

TDA-fall 2025

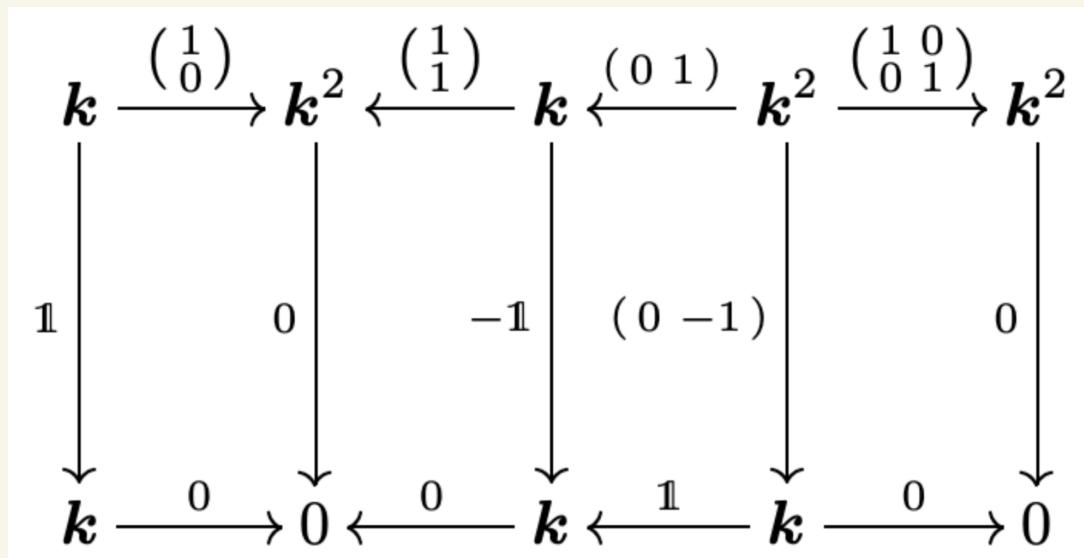
Rank invariants &
generalized
Betti #s & PDs.



Last time: Representation theory

Directed graphs, $Q_0 \rightarrow Q_1 \leftarrow$

↳ Realization:



Quiver representations are like vector spaces:

- They contain a 0 object
 \hookrightarrow all spaces & maps $= 0$
- They have a br-product, the direct sum:
 $V \oplus W$: spaces $V_i \oplus W_i, i \in Q_0$
maps $V_a \oplus W_a = \begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix}$
- Every morphism $\phi: V \rightarrow W$ has a kernel: $(\ker \phi)_i = \ker \phi_i$
(as well as image & cokernel)

A non-trivial representation V is called **decomposable** if it is isomorphic to the direct sum of 2 non-trivial representations.

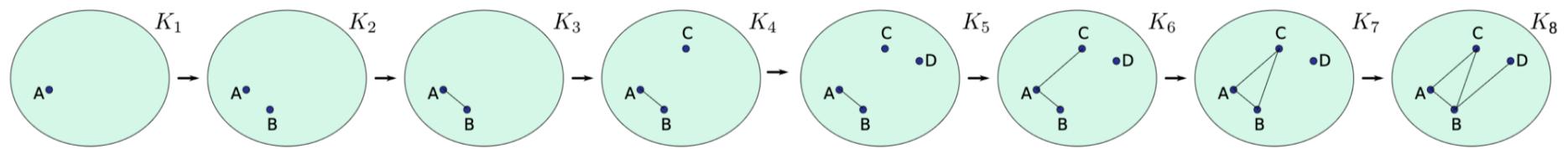
(Otherwise indecomposable.)

Back to persistence for a minute...

$$V = W \oplus X$$

neither or 0

Let $k = \mathbb{Z}_2$ & do H_0 :



$$H_0(K_1) \rightarrow H_0(K_2) \rightarrow H_0(K_3) \rightarrow H_0(K_4) \rightarrow H_0(K_5) \rightarrow H_0(K_6) \rightarrow H_0(K_7) \rightarrow H_0(K_8)$$

$$\mathbb{Z}_2 \rightarrow \mathbb{Z}_2^2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2^2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$$

$$\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\sim} \quad \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\sim} \quad \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\sim} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \end{bmatrix}$$

decompose

$$\mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2$$

$$\oplus$$

$$0 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0$$

$$\oplus$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0$$

$$\oplus$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} 0$$

Central question

Classify representations of a given quiver, up to isomorphism.

→ [usually all finite + finite dim]

Define $\underline{\dim} V = (\dim V_1, \dots, \dim V_n)^T$

→ a vector $\in \mathbb{N}^n$

and $\dim V = \sum_{i=1}^n \dim V_i$

How to get a handle on this?

even when \dim is finite,
these get wild

Krull-Remak-Schmidt Theorem

Wedderburn 1909, Remak 1911

Schmidt 1913, Krull 1925

Assuming \mathbb{Q} is finite, for any $V \in \text{rep}_k(\mathbb{Q})$, \exists indecomposable representations V_1, \dots, V_r st.

$$V = V_1 \oplus \dots \oplus V_r.$$

Moreover, for any other indecomposable rep. W_1, \dots, W_s with $W = W_1 \oplus \dots \oplus W_s$ must have $r \geq s$ and the W_i 's & V_j 's are permutations.

So \rightarrow to classify, need to understand
 & characterize indecomposables.

Gabriel's theorem 1972

Let Q be finite quiver +
 k a field. Then, Q has
 a finite # of classes of
 indecomposables

$\Leftrightarrow Q$ is Dynkin.

Why surprising?

K & \hookleftarrow are
 irrelevant

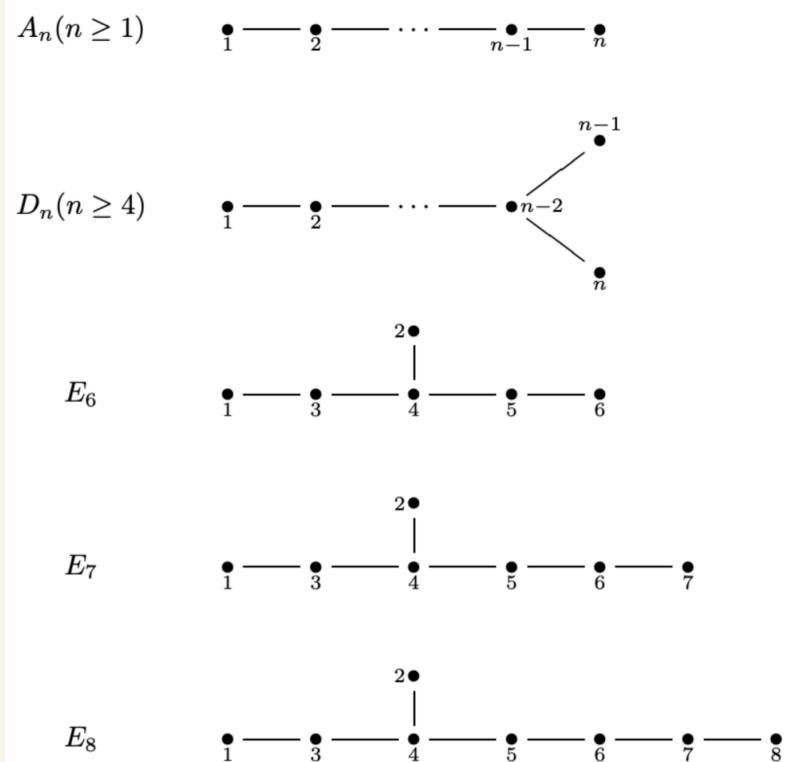
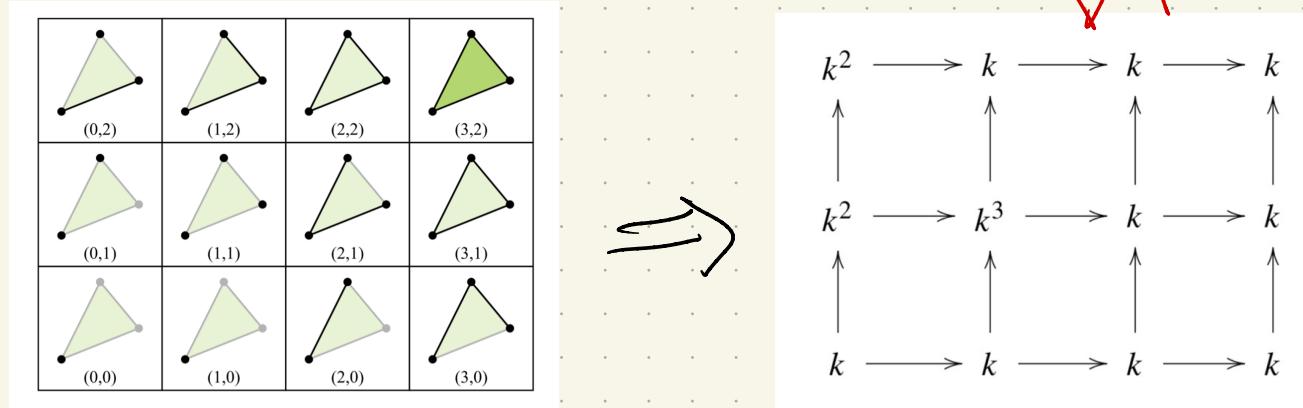


FIGURE A.2. The Dynkin diagrams.

Back to persistence

Why we still care?



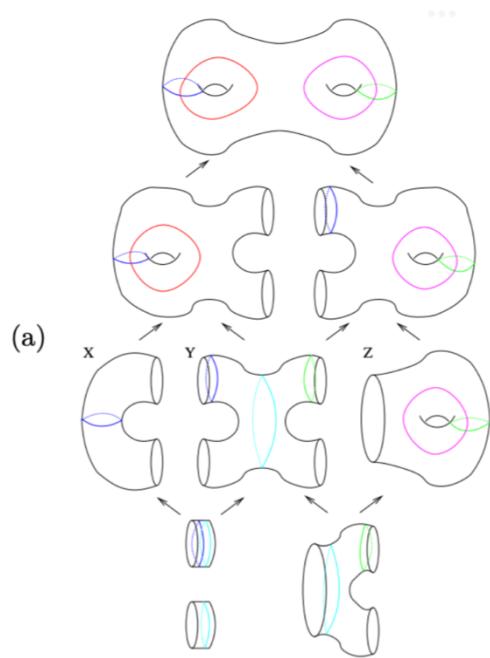
since not Dynkin in general,
Gabriel's theorem doesn't apply
 $\Rightarrow \mathbb{Q}$ has infinite # of
isomorphism classes of
indecomposables.

Result:

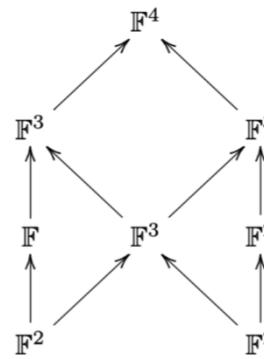
Invariants

Carrier Subgraph:

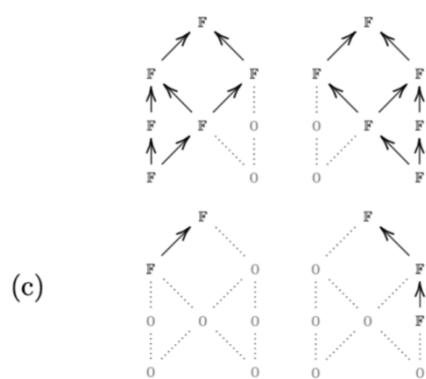
C.-Litscher 2019



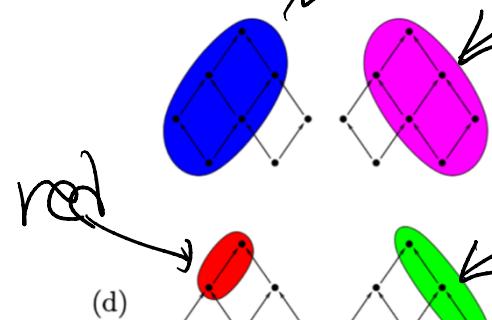
(a)



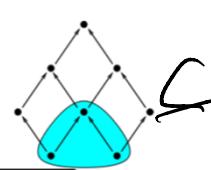
(b)



(c)



(d)



Up to choice on generators!

dark blue cycle

green

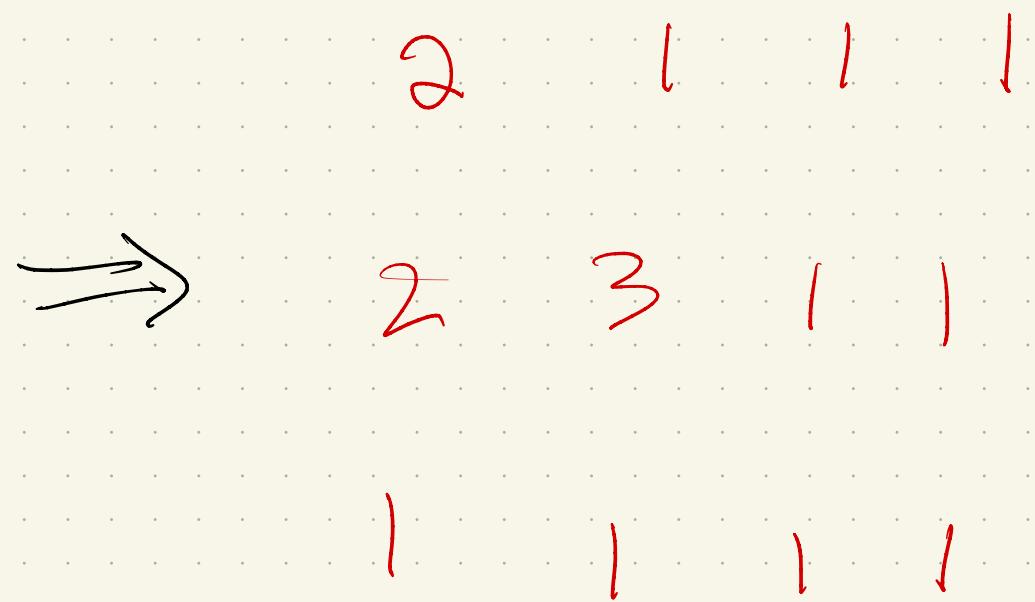
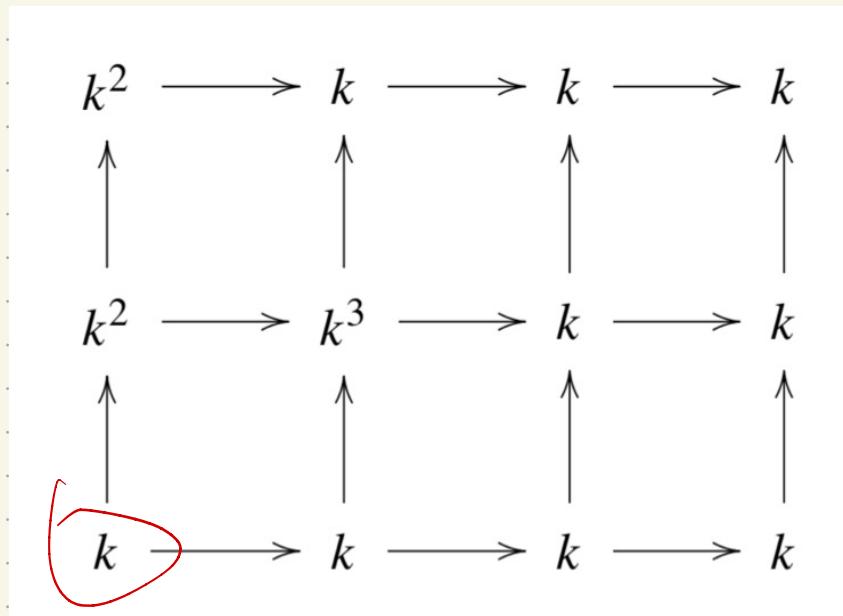
pink

turquoise

Biparameter filtrations again

① Dimension Function

Simply map each $a \in \mathbb{R}^2$ to $\dim(M_a)$



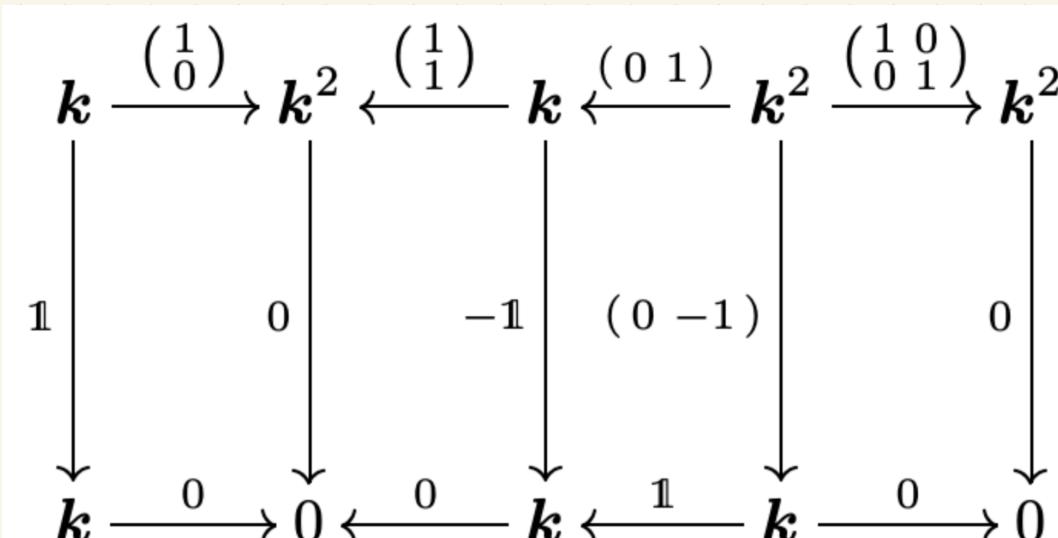
pros: easy to compute & visualize!

cons: no notion of lifespan or persistence

② Rank invariant

For each linear map $M_a \rightarrow M_b$,
compute $\text{rank}(M_a \rightarrow M_b)$.

Note: lots of these!



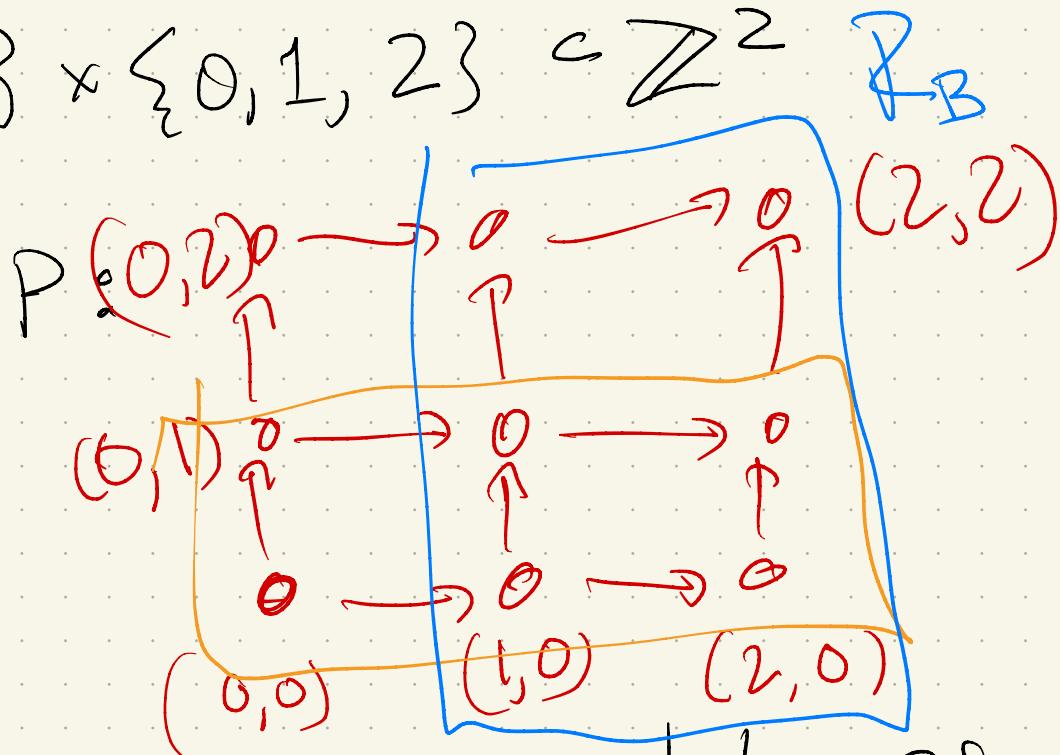
An example to illustrate issues + concept.

Let $P = \{0, 1, 2\} \times \{0, 1, 2\} \subset \mathbb{Z}^2$ P_B

with usual
partial order:

$$(i, j) \leq (i', j')$$

$$\Leftrightarrow i \leq i' \wedge j \leq j'$$



Let's build a dipersistent module as
direct sum of 2 rectangles:

R_A

$$\{ (i, j) \mid i \in \{0, 1, 2\}, j \in \{0, 1\} \}$$

$$\underline{\underline{P_B}} = \{ (i, j) \mid i \in \{1, 2\}, j \in \{0, 1, 2\} \}$$

Example continued:

For each $p = (i, j)$,

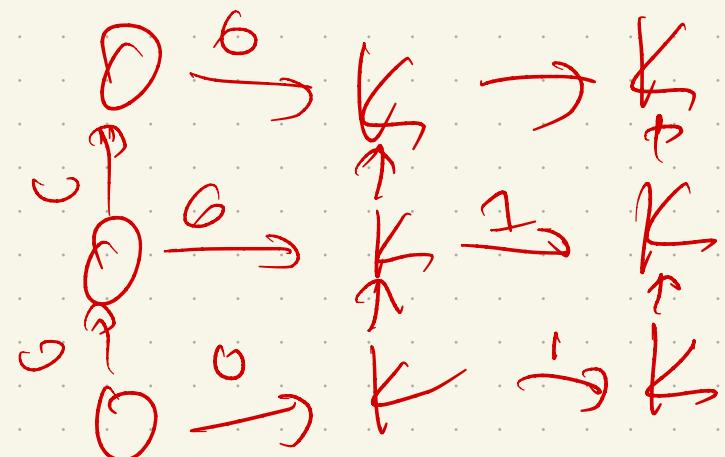
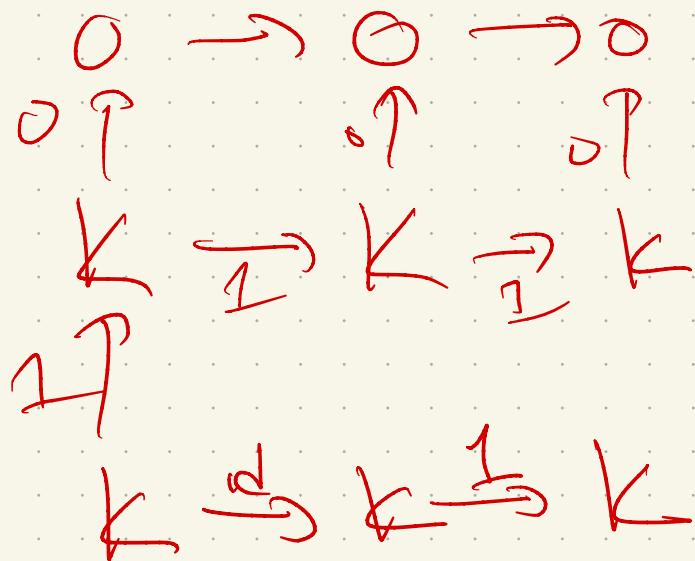
$$\underline{A(p)} = \begin{cases} K & \text{if } p \in R_A \\ O & \text{if not} \end{cases}$$

$$\underline{B(p)} = \begin{cases} K & \text{if } p \in R_B \\ O & \text{if not} \end{cases}$$

& all maps either

O or 1

Let $M = A \oplus B$



dimension grad here:

$$0 \quad k \quad k$$

$$k \quad k^2 \quad k^2 \quad \Rightarrow$$

$$k \quad k^2 \quad k^2$$

$$0 \quad 1 \quad 1$$

$$1 \quad 2 \quad 3$$

$$1 \quad 2 \quad 2$$

Rank invariant: $\forall p \leq q$, defined as

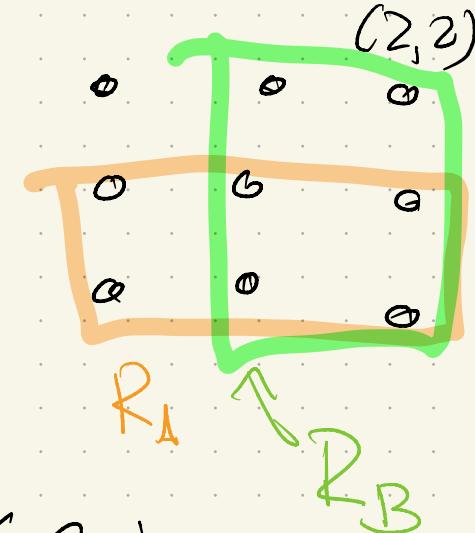
$$\text{rank}_n(p, q) = \text{rank}(p \rightarrow q)$$

Here, $\text{rank}(p \rightarrow q) =$

So, trying to compute:

$$\text{Fix } q = (2, 2)$$

$q \notin R_A$, but $q \in R_B$

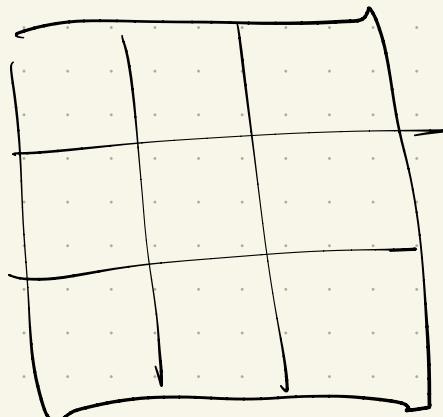


Then consider all P s.t. $P \leq q$:

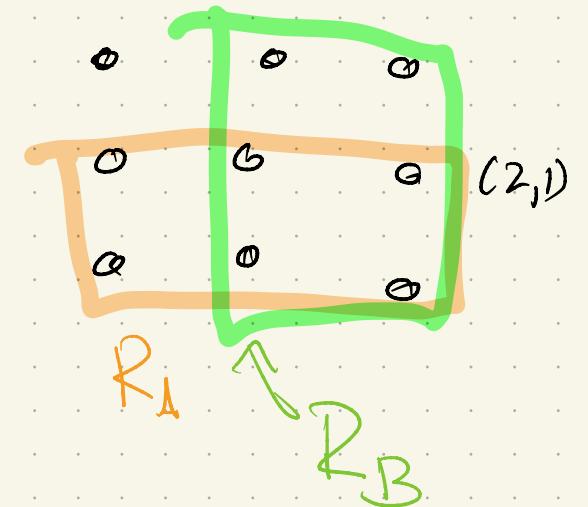
$$\text{rank}(P \rightarrow q) = \text{rank}_A(P \rightarrow q) + \text{rank}_B(P \rightarrow q)$$

here:

$$\text{so: } \text{rank}_q(P) =$$



Another: $f_x \ q = (2,1)$

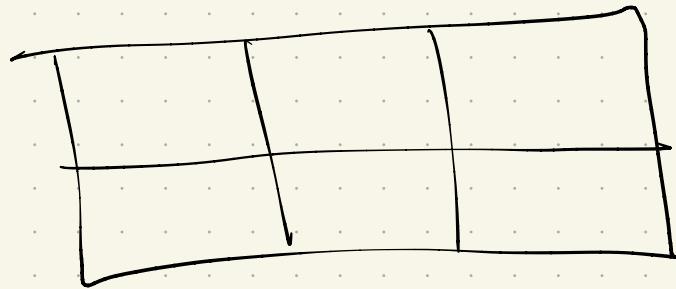


Now, all $P \leq q$!

$$\text{rank}_q(P) =$$

$$= \text{rank}_A(P \rightarrow S)$$

$$+ \text{rank}_B(P \rightarrow S)$$



This still (in a sense) measures
"homological features in P that
persist until q "

But: non-isomorphic modules can
share rank invariants
& can't have "good barcodes"

Let's unpack why ...

"Good" barcodes: what we mean

Consider a multiset of subsets of

$P \rightsquigarrow$ a "barcode"

in standard:

in multid:

Say it is good if $\forall x \leq y$,

$$\text{Rank}(M_x \rightarrow M_y) = |\{S \in B \mid x, y \in S\}|$$

↳ # elements w/ $x \neq y$

2D modules are

not good:

An example:

Suppose B is
a good barcode:

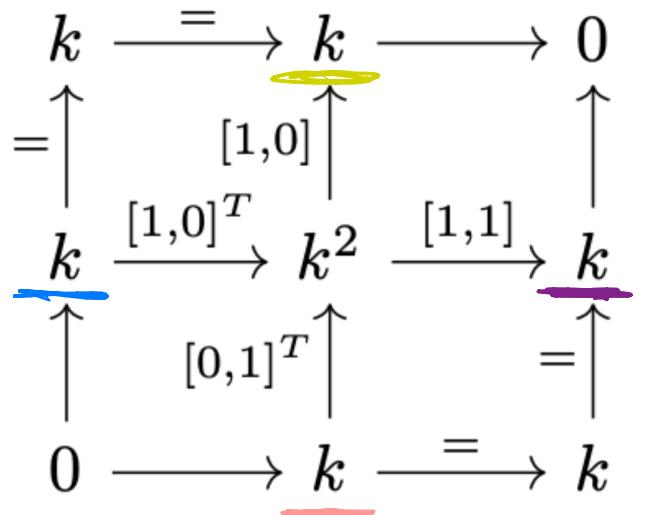
Know:

$$\text{Rank}(\underline{M_{(0,1)}} \xrightarrow{\quad} \underline{M_{(2,1)}}) = \text{Rank}(\underline{M_{(0,1)}} \xrightarrow{\quad} \underline{M_{(1,2)}})$$
$$= \text{Rank}(\underline{M_{(1,0)}} \xrightarrow{\quad} \underline{M_{(2,1)}}) =$$

What must B contain?

Bochner-Lesnick 2023

$$\begin{array}{ccccccc} k & \xrightarrow{=} & k & \longrightarrow & 0 \\ \uparrow & & [1,0] \uparrow & & \uparrow \\ k & \xrightarrow{[1,0]^T} & k^2 & \xrightarrow{[1,1]} & k \\ \uparrow & [0,1]^T \uparrow & & & \uparrow \\ 0 & \longrightarrow & k & \xrightarrow{=} & k \end{array}$$



Need

$$\begin{aligned}
 & \underline{(0,1)}, \underline{(2,1)} \in \mathcal{I} \\
 & \underline{(0,1)}, (1,2) \in \mathcal{J} \\
 & (1,0), \underline{(2,1)} \in \mathcal{K}
 \end{aligned}$$

$$+ \quad \mathcal{I}, \mathcal{J}, \mathcal{K} \in \mathcal{B}$$

$$\text{But } \dim \underline{M_{0,1}} = \dim \underline{M_{2,1}} = 1$$



Contradiction!

What is $\text{Rank}(\underline{M_{(0,1)}} \rightarrow \underline{M_{(1,2)}})$?

Note: This is a module whose rank invariant is not equal to the rank invariant of any interval-decomposable module.

But: is the difference between rank invariants of 2 interval decomposable modules!

$$\text{Rk} \left(\begin{array}{ccccc} k & \xrightarrow{\text{id}} & k & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \text{id} & & [1 \ 0] & & \\ k & \xrightarrow{[1 \ 0]} & k^2 & \xrightarrow{[1 \ 1]} & k \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & k & \xrightarrow{\text{id}} & k \end{array} \right) = \text{Rk} \left(\begin{array}{c} \text{blue shaded grid} \\ \oplus \\ \text{blue shaded grid} \\ \oplus \\ \text{blue shaded grid} \end{array} \right) - \text{Rk} \left(\begin{array}{c} \text{red shaded grid} \end{array} \right)$$

Fig. 2 The indecomposable module M on the left-hand side does not have the same rank invariant as any direct sum of interval modules on the 3×3 grid. However, $\text{Rk } M$ is equal to the difference between the rank invariants of two direct sums of interval modules, as shown on the right-hand side. Blue is for intervals counted positively in the decomposition, while red is for intervals counted negatively (Color figure online)

This can be useful

↳ but not unique:

③

Bigraded Betti Numbers

Essentially, for each $(a, b) \in \mathbb{N}^2$
record new generators or relations
that appear at (a, b) .

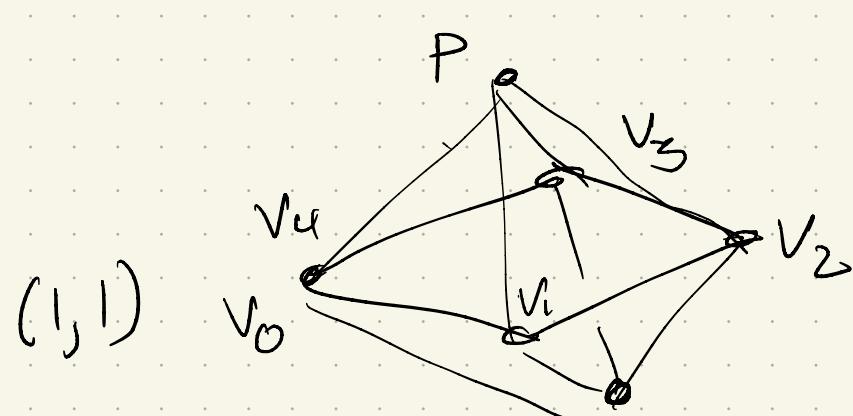
$\beta_{0,(a,b)} = \# \text{ of new homology classes born}$

$\beta_{1,(a,b)} = \# \text{ of independent relations (ie deaths) at } (a,b)$

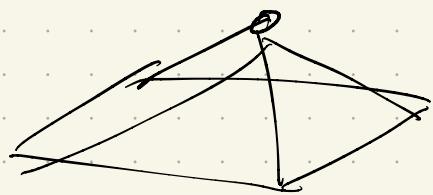
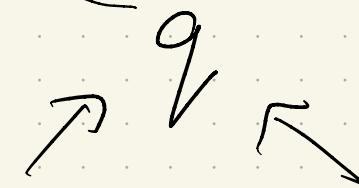
$\beta_{2,(a,b)} = \# \text{ of relations among relations at } (a,b)$

etc.

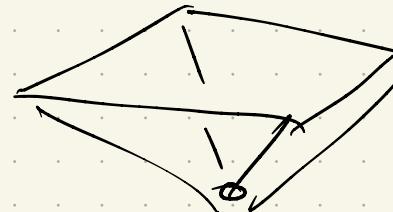
What is β_2 ? More complex relations!



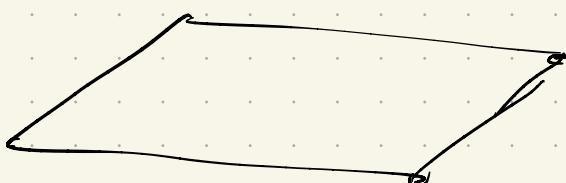
$$H_1 + \beta_i^0 \circ$$



$$(1,0)$$



$$(0,1)$$



$$(0,0)$$

New trends Connect this to something called the Möbius inversion.

Patel 2018

Patel & Skraba '23

- USES INVARIANTS from classical combinatorics

In addition, algebraic generalizations are being developed

Kim & Memoli 2023

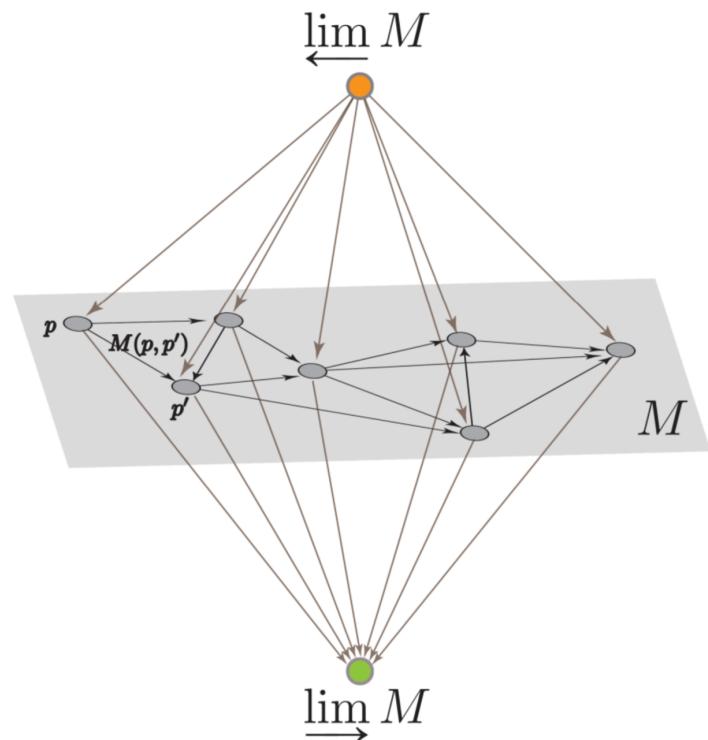


Figure 4. The notions of limit and colimit of a diagram of vector spaces are used in a fundamental way to generalize the notion of rank invariant and persistence diagram.

Vizualization: Rivet

Lesnick-Wright 2015

Interactive tool for 2d filtrations.

Bigraded Betti numbers :

User selects a line $L \in \mathbb{R}^2$

Software displays the barcode

$$B(M^L)$$

- User can then drag the line, & barcode will update



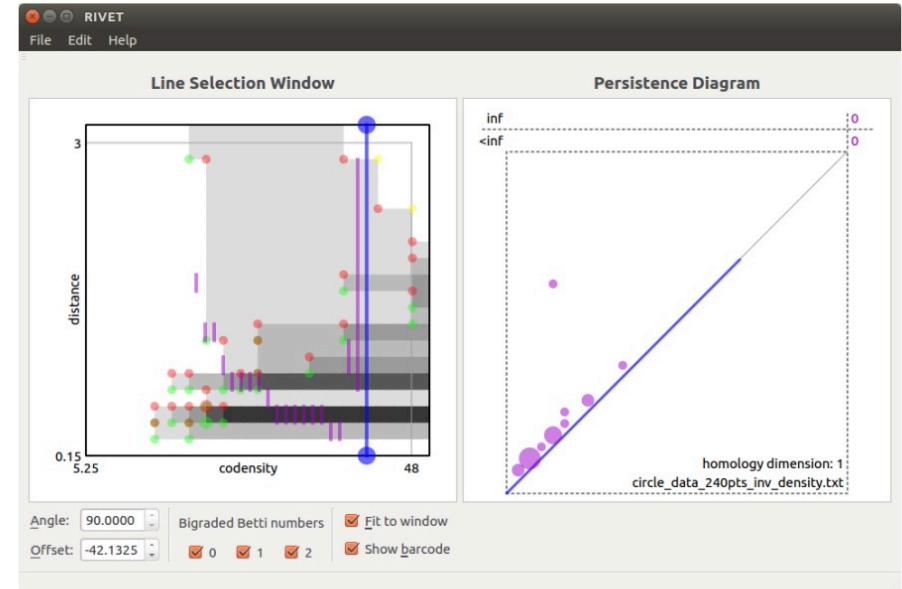
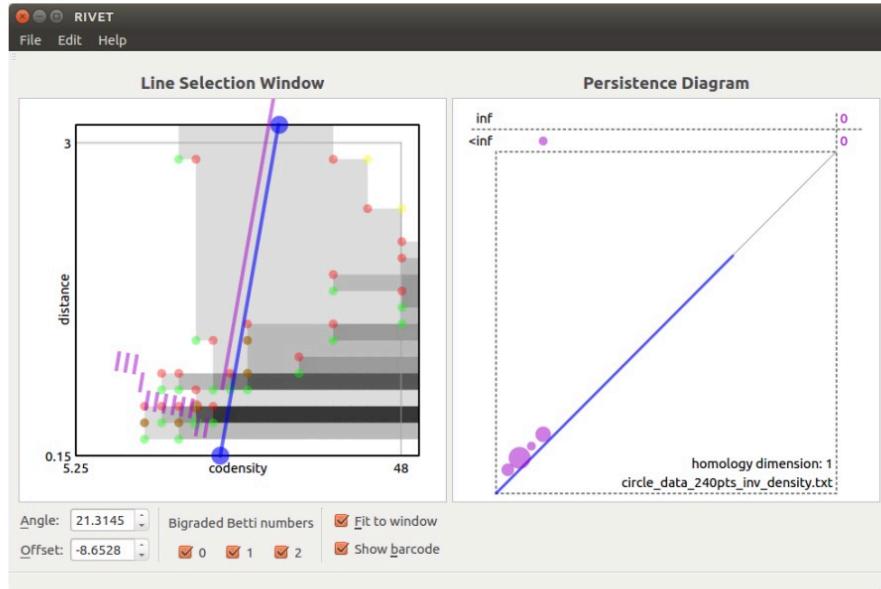
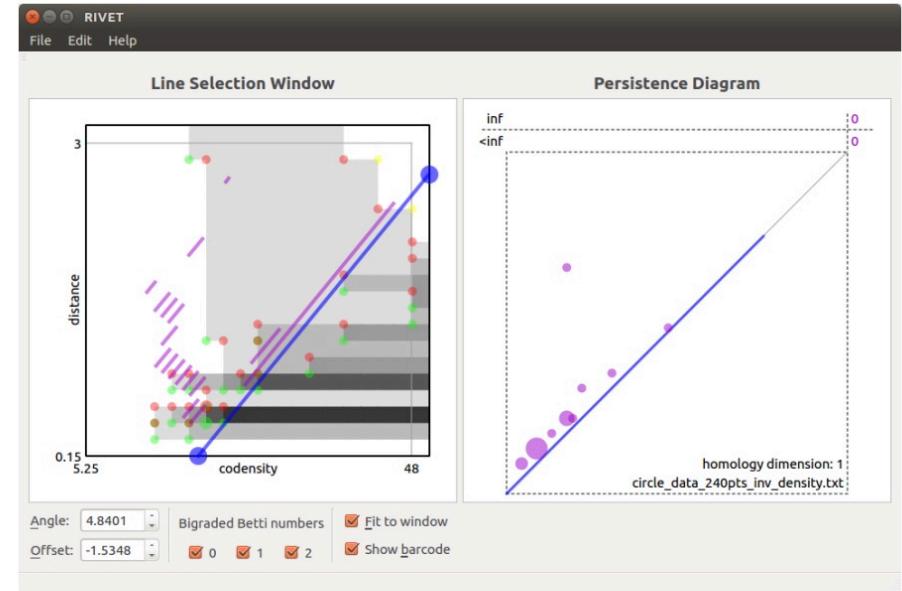
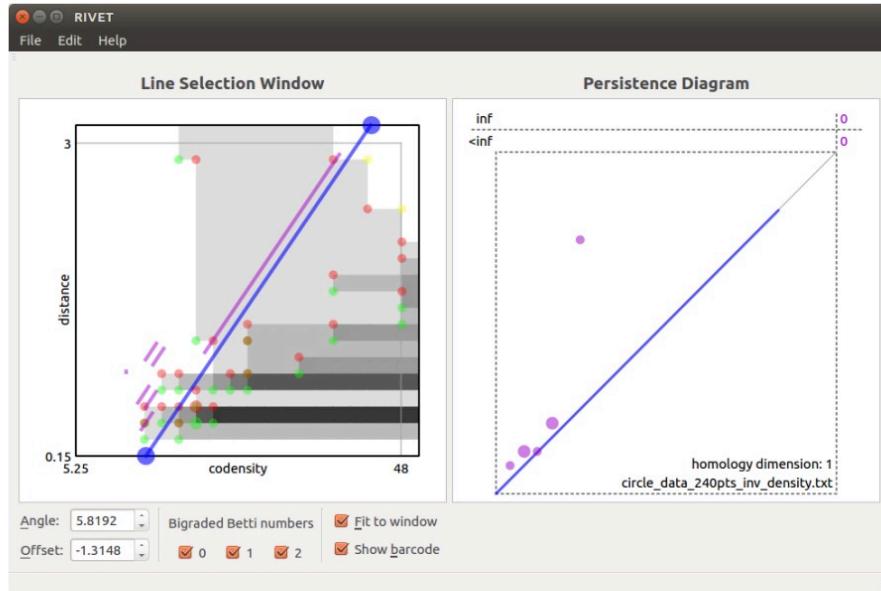


Figure 3: Screenshots of RIVET for a single choice of 2-D persistence module M and four different lines L . RIVET provides visualizations of the dimension of each vector space in M (greyscale shading); the 0th, 1st, and 2nd bi-graded Betti numbers of M (green, red, and yellow dots); and the barcodes of the 1-D slices M^L , for each L (in purple).