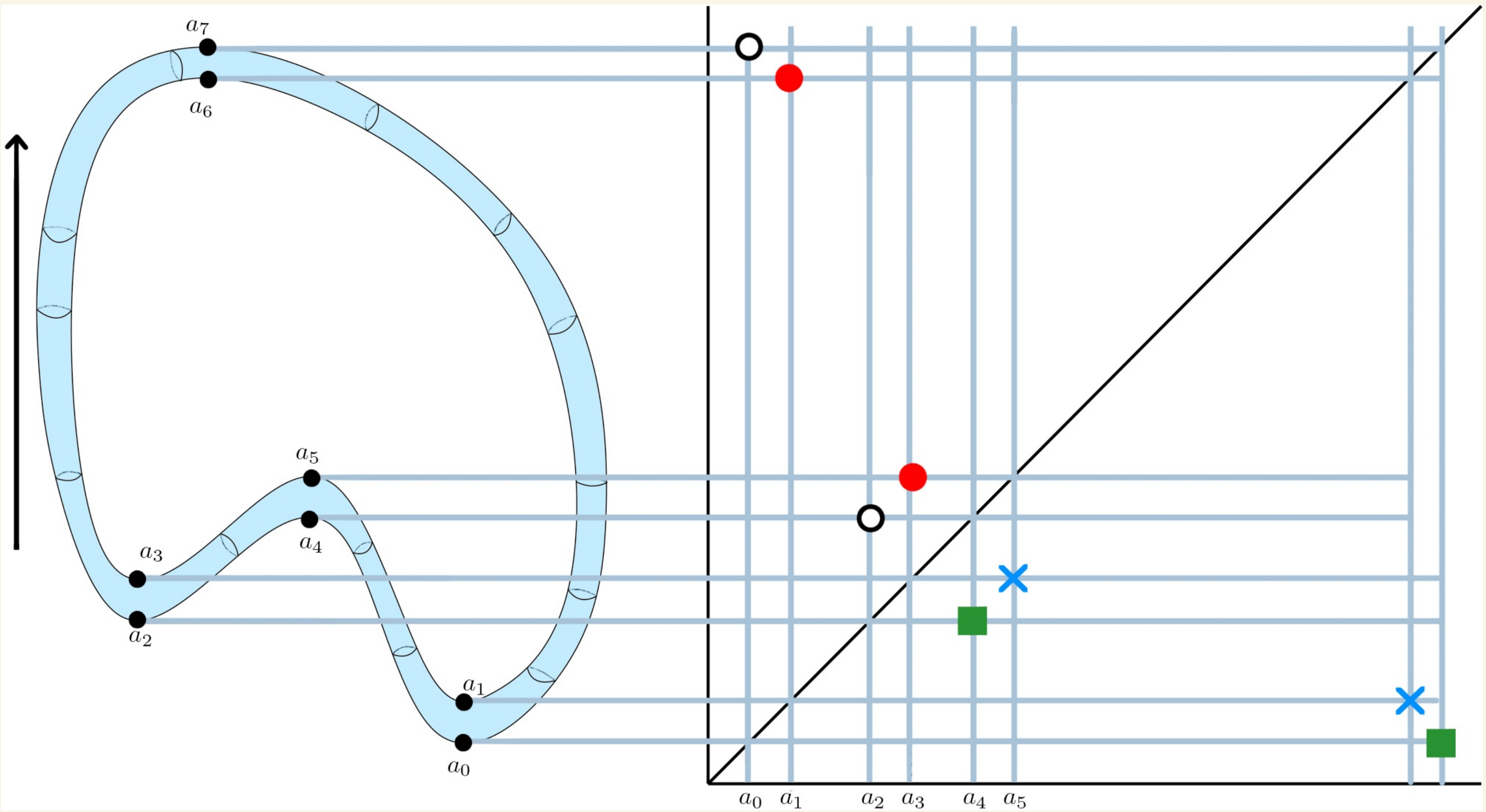
- Ordinary : usual in PD
- Relative : born & die on way down
- Extended : use both

On graphs



$\tilde{f}$
$\mathrm{ExDgm}(f)$
a_{10}
a_9
a_8
a_7
a_6
a_5
a_4
a_3
a_2
a_1
$\mathbb{R}$
$\mathcal{R}_f$
$\mathrm{Ord}_0(f)$
$\mathrm{Ext}_0(f)$
$\mathrm{Ext}_1(f)$
$\mathrm{Rel}_1(f)$

More generally



- H_0 (ordinary & extended)
- ordinary H_1
- relative H_2
- relative H_1

Under the hood:

- Very beautiful combination of Lefschetz & Poincare duality (that 2010 paper)
- Some more algebraic conections $\boxed{\text{Turner-Robins-Morgan 2022}}$