TDA - fall 2025

Induced & relative homology

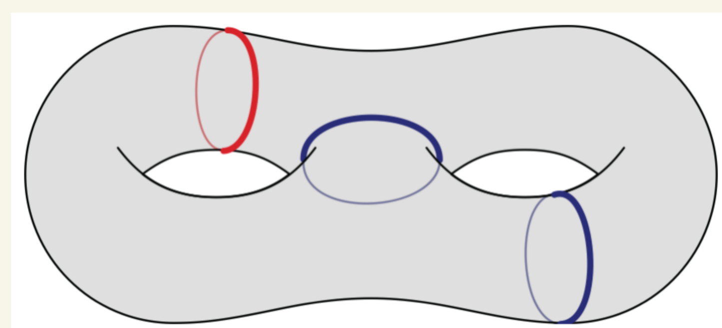
Last time: Homology!

$$\cdots \to C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \to \cdots$$

$$H_p(K) = Z_p / B_p =$$

Why?

- Computable! & homologous cycles are somehow "the same"
- But → <u>not</u> homotopy or isotopy.



Computing homology groups

To compute Betti number:

$$\beta_p = \dim(H_p(K))$$

Well, for any linear transformation $f: U \to V$,

$$\dim(U) = \underbrace{\dim(\ker f)}_{\ker \partial_K} + \underbrace{\dim(\operatorname{im} f)}_{\operatorname{im} \partial_K}$$

set $f = \partial_p$:

$$\dim(C_p) = \dim(Z_p) + \dim(B_{p-1})$$

Also, for a quotient space V/W,

$$\dim(V/W) = \dim(V) - \dim(W)$$

$$\Longrightarrow \beta_p = \dim(Z_p) - \dim(B_p)$$

So: computing!

Back to boundary matrices:

$$\partial_p \circ \underbrace{\alpha}_{p} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n_p} \\ b_{2,1} & & & b_{2,n_p} \\ \vdots & & & \\ b_{n_{(p-1)},1} & \cdots & & b_{n_{p-1},n_p} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{n_p} \end{bmatrix}$$

where $\alpha =$

$= p-1$ chain

Rows are a basis for C_{p-1}

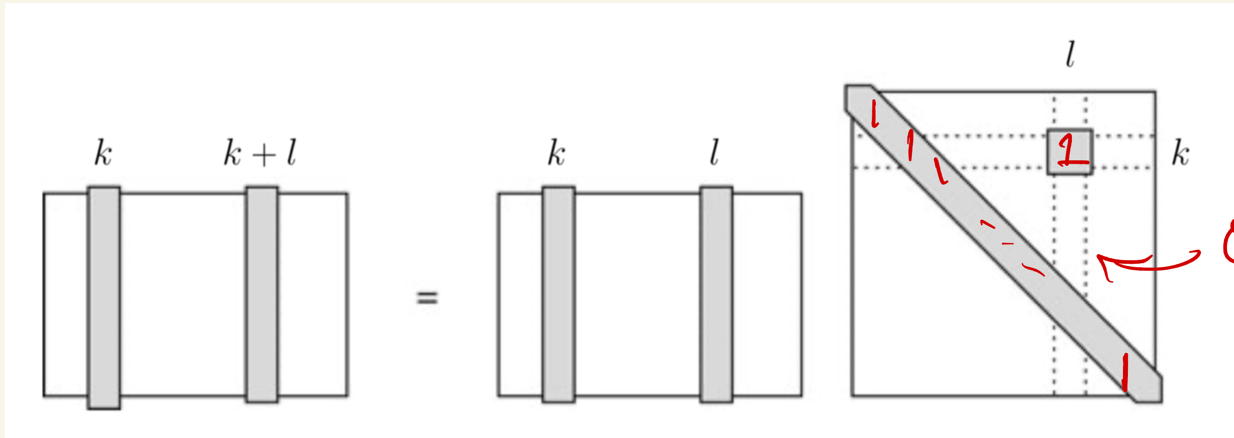
Columns are a basis for C_p

How to find rank?

Operations on matrices

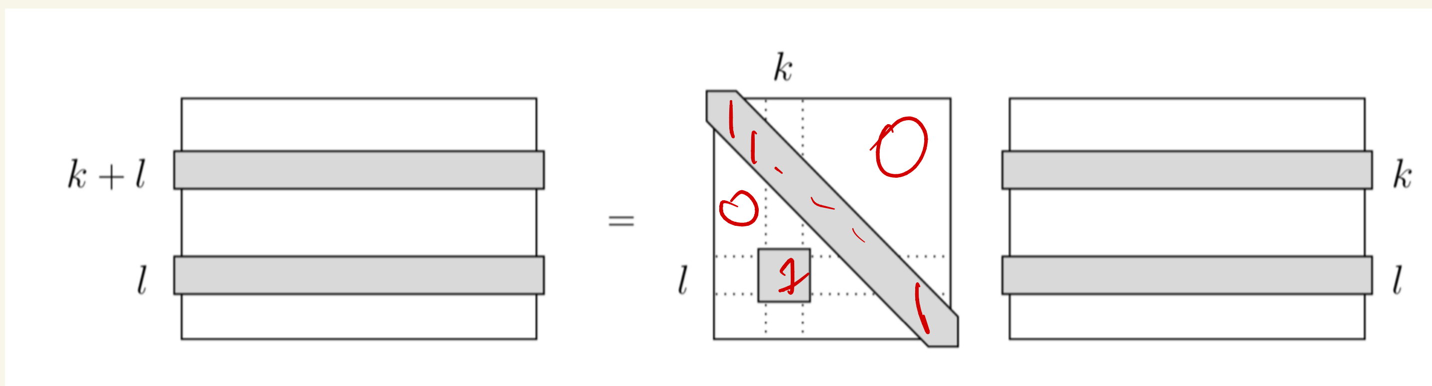
Simplify to Smith-Normal form. How?

Add columns



0 all other spots

Add rows

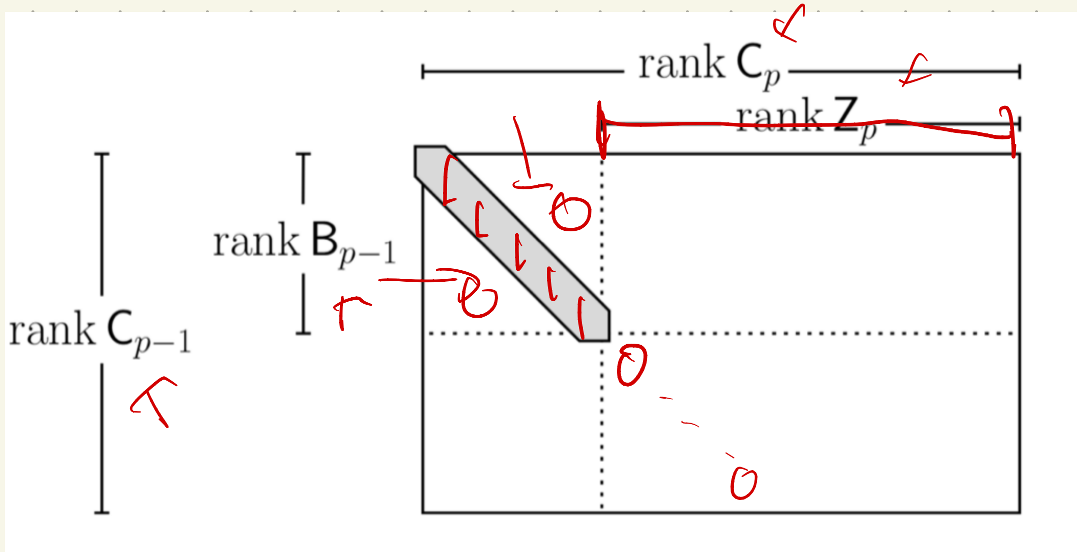


(& exchange rows/columns with 0 on diagonal)

Goal: Move 1's to diagonal

Smith-Normal form: $N_p = U_{p-1} \partial_p V_p$

$N_p =$

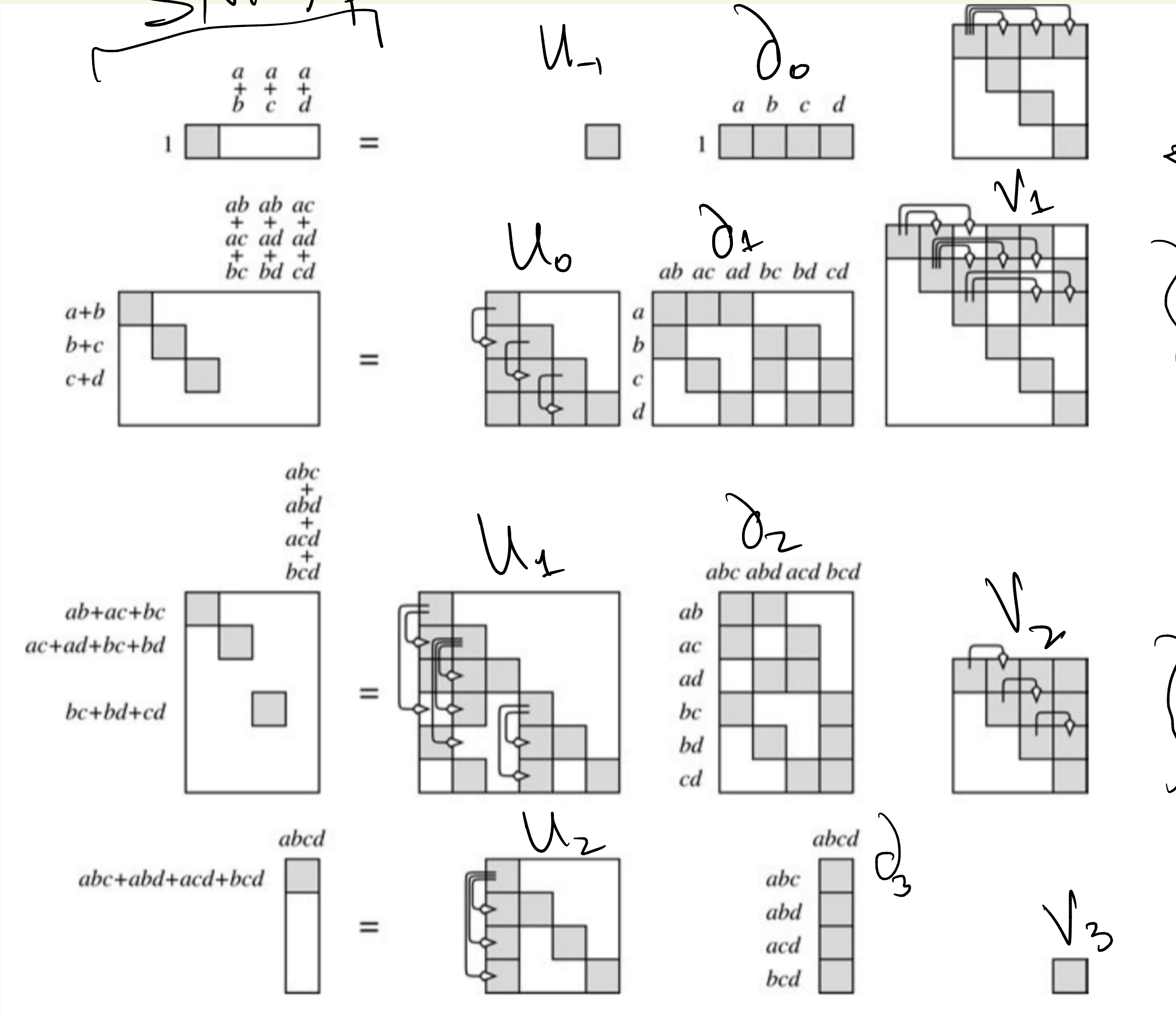


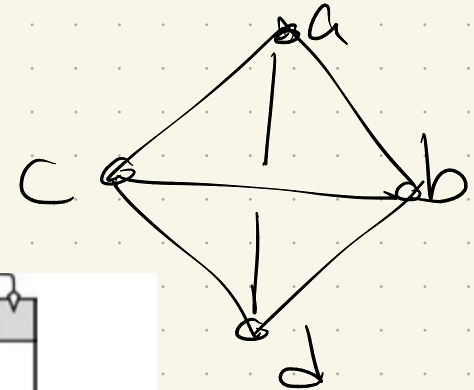
then $\beta_p = \text{rank}(Z_p) - \underbrace{\text{rank}(B_p)}_{?}$

$N_{p+1} =$ I ← rank B_p

An example: solid tetrahedron

SNF N_p





rank Z_0 =

rank B_0 =

rank Z_1 =

rank B_1 =

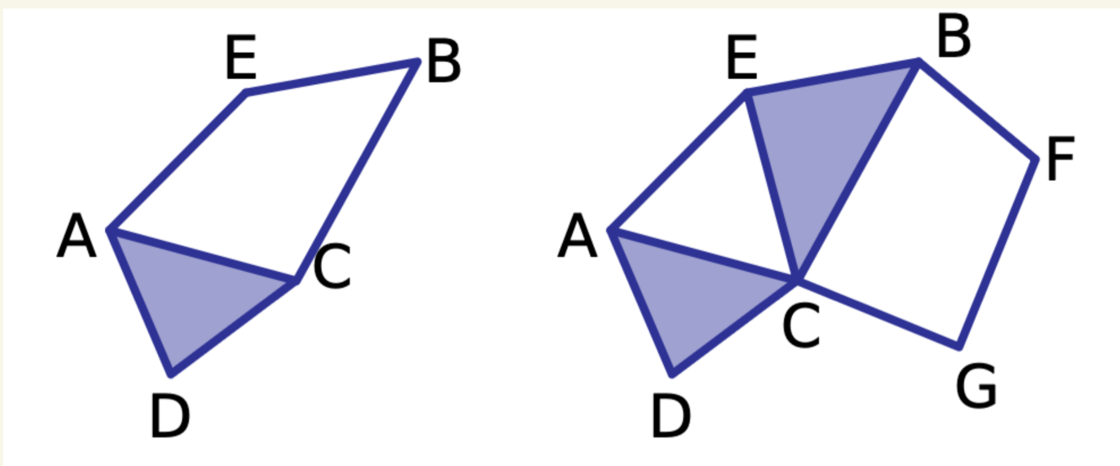
rank Z_2 =

rank B_2 =

Recall: A simplicial map between abstract simplicial complexes $f: K \to L$ is induced by a map on vertices $V(K) \to V(L)$

Inclusion maps: $i: K \to L$, $K \subseteq L$

$$i(\sigma) = \sigma$$



Passing to chain complexes

Any $f: K \to L$ naturally extends to a map on chain complexes:

$$f_{\#}: C_p(K) \longrightarrow C_p(L)$$

Results in a diagram:

$$\cdots \to C_{p+1}(K) \longrightarrow C_p(K) \longrightarrow C_{p-1}(K) \to \cdots$$

$$\cdots \to C_{p+1}(L) \longrightarrow C_p(L) \longrightarrow C_{p-1}(L) \to \cdots$$

Claim: $f_\# \partial^K = \partial^L \circ f_\#$.

Why? remember how f worked on vertices.

$$\begin{tikzcd}
C_p(K) \arrow[r, "\partial_p^K"] \arrow[d, "f_\#"'] & C_{p-1}(K) \arrow[d, "f_\#"] \\
C_p(L) \arrow[r, "\partial_p^L"] & C_{p-1}(L)
\end{tikzcd}$$

Consider a $\sigma \in C_p(K)$.

Claim: $f_\#(\text{cycle in } K) = \text{cycle in } L$

$f_\#(\text{boundary in } K) = \text{boundary in } L$

Why?

Because it commutes!

Consider a cycle:

$\partial_p^K(\alpha) = 0$

$$\begin{tikzcd} C_p(K) \arrow[r, "\partial_p^K"] \arrow[d, "f_\#"'] & C_{p-1}(K) \arrow[d, "f_\#"] \\ C_p(L) \arrow[r, "\partial_p^L"] & C_{p-1}(L) \end{tikzcd}$$

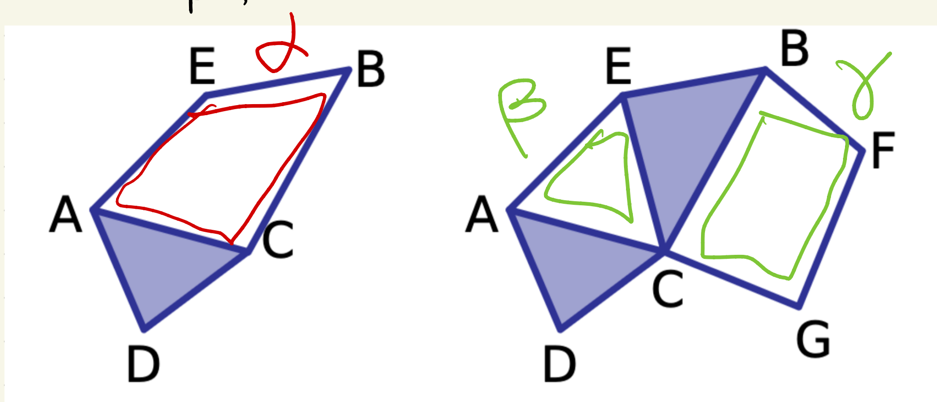
This induces a map on homology:

$$f_* : H_p(K) \longrightarrow H_p(L)$$
$$[\alpha] \longmapsto [f_\#(\alpha)]$$

Example:

$$H_1(K) = \{ \qquad \}$$
$$H_1(L) = \{ \qquad \}$$

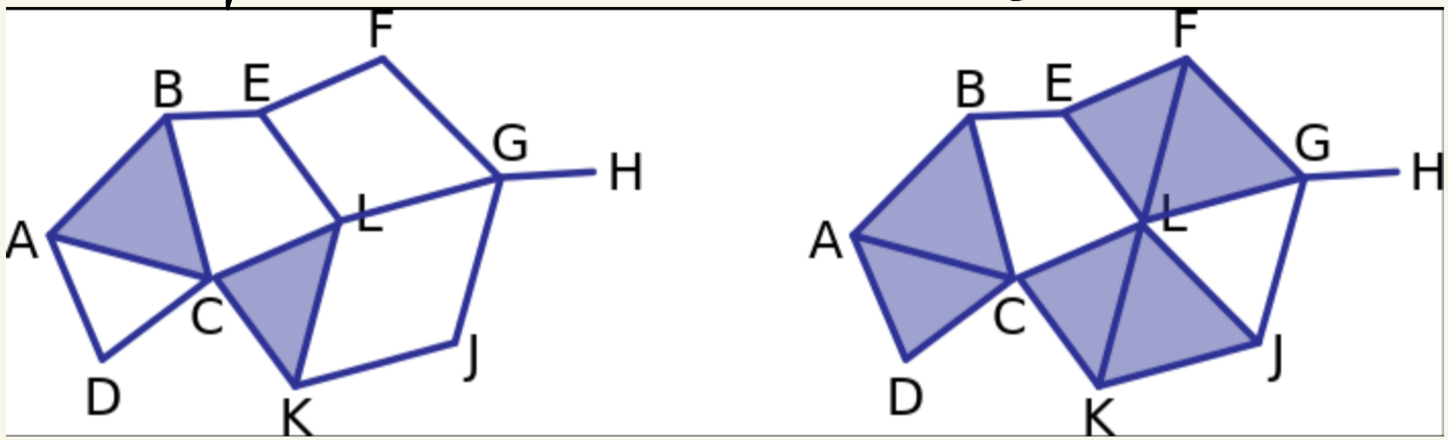
K L



$$f_* : \quad \left(\qquad \right)$$

Another!

$K \hookrightarrow L$



$H_1(K)$ generated by: $[\alpha_1] = [AC + AD + CD]$

$[\alpha_2] = [BC + BE + CL + EL]$

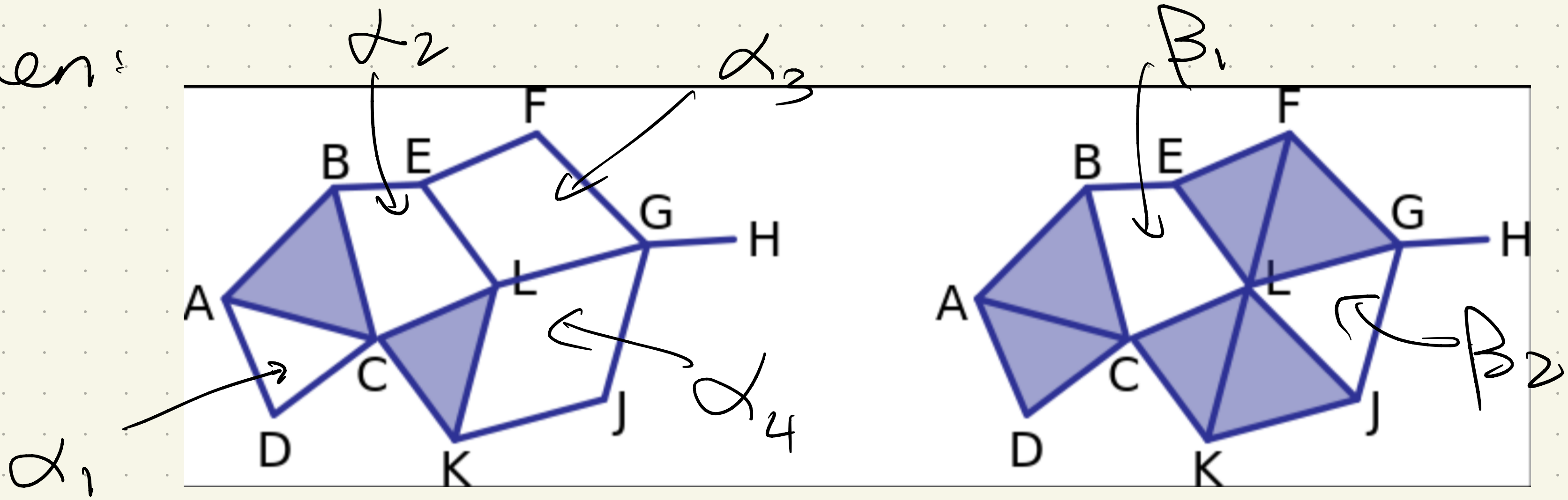
$[\alpha_3] = [EF + EL + FG + GL]$

$[\alpha_4] = [GJ + GL + JK + KL]$

& $H_1(L)$ by $[\beta_1] =$

$[\beta_2] =$

Then:



$$\begin{array}{c c} & \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{array}{c} \beta_1 \\ \beta_2 \end{array} & \begin{pmatrix} & & & \\ & & & \end{pmatrix} \end{array}$$

Relative Homology

Idea: compute homology of a complex relative to a subcomplex

Take L a subcomplex of K.

$\Rightarrow$ $C_p(L)$ is a subgroup of $C_p(K)$.

Quotients again!

$$C_p(K)/C_p(L) = C_p(K,L)$$

Relative chain group

Boundaries extend naturally:

$$\partial_p^{K,L} : [c_p] \longrightarrow [\partial_p c_p]$$

Can check all the same things:

$$\partial_{p-1}^{K,L} \partial_p^{K,L} = 0$$

so $Z_p(K,L) = \ker \partial_p^{K,L}$

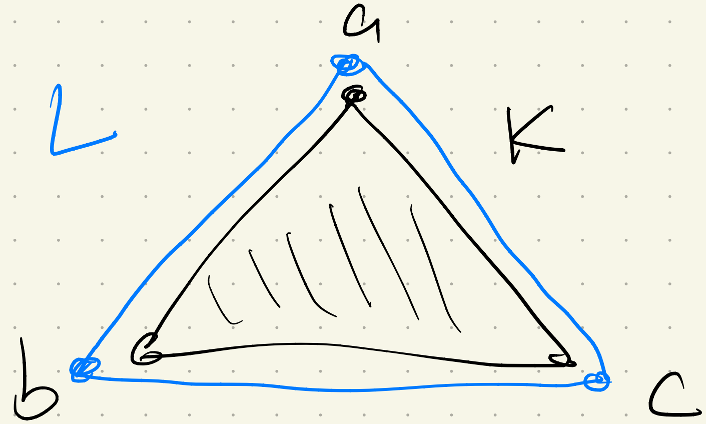
$B_p(K,L) = \operatorname{im} \partial_{p+1}^{K,L}$

+ $H_p(K,L) = Z_p(K,L) / B_p(K,L)$

But why?

Essentially, equivalent to "coning off" L, so L has no topology.

Example:



$$C_2(K) = \langle 0, [a_0 a_1 a_2] \rangle$$

$$C_2(L) = 0$$

$$\Rightarrow C_2(K, L) =$$

$$C_1(K) = \langle 0, [ab], [ac], [bc] \rangle$$

$$C_1(L) = \langle 0, [ab], [ac], [bc] \rangle$$

$$\Rightarrow C_1(K, L) =$$

$$\& \quad C_0(K) = C_0(L) = \langle 0, [a], [b], [c] \rangle$$

$$\Rightarrow C_0(K, L) =$$

Then:

$$
\begin{array}{ccccccc}
0 \xrightarrow{\partial_3} & C_2(K,L) & \xrightarrow{\partial_2} & C_1(K,L) & \xrightarrow{\partial_1} & C_0(K,L) \\
& \| & & \| & & \| \\
& \mathbb{Z}_2 & & 0 & & 0
\end{array}
$$

(generated by [abc])

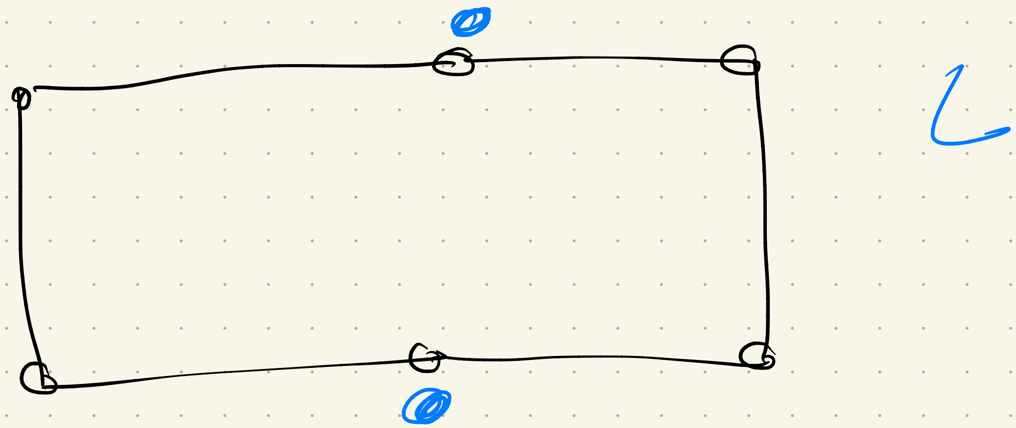
So $H_1(K,L)$ + $H_0(K,L)$ are both 0.

$$H_2(K,L) = \ker \partial_2 / \operatorname{im} \partial_3$$

$$=$$

Faster!

"cone off" L



Book also covers singular homology,
as well as cohomology.

I'm skipping these for now, but
we might revisit...

Next time: filtrations, &
using this all for persistent
homology