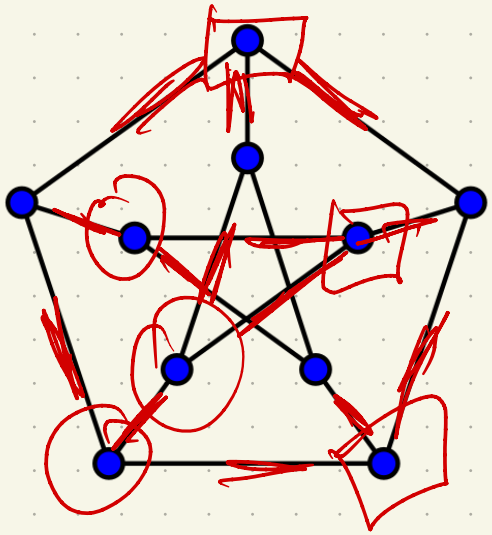
Algorithms + Complexity
Spring 2026

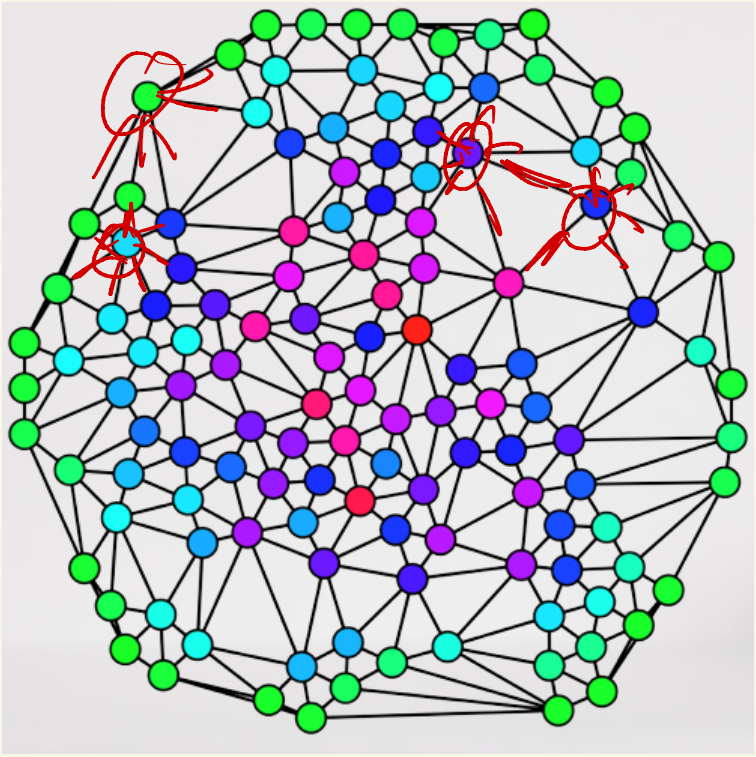
Greedy Approx, part 2

First example

Vertex cover: Given a graph $G = (V, E)$, choose a set of vertices $S \subseteq V$ such that every $e \in E$ is incident to some $v \in S$.

Examples:





How hard?

Easy to find a cover:

$S = V$ $|V|$ or V

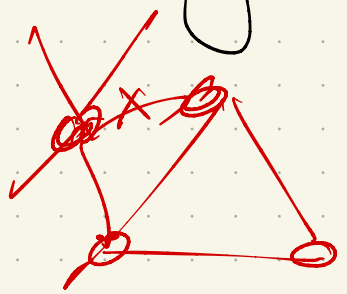
Challenge: make it smaller

↳ minimization problem

Note: In general, NP-Hard. (More later...)

One idea: Use vertices with high degree.

Why? Take lots of edges!



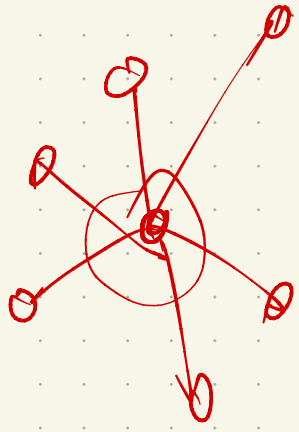
Greedy algorithm:

```
GREEDYVERTEXCOVER(G):
  C ← ∅
  while G has at least one edge
      v ← vertex in G with maximum degree
      G ← G \ v
      C ← C ∪ v
  return C
```

Why?

these edges are covered ↳ remove v & all incident edges

Question: does this ever give the min set?

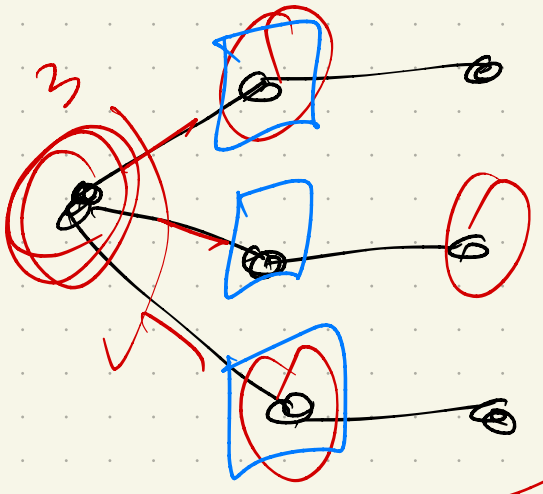


:)

Yes

Question: how to make it fail?

Need high degree vertices that are not optimal.



size = 4

OPT = 3

deg ≤ 3

But:

only +1 apart

Can we prove this is an approximation to optimal?

last slide: example with $|C| = opt + 1$

ie $|C| > |opt|$ (see last slide)

but $|C| \leq \alpha \cdot |OPT|$?

Note: Nothing in our algorithm tells us what to aim for!

prev. example $\Rightarrow$ $\alpha > 1$

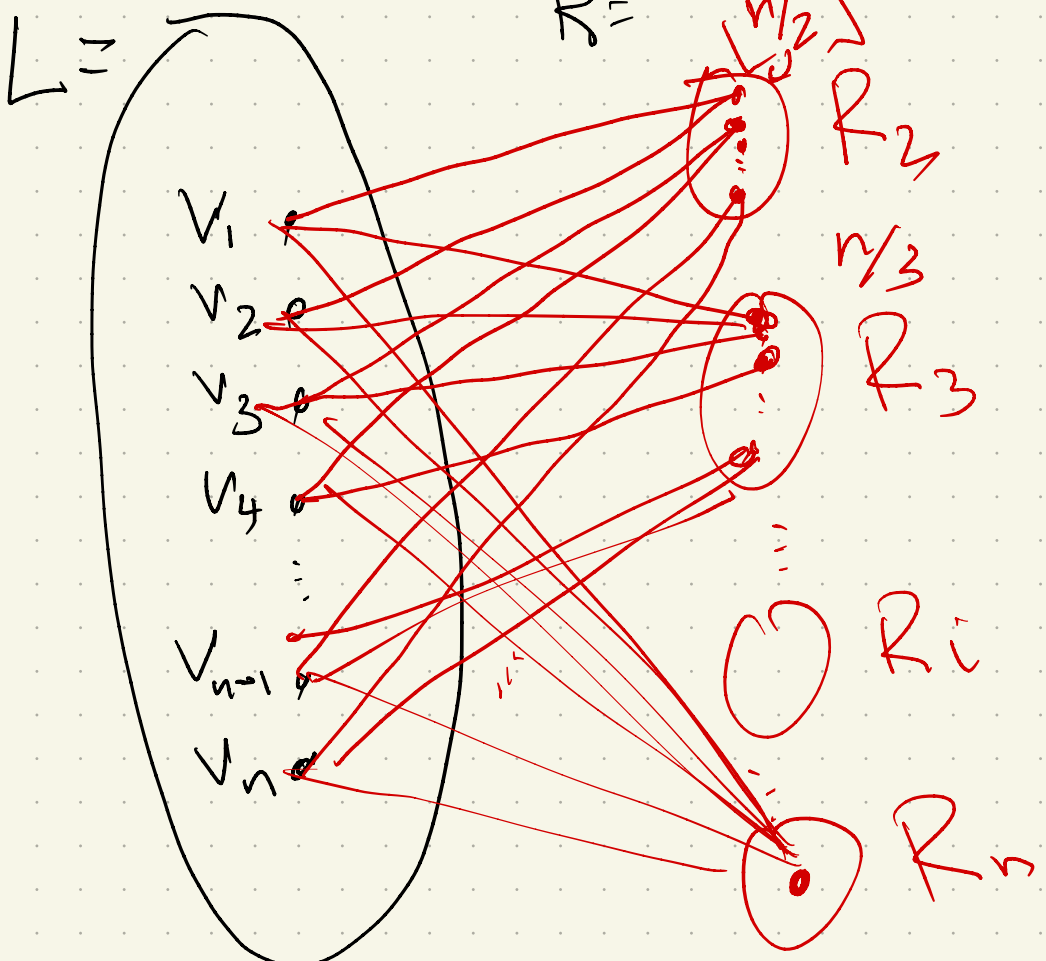
Let's check some notation here...

Back to VC

Question: is it a 2-approximation?

No. (But not obvious...)

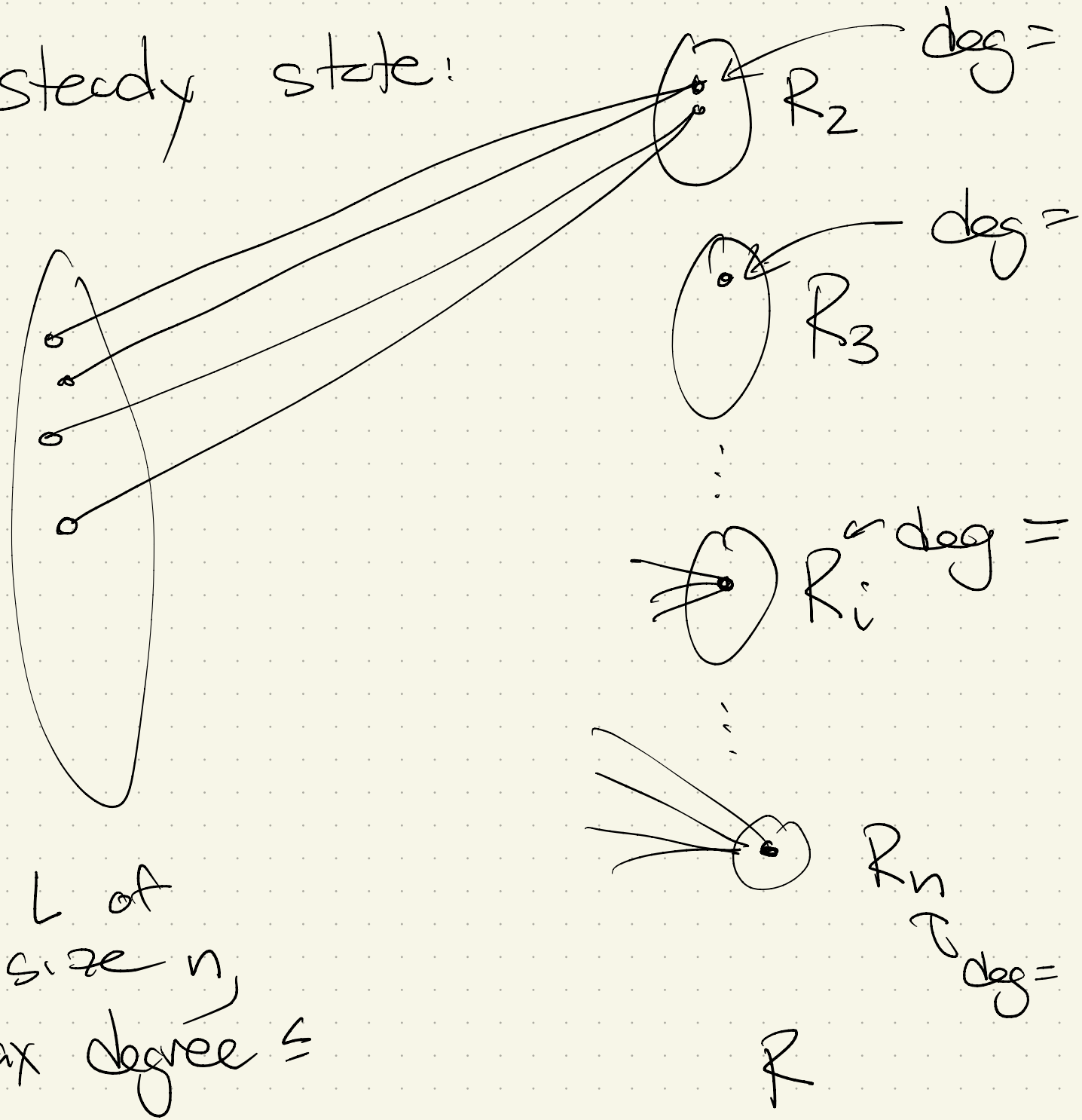
Construction: bipartite graph $G = (V, E)$ where $V = L \cup R$



For R: for each $i \leftarrow 2..n$, add $\lfloor \frac{n}{i} \rfloor$ vertices, each degree i, & connect to <u>different</u> vertices in L.

↳ call these $R_i \subseteq R$

In steady state:



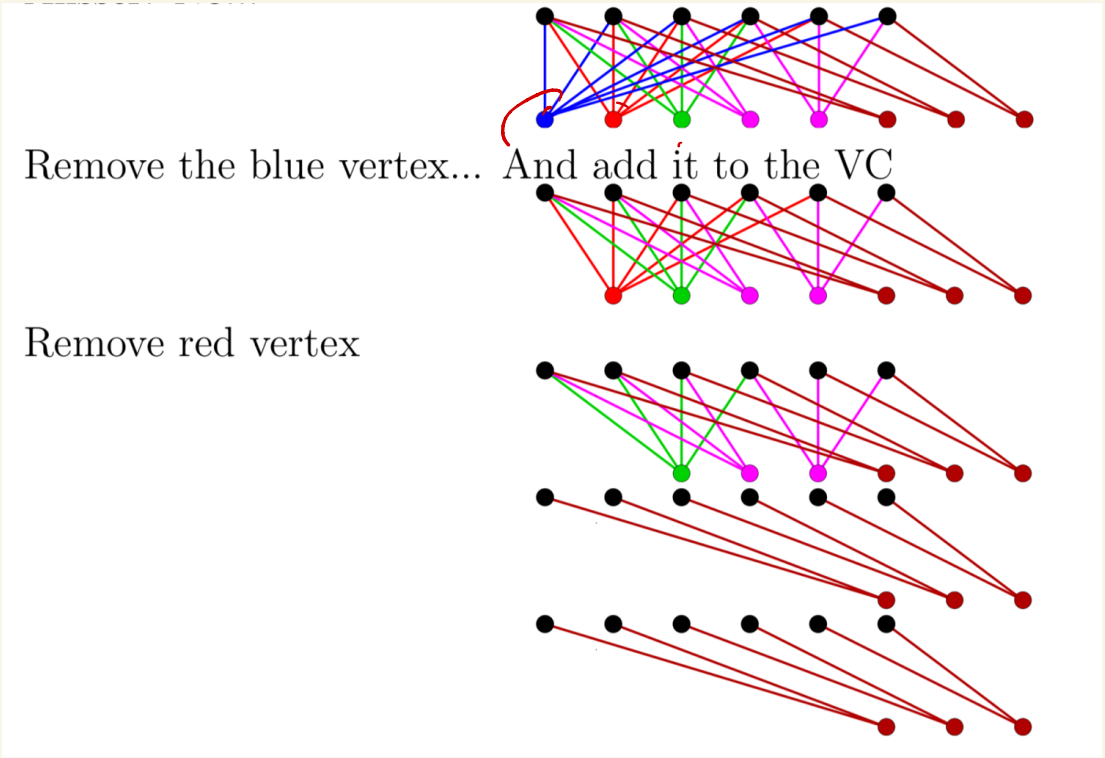
L of size n,

max degree ≤

What does our algorithm do?

```
GreedyVertexCover(G):
  C ← ∅
  while G has at least one edge
      v ← vertex in G with maximum degree
      G ← G \ v
      C ← C ∪ v
  return C
```

Highest degree vertex?
↳ in R_1, one of degree n.
When removed:



So, in end, all R vertices chosen.

What is $|R|$?

$$|R| = \sum_{i=2}^{n} |R_i| = \sum_{i=2}^{n} \left\lfloor \frac{n}{i} \right\rfloor$$

$$\geq$$

Recall that "cheat sheet":

at

Harmonic numbers:

$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{71}{25}$

$$\ln n < H_n < \ln n + 1,$$

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i},$$

So, back to $\alpha(n)$ stuff:

$$|R| \geq n(H_n - 2)$$

$$|L| = n$$

so, greedy factor $\alpha(n) \geq \frac{|R|}{|L|}$

& $\frac{|R|}{|L|} \geq$

Note: lower bound! Can we show it always gets at least this?

Theorem Greedy algorithm always chooses a set of size $\leq (\log n) \cdot OPT$

To prove: Rewrite slightly:

```
GreedyVertexCover(G):
  C ← ∅
  G_0 ← G
  i ← 0
  while G_i has at least one edge
      i ← i + 1
      v_i ← vertex in G_{i−1} with maximum degree
      d_i ← deg_{G_{i−1}}(v_i)
      G_i ← G_{i−1} \ v_i
      C ← C ∪ v_i
  return C
```

Let G_i = graph in i^{th} iteration.

Let d_i = max degree in G_i

Let C^* = optimal vertex cover in G
(which must exist but which we don't know)

We do know that C^* is a vertex cover for each G_i.

So:

$$\sum_{v \in C^*} \text{degree of } v \text{ in } G_i \geq \#\text{ edges in } G_i$$

Why?

Since $\sum_{v \in C^*} \deg_{G_i}(v) \geq |E(G_i)|$

$\Rightarrow$ average degree in G_i of C^* is $\geq \frac{|E(G_i)|}{|C^*|}$

Why?

But: this means max degree in G_i is at least this size.

$\Rightarrow d_i \geq \frac{|E(G_i)|}{|C^*|} = \frac{|E(G_i)|}{OPT}$

Also: # of edges in G_i decreases

$$d_i \geq \frac{|E(G_i)|}{OPT} \geq \frac{|E(G_j)|}{OPT}$$

for $j \geq i$

Now, consider first OPT iterations of loop:

$$G_1 \to G_1 \to G_2 \to \cdots \to G_{OPT}$$

How many edges get removed?

$$\sum_{i=1}^{OPT} d_i \geq$$

So: $\sum_{i=1}^{OPT} d_i \geq |E(G_{OPT})|$

But: $|E(G_{OPT})| = |E(G)| - \sum_{i=1}^{OPT} d_i$

Why?

Crazy sums: $\sum_{i=1}^{OPT} d_i \geq |E(G)| - \sum_{i=1}^{OPT} d_i$

In other words:

OPT iterations removes at least half the edges.

$|E| \longrightarrow \frac{|E|}{2} \longrightarrow$

Keep going: OPT iterations more

How many times?

After $\log(|E|)$ rounds, done.

How many per round?

Runtime & space:

A different approximation - simpler idea:

- pick any edge + add its endpoints to the cover
- delete all "covered" edges
- Repeat

Seems worse, right?

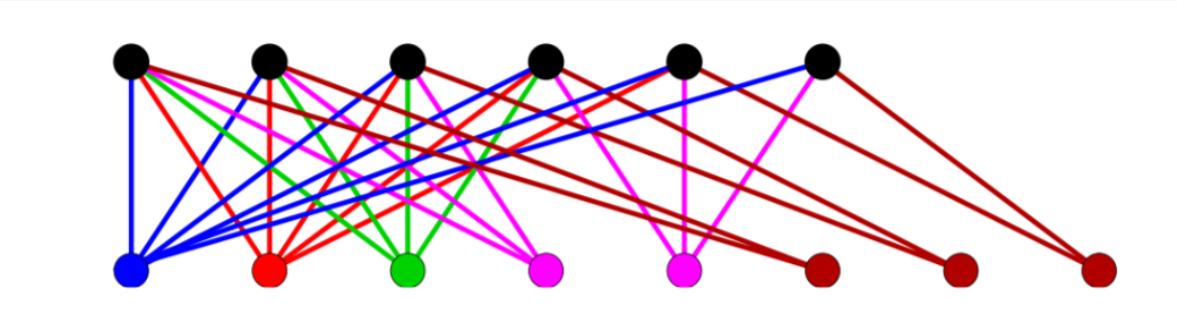
```
DUMBVERTEXCOVER(G):
  C ← ∅
  while G has at least one edge
      (u, v) ← any edge in G
      G ← G \ {u, v}
      C ← C ∪ {u, v}
  return C
```

Theorem Dumb vertex cover is a 2-approximation.

Proof Let C be greedy cover here, & C^* be OPT.

For each edge $e = \{uv\}$:

Hub?



Parameterized Complexity

Next section considers: can we get an exact solution that is exponential, but in some other parameter?

Example: in Vertex Cover, can get exact answer in $O(n^{k+2})$, where $k = |Opt|$

↳ Can improve to $O(3^k n^2)$

Traveling Salesmen (TSP)

Given n cities with pairwise distances between them, find the shortest cycle visiting all cities.

This is NP-Hard: more next week!

But idea: Take some problem X where we have reason to believe it will not have a polynomial solution.

Show any alg for TSP would be a subroutine to solve X.

So:

Additional benefit:

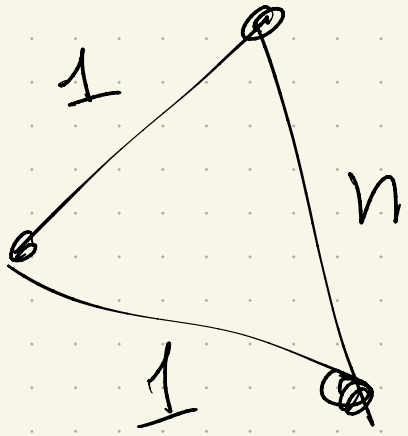
The reduction for TSP is from Hamiltonian cycle:

Any approximation algorithm for TSP would give an exact solution to Ham cycle

$\Rightarrow$ Hard to approximate (unless $P = NP$)

So, why study?

The reduction builds strange graphs!

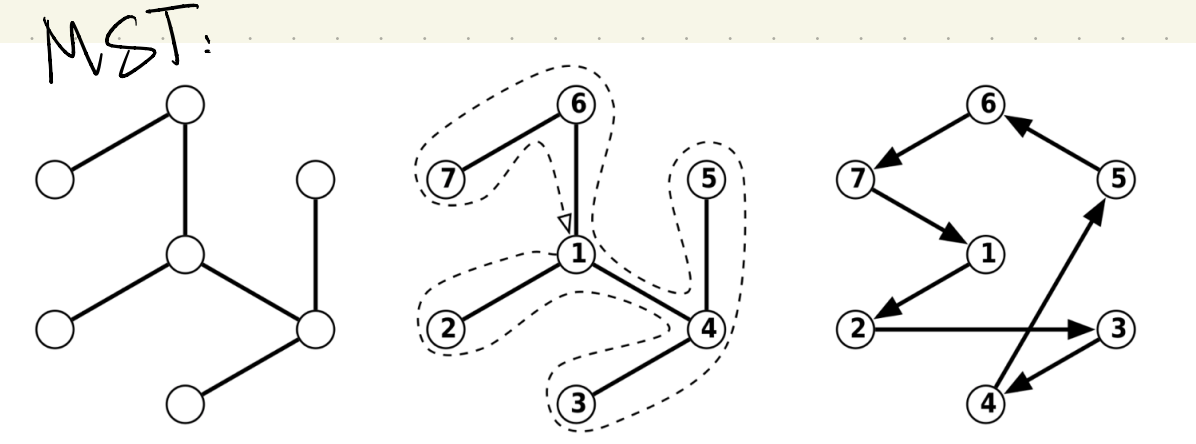


Why strange?

Triangle inequality:

Theorem: If G satisfies the triangle inequality, can compute a 2-approx for TSP.

Idea: Use Minimum Spanning Tree (MST):



Then:

Proof:

Let OPT be the cost of optimum solution to TSP.

Let MST be the weight of the Minimum spanning tree.

And let X be the weight of our constructed cycle.

Need to show:

$$X \leq 2 \cdot OPT$$

Step 1: Prove $X \leq 2$ MST.

Step 2: Prove $MST \leq OPT$:

Why? TSP solution is a cycle, so:

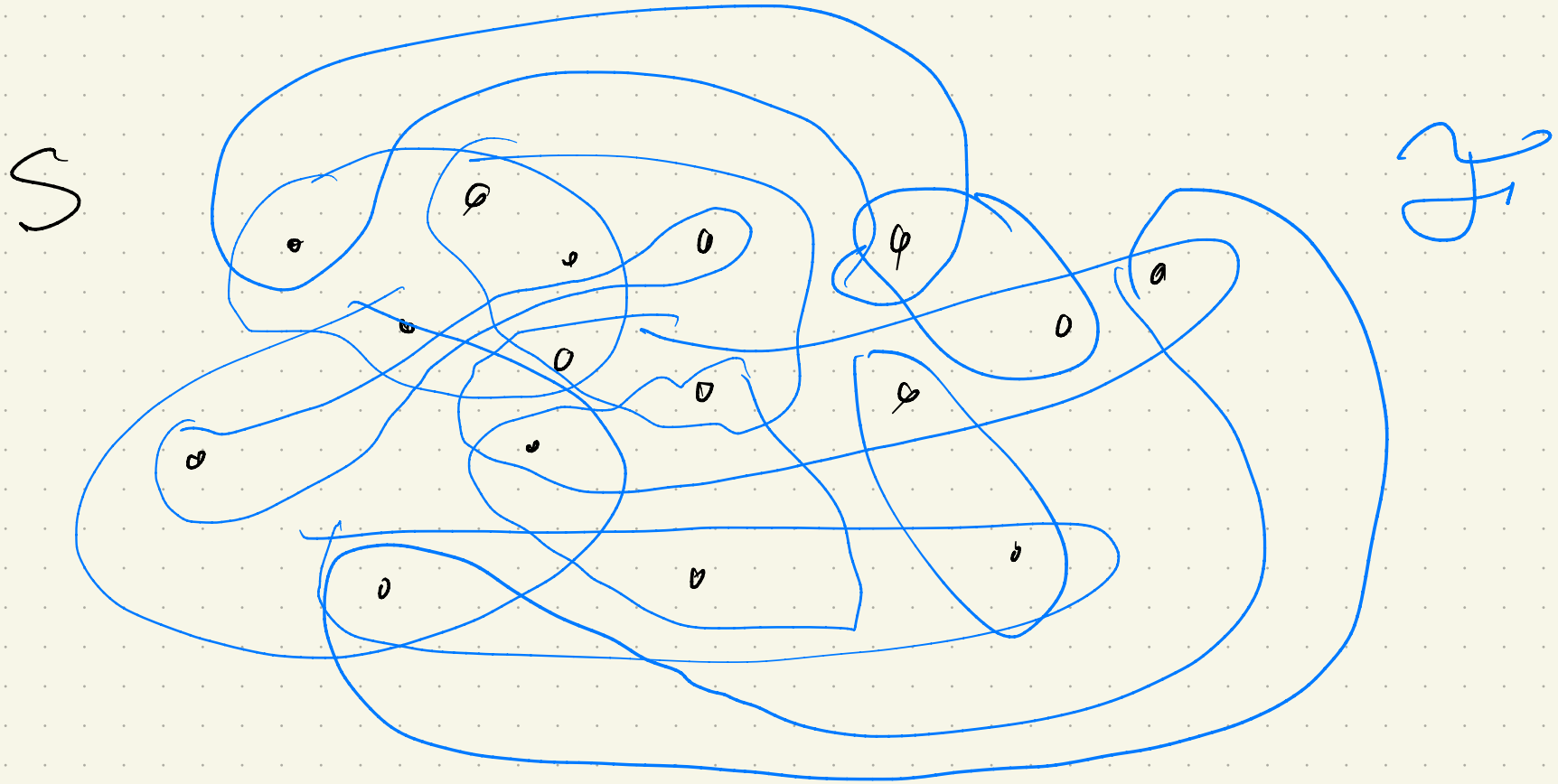
Set Cover

Another classic NP-Hard Problem.

Set Cover

Instance: $(S, \mathcal{F})$:
S - a set of n elements
$\mathcal{F}$ - a family of subsets of S, s.t. $\bigcup_{X \in \mathcal{F}} X = S$.

Question: The set $\mathcal{X} \subseteq \mathcal{F}$ such that $\mathcal{X}$ contains as few sets as possible, and $\mathcal{X}$ covers S. Formally, $\bigcup_{X \in \mathcal{X}} X = S$.



Greedy Set Cover

How should we be greedy?

Sanity Check: does it work?

```
GreedySetCover(S, F)
    X ← ∅;  T ← S
    while T is not empty do
        U ← set in F covering largest
            # of elements in T
        X ← X ∪ {U}
        T ← T \ U

    return X.
```

Observation: Let α_i be # of <u>new</u> elements covered in iteration i of loop.

Then, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$, where m is total # of iterations.

Why?

Notation:

Let $\{V_1, \ldots, V_k\}$ be OPT set cover.

Let T_i = uncovered elements at beginning of iteration i

and U_i = set chosen in i^{th} iteration

```
GreedySetCover(S, F)
    X ← ∅;  T ← S
    while T is not empty do
        U ← set in F covering largest
            # of elements in T
        X ← X ∪ {U}
        T ← T \ U

    return X.
```

Lemma: $\alpha_i \geq \frac{|T_i|}{k}$.

Proof: Consider OPT again → k sets, $\forall$ covers T_i.

Some set in OPT must have size $\frac{|T_i|}{k}$ → why?

Greedy picks biggest coverage, so

$|U_i| \geq$

Rewrite: if $\alpha_i \geq \frac{|T_i|}{k}$ and

$$|T_{i+1}| = |T_i| - \alpha_i$$

$$\Rightarrow |T_{i+1}| \leq$$

Thm: GreedySet cover is $O(\log n)$ approx.

Proof: Need to know how many times the loop runs (since adds 1 set per iteration)

GreedySetCover(S, $\mathcal{F}$)
    $\mathcal{X} \leftarrow \emptyset$; $T \leftarrow$ S
    while T is not empty **do**
        $U \leftarrow$ set in $\mathcal{F}$ covering largest # of elements in T
        $\mathcal{X} \leftarrow \mathcal{X} \cup \{U\}$
        $T \leftarrow T \setminus U$

    return $\mathcal{X}$.

Well,

$$|T_i| \leq \left(1 - \frac{1}{k}\right) |T_{i-1}| \leq$$

When do we reach an iteration M where this bound shows $|T_M| < 1$? (Loop would then stop).

Math tricks: $1 - x < e^{-x}$ for $x \geq 0$.

Let $M = \lceil 2k \ln n \rceil$:

$$|T_M| \leq$$

(Math, cont):

End recap: If $|OPT| = k$:

After $2k \ln n$ repetitions, T is empty.

```
GreedySetCover(S, F)
    X ← ∅;  T ← S
    while T is not empty do
        U ← set in F covering largest
            # of elements in T
        X ← X ∪ {U}
        T ← T \ U

    return X.
```

So: $|X| \leq \lceil 2k \ln n \rceil$

$$\Rightarrow \frac{|X|}{|OPT|} \leq O(\log n)$$

$\Rightarrow O(\log n)$ approximation.