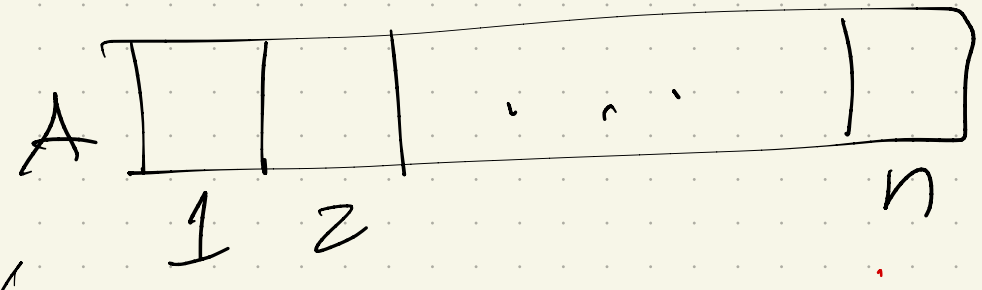
# Algorithms - Spring '25

## Dynamic Programming: LIS

# Recap: Longest Increasing Subsequence

Why "jump to the middle"?
Need a recursion!

First: how many subsequences?




↳ could use or skip each #,
⇒ $2^n$ worst case

Backtracking approach:

At index $i$:

# Result:

Given two indices $i$ and $j$, where $i < j$, find the longest increasing subsequence of $A[j..n]$ in which every element is larger than $A[i]$.

Store last "taken" index i.

Consider including A[j]:

- if $A[i] \geq A[j]$
  - ↳ must skip!
- if A[i] is less:
  - try both options

# Recursion:

$$LISbigger(i,j) = \begin{cases} 0 & \text{if } j > n \\ LISbigger(i, j+1) & \text{if } A[i] \geq A[j] \\ \max \begin{Bmatrix} LISbigger(i, j+1) \\ 1 + LISbigger(j, j+1) \end{Bmatrix} & \text{otherwise} \end{cases}$$

Code version: (helper function)

```
LISBIGGER(i, j):
  if j > n
      return 0
  else if A[i] ≥ A[j]
      return LISBIGGER(i, j + 1)
  else
      skip ← LISBIGGER(i, j + 1)
      take ← LISBIGGER(j, j + 1) + 1
      return max{skip, take}
```

Problem – what did we want??

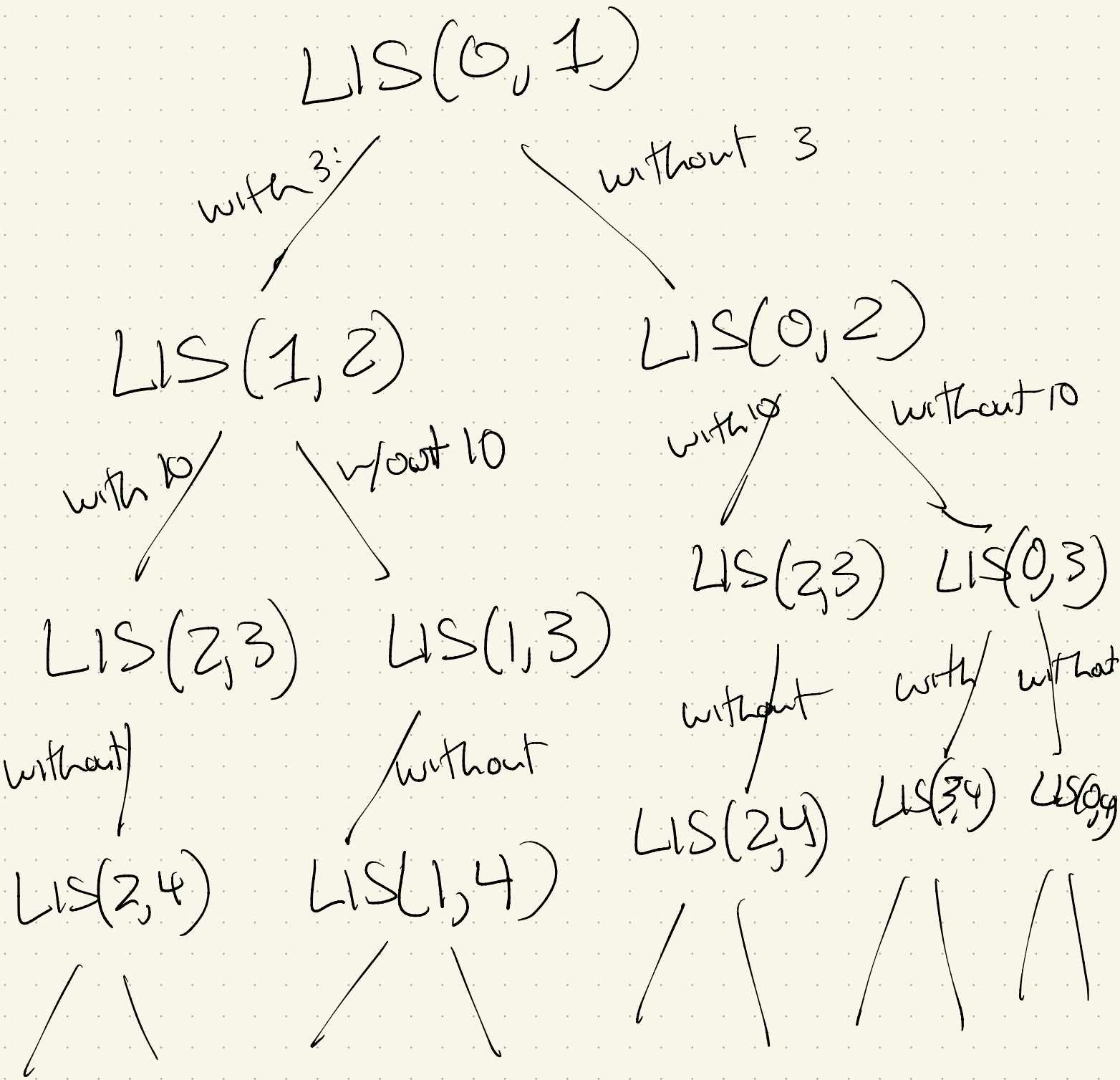
LIS(A[1..n]).

So: don't forget our "main":

```
LIS(A[1..n]):
  A[0] ← −∞
  return LISBIGGER(0, 1)
```

Example: A: [3, 10, 2, 11, 5, 16]

↳ [-∞, 3, 10, 2, 11, 5, 16]



# Next: memoize?

What sort of calls are we making often? Can we save them, & avoid recomputing over and over?

$$LISbigger(i,j) = \begin{cases} 0 & \text{if } j > n \\ LISbigger(i, j+1) & \text{if } A[i] \geq A[j] \\ \max \begin{Bmatrix} LISbigger(i, j+1) \\ 1 + LISbigger(j, j+1) \end{Bmatrix} & \text{otherwise} \end{cases}$$

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  else
      skip ← LISBIGGER(i, j + 1)
      take ← LISBIGGER(j, j + 1) + 1
      return max{skip, take}
```

Here:

$$LISbigger(i,j) = \begin{cases} 0 & \text{if } j > n \\ LISbigger(i, j+1) & \text{if } A[i] \geq A[j] \\ \max \begin{Bmatrix} LISbigger(i, j+1) \\ 1 + LISbigger(j, j+1) \end{Bmatrix} & \text{otherwise} \end{cases}$$

This is a recursion, but think for a moment of it as a function.

After computing, store values!

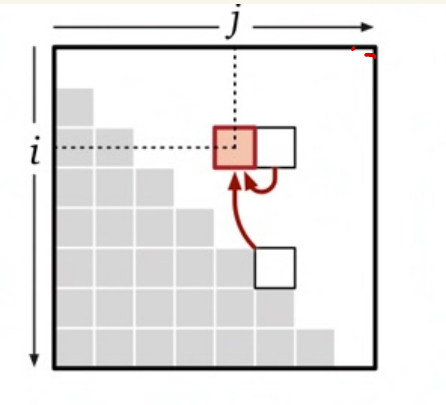
How many values to store?

How long to compute each?

Now, can we do the same trick as Fibonacci memoization, & convert to something loop-based?

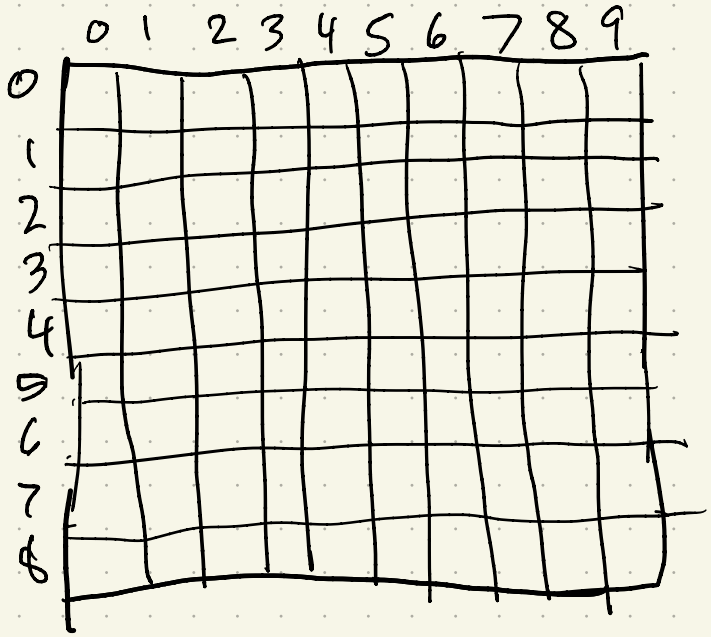
Rethink:

To fill in $L[i][j]$, what do I need?




So, go in that order!

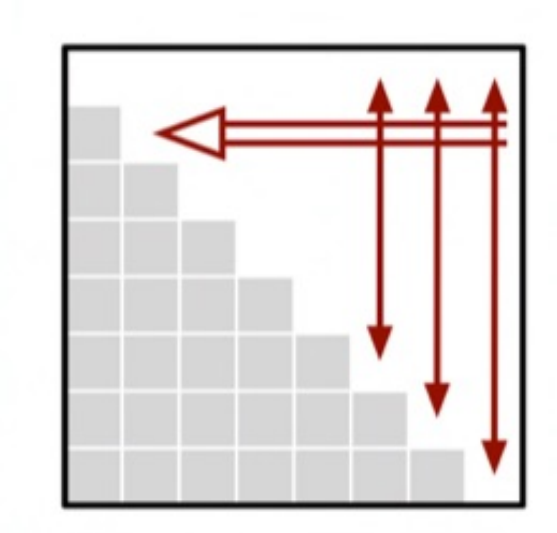
Ex: A = [10 2 4 1 6 11 7 9]



Result:

```
FastLIS(A[1..n]):
  A[0] ← −∞                                  ⟨⟨Add a sentinel⟩⟩
  for i ← 0 to n                             ⟨⟨Base cases⟩⟩
      LISbigger[i, n+1] ← 0
  for j ← n down to 1
      for i ← 0 to j − 1                     ⟨⟨...or whatever⟩⟩
          keep ← 1 + LISbigger[j, j+1]
          skip ← LISbigger[i, j+1]
          if A[i] ≥ A[j]
              LISbigger[i, j] ← skip
          else
              LISbigger[i, j] ← max{keep, skip}
  return LISbigger[0, 1]
```

Picture:



# Edit distance:

HUGE in bioinformatics!

One of the basic tools in sequence alignment.

(I have a book with an entire chapter on how to optimize.)

Also: spell checkers, word prediction, etc.

How to begin? (Recursively!)

ALGORITM

ALTRUISTIC

Start at end, + ask "obvious" question:

Let's try: A: ALGORITM
B: ALTRUISTIC

Start at end:

Align:

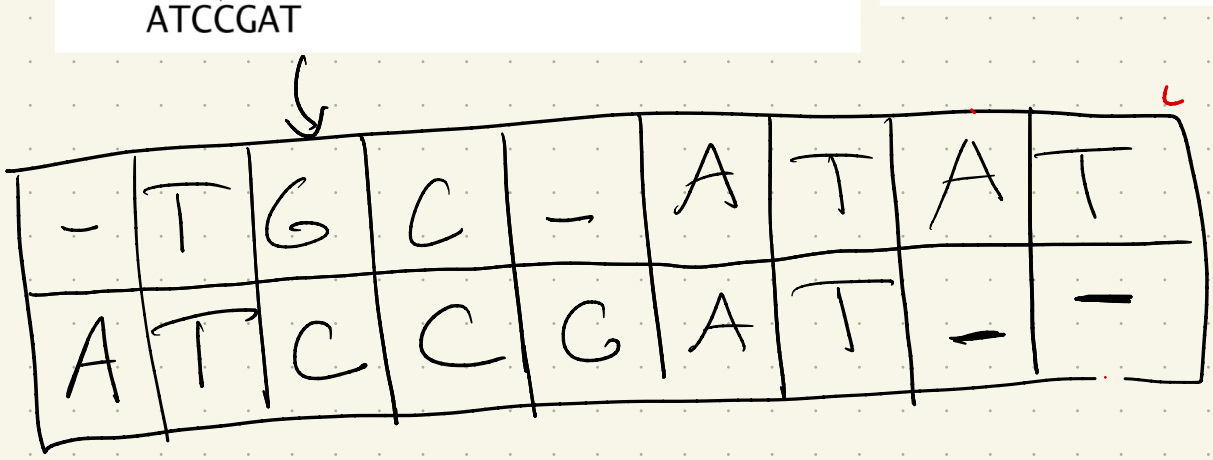
Insert

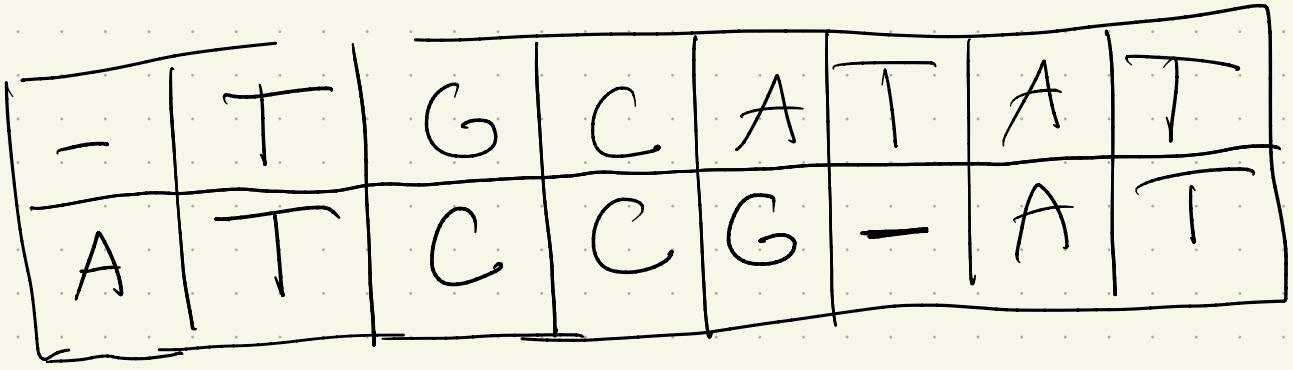
Delete:

Example: TGCATAT to ATCCGAT

TGCATAT
↓ delete last T
TGCATA
↓ delete last A
TGCAT
↓ insert A at the front
ATGCAT
↓ substitute C for G in the third position
ATCCAT
↓ insert a G before the last A
ATCCGAT

TGCATAT
↓ insert A at the front
ATGCATAT
↓ delete T in the sixth position
ATGCAAT
↓ substitute G for A in the fifth position
ATGCGAT
↓ substitute C for G in the third position
ATCCGAT



Input: $A[1..m]$
$B[1..n]$

$Edit(\ ,\ )$

$= \min$ {

+ Base cases:

Hs way:

$$Edit(i,j) = \begin{cases} i & \text{if } j = 0 \\ j & \text{if } i = 0 \\ \min \begin{cases} Edit(i, j-1) + 1 \\ Edit(i-1, j) + 1 \\ Edit(i-1, j-1) + [A[i] \neq B[j]] \end{cases} & \text{otherwise} \end{cases}$$

So: what's our "memory" data structure?

Then, our algorithm:

- start w/ base case (row & column)
- Fill in:

$$Edit(i,j) = \begin{cases} i & \text{if } j = 0 \\ j & \text{if } i = 0 \\ \min \begin{cases} Edit(i, j-1) + 1 \\ Edit(i-1, j) + 1 \\ Edit(i-1, j-1) + [A[i] \neq B[j]] \end{cases} & \text{otherwise} \end{cases}$$

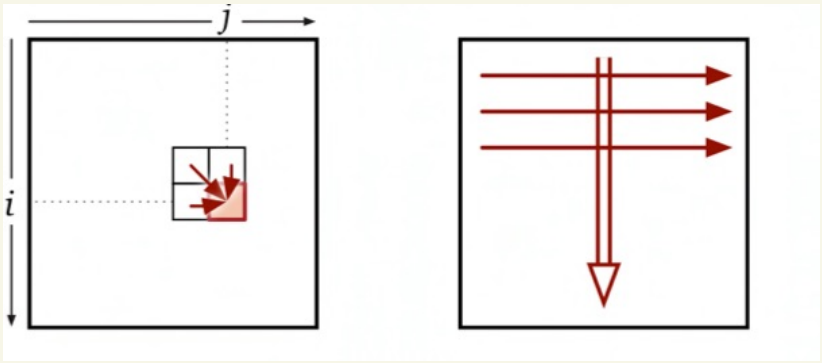
Result:

```
EDITDISTANCE(A[1..m], B[1..n]):
    for j ← 0 to n
        Edit[0, j] ← j

    for i ← 1 to m
        Edit[i, 0] ← i
        for j ← 1 to n
            ins ← Edit[i, j − 1] + 1
            del ← Edit[i − 1, j] + 1
            if A[i] = B[j]
                rep ← Edit[i − 1, j − 1]
            else
                rep ← Edit[i − 1, j − 1] + 1
            Edit[i, j] ← min {ins, del, rep}

    return Edit[m, n]
```

Picture:



Question:

Can we do better?

A really good question!

Lots of attention in bioinformatics.

Clever divide an conquer can reduce space.

↳ but will give #, not sequence, w/out some nice tricks

# Subset Sum (revisited)

Key takeaway (I think):
Sometimes, our backtracking recurrences can be memoized
(Note: sometimes, they can't! Think n queens.)

## Recall:

Given a set $X[1..n]$ of numbers & a target $T$, find a subset of $X$ whose sum is $= T$.

# Ch 2 solution

```
⟨⟨Does any subset of X sum to T?⟩⟩
SUBSETSUM(X, T):
  if T = 0
      return TRUE
  else if T < 0 or X = ∅
      return FALSE
  else
      x ← any element of X
      with ← SUBSETSUM(X \ {x}, T − x)     ⟨⟨Recurse!⟩⟩
      wout ← SUBSETSUM(X \ {x}, T)         ⟨⟨Recurse!⟩⟩
      return (with ∨ wout)
```

```
⟨⟨Does any subset of X[1..i] sum to T?⟩⟩
SUBSETSUM(X, i, T):
  if T = 0
      return TRUE
  else if T < 0 or i = 0
      return FALSE
  else
      with ← SUBSETSUM(X, i − 1, T − X[i])   ⟨⟨Recurse!⟩⟩
      wout ← SUBSETSUM(X, i − 1, T)          ⟨⟨Recurse!⟩⟩
      return (with ∨ wout)
```

The recursion:

(Note: same thing as code!!)

$$SS(i,t) = \begin{cases} \text{TRUE} & \text{if } t = 0 \\ \text{FALSE} & \text{if } t < 0 \text{ or } i > n \\ SS(i+1,t) \vee SS(i+1,t-X[i]) & \text{otherwise} \end{cases}$$

$SS(i,t)$ = T or F

$0 \le i \le n$

$0 \le t \le T$

So: another 2-d table!

To decide:

$$SS(i,t) = \begin{cases} \text{TRUE} & \text{if } t = 0 \\ \text{FALSE} & \text{if } t < 0 \text{ or } i > n \\ SS(i+1,t) \vee SS(i+1,t-X[i]) & \text{otherwise} \end{cases}$$

look at these 2 cells.

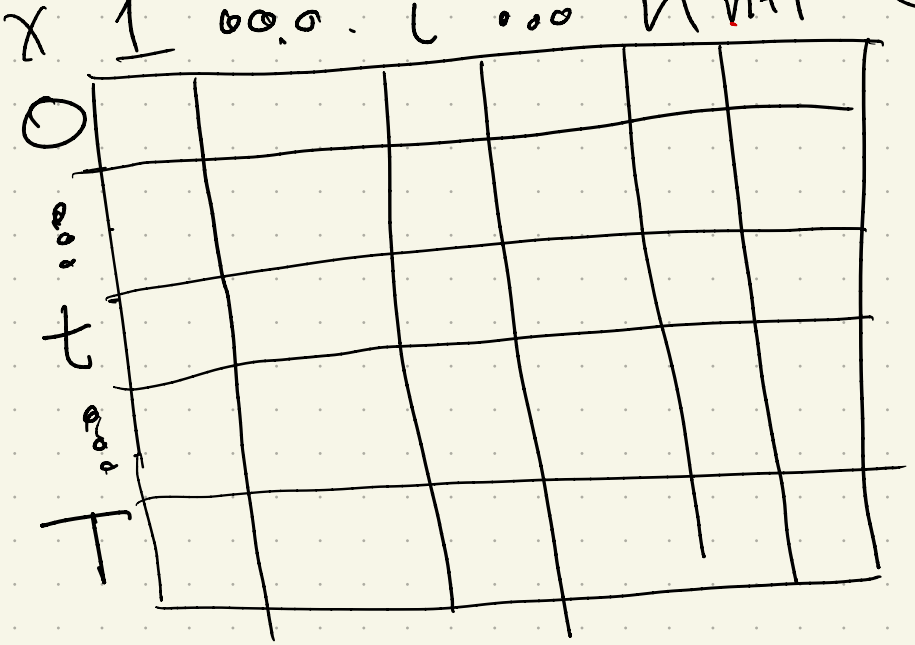
one note: if $t - X[i] < 0$, wasting time! Equivalent to:

$$SS(i,t) = \begin{cases} \text{TRUE} & \text{if } t = 0 \\ \text{FALSE} & \text{if } i > n \\ SS(i+1,t) & \text{if } t < X[i] \\ SS(i+1,t) \vee SS(i+1,t-X[i]) & \text{otherwise} \end{cases}$$

Now – need to code this:

$$SS(i,t) = \begin{cases} \text{TRUE} & \text{if } t = 0 \\ \text{FALSE} & \text{if } i > n \\ SS(i+1,t) & \text{if } t < X[i] \\ SS(i+1,t) \vee SS(i+1,t-X[i]) & \text{otherwise} \end{cases}$$

How should our loops go?




Fill:

Its code:

```
FastSubsetSum(X[1..n], T):
  S[n+1, 0] ← True
  for t ← 1 to T
      S[n+1, t] ← False

  for i ← n downto 1
      S[i, 0] = True
      for t ← 1 to X[i] − 1
          S[i, t] ← S[i+1, t]          ⟨⟨Avoid the case t < 0⟩⟩
      for t ← X[i] to T
          S[i, t] ← S[i+1, t] ∨ S[i+1, t − X[i]]

  return S[1, T]
```

Correctness:

Time/Space Analysis:

Note:

How big is this, & is it even a good idea??

input: number T and array X[1..n]

table has a column for every number 1..T.

How bad?

Well, X could be a list of 5000 #s, but T could be in the millions!

(lots of empty columns, many of which are impossible to hit!)