TDA - fall 2025

Extended
Persistence (cont)
+ Reeb graphs

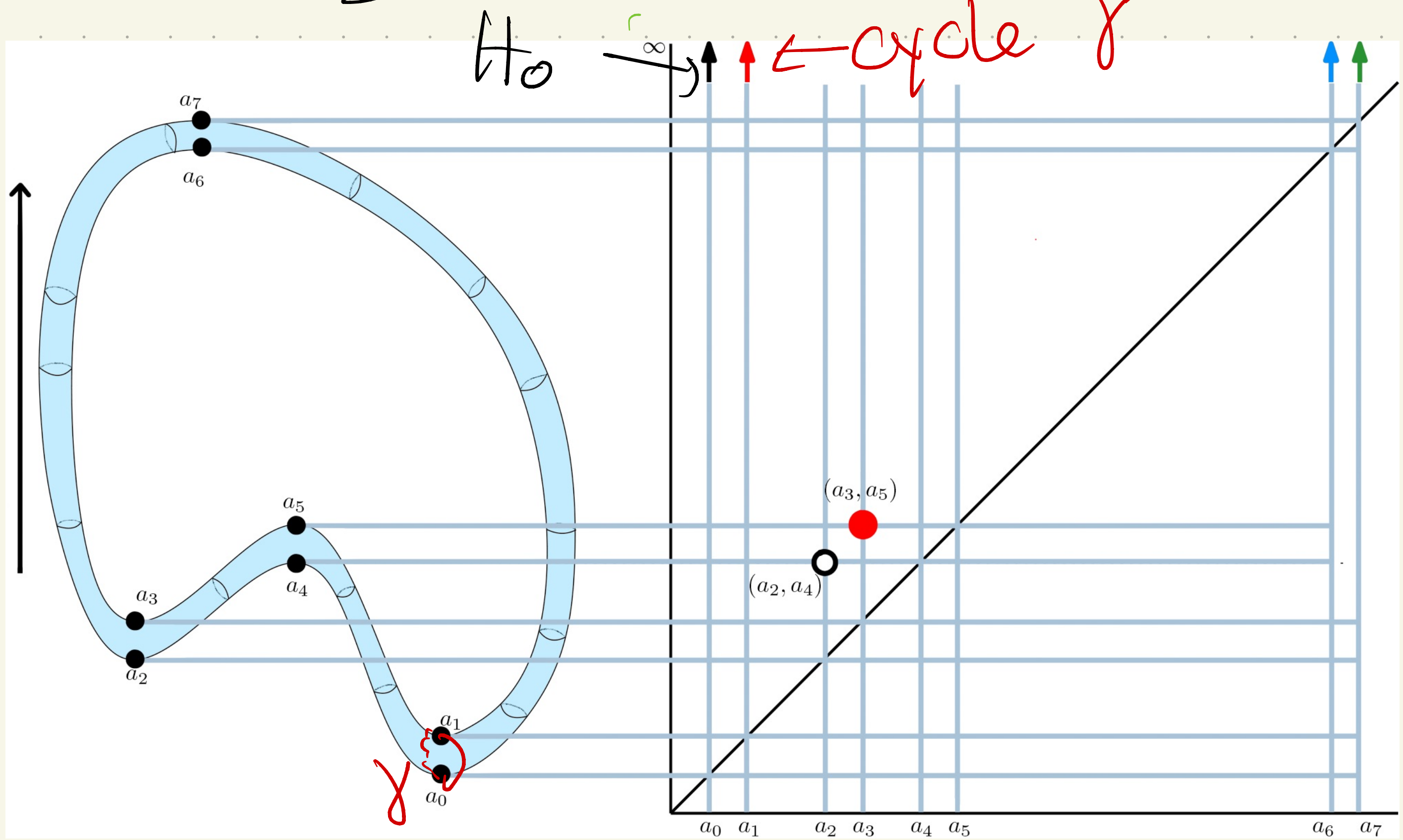
Recap

- Paper chase: due October 17
  (sorry for typo on webpage!)
- Next assigment will be final
  project proposal

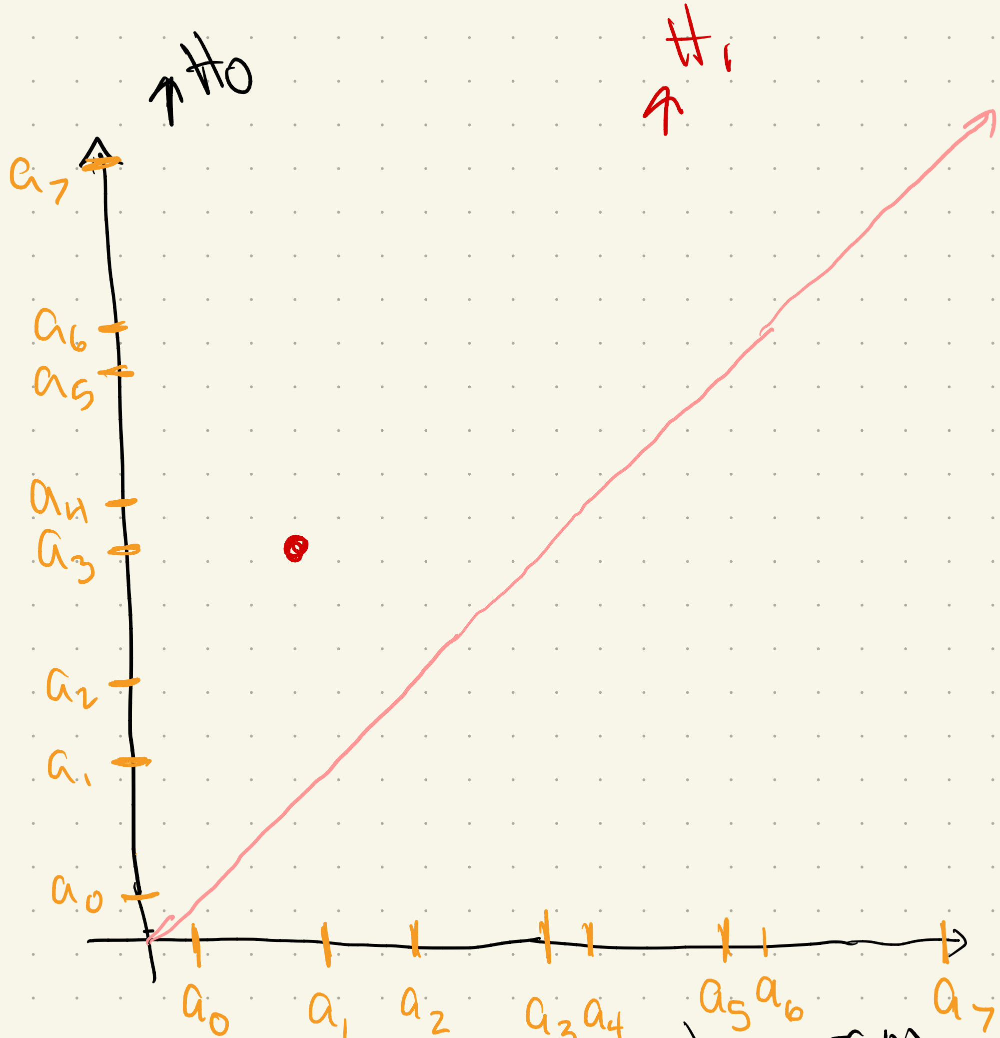
Extended Persistence

Odd parts of persistence:

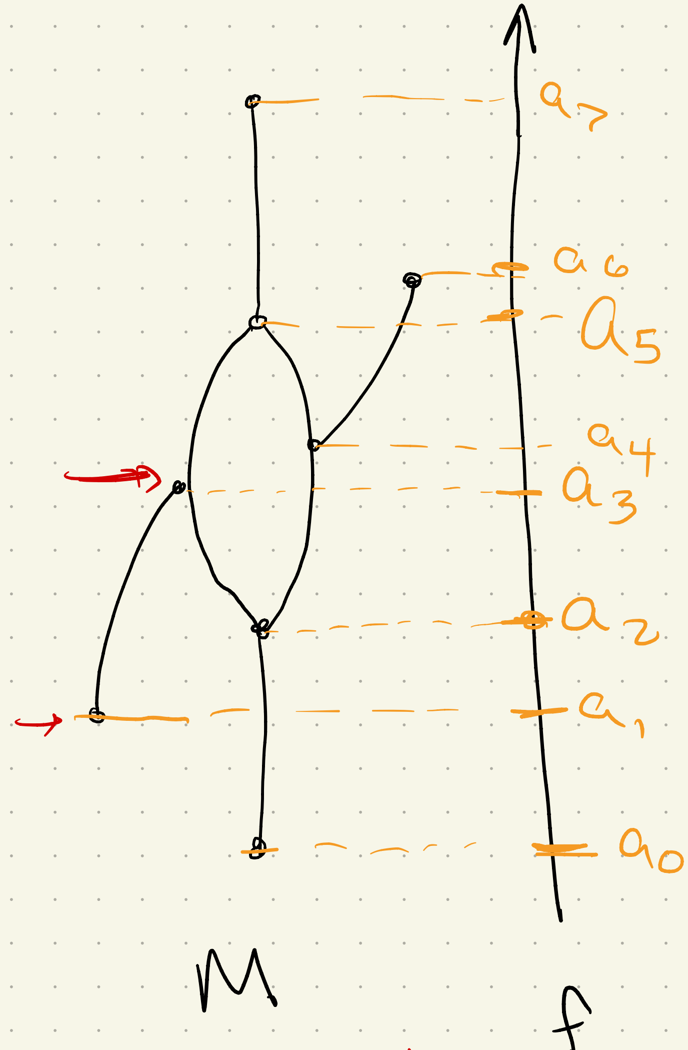
- points at infinity
- some Morse critical points don't seem to matter



Really becomes obvious on Reeb graphs
(more next week)



Persistence diagram



what about the rest!

Agarwal-Edelsbrunner-Harer-Wang 2006

⇒ Cohen-Steiner, Edelsbrunner, Harer 2009

Use relative homology to find other critical points, + get better pairings.

Relative homology   Fix $L \subseteq K$

Define $C_p(K, L) := C_p(K) / C_p(L)$

$+ \ [\alpha] = \{ \gamma \in C_p(K) \mid \alpha + \gamma \in C_p(L) \}$

Then maps extend to homology.

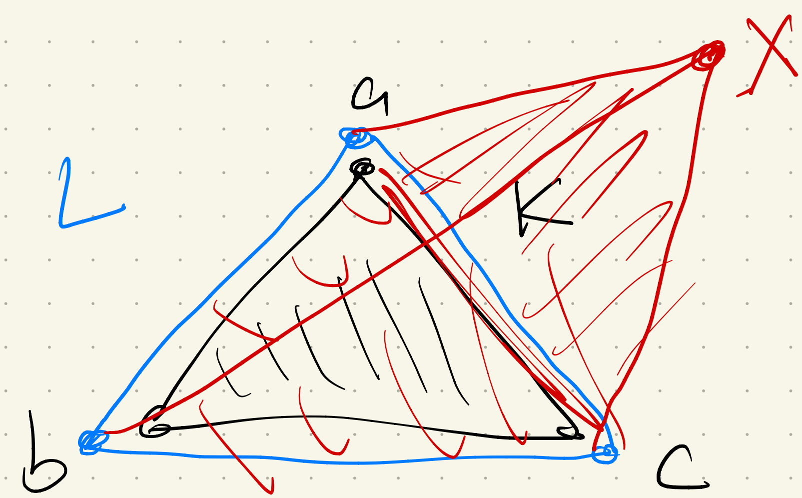
Remember a month ago? →

Fun fact

Let $K^* = K \cup \{x\} \cup \{\sigma \cup \{x\} \mid \sigma \in L\}$

"Coned off"

tetrahedra



Theorem:

$$H_p(K,L) = H_p(K^*) \text{ for } p>0$$

$$\& \quad \beta_0(K,L) = \beta_0(K^*) - 1$$

Here, we want to look at relative homology of the superlevel sets:

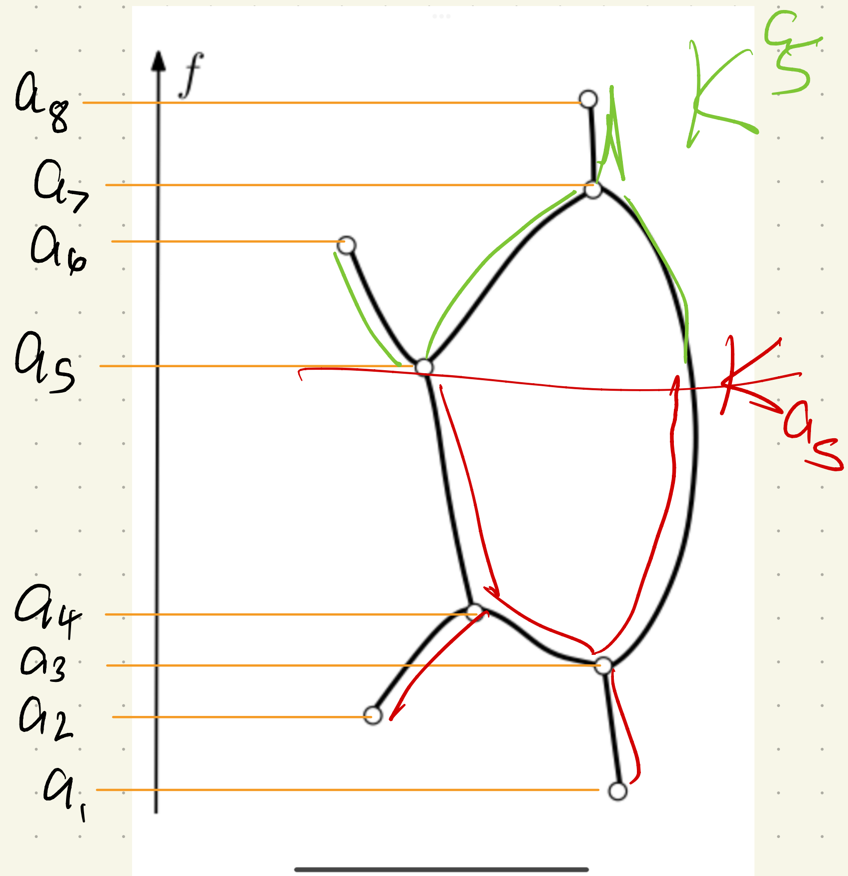
Given $f: K \to \mathbb{R}$

$$K_a = \{\sigma \in K \mid f(\sigma) \leq a\}$$

$$K^a = \{\sigma \in K \mid f(\sigma) \geq a\}$$

& study $H_p(K, K^a)$

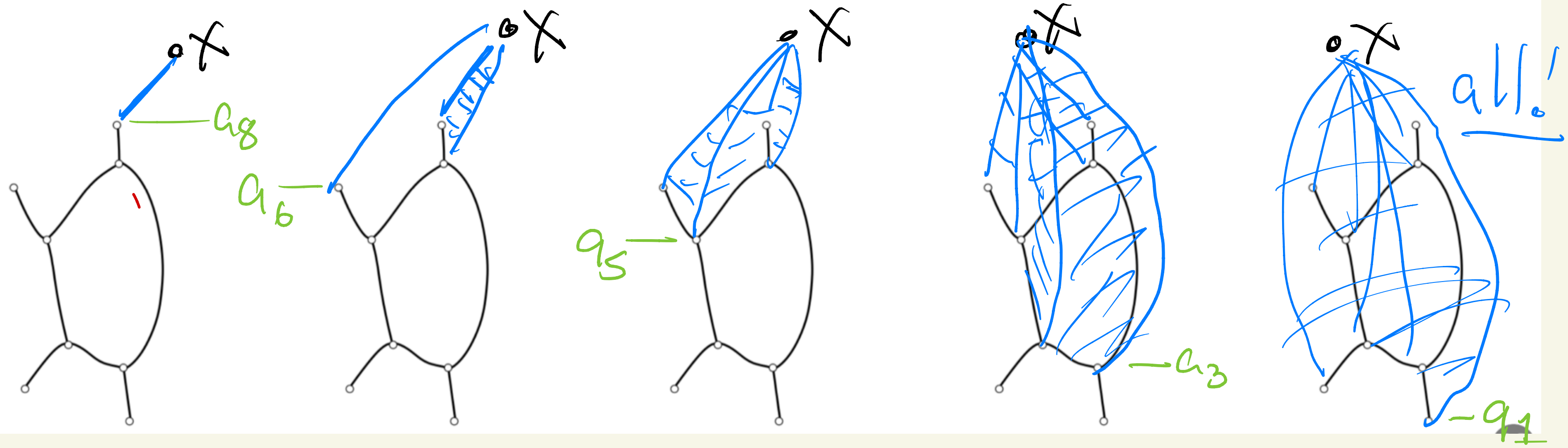
(as well as $H_p(K_a)$)



What are important bits? ("cone off" K^{a_i}) (revisited)

$H_0(K^{a_i})$ minus 1

$H_0(K, K^{a_8})$	$H_0(K, K^{a_6})$	$H_0(K, K^{a_5})$	$H_0(K, K^{a_3})$	$H_0(K, K^{a_1})$
$H_1(K, K^{a_8})$	$H_1(K, K^{a_6})$	$H_1(K, K^{a_5})$	$H_1(K, K^{a_3})$	$H_1(K, K^{a_1})$



Extended persistence Module

regular persistence

$$H_p(K_{a_1}) \to H_p(K_{a_2}) \to \cdots \to H_p(K_{a_n})$$

$$\hookrightarrow H_p(K, K^{a_n}) \to \cdots \to H_p(K, K^{a_1})$$

relative superlevel set homology

3 kinds of points:

- Ordinary : usual in PD
- Relative : born & die on way down
- Extended : use both

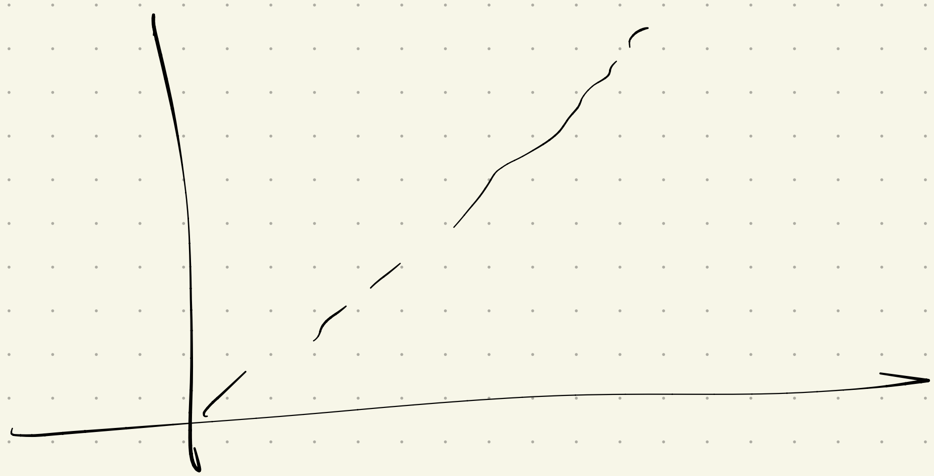
Points in diagram:

$$H_p(K_{a_1}) \to \cdots \to H_p(K_{a_n}) \to H_p(K, K^{a_n}) \to \cdots \to H_p(K, K^{a_1})$$

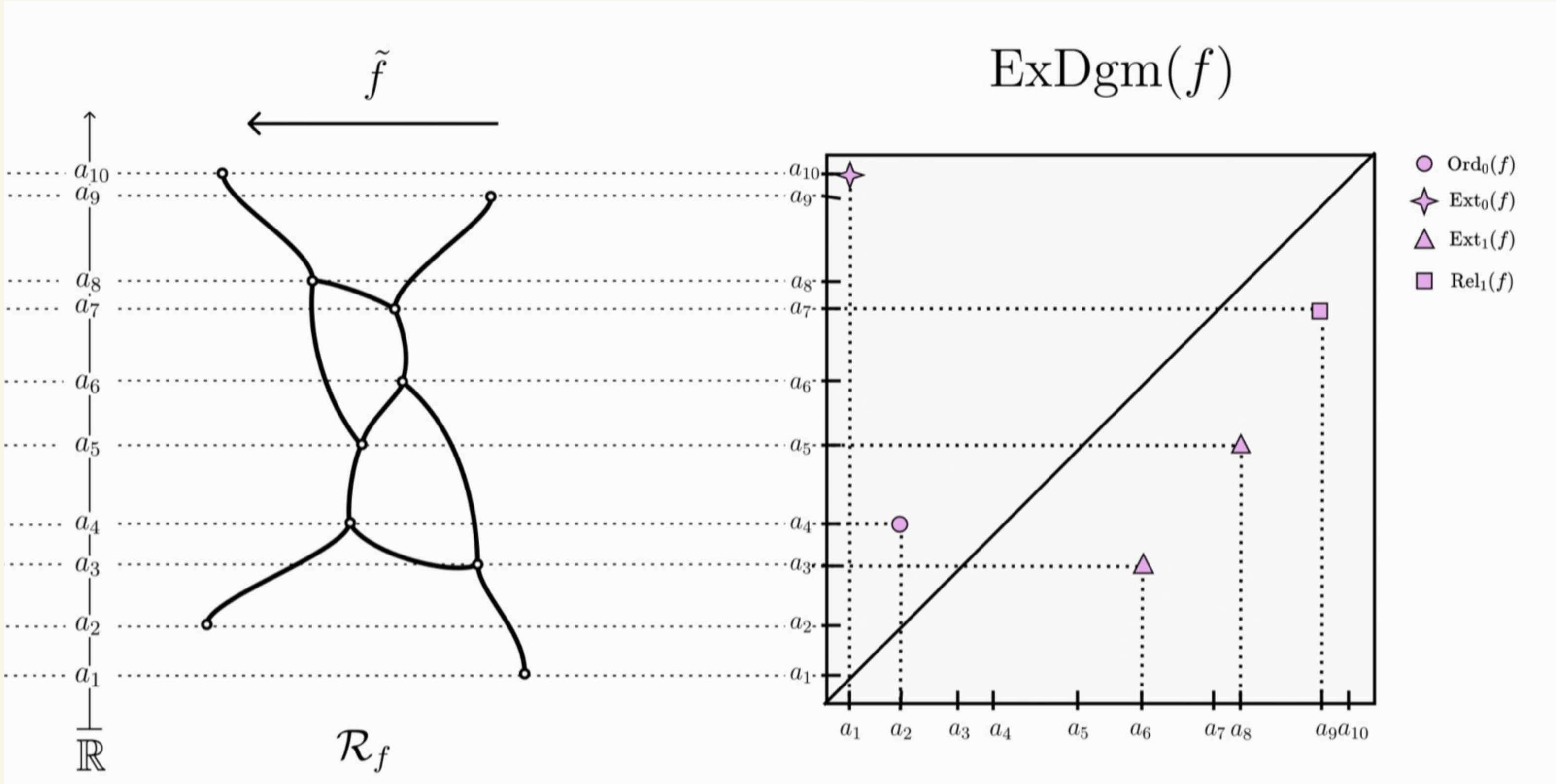
Ord: Starts & ends in 1^{st} half, so $(a,b) \Rightarrow$

Rel: Starts & ends in 2^{nd} half, $(a,b) \Rightarrow$

Ext: starts in 1^{st} half, ends in 2^{nd} half $\Rightarrow (a,b)$

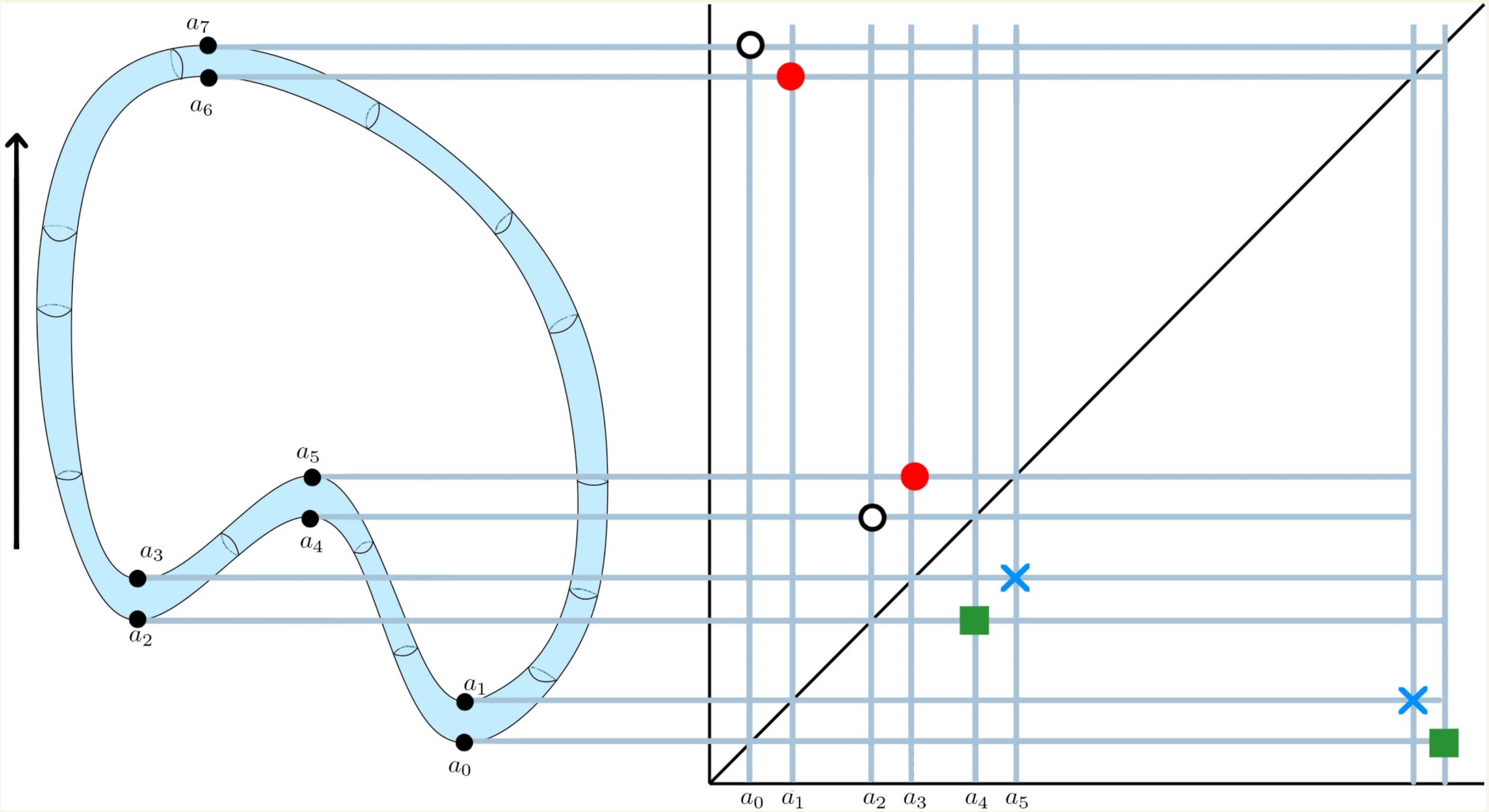


On graphs



$\tilde{f}$
a_{10}
a_9
a_8
a_7
a_6
a_5
a_4
a_3
a_2
a_1
$\mathbb{R}$
$\mathcal{R}_f$
$\mathrm{ExDgm}(f)$
a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}
$\mathrm{Ord}_0(f)$
$\mathrm{Ext}_0(f)$
$\mathrm{Ext}_1(f)$
$\mathrm{Rel}_1(f)$

More generally: on manifolds, get symmetry



- H_0 (ordinary & extended)
- ordinary H_1
- relative H_2
- relative H_1

Under the hood:

- Very beautiful combination of Lefschetz & Poincare duality (that 2010 paper)
- Some more algebraic conections Turner-Robins-Morgan 2022

Reeb Graphs (named for Georges Reeb, 1946 by René Thom)

Recall: level sets

Given (PL) $f : |K| \to \mathbb{R}$,

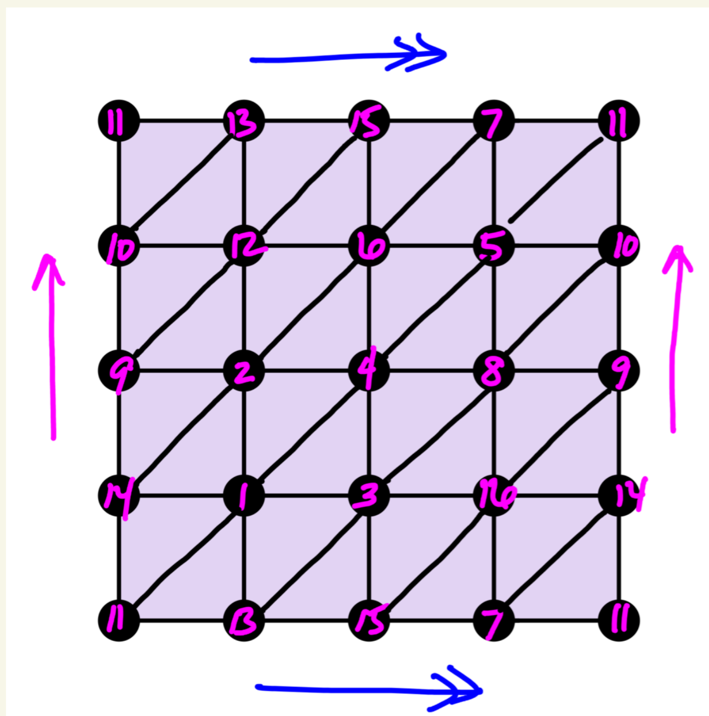
a level set is $f^{-1}(a)$

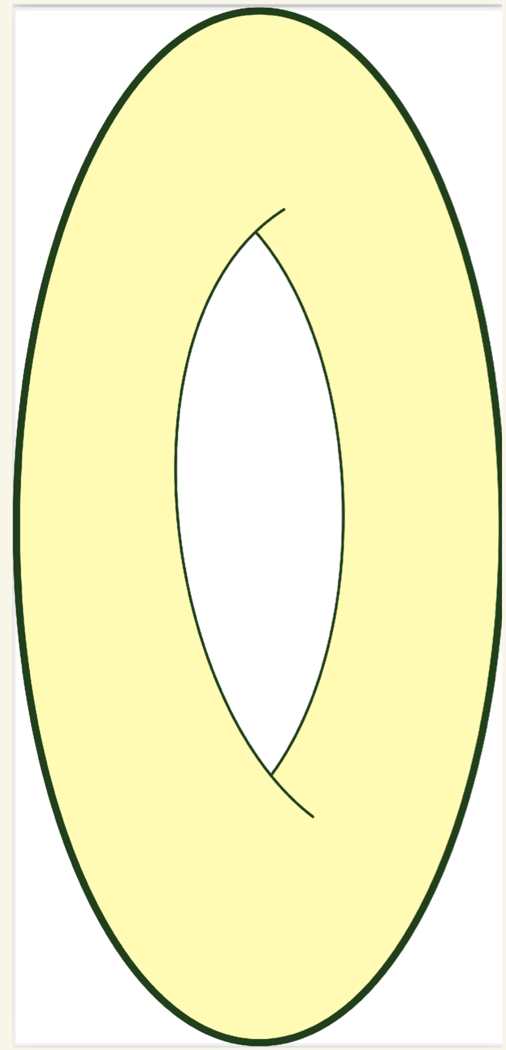
for $a \in \mathbb{R}$

Ex: Consider

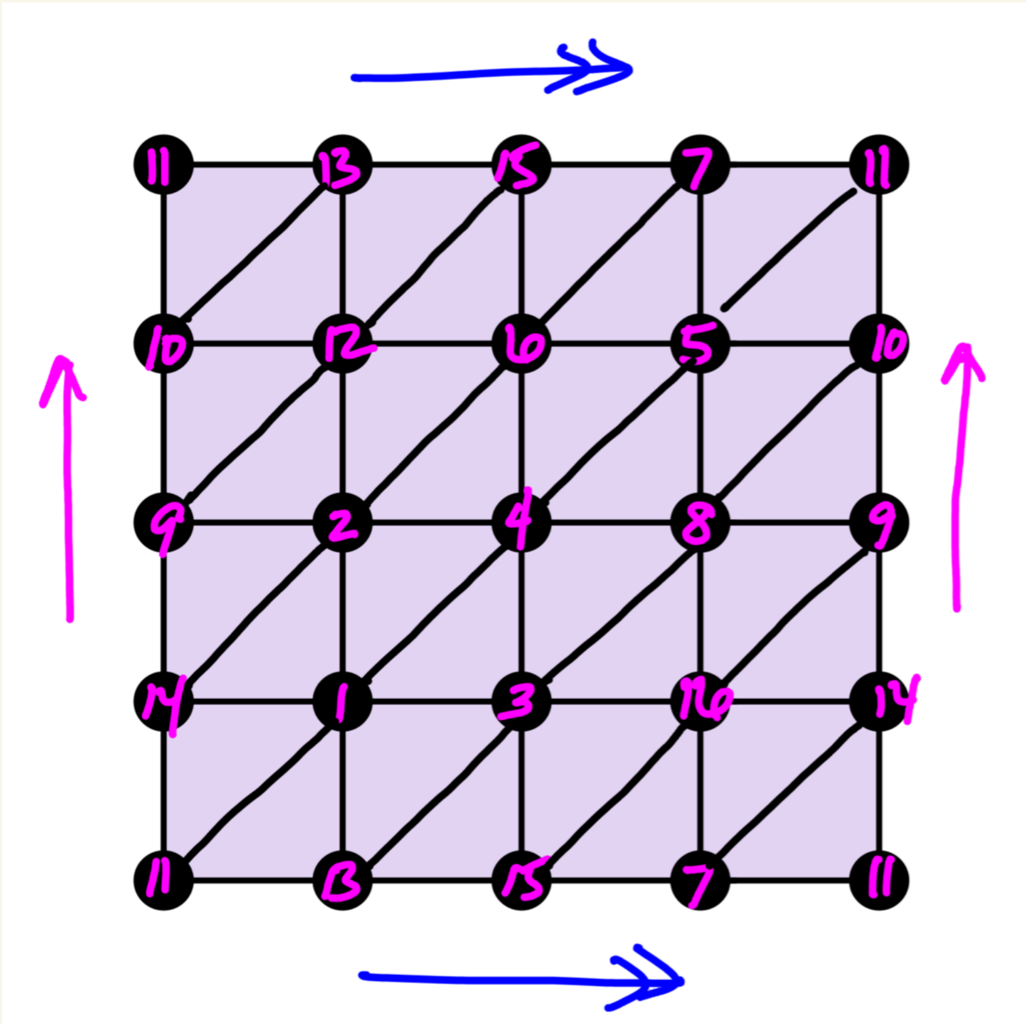
$f^{-1}((-\infty, 4])$

& $f^{-1}(4)$

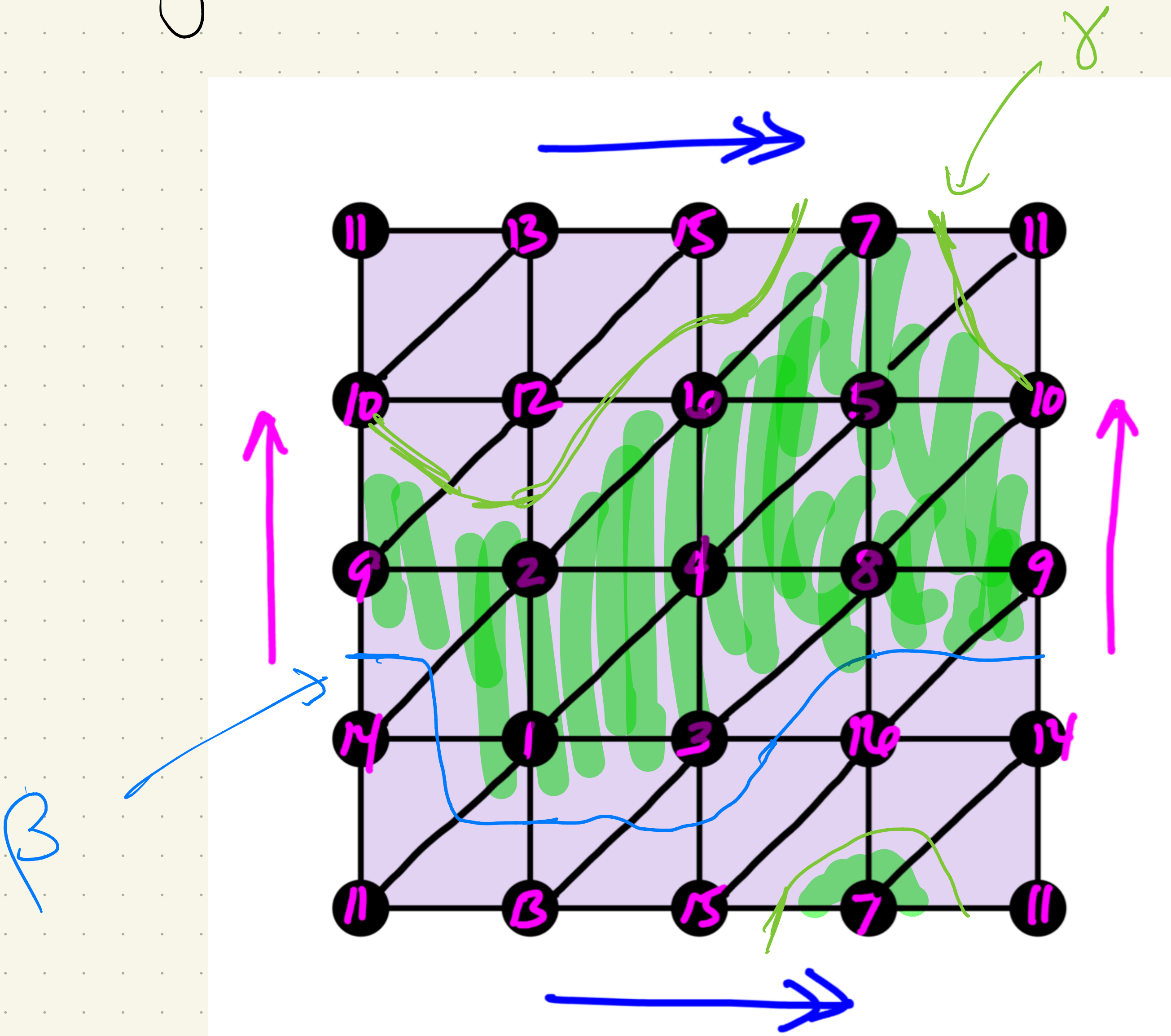




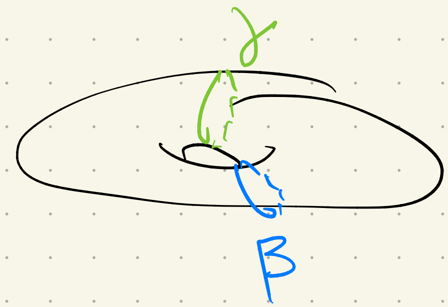
Practice: What is $f^{-1}(10)$?



Answer: just in case! ☺



On Torus:



Sublevel set $f^{-1}((-\infty, 10])$

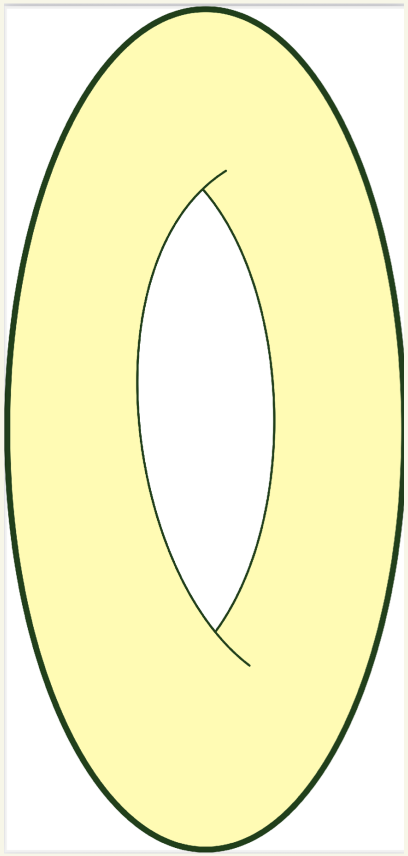
Recall:

A space is disconnected if $\exists$ disjoint open sets U, V with $X = U \cup V$

↳ otherwise, connected.

A connected component is a maximal subset that is connected

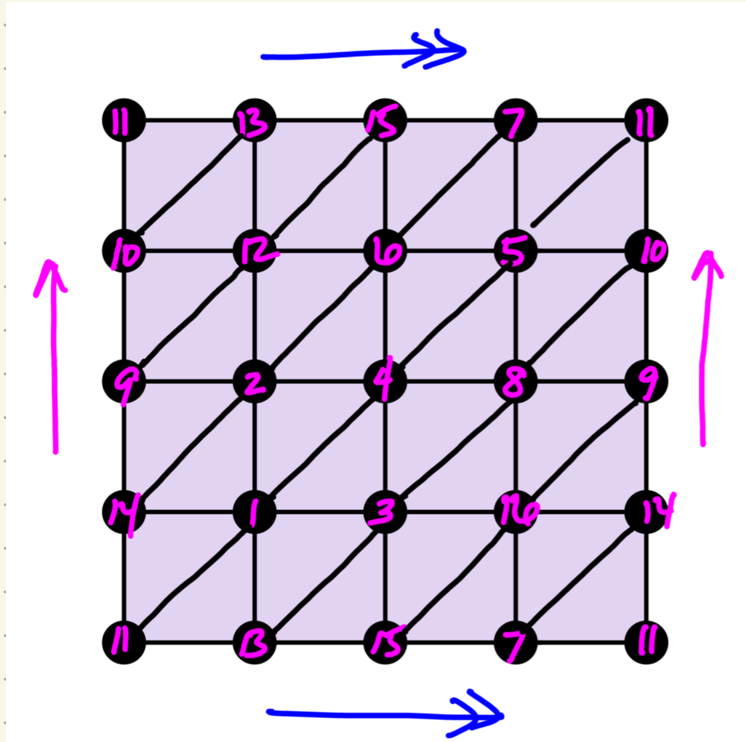
A contour is a connected component of a level set



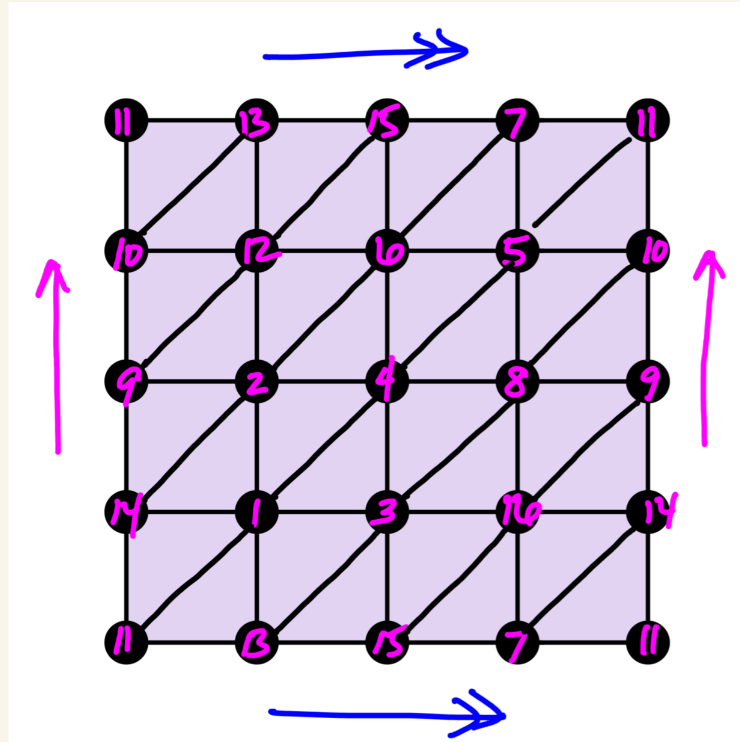
Q: How do contours change as a level set changes?

Back to triangulation:

$f^{-1}(8)$



$f^{-1}(9)$



What shape, or how many components?

(& recall $f^{-1}(10)$ a few slides back had 2 contours)

Equivalen relations + quotients

A binary relation $\sim$ on a set T is an equivalence relation if $\forall a, b, c \in T$,

- $a \sim a$ [relexive]
- $a \sim b \iff b \sim a$ [symmetric]
- if $a \sim b$ & $b \sim c \Rightarrow a \sim c$ [transitive]

Equivalence class of a under $\sim$

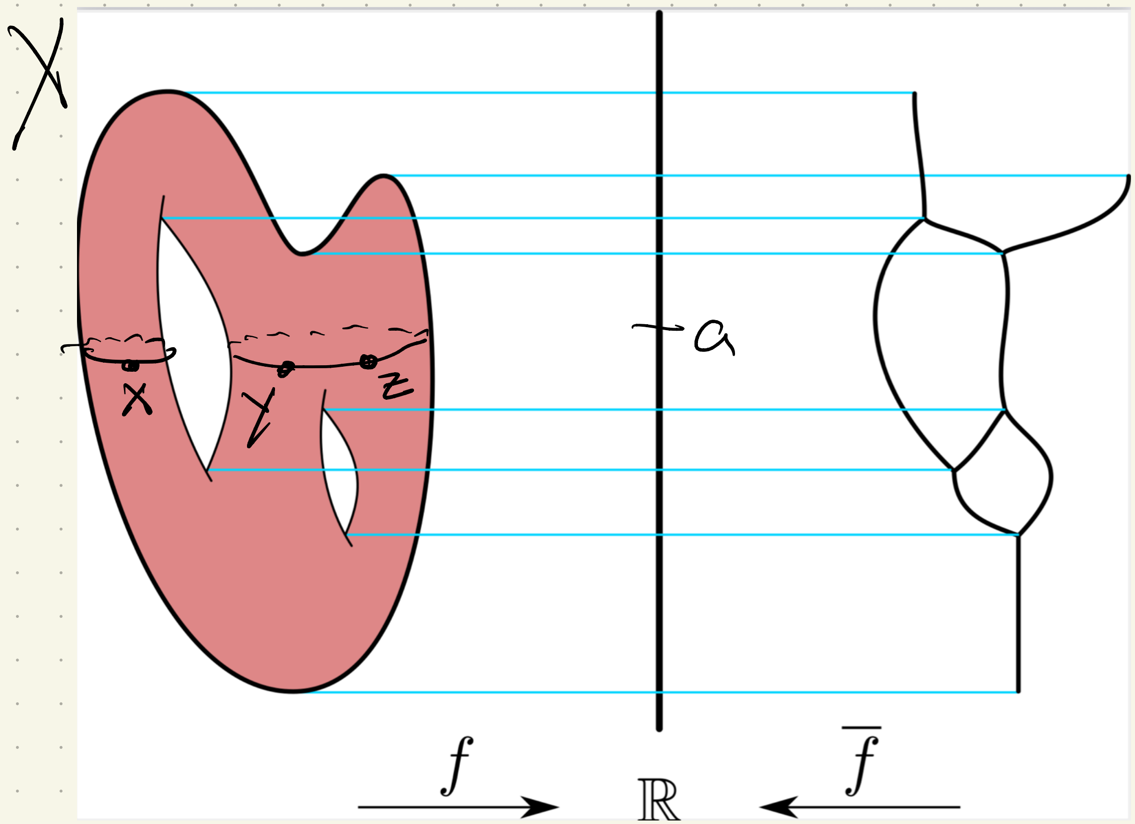
$$[a] = \{x \in T \mid x \sim a\}$$

On a topological space with an equivalence relation, get quotient map $q(a) = [a]$

(and U is open $\iff q^{-1}(U)$ is open)

Reeb graph equivalence relation

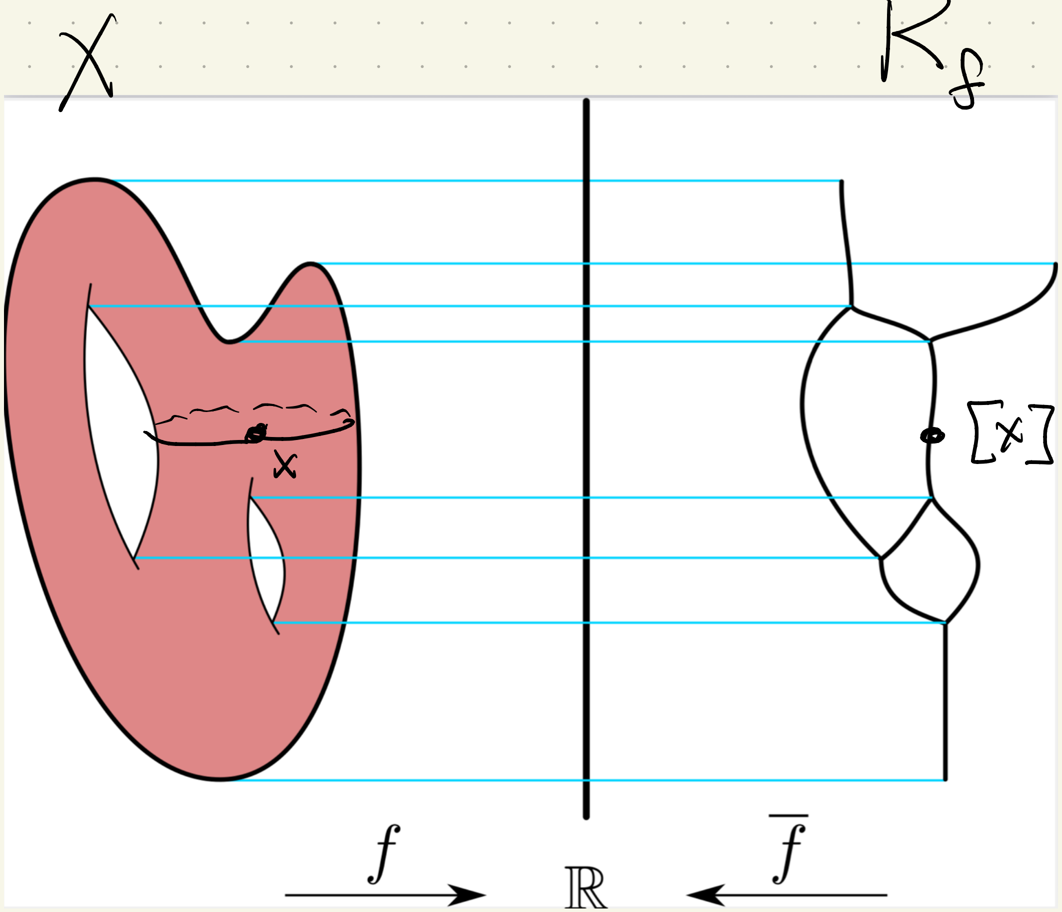
Given function $f: X \to \mathbb{R}$.



Define equiv. rel. $\sim$ by $x \sim y$ iff:

- $f(x) = f(y) = a$
- x + y are in same connected component of level set $f^{-1}(a)$.

Reeb graphs



- Let $[x]$ denote the equivalence class of $x \in X$.
- The Reeb graph $R_f = R(X, f)$ is the quotient space $X/\sim$
- Let $\Phi : X \to R_f$, $x \longmapsto [x]$ be the quotient map

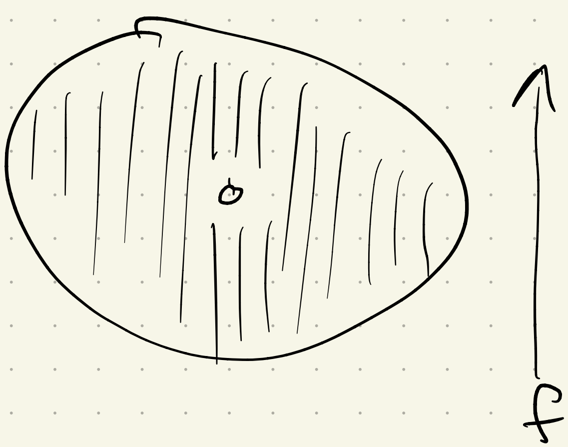
Niceness assumptions

If input is "nice" (both X + f), then
R_f is a graph.

- Morse function on compact manifold
- PL function of compact polyhedron
- others...

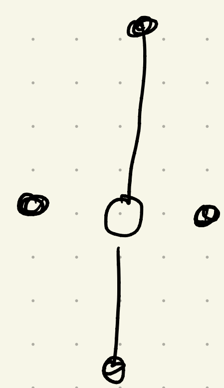
Bad example:

$X =$ disk/pt



$\Longrightarrow$

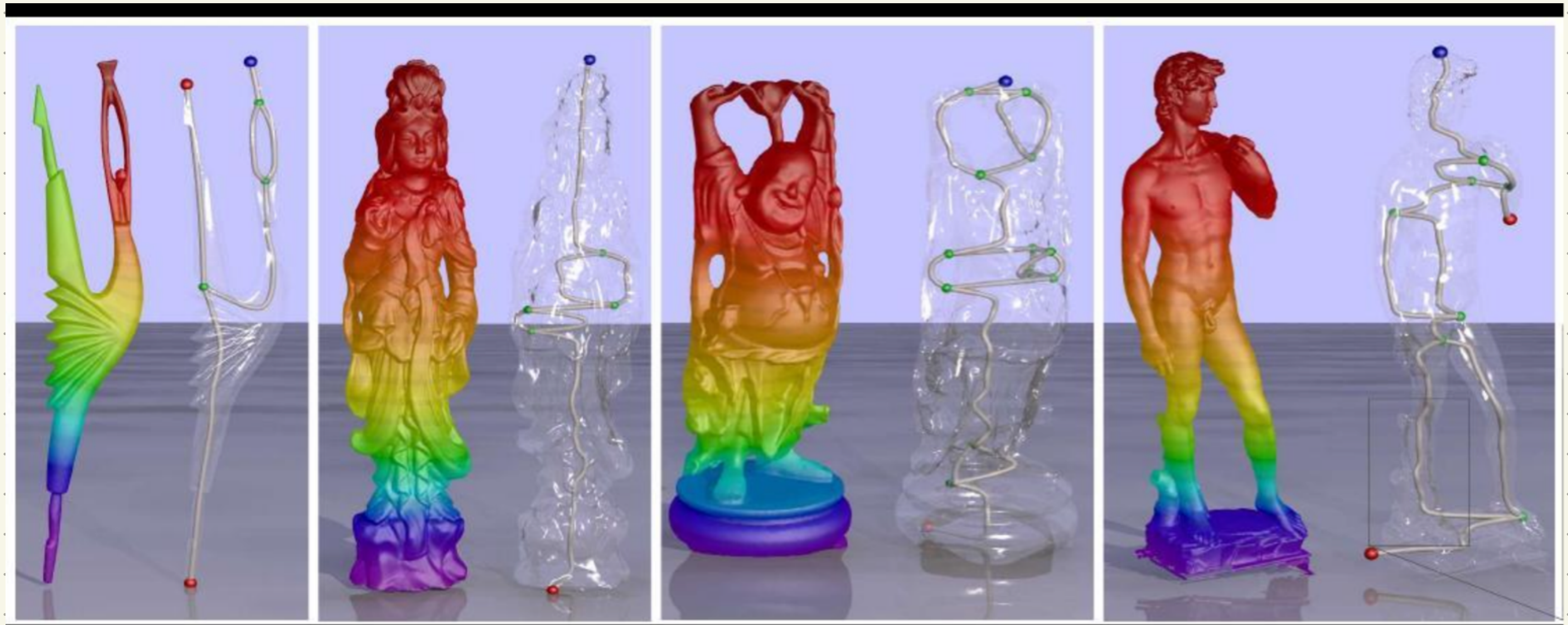
R_f



Non-Hausdorff

Applications: a topological signature

$\boxed{\text{Pascucci et al}}$



Note: a bit of a lie in this figure: