Algorithms - Spring '25

MSTs

Recap

Minimum Spanning Trees

undirected

Goal: Given a weighted Graph G, $w: E \to \mathbb{R}^+$ the weight function, find a spanning tree T of G that minimizes:

$$w(T) = \sum_{e \in T} w(e)$$

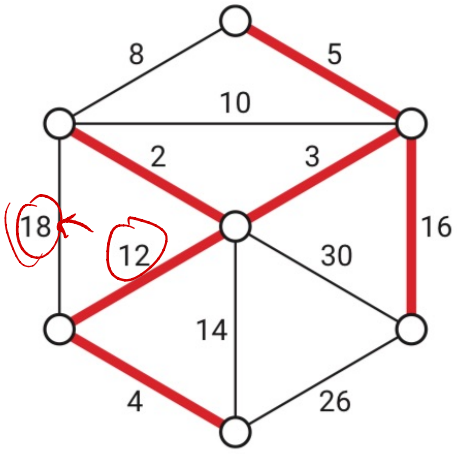


Figure 7.1. A weighted graph and its minimum spanning tree.

The magic truth of MSTs:

You can be SUPER greedy!

Almost any natural idea will work!

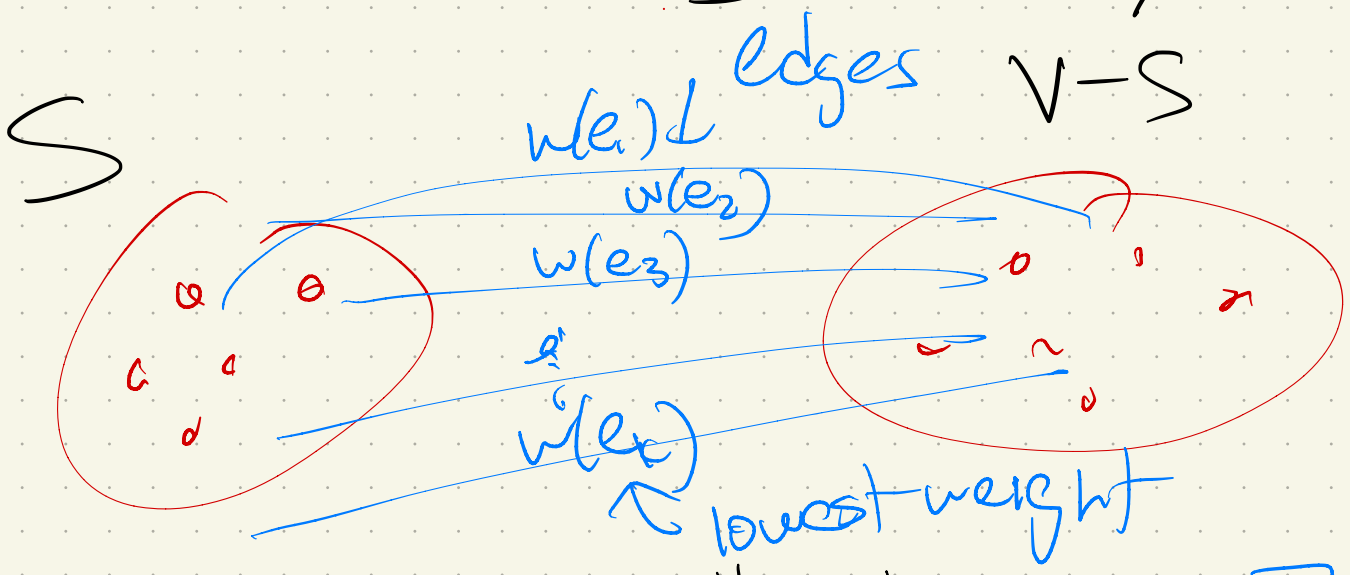
This is highly unusual, & there's a reason for it:

these are a (rare) example of something called a matroid.

(Way beyond this class...)

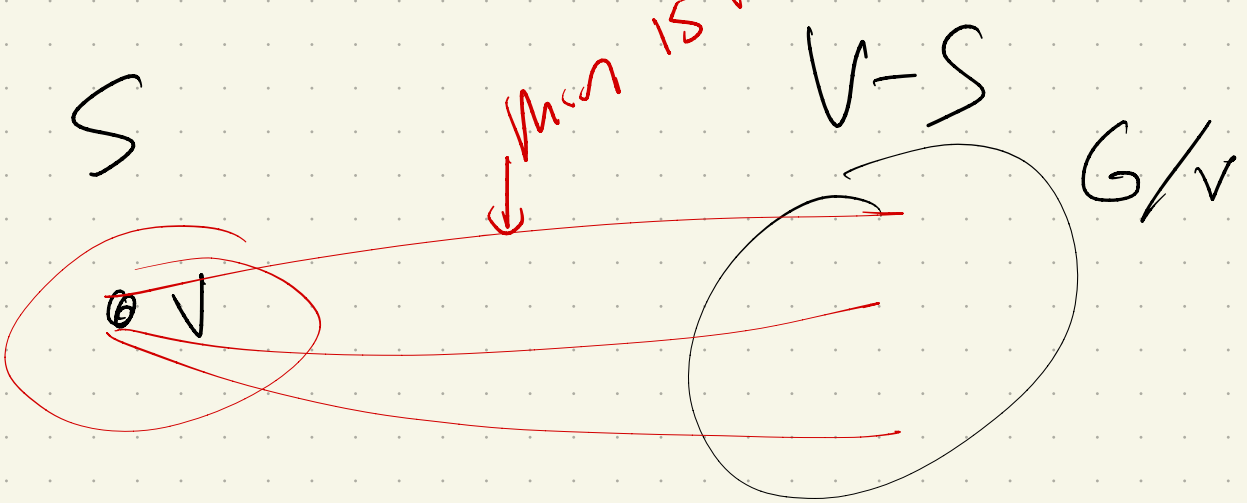
Key property:

Consider breaking G into two sets: S and V/S



The MST will always contain the lowest edge connecting the two sides.

No matter what!



Generic Algorithm:

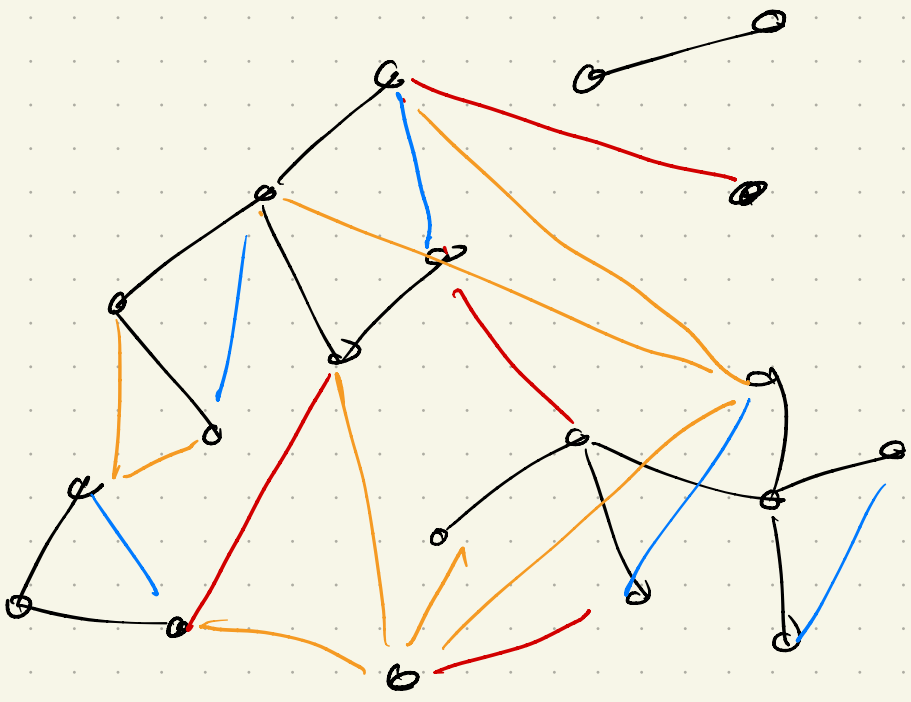
Build a forest: an acyclic subgraph.

Dfn: An edge is useless if it connects 2 endpts in same component of F

also edges that are neither.

An edge is safe if it is minimum edge from some component of F to another.

F:



So idea:

Add safe edges until you get a tree

If everything isn't connected, must have some safe edge.

Why?

Add it & recurse.

We'll see 3 ways:

1. Find all safe edges. Add them + recurse.

2. Keep a single connected component. At each iteration, add 1 safe edge.

3. Sort edges + loop through them. If edge is safe, add it.

First one: (1926-ish)

BORŮVKA: Add ***ALL*** the safe edges and recurse.

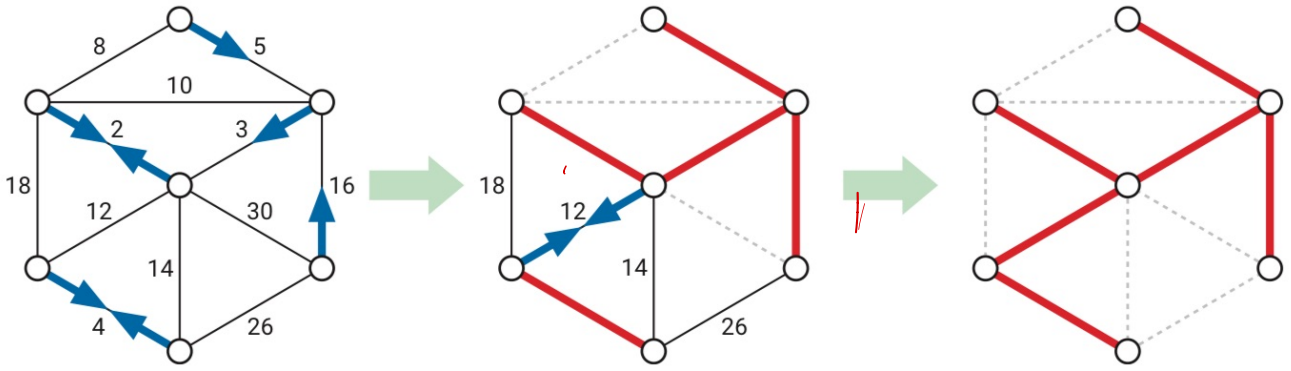


Figure 7.3. Borůvka's algorithm run on the example graph. Thick red edges are in F; dashed edges are useless. Arrows point along each component's safe edge. The algorithm ends after just two iterations.

So we need to:

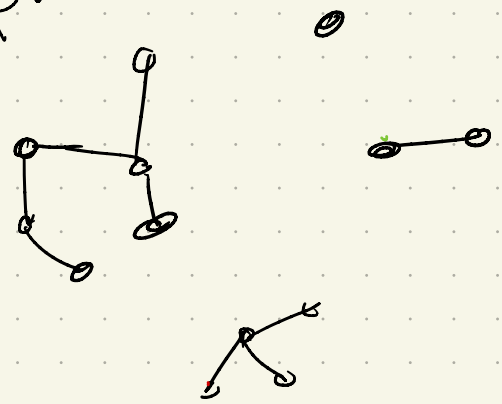
While more than 1 component:

- Track components
- Find all safe edges
- Add them

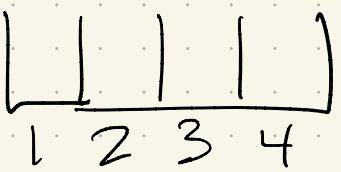
More formally:

```
BORŮVKA(V, E):
  F = (V, ∅)
  count ← COUNTANDLABEL(F)
  while count > 1
      ADDALLSAFEEDGES(E, F, count)
      count ← COUNTANDLABEL(F)
  return F
```

Graph



Safe:




```
ADDALLSAFEEDGES(E, F, count):
  for i ← 1 to count
      safe[i] ← NULL
  for each edge uv ∈ E
      if comp(u) ≠ comp(v)
          if safe[comp(u)] = NULL or w(uv) < w(safe[comp(u)])
              safe[comp(u)] ← uv
          if safe[comp(v)] = NULL or w(uv) < w(safe[comp(v)])
              safe[comp(v)] ← uv
  for i ← 1 to count
      add safe[i] to F
```

Uses WFS-variant from Ch 5:

```
COUNTANDLABEL(G):
  count ← 0
  for all vertices v
      unmark v
  for all vertices v
      if v is unmarked
          count ← count + 1
          LABELONE(v, count)
  return count
```

```
⟨⟨Label one component⟩⟩
LABELONE(v, count):
  while the bag is not empty
      take v from the bag
      if v is unmarked
          mark v
          comp(v) ← count
          for each edge vw
              put w into the bag
```

Correctness:

- MST must have any safe edge
- We keep computing safe edges & adding
- Stop when # connected components = 1

$\Rightarrow$ Have the MST!

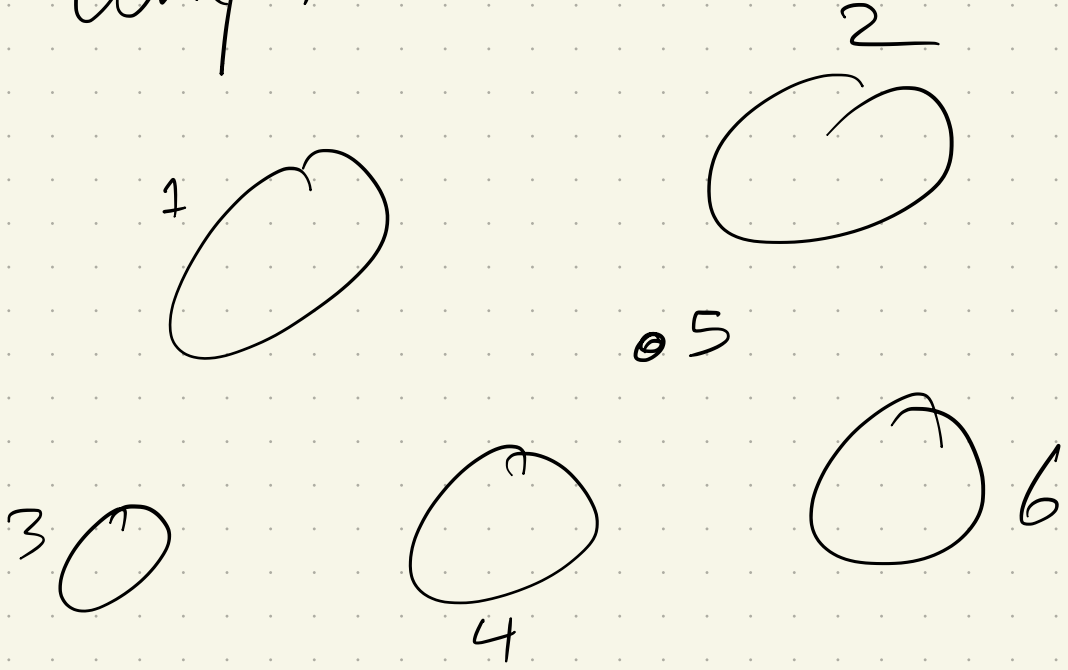
Run time:

A bit trickier!

Depends on how many safe edges we get.

Claim: There are at least $\#\text{components}/2$ safe edges each time.

Why?



So: runtime:

```
AddAllSafeEdges(E, F, count):
  for i ← 1 to count
      safe[i] ← Null
  for each edge uv ∈ E
      if comp(u) ≠ comp(v)
          if safe[comp(u)] = Null or w(uv) < w(safe[comp(u)])
              safe[comp(u)] ← uv
          if safe[comp(v)] = Null or w(uv) < w(safe[comp(v)])
              safe[comp(v)] ← uv
  for i ← 1 to count
      add safe[i] to F
```

↑ Looks at each vertex + edge in worst case:

```
Borůvka(V, E):
  F = (V, ∅)
  count ← CountAndLabel(F)
  while count > 1
      AddAllSafeEdges(E, F, count)
      count ← CountAndLabel(F)
  return F
```

→ BFS/DFS on tree:

How many iterations?

Prim's algorithm:

(really Jarník, we think)

Keep one spanning sub tree.

While $|T| \neq V$:

add next safe edge

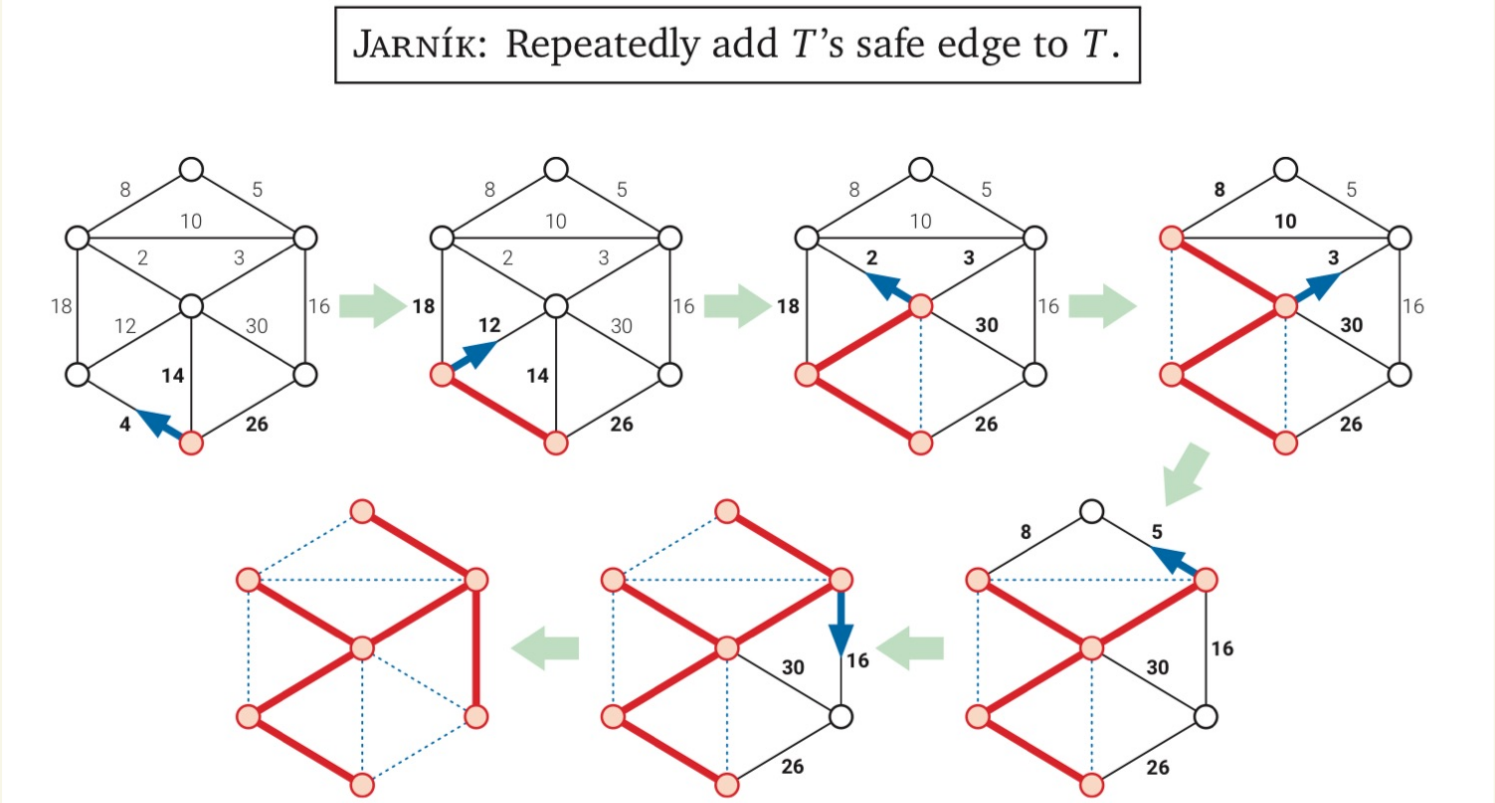


Figure 7.4. Jarník's algorithm run on the example graph, starting with the bottom vertex. At each stage, thick red edges are in T, an arrow points along T's safe edge; and dashed edges are useless.

Implementation:

From all edges going from $V(T)$ to $V(G)-V(T)$, add safe one.

↳ how??

Q: Which data structure?

runtime?

Runtime:

Aside:
Book goes over alternative —
uses a cooler data structure.

Comparison to Boruvka:
Faster (worst case), unless $E = O(V)$
But practically?

Kruskal's Algorithm:

KRUSKAL: Scan all edges by increasing weight; if an edge is safe, add it to F.

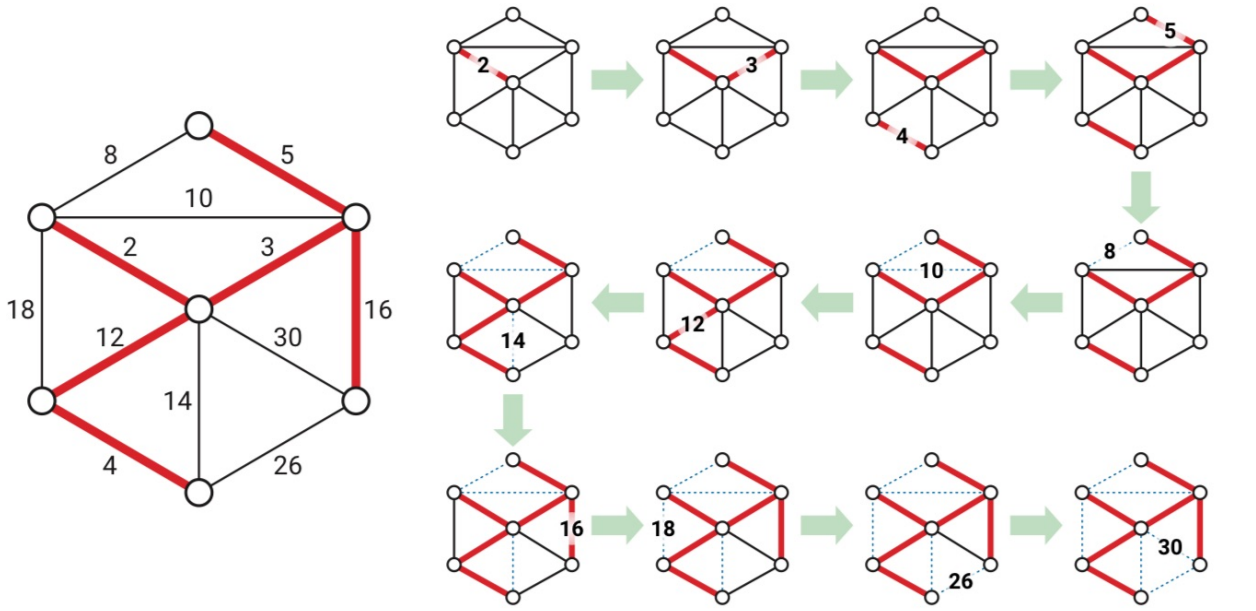


Figure 7.6. Kruskal's algorithm run on the example graph. Thick red edges are in F; thin dashed edges are useless.

How to implement?

Algorithm :

```
KRUSKAL(V, E):
  sort E by increasing weight
  F ← (V, ∅)
  for each vertex v ∈ V
      MAKESET(v)
  for i ← 1 to |E|
      uv ← ith lightest edge in E
      if FIND(u) ≠ FIND(v)
          UNION(u, v)
          add uv to F
  return F
```

Data structure : Union find

- MAKESET(v) — Create a set containing only the vertex v.
- FIND(v) — Return an identifier unique to the set containing v.
- UNION(u, v) — Replace the sets containing u and v with their union. (This operation decreases the number of sets.)

Comparison:

- Boruvka: $O(E \log V)$
- "Prim": $O(E + V \log V)$
- Kruskal: $O(E \log V)$

- remember: worst case! (plus - hidden constants)
- also: $E \neq V$. Might have $E \approx V^2$, but also could be smaller.