

TDA - Fall 2025

Morse-Smale
Complexes

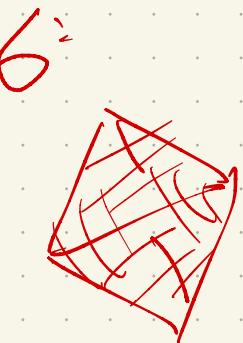


Recap

- Welcome back!
- Next assignment: project proposals
 - ↳ submit individually by Nov. 5
 - ↳ 2-3 pages: proposed topic, brief survey of known results, potential plan of attack + some half-baked ideas
- Projects will be in groups of 1-3 students
 - ↳ presentation (~15 minutes) last week for so of class
 - ↳ 10-15 page document by Dec. 14

A discrete Morse function f on a complex K is a function $f: K \rightarrow \mathbb{R}$ st.
for every p -simplex $\sigma \in K$

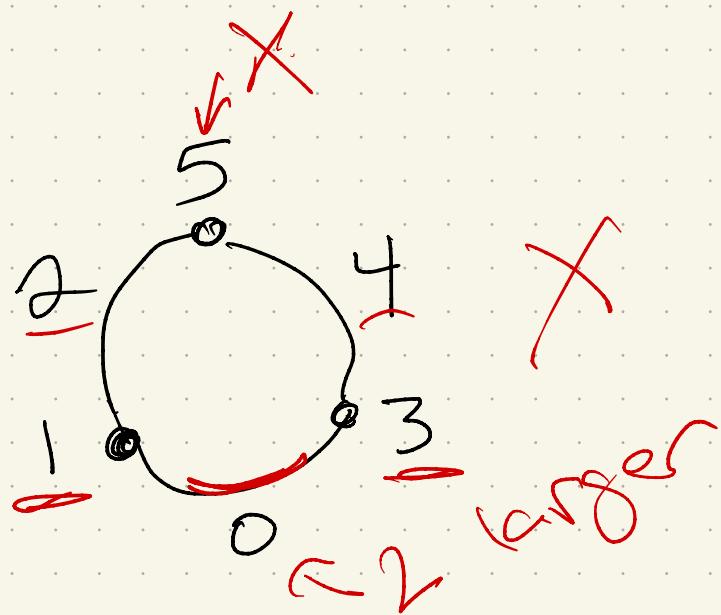
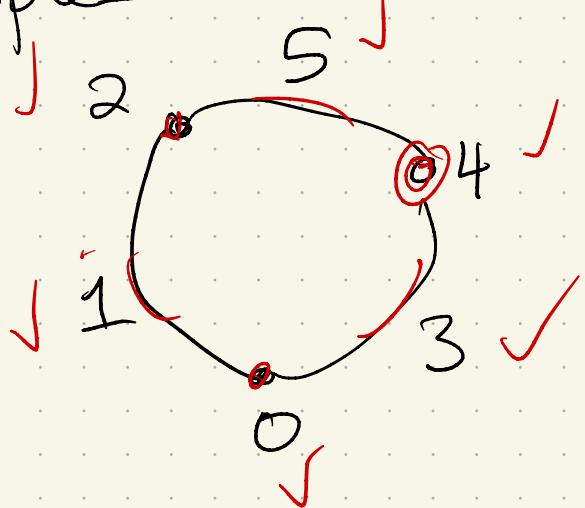
$$| \{ \tau^{(p+1)} < \sigma : f(\tau) \geq f(\sigma) \} | \leq 1$$



and

$$| \{ \tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma) \} | \leq 1$$

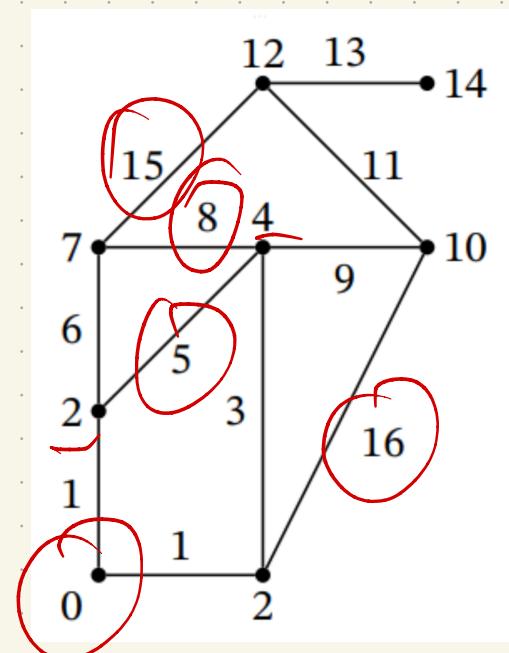
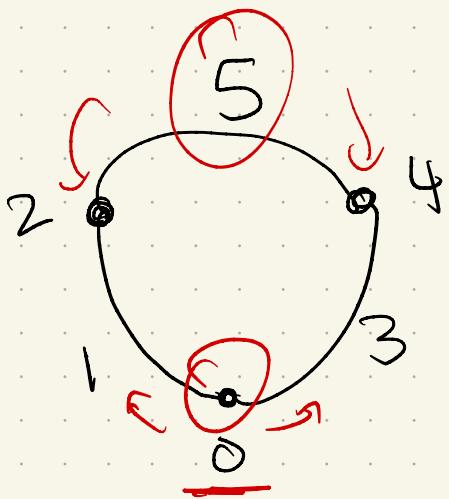
Examples: Yes/no?



Critical Simplices

A p -simplex is critical with respect to f if $|\{x^{(p-1)} \leq g : f(x) \geq f(g)\}| = 0$
and $|\{x^{(p+1)} \geq g : f(x) \leq f(g)\}| = 0$

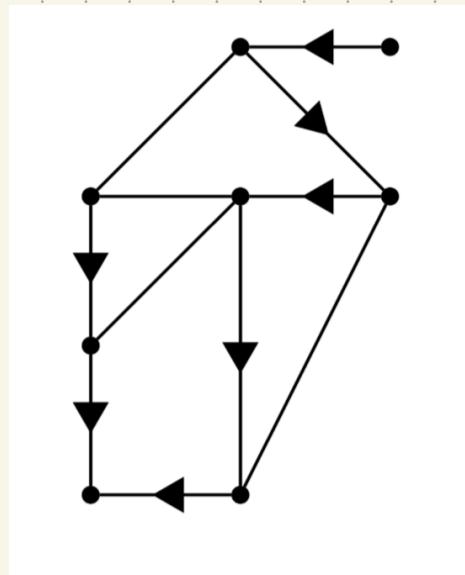
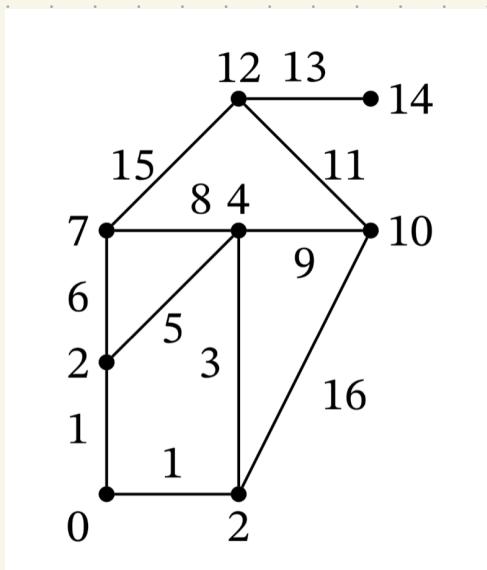
Example:



Regular points + discrete gradients

Any simplex that is not critical is regular, & will have one higher dim incident simplex with lower value or one lower dim simplex with higher

Pair these!



Exclusion lemma: Can't have both

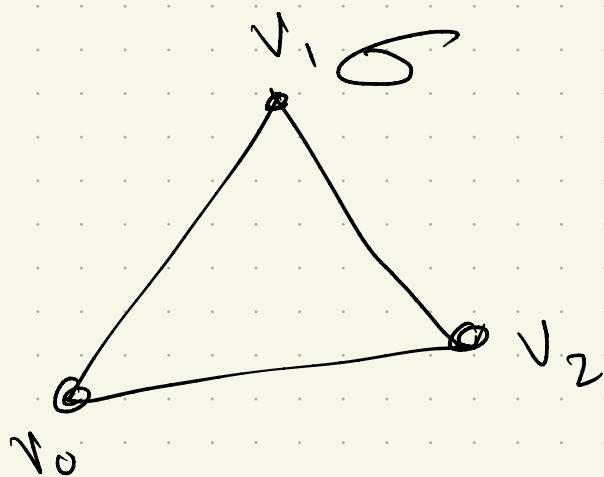
Proof: $\sigma = v_0 - v_p + \dots$ spps both:

$x = v_0 - v_p v_{p+1}, y = v_0, \dots, v_{p-1} + f(x) \leq f(\sigma) \leq f(y)$

Let $\tilde{\sigma} = v_0, \dots, v_{p-1}, v_{p+1} \Rightarrow f(y) < f(\tilde{\sigma})$

+ $f(\tilde{\sigma}) < f(x)$. But:

$f(x) \leq f(\sigma) \leq f(y) < f(\tilde{\sigma}) < f(y) \quad \checkmark$



Back to motivation

In Foreman's original work, goal was to identify a Simpler complex with same homology:

Let $M_p \subseteq C_p(K)$ be critical psimplices

Then \exists maps $\tilde{\delta}_p$ s.t. $\tilde{\delta}_{p+1} \circ \tilde{\delta}_p = 0$
with $M_d \xrightarrow{\tilde{\delta}_d} M_{d-1} \xrightarrow{\tilde{\delta}_{d-1}} \dots \xrightarrow{\tilde{\delta}_1} M_0$

s.t. $H_d(M, \tilde{\delta}) \cong H_d(K)$

(Uses CW complexes + homotopy equivalence)

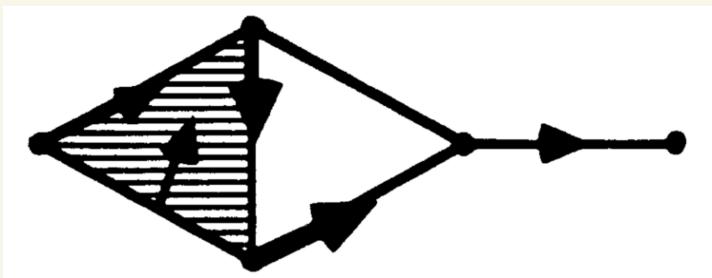
Theorem (Forman 1995)

Suppose $\sigma^{(p)}$ is a critical simplex with $f(\sigma) \in [a, b]$, & there are no other critical simplices in $[a, b]$. Then

$M_{\leq b}$ is homotopy equivalent to

$M_{\leq a} \cup e^{(p)}$, where $e^{(p)}$ is a p-cell glued to ' $M(c)$ ' along its boundary.

Can prove via collapses:

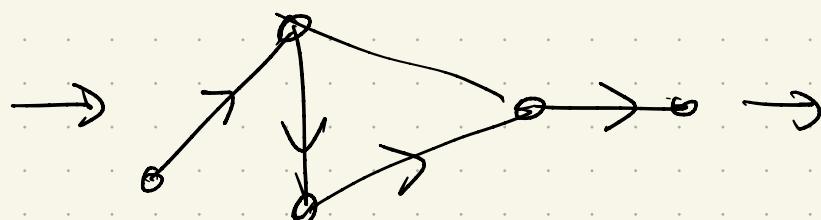
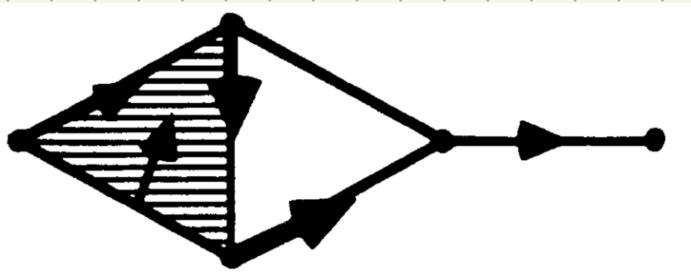


Why does this work?

Exclusion lemma!

Any pair is call an elementary collapse \rightarrow won't change topology.

No conflicts, so end with simpler complex, ie



Discrete Morse inequalites

Let $f: K \rightarrow \mathbb{R}$ be a discrete Morse function with m_i critical values in \dim_i , $i=0-d$.

Then: $B_i(K) \leq m_i + f_i$

and $\sum_{i=0}^d (-1)^i m_i = \chi(K)$

Euler characteristic: $\sum (-1)^i (\# \dim_i \text{ cells})$

Proof: induction on # of simplices

Base case:

$$\beta_0(K) = 1 \quad (\text{all others} = 0)$$

Inductive Step: Suppose true for any complex with $1 \leq j \leq l$ simplices.

Consider K with $l+1$ simplices.

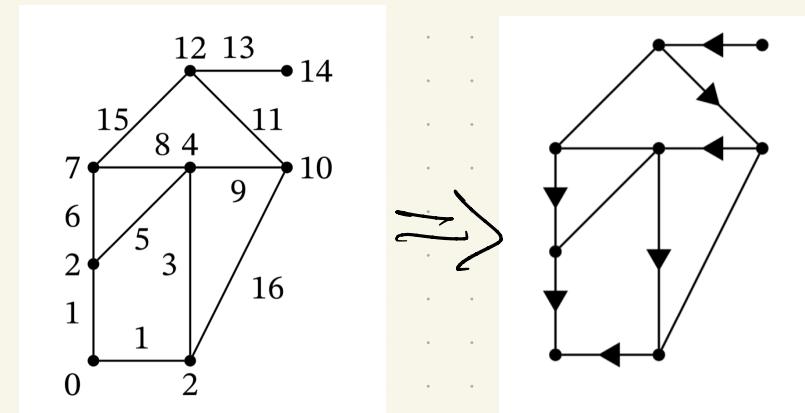
Let σ be maximum simplex of K ,
+ K' be $K - \sigma$.

σ regular:

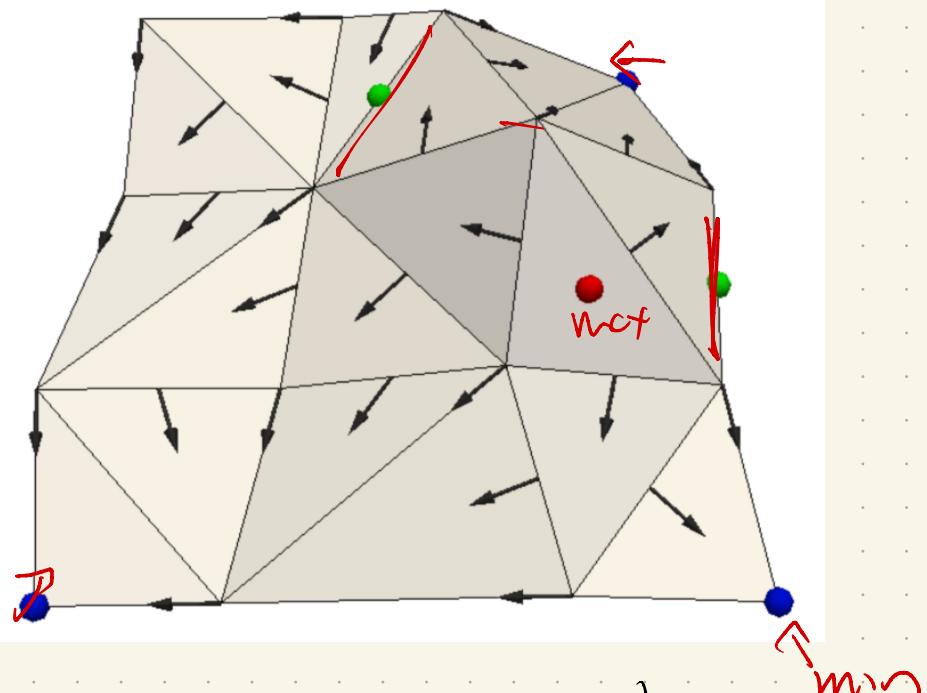
σ critical:

Result: Discrete Gradient Vector field:

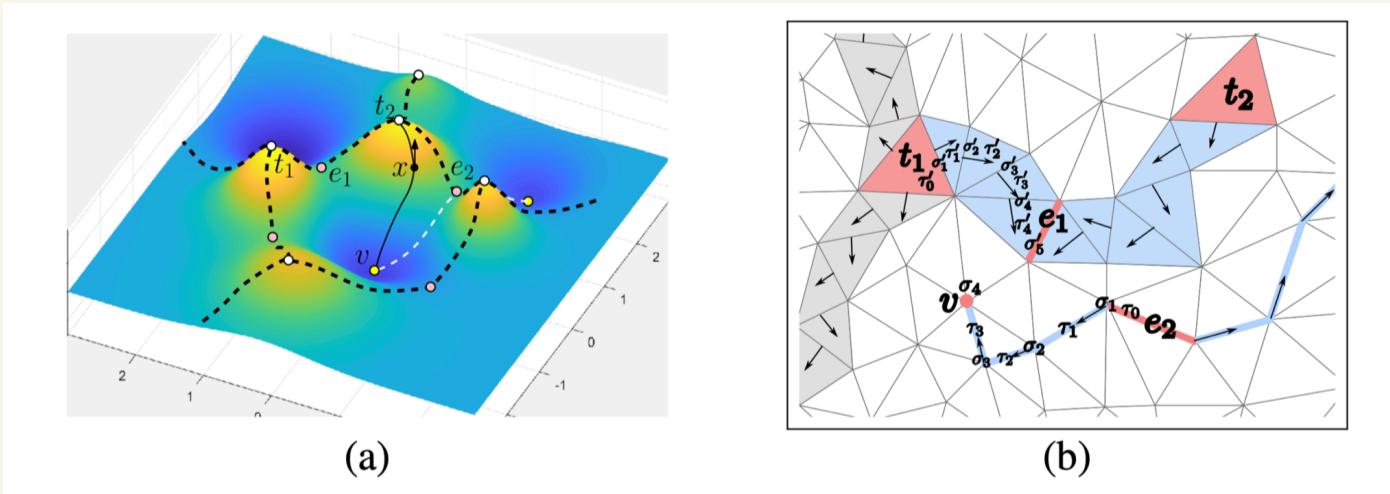
Draw arrow from
6 to higher dim
nbr with lower
value



- Each simplex "flows" to at most one nbr
- Flow lines go down
- Flow vanishes at ^{mn} critical simplices
 - Why? not paired



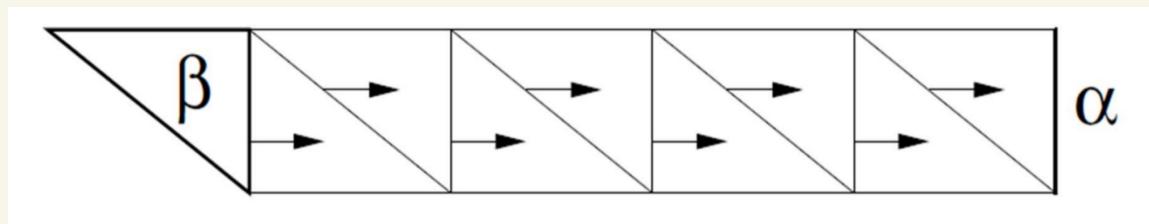
Back to V-paths: either face-edge
or edge-vertex



continuous
flow lines

discretized
V-paths

i.e.



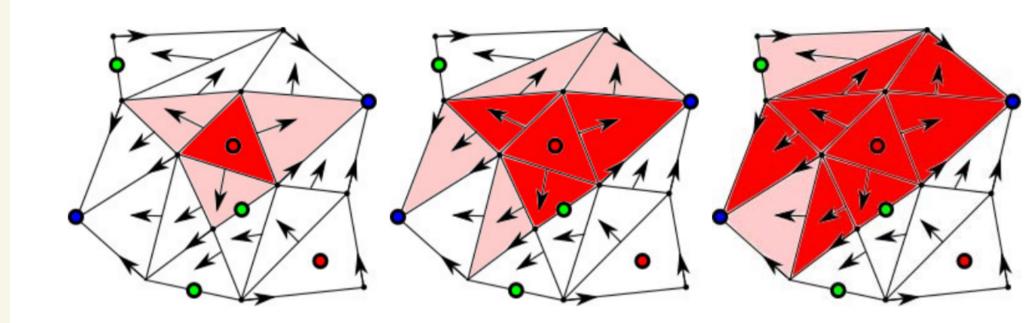
One difference: discrete flow goes down
(not up like continuous)

So, in discrete setting: For critical edge e :

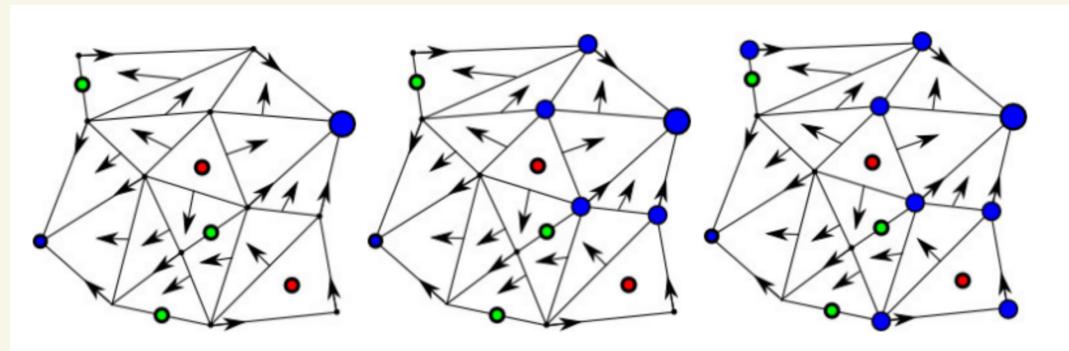
- Stable manifold is union of edge-triangle gradient paths
- Unstable manifold is union of vertex edge gradient paths

ie:

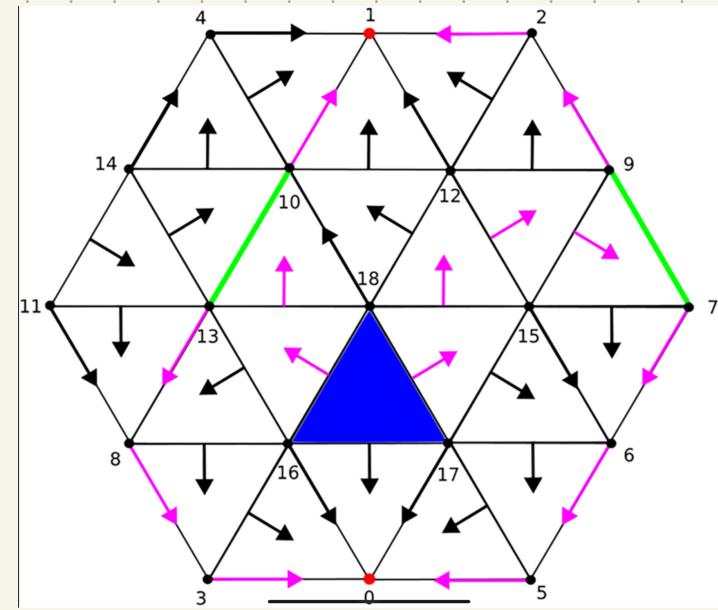
descending:



ascending:



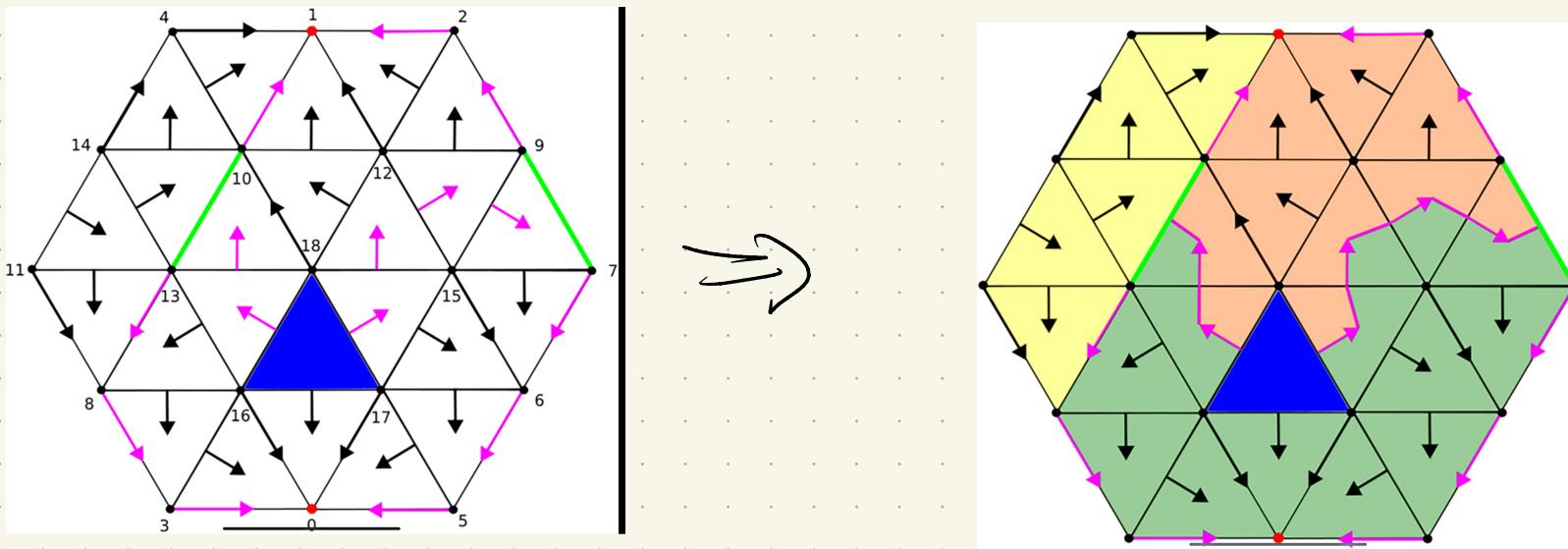
Separatrices:
V-paths between
critical simplices
(marked pink)



How to find?

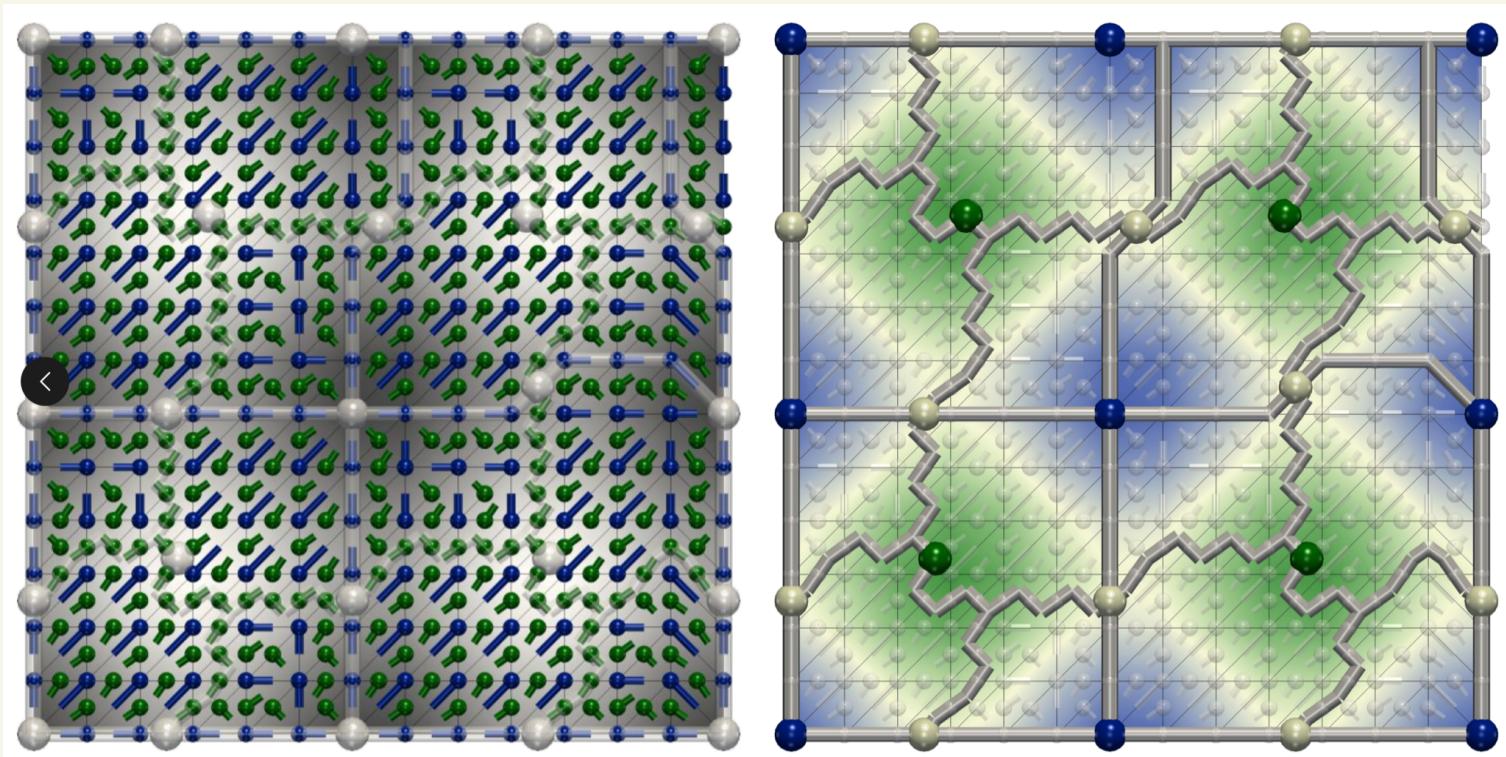
- easy starting from critical edges
- From critical faces: try all options

Discrete Morse-Smale Complex



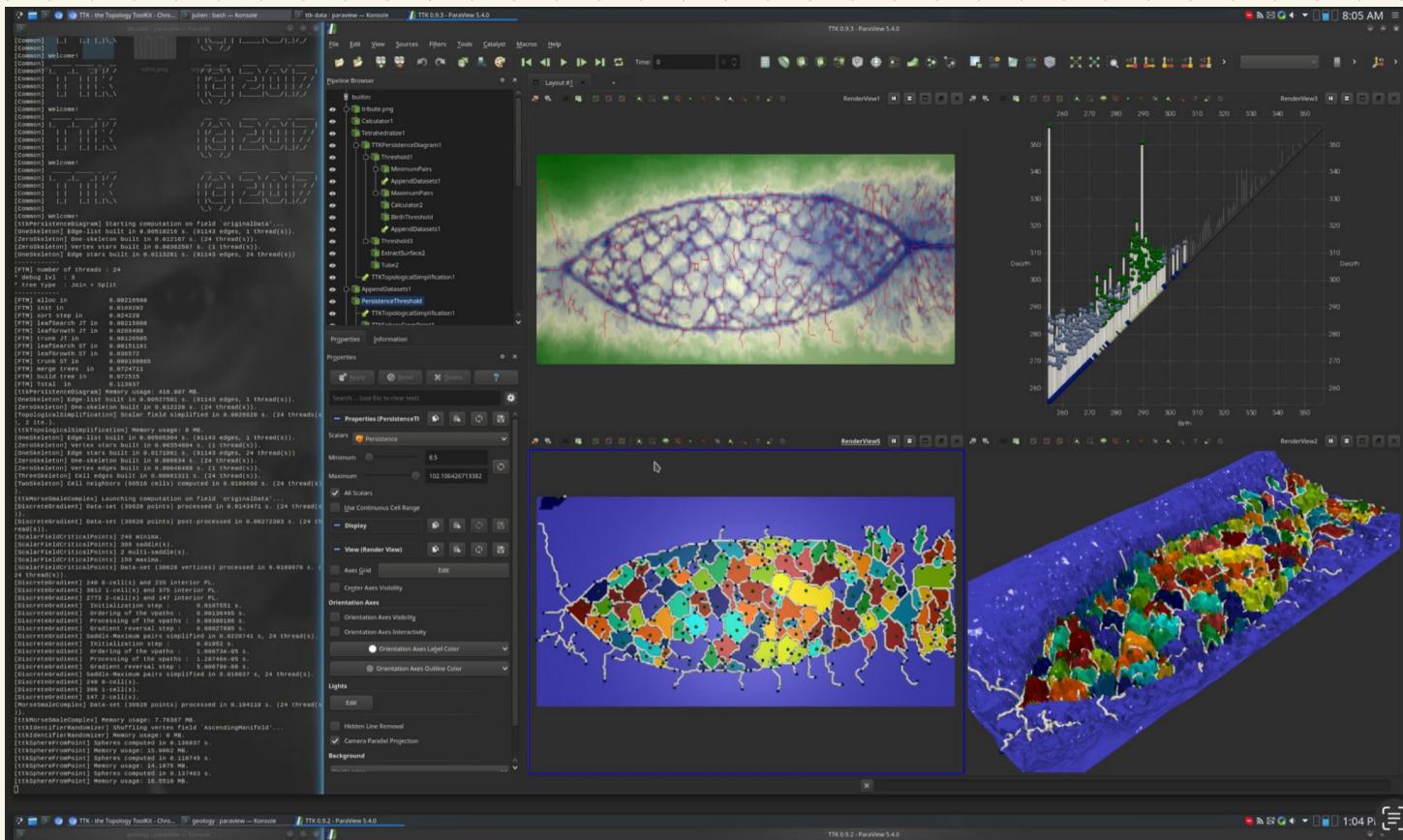
Algorithm: Collect ascending &
descending manifolds of each
critical simplex

Result: Partition of Complex



In TTK

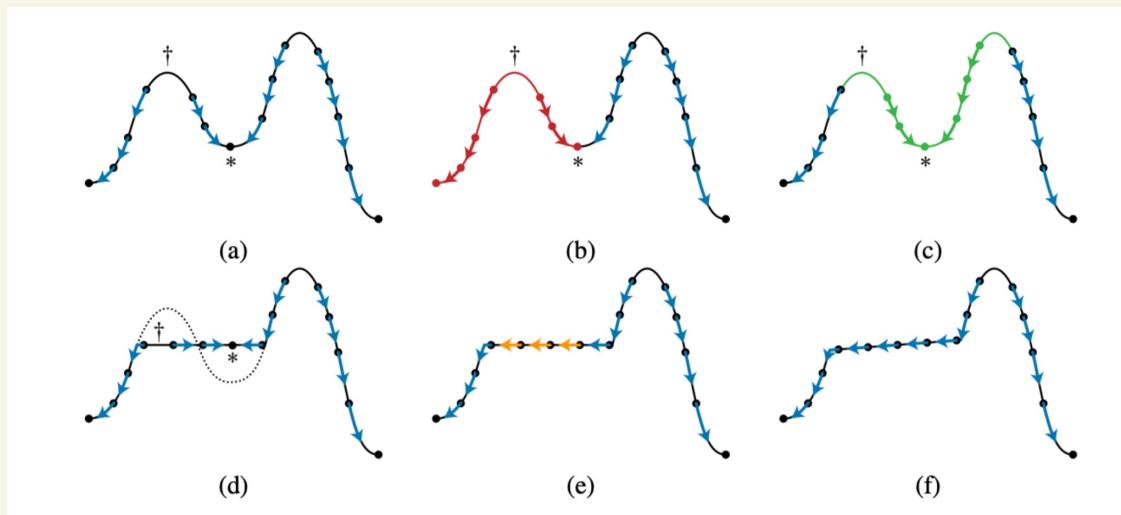
Methods to also apply this to
greyscale data → partitioning +
skeletonizing



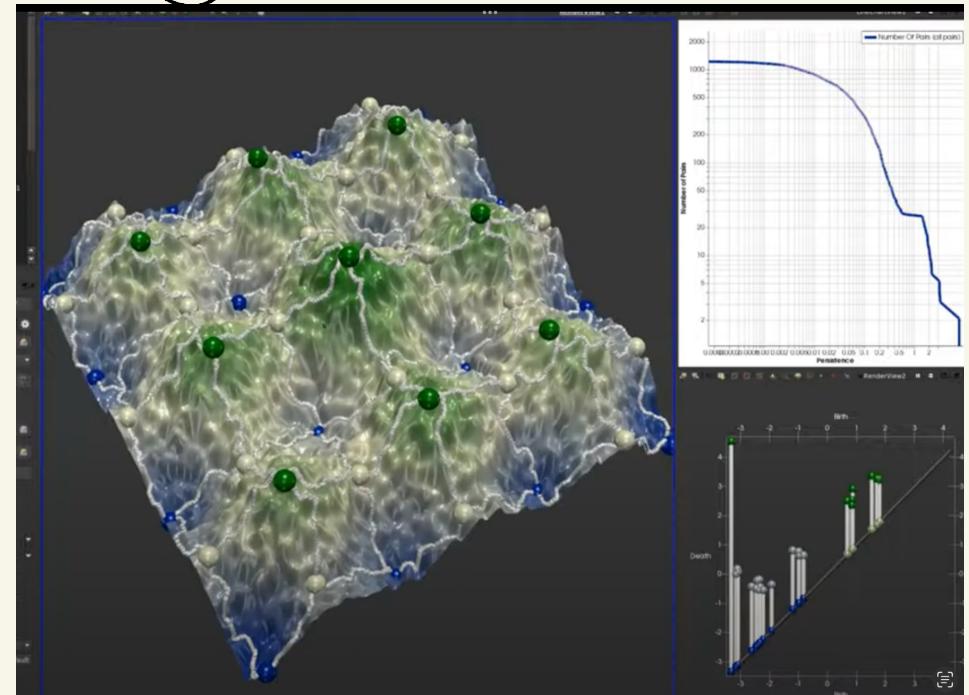
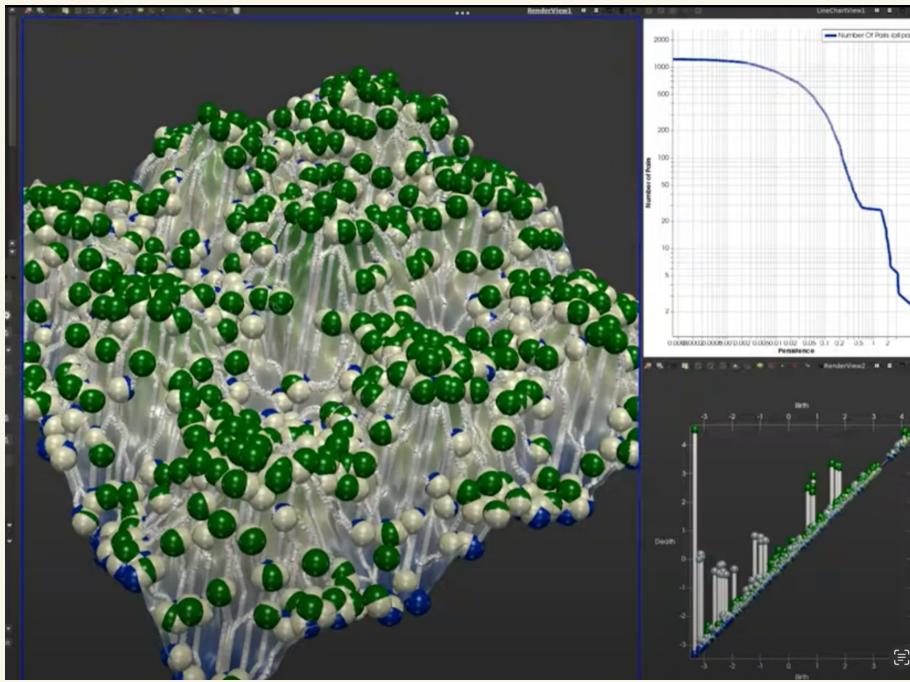
Persistence Simplifications

Bauer, Lange, Mordzkiy 2010 proposed a method to simplify 1+2 manifolds using these ideas.

Focus on critical points!

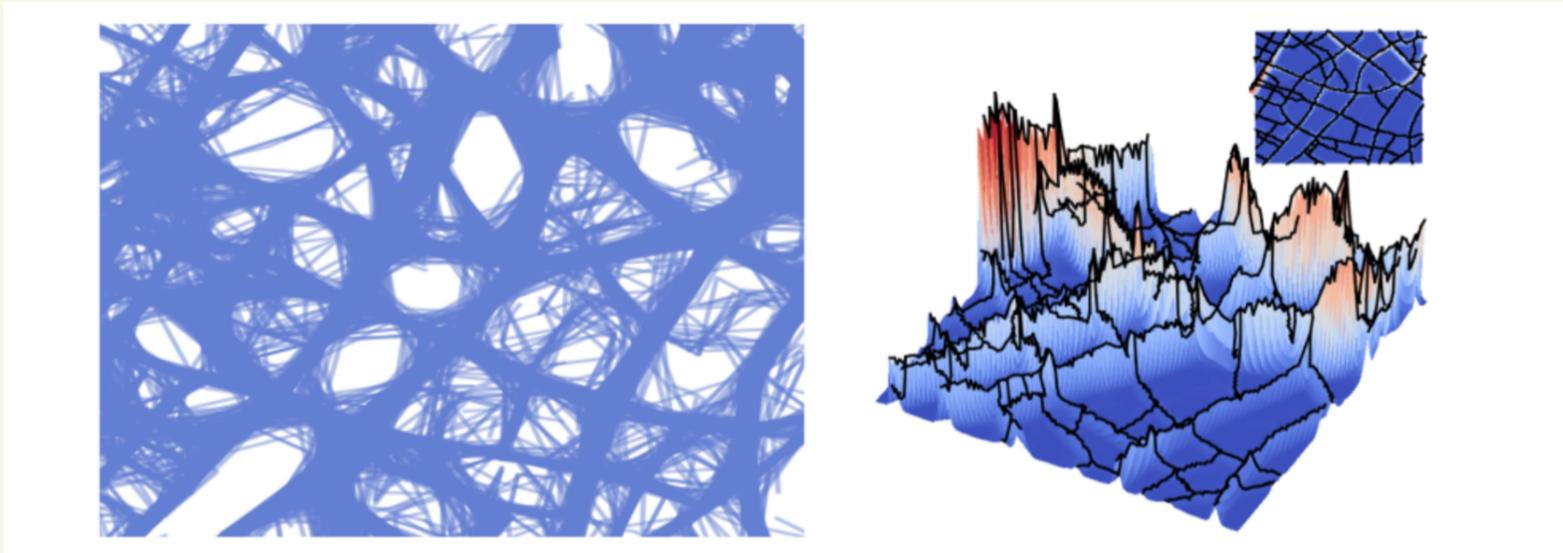


In achor



Uses: Road network analysis

[Wang, Wang & Li 2018]

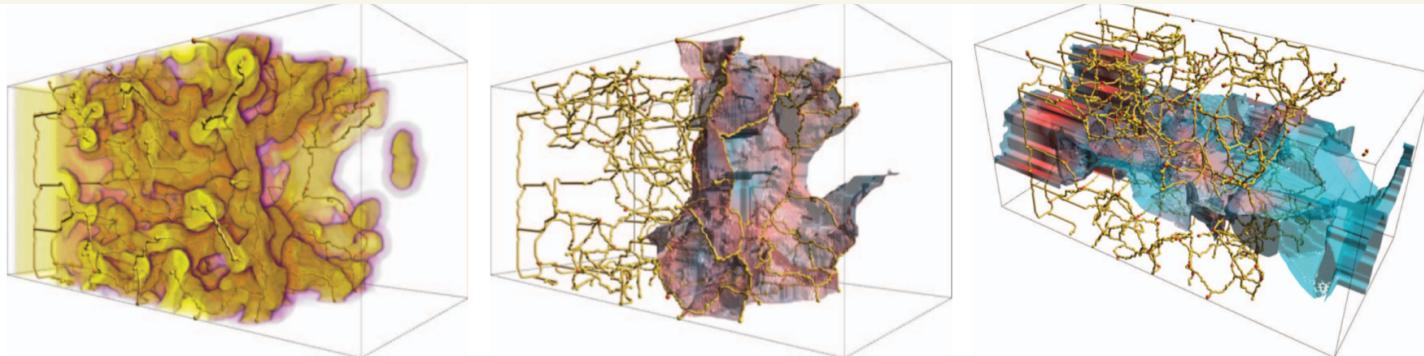


Goal: Reconstruct map from GPS data

• Analyze traffic patterns

Another: Analyze porous solids

Gyulassy et al 2012



Porous Solid (left), $115 \times 115 \times 237$, float: A signed distance field from the material boundary represents a porous copper foam. This time-step ends a sequence where a micro-meteoroid impacts the foam from the right. We identify core structure, and additionally we extract a surface representing the impact crater (middle), and shear surface through the weakest points connecting the top half of the material to the bottom (right). Impact Crater: (middle) Simplify to 5.0% total persistence using the threshold values found by Gyulassy et al. [41]. Select 2-saddle-maximum arcs in the range $[-2.0, 30]$ and incident nodes for rendering. Select the lowest minimum (located outside the crater), and select neighboring 1-saddles (using two incidence selectors). Extract ascending 2-manifolds from these saddles, smooth, and apply a blue (low) to red (high) ramp transfer function with transparency. Fracture Surface: (right) Use the same hierarchy and arcs/nodes for rendering as in (Middle). Select all 2-saddle-maximum arcs that cross y-coordinate = 57.5. Select 2-saddles from their incident nodes. Extract descending 2-manifolds from these, smooth, and apply a blue (low) to red (high) transfer function with transparency.

Also see Delgado-Friedricks et al 2015

& Robins et al 2011

Also in textbook:

- Detailed algorithms for persistence cancellation

Recommendation if you want basics:

Discrete Morse theory by Scoville

Otherwise, go play with
TTK jets!

