

TDA - fall 2025

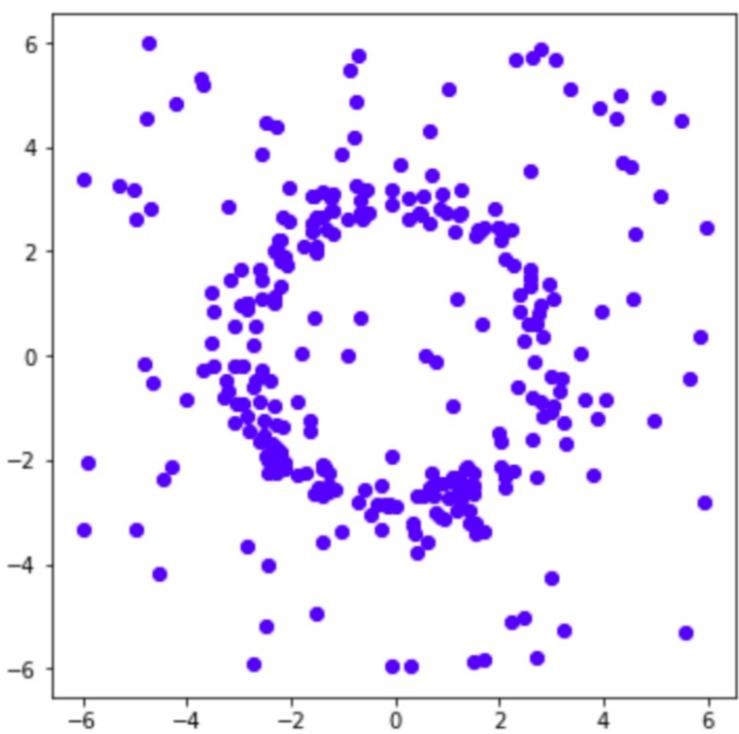
Mult parameter
Persistence



1-parameter persistence:

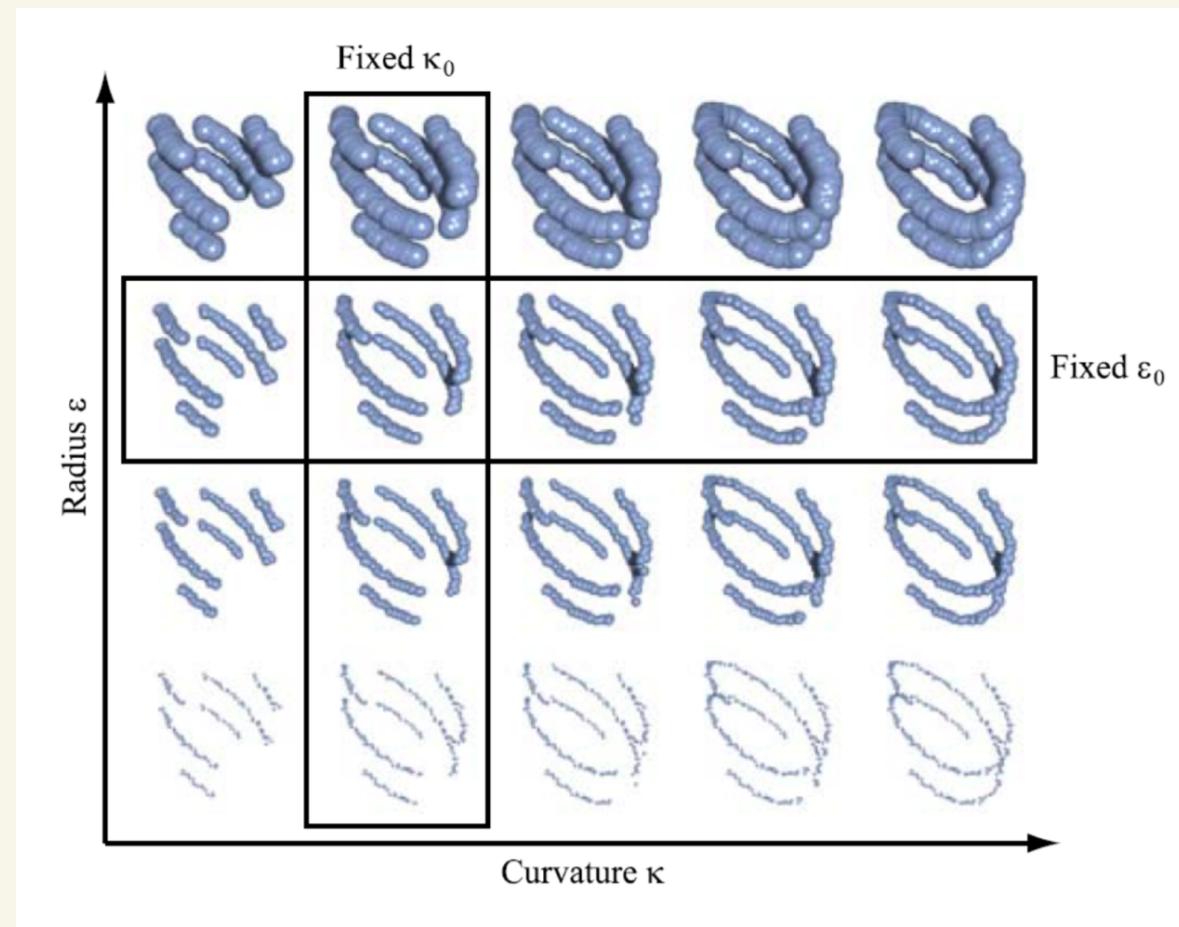
One problem: outliers

What will PD be?



Bifibrations

2 different fibrations



Common example

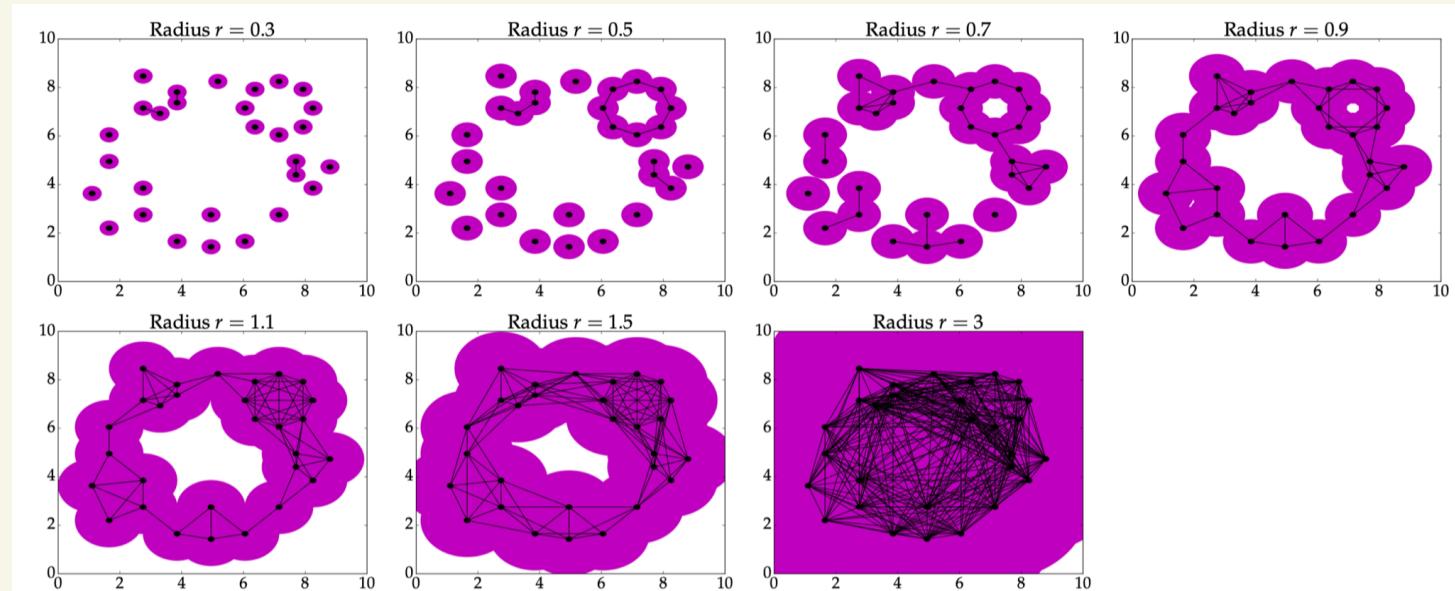
$P = \text{finite metric space}$

$N(P)_r = r\text{-neighborhood graph of } P:$

- Vertex set P , and edge $(i,j) \in N(P)_r$ iff $d(i,j) \leq r$
- If $r < 0$, $N(P) = \emptyset$

$R(P)_r$!

Vietoris-Rips
complex



Then, add some other filtration:

- Curvature estimate K

- Density: $\gamma_r(x) = C \cdot (\# \text{ pts within distance } r \text{ of } x)$

Where C is normalization constant

$$\text{so } \sum \gamma(x) = 1$$

- Gaussian density: $\sigma > 0$ parameter,
 C again normalization constant:

$$\gamma(x) = C \sum_{y \in P} \exp\left(\frac{-d(x, y)^2}{2\sigma^2}\right)$$

Or, on voxelized data:

often have intensity

↳ filter based on voxels with $\leq t$ intensity

↳ or, fixing t , can filter based on distance to nearest voxel.

More formally:

A bifiltration is a collection of simplicial complexes indexed by \mathbb{Z}^2

s.t. $F_a \subseteq F_b$ when $a \leq b$

Here, $(a_1, a_2) \leq (b_1, b_2)$ if

\int

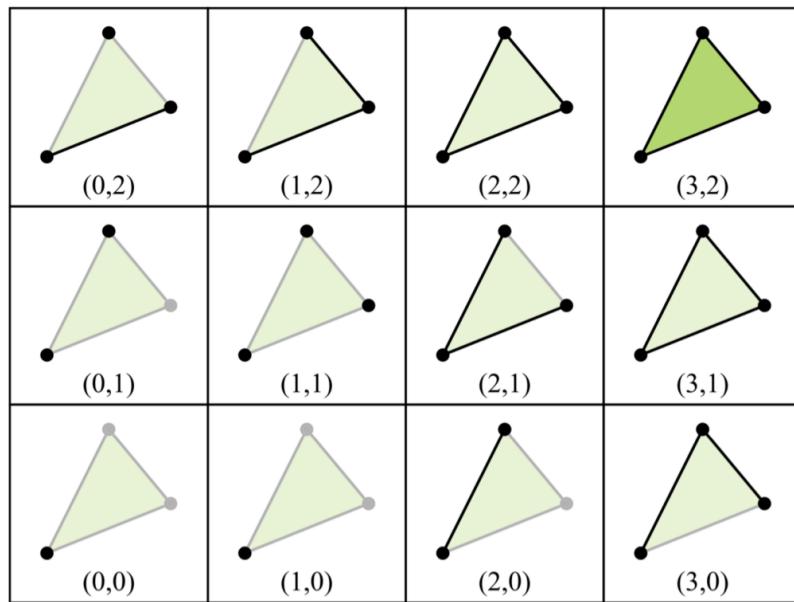
$F_{0,0} \hookrightarrow$

Bipersistence Module

Let M be a collection of vector spaces
 $\{M_a\}_{a \in \mathbb{Z}^2}$ with maps $\{M_{a,b} : M_a \rightarrow M_b\}$

M_{go}

Simple example



maps

\mathcal{O} -homology
w/ k the field

$$\begin{array}{ccccccc}
 k^2 & \longrightarrow & k & \longrightarrow & k & \longrightarrow & k \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 k^2 & \longrightarrow & k^3 & \longrightarrow & k & \longrightarrow & k \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 k & \longrightarrow & k & \longrightarrow & k & \longrightarrow & k
 \end{array}$$

Recall Barcodes / diagrams are unique representations, b/c of Gabriel's theorem:

$$H_d(M_0) \rightarrow H_d(M_1) \rightarrow \dots \rightarrow H_d(M_{n-1}) \rightarrow H_d(M_n)$$

If we consider the field $= k$,
 can look at maps explicitly:

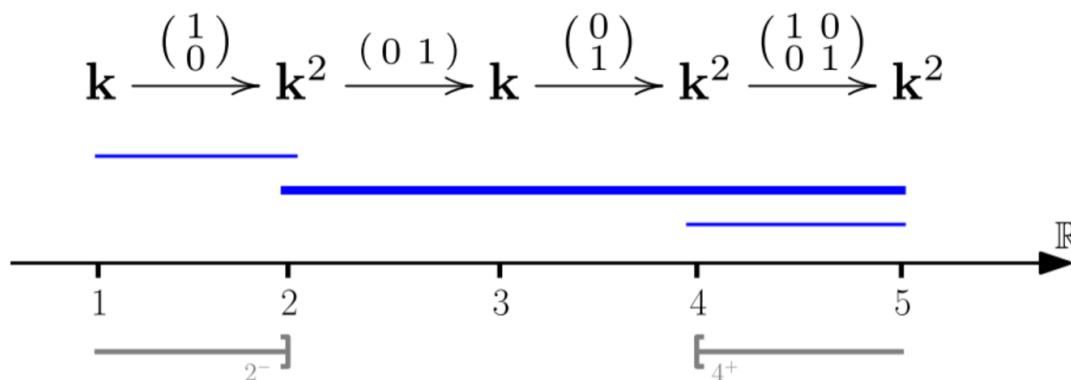


Fig. 1 A one-parameter persistence module M (top) indexed over $\{1, 2, 3, 4, 5\} \subset \mathbb{R}$, and the graphical representation of its barcode (in blue). The corresponding rank decomposition $\text{Rk } M = \text{Rk } k_{[1,2]} + \text{Rk } k_{[2,5]} + \text{Rk } k_{[4,5]}$ is readily available, and the ranks can easily be read from it: for instance, the rank $\text{Rk } M(2, 4) = 1$ is given by the one bar (thickened) that connects the down-set 2^- to the up-set 4^+ (Color figure online)

figure from:

Botnan, Oppermann + Oudot
2025

The bad news

Theorem

Carlsson & Zomorodian 2009

(paraphrased a bit)

The algebraic classification of
indecomposables of multiparameter
persistence modules contain both
discrete & continuous portions

→ no PD-diagram-like representation
is possible.

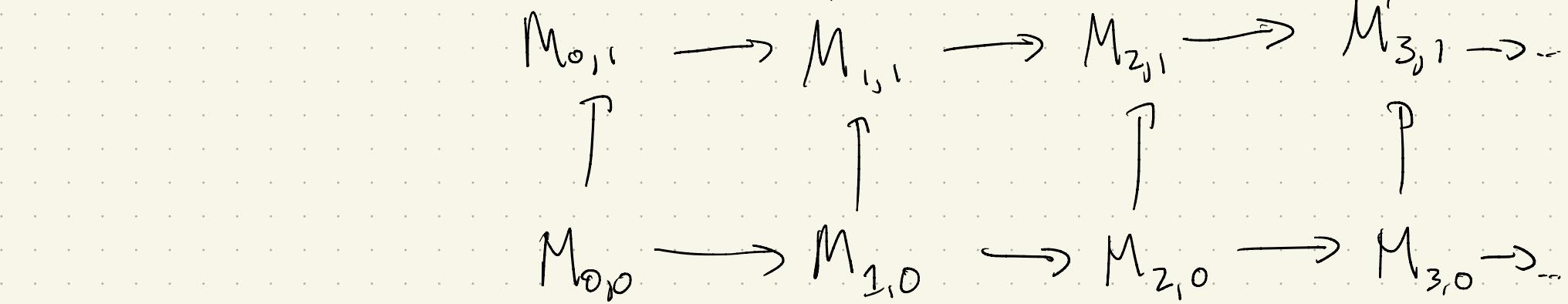
(It is of "wild representation type")

Lowering the dimension

One common approach is

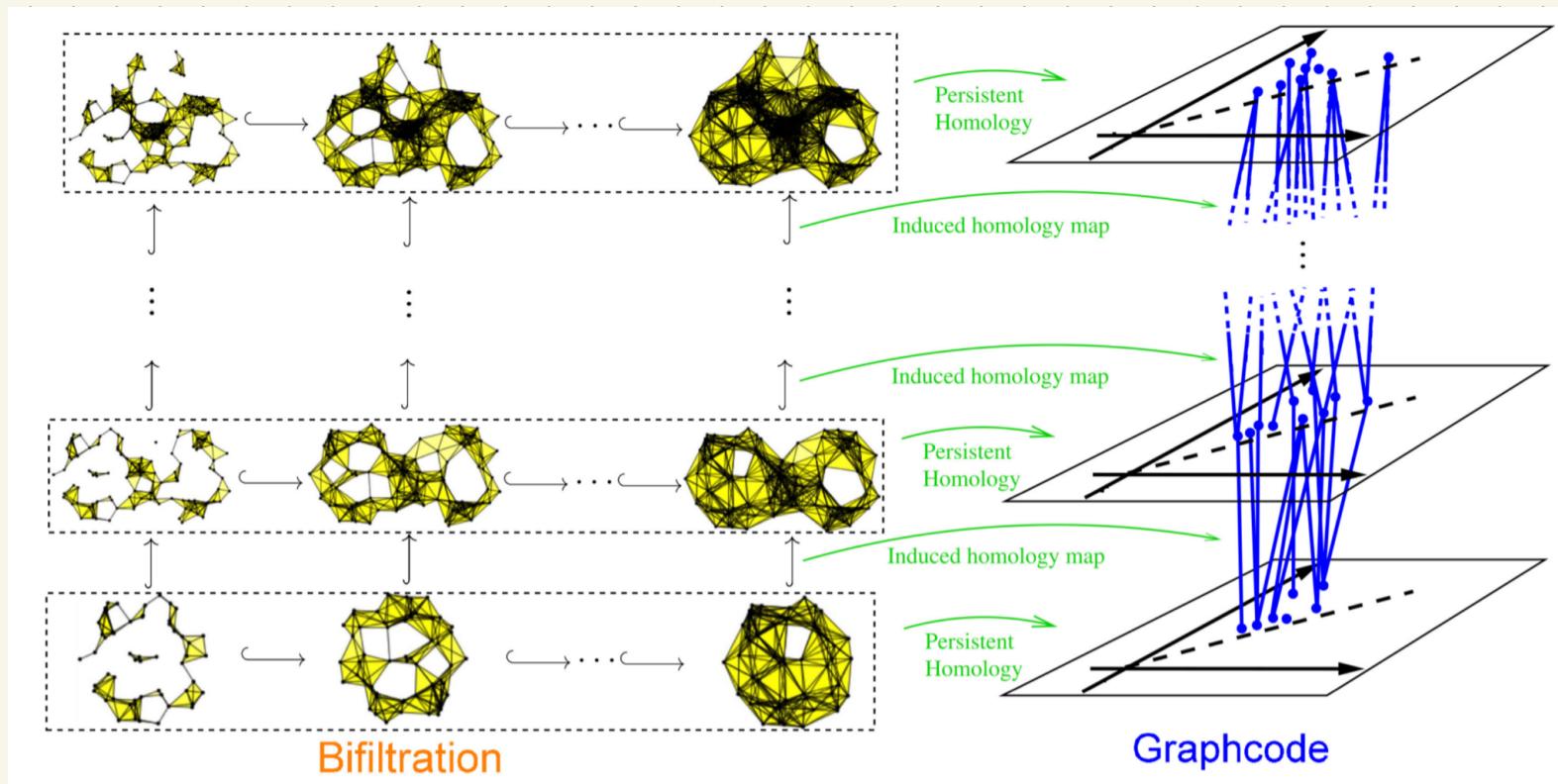
to restrict

to a "line":



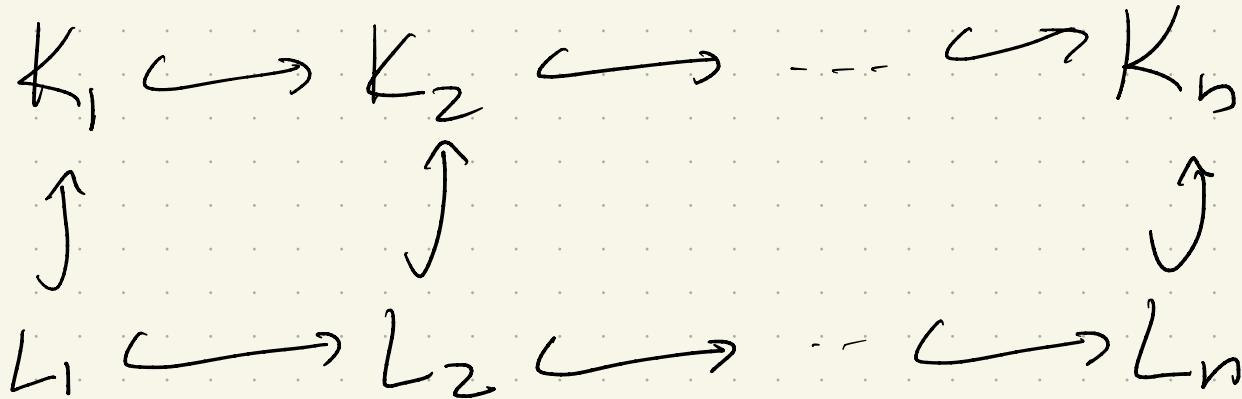
Graph codes (for ML pipelines)

Kerber & Russold 2024



Each PD is easy, but "vineyard" maps are a bit trickier.

Here:



Each inclusion $L_i \hookrightarrow K_i$ extends
to a map on chain complexes
(+ hence cycles) :

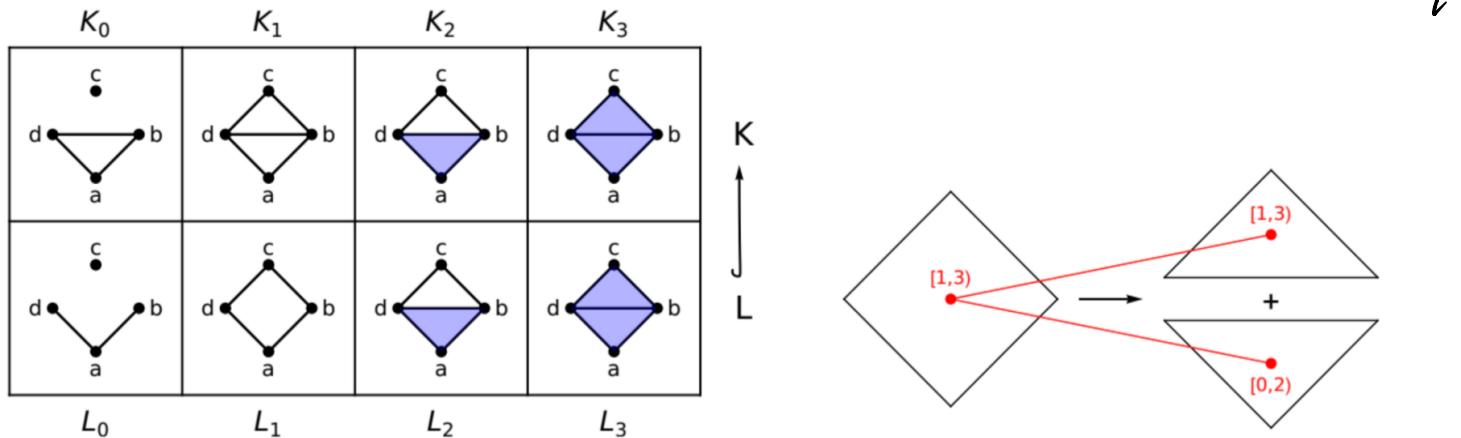


Figure 3: Left, lower row: $Z_1(L)$ is generated by the cycles $abcd$ and abd . They form a barcode basis, with attached bars $[1, 3]$ and $[2, 2]$, respectively. Note that also abd and bcd form a basis of $Z_1(L)$, but that is not a barcode basis as none of these cycles is already born at L_1 , so they do not induce a basis of $Z_1(L_1)$. Left, upper row: Here, abd and bcd form a barcode basis with attached bars $[0, 2]$ and $[1, 3]$, respectively, and abd and $abcd$ as well (with identical barcode).

Right: Choosing the basis $abcd$, abd for $Z_1(L)$ and abd and bcd for $Z_1(K)$, we have $abcd = abd + bcd$, hence the cycle $abcd$ has two outgoing edges, to both basis elements in K . We ignore the basis vector abd of L in the figure, since its birth and death index coincide, so the corresponding

Next time: Dr. Danny Chen

Guest lecture: TDA in medical
imaging

Friday:

Representation theory & rank invariants!
(Will get mathier.)

Now:

Project proposals & time to discuss