

TDA - fall 2025

Discrete
Morse Theory



Recall: Morse Theory

First few weeks of class, we discussed (continuous) Morse theory.

Let's quickly recall...

[Begin old slides]

Smooth

Topological manifolds are spaces
But usually, consider an embedding
into Euclidean space \Rightarrow geometry.

Given a smooth function $f: \mathbb{R}^d \rightarrow \mathbb{R}$,
the gradient vector field $\nabla f: \mathbb{R}^d \rightarrow \mathbb{R}^d$
at a point x is:

$$\nabla f = \left[\underbrace{\frac{\partial f}{\partial x_1}(x)}, \dots, \underbrace{\frac{\partial f}{\partial x_d}(x)} \right]$$

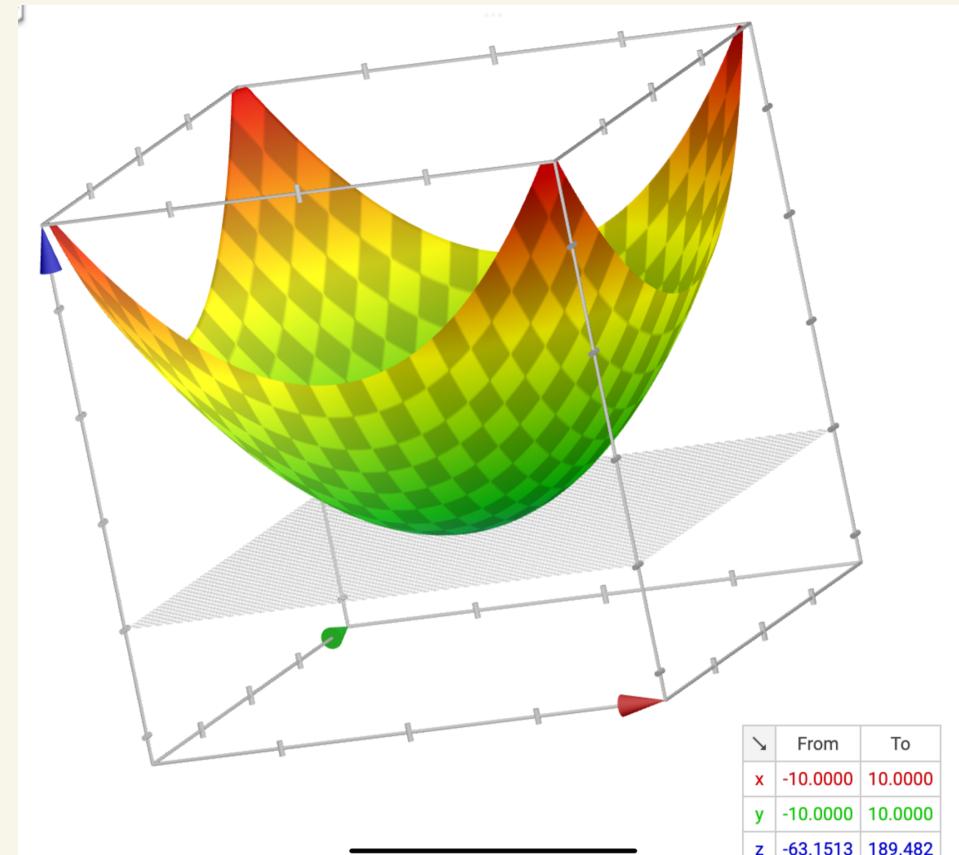
Ex: $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$\nabla f =$$

$$\left[\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f \right]$$

$$\Rightarrow [2x_1, 2x_2]$$



Then $\nabla f(0,0) = [0,0]$

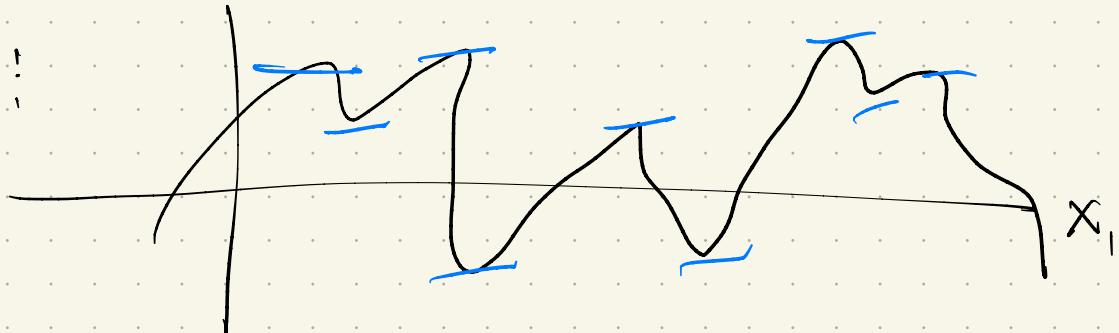
$$\nabla f(1,0) = [2,0]$$

Critical point

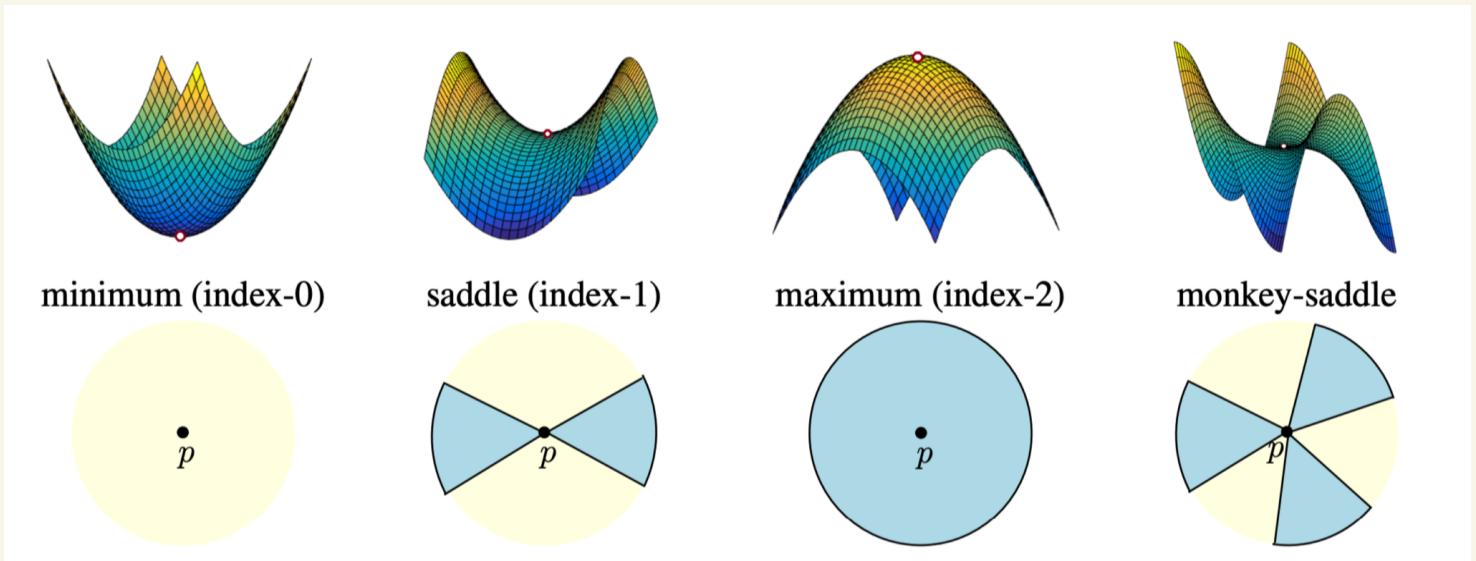
Any $p \in \mathbb{R}^d$ where $\nabla f(p) = \vec{0}$
(Otherwise we say p is regular)

On 1 manifolds:

$$\frac{\partial f}{\partial x} \cdot x = 0$$



On 2 manifolds:



Extending to manifolds:

Given $\phi: U \rightarrow W$, $U \subseteq \mathbb{R}^k$ & $W \subseteq \mathbb{R}^d$
open sets, where

$$\phi(x) = (\phi_1(x), \dots, \phi_d(x))$$

The Jacobian of ϕ is a $d \times k$
matrix of partial derivatives:

$$\begin{bmatrix} \frac{\partial \phi_1(x)}{\partial x_1} & \cdots & \frac{\partial \phi_1(x)}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_d(x)}{\partial x_1} & \cdots & \frac{\partial \phi_d(x)}{\partial x_k} \end{bmatrix}$$

Types of critical points

For a smooth m -manifold, the Hessian matrix of $f: M \rightarrow \mathbb{R}$ is the matrix of 2nd order partial derivatives:

$$\text{Hessian}(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix}$$

A critical point is non-degenerate if Hessian is nonsingular ($\det \neq 0$); otherwise degenerate.

An example: $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x_1, x_2) = \underline{x_1^3 - 3x_1x_2^2}$$

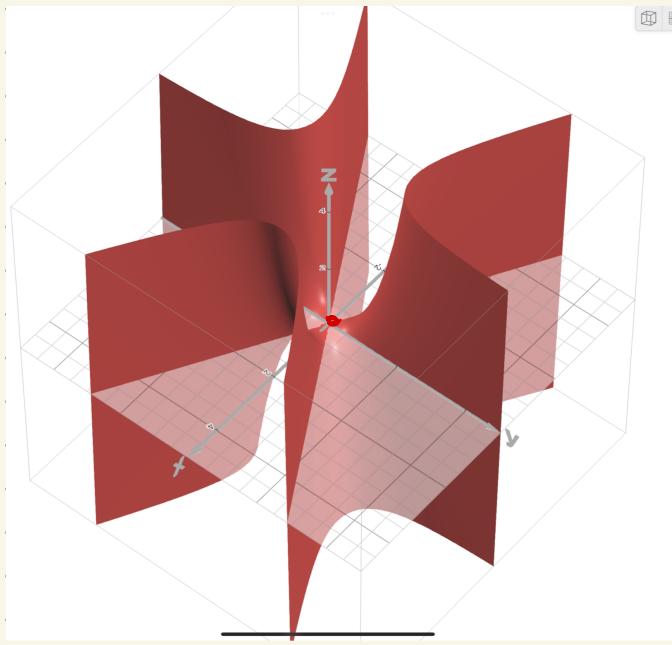
$$\nabla f = [3x_1^2 - 3x_2^2, -6x_1x_2]$$

$\Rightarrow (0,0) \quad \{0,0\}$

Is it degenerate?

Hessian:

$$\begin{pmatrix} \frac{\partial}{\partial x_1 \partial x_1} & \frac{\partial}{\partial x_1 \partial x_2} \\ \frac{\partial}{\partial x_2 \partial x_1} & \frac{\partial}{\partial x_2 \partial x_2} \end{pmatrix}$$



$$= \begin{bmatrix} 6x_1 & -6x_2 \\ -6x_2 & -6x_1 \end{bmatrix}$$

So at $(0,0)$, $\det = 0$

Degenerate & critical at $(0,0)$

Morse Lemma

Given a smooth function $f: M \rightarrow \mathbb{R}$ defined on a smooth manifold M , let p be a non-degenerate critical point of f . Then \exists a local coordinate system in a neighborhood $U(p)$ s.t.

- p 's coordinate is $\overset{\rightharpoonup}{0}$

- locally, any x is in the form

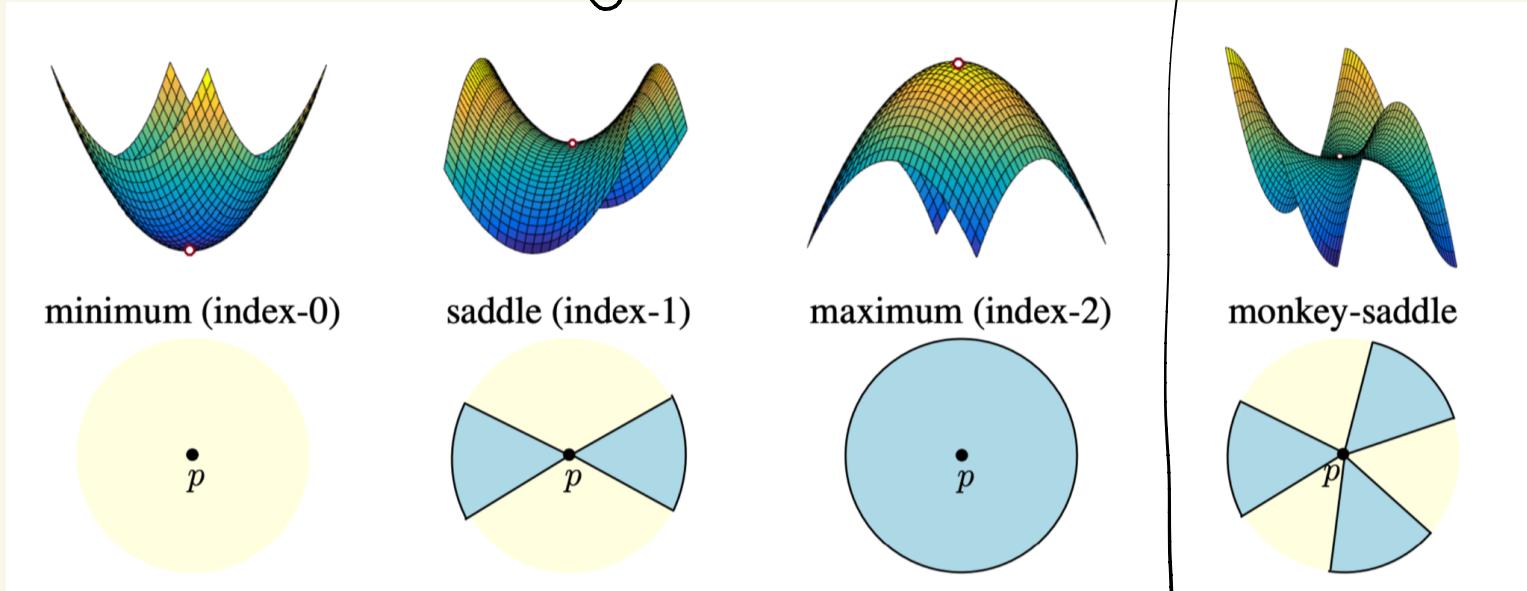
$$f(x) = f(p) - x_1^2 - \dots - x_s^2 + x_{s+1}^2 + \dots + x_m^2$$

for some $s \in [0, m]$

s is called the **index** of p .

Back to that picture...
non-degenerate

degenerate



↑
everything is
bigger around p

↑
everything is
smaller around p
One coordinate bigger,
one smaller

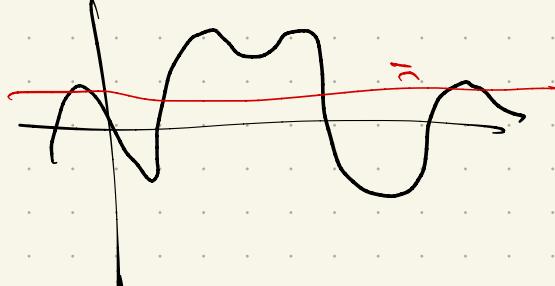
Morse functions

A smooth function $f: M \rightarrow \mathbb{R}$ (on a smooth manifold M) is a **Morse function** if

- none of f 's critical points are degenerate
- the critical points have distinct function values

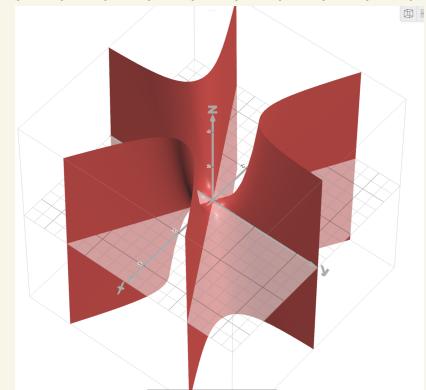
Some examples: Morse?

$f: \mathbb{R} \rightarrow \mathbb{R}$
No



$$\begin{cases} g: \mathbb{R}^2 \rightarrow \mathbb{R} \\ g((x_1, x_2)) = x_1^3 - 3x_1x_2^2 \end{cases}$$

NO



Begin new
slides.

Discrete Morse theory

A few motivations:

- Attempt to simplify representation
into a combinatorial format

↳ Why?

- Provide a tool to simplify simplicial
complexes

↳ Why?

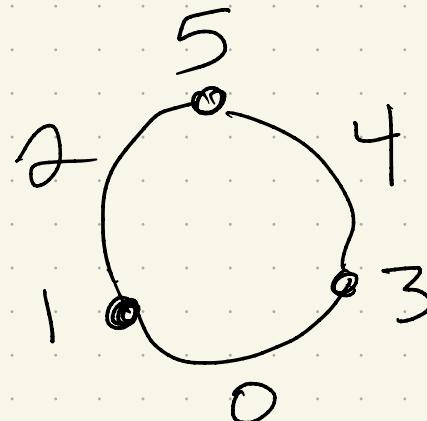
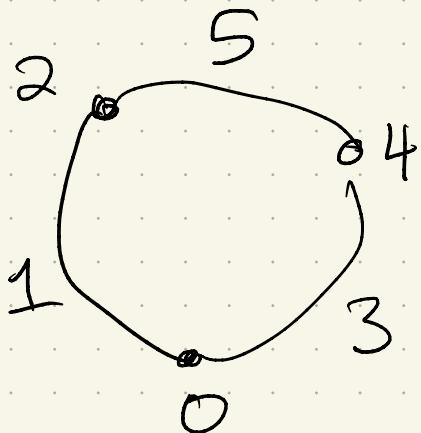
A discrete Morse function f on a complex K is a function $f: K \rightarrow \mathbb{R}$ st.
 for every p -simplex $\sigma \in K$

$$|\{\tau^{(p+1)} < \sigma : f(\tau) \geq f(\sigma)\}| \leq 1$$

and

$$|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| \leq 1$$

Examples: Yes/no?

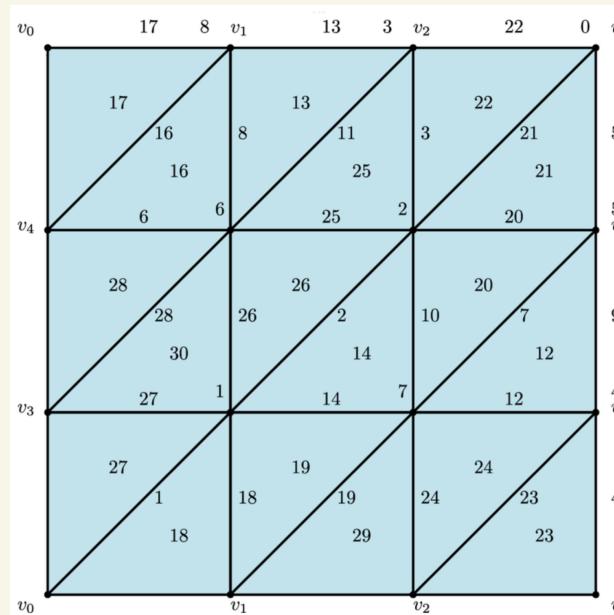


In other words,

- higher dimensional neighbors have higher values (with ≤ 1 exception)
- lower dimensional neighbors have lower values (with ≤ 1 exception)

Example:

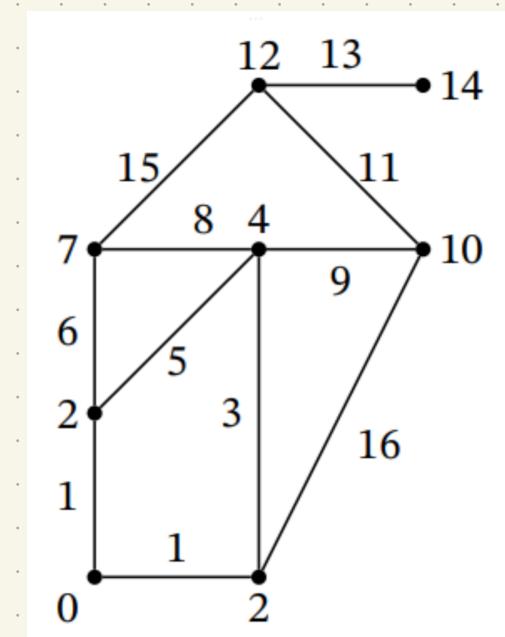
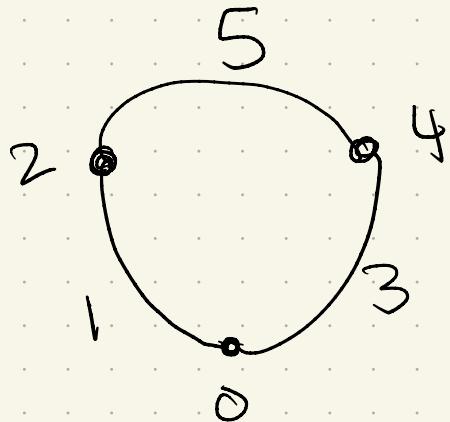
9 vertex triangulation of the torus



Critical Simplices

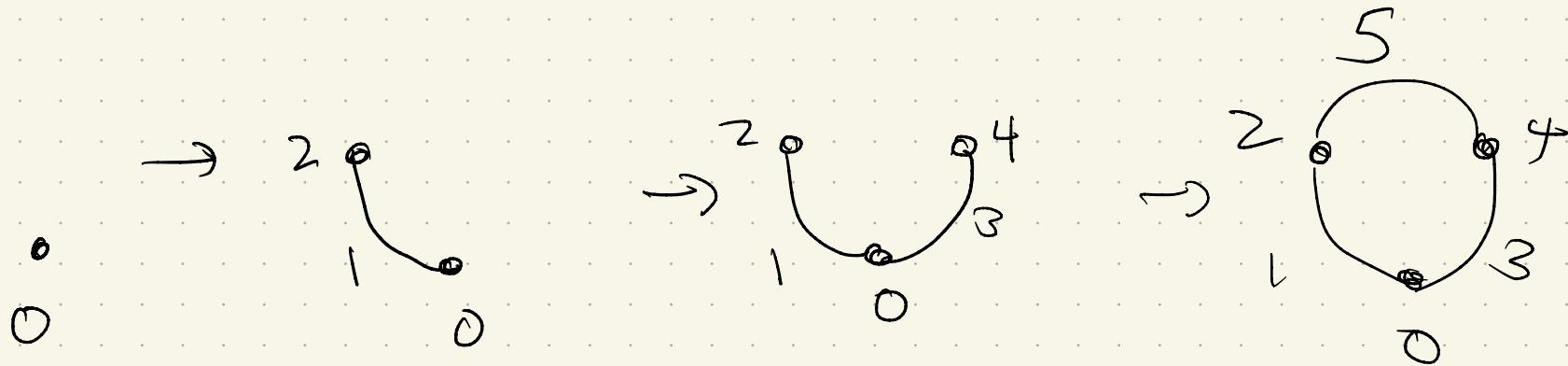
A p -simplex is critical with respect to f if $|\{x^{(p-1)} \leq g : f(x) \geq f(g)\}| = 0$
and $|\{x^{(p+1)} > g : f(x) \leq f(g)\}| = 0$

Example:



Why is this intuitive?

Think of levelsets & fibrations again:



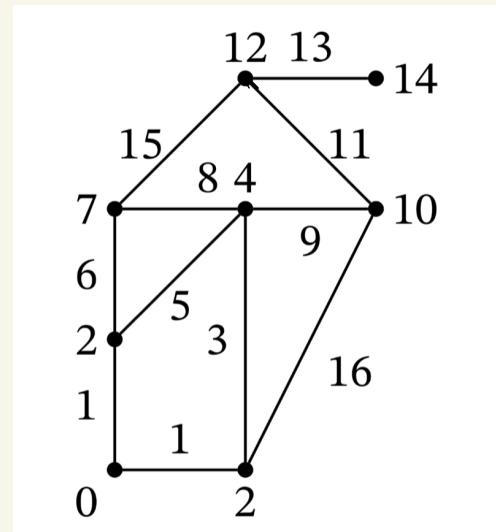
- If $f(e) > f(v), f(w)$, with $e = vw$ then
e connects two existing vertices
↳ changes homotopy type
- If $f(v) < f(e)$ for all incident edges e,
then v is a new vertex not on any
existing edge → changes homotopy type.

Regular points + discrete gradients

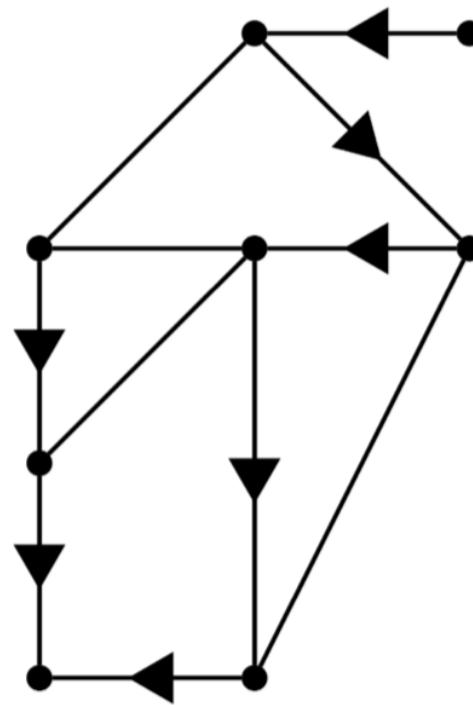
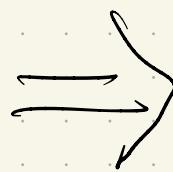
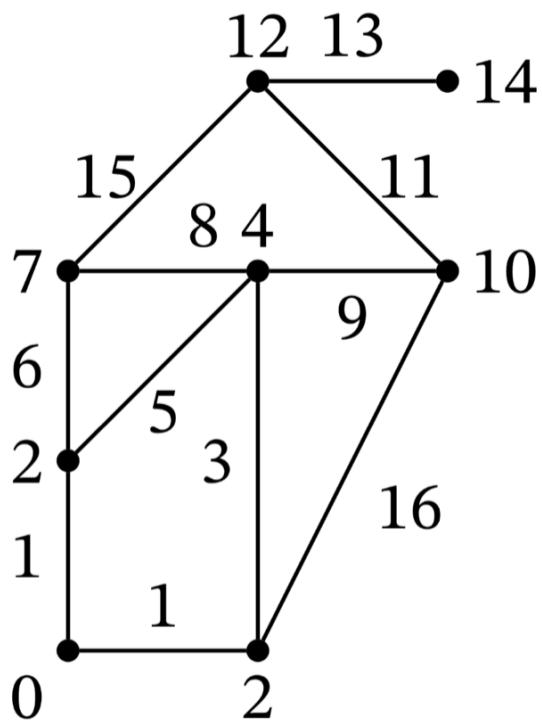
Any simplex that is not critical is regular, & will have one higher dim incident simplex with lower value or one lower dim simplex with higher

Discrete Gradient Vector field:

Draw arrow from 6 to higher dim nbr with lower value

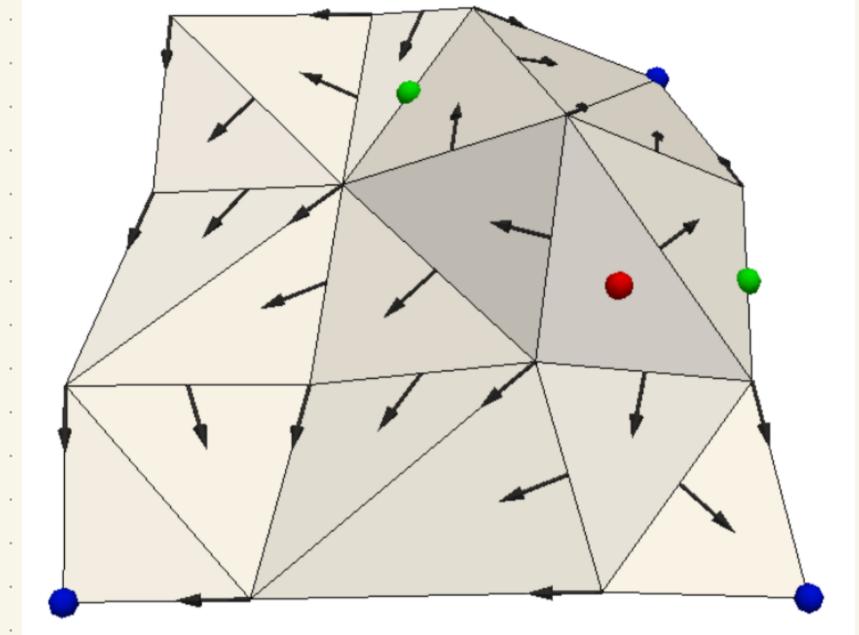


Result (check) :



Result:

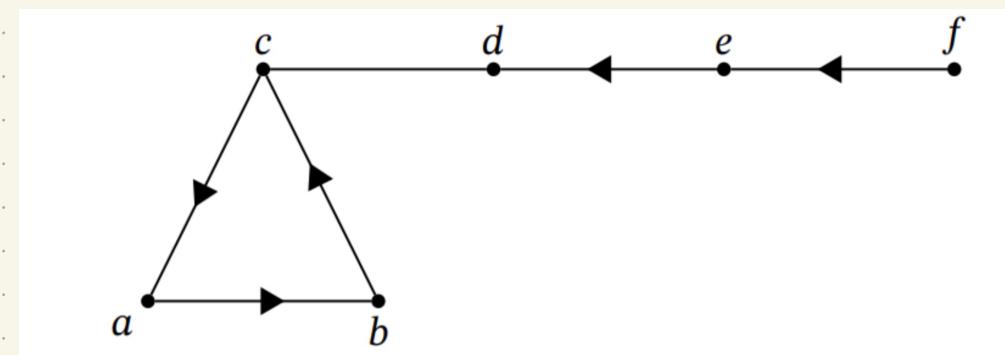
- Each simplex "flows" to at most one nbr
- Flow lines go down
- Flow vanishes at critical simplices
 - ↳ Why?



Definition (more general)

A discrete vector field on K is
a collection of pairs $\{\alpha^{(p)} \in \mathbb{R}^{(p+1)}\}$
such that each simplex is in at
most one pair.

Question: Are these the same as gradients?



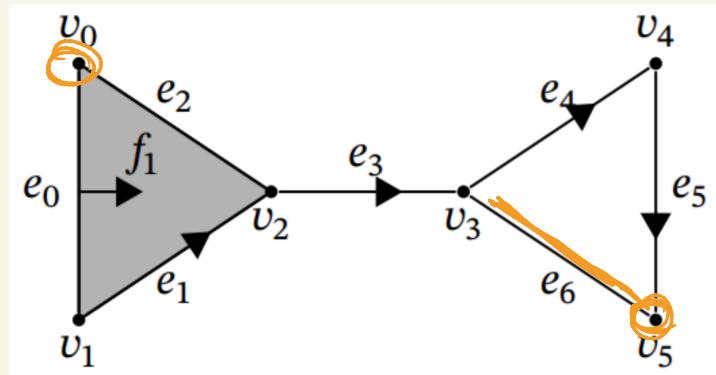
V-paths:

Sequence of simplices

$$[\tau_{-1}^{(P+)}] \alpha_0^{(P)}, \beta_0^{(P+1)}, \alpha_1^{(P)}, \dots, \beta_r^{(P+1)}, \alpha_{r+1}^{(P)}$$

such that $\alpha_i < \beta_i$ & $\beta_i > \alpha_{i+1} \neq \alpha_i$
or either τ_{-1} is critical or α_0 is regular

Ex:



maximal:

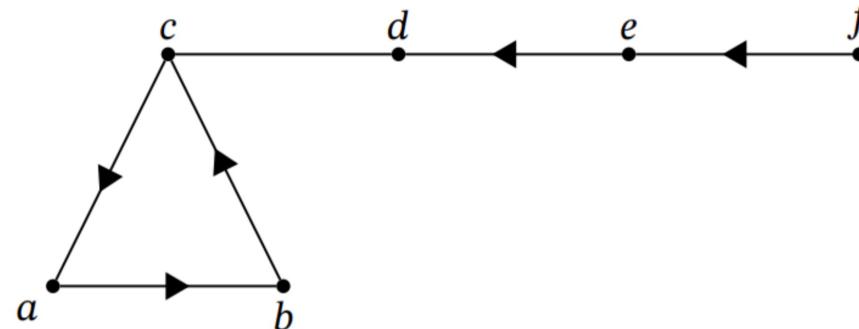
Theorem:

A discrete vector field V is the gradient of a discrete Morse function



no non-trivial closed V -paths.

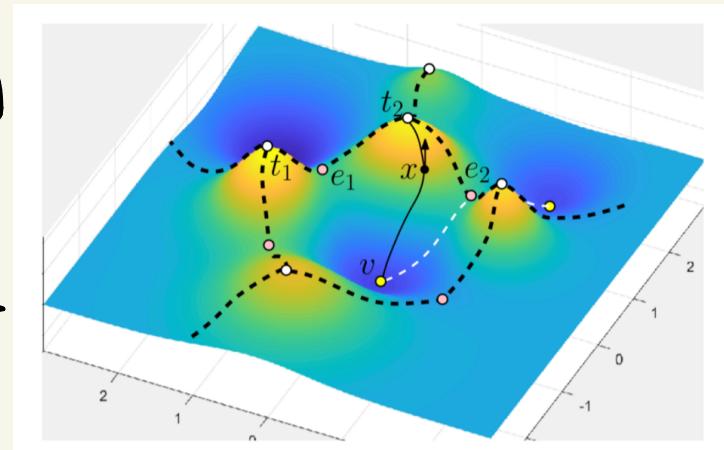
Proof
(by picture)



Continuous Morse functions

In continuous Morse theory,
integral flow lines start
→ end at critical points

Why?

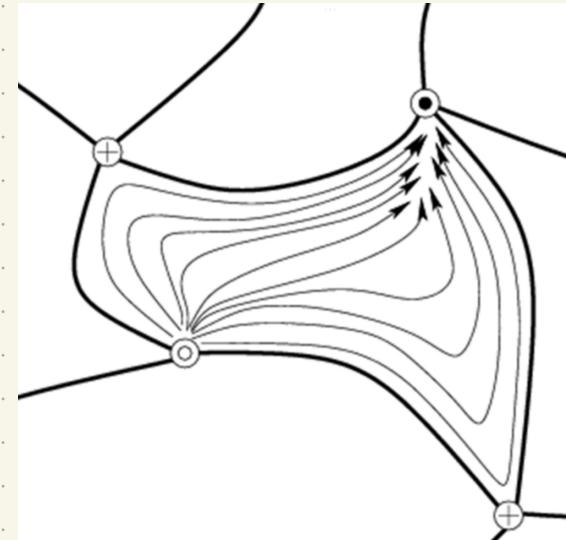


For a critical point p , $\mathcal{P} \cup$ all integral
lines into p = stable manifold
& all flow lines out of a critical
point q is unstable manifold

[also called ascending & descending manifolds]

Morse-Smale Complex

Partition manifold
into cells, where
each cell shares
same source & destination
points for integral lines

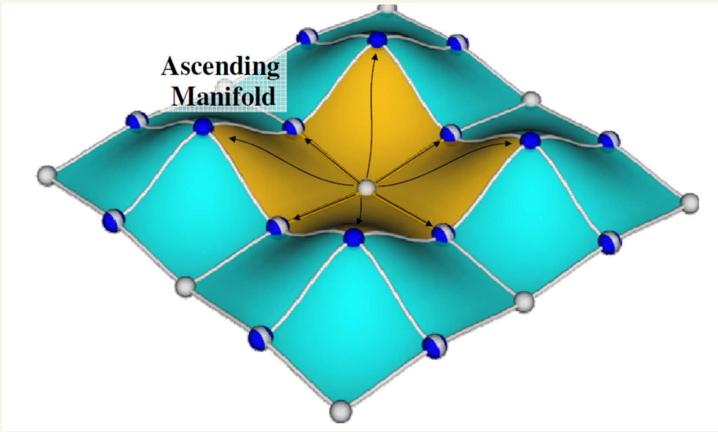


Vertices: critical points

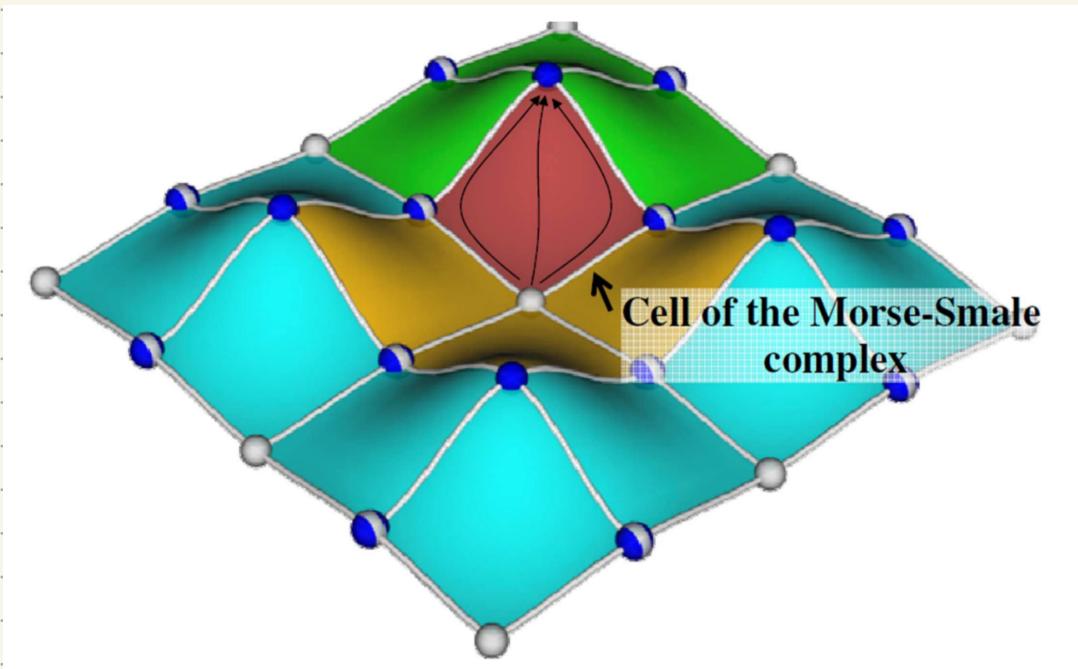
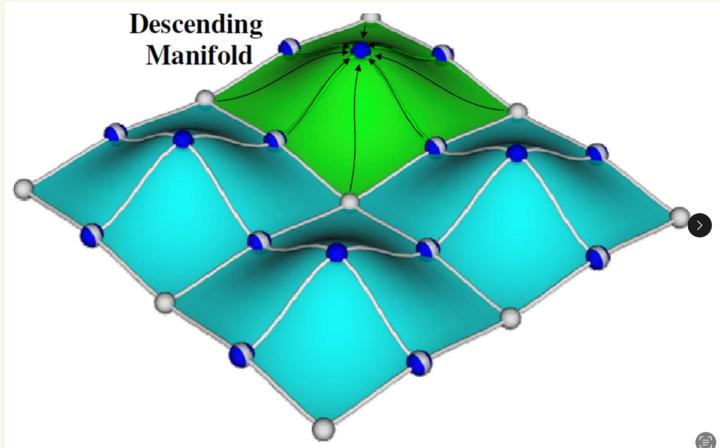
Arcs: integral lines connecting critical points

Faces: intersection of stable & unstable manifolds

For 2-manifolds: If generic, get:

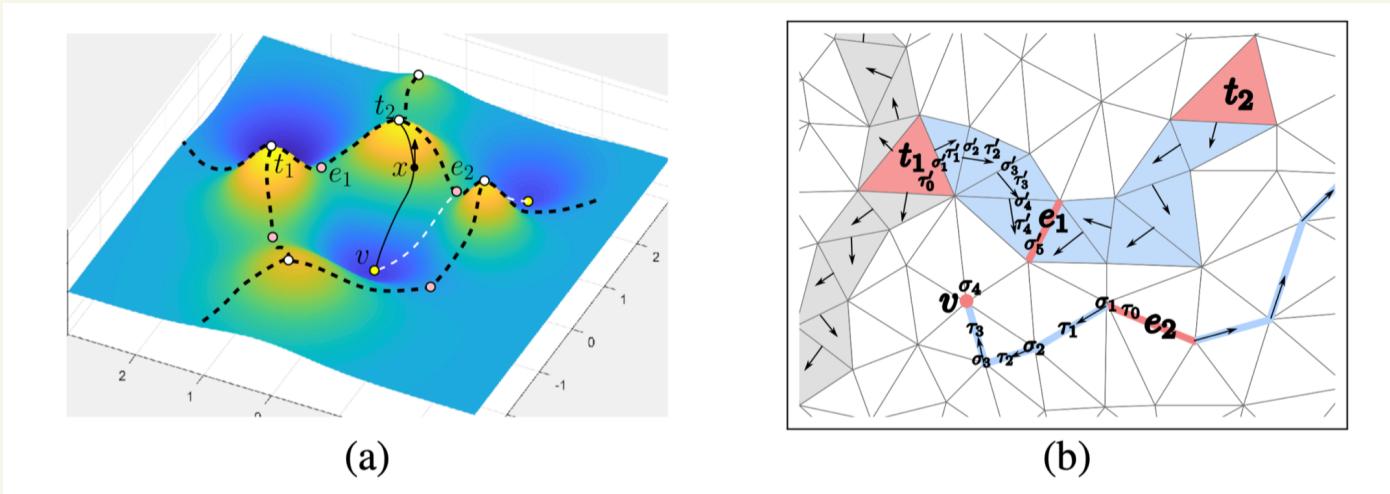


and



- Each saddle has 4 acs
- Each face :

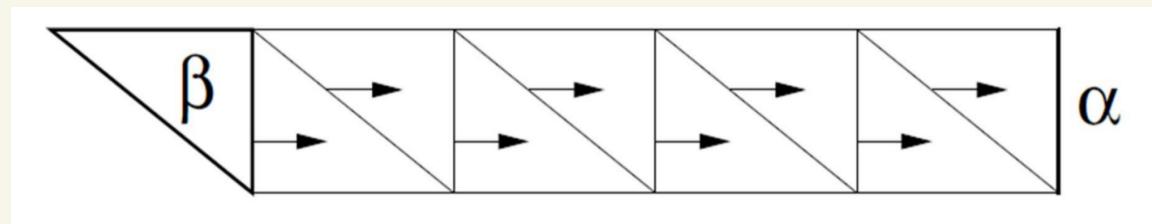
Back to V-paths: either face-edge
or edge-vertex



continuous
flow lines

discretized
V-paths

i.e.



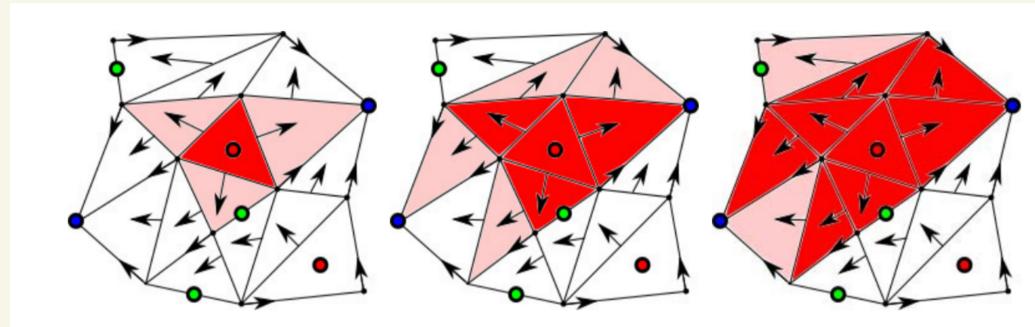
One difference: discrete flow goes down
(not up like continuous)

So, in discrete setting: For critical edge e :

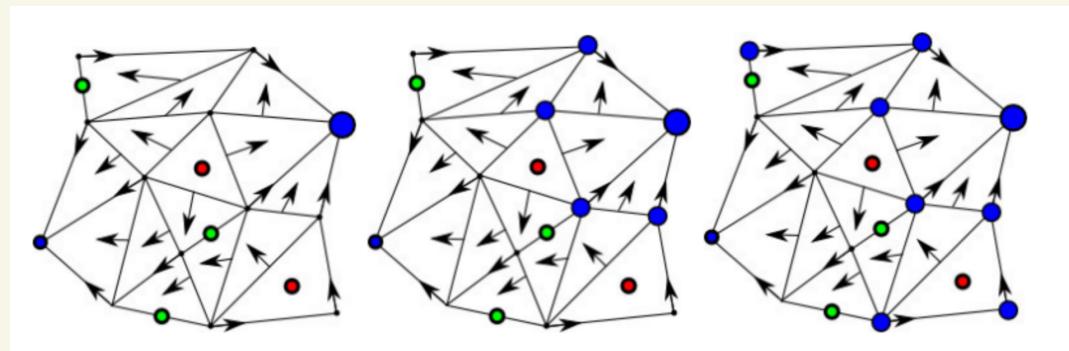
- Stable manifold is union of edge-triangle gradient paths
- Unstable manifold is union of vertex edge gradient paths

ie:

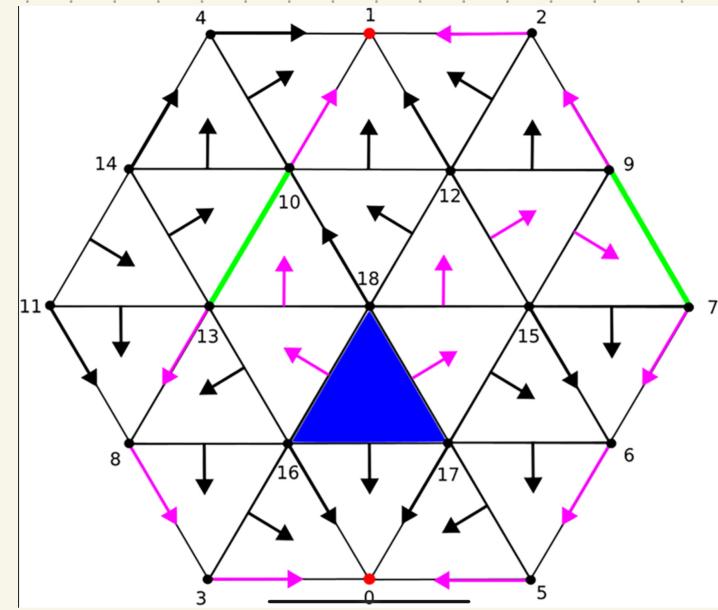
descending:



ascending:



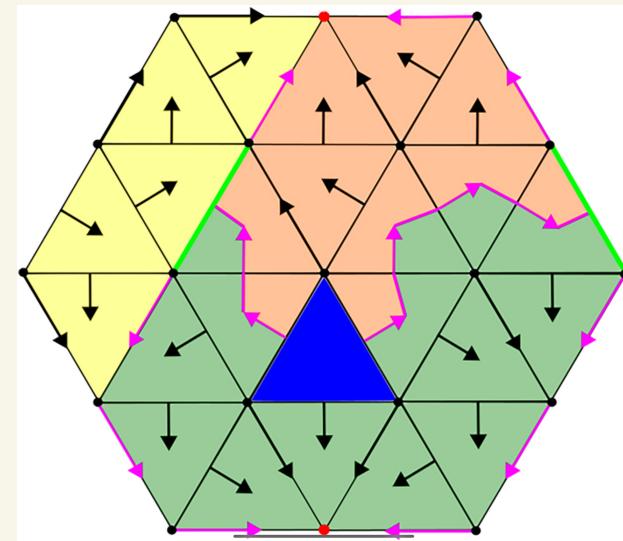
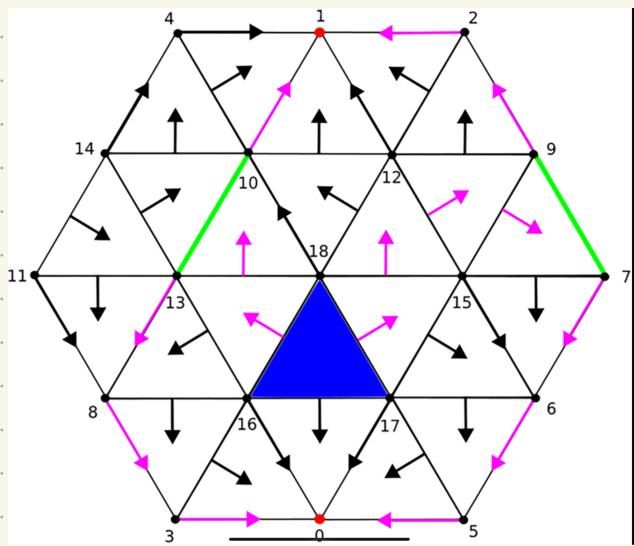
Separatrices:
V-paths between
critical simplices
(marked pink)



How to find?

- easy starting from critical edges
 - From critical faces: try all options

Discrete Morse-Smale Complex



Consider Chain complex:

$$C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K)$$

But: only critical simplices + separatrices
between them

Why?

Back to Motivation

In Foreman's original work, goal was
to identify a Smaller complex with
same homology