

TDA - Fall 2025

Extended  
Persistence (cont)  
+ Reeb graphs

# Recap

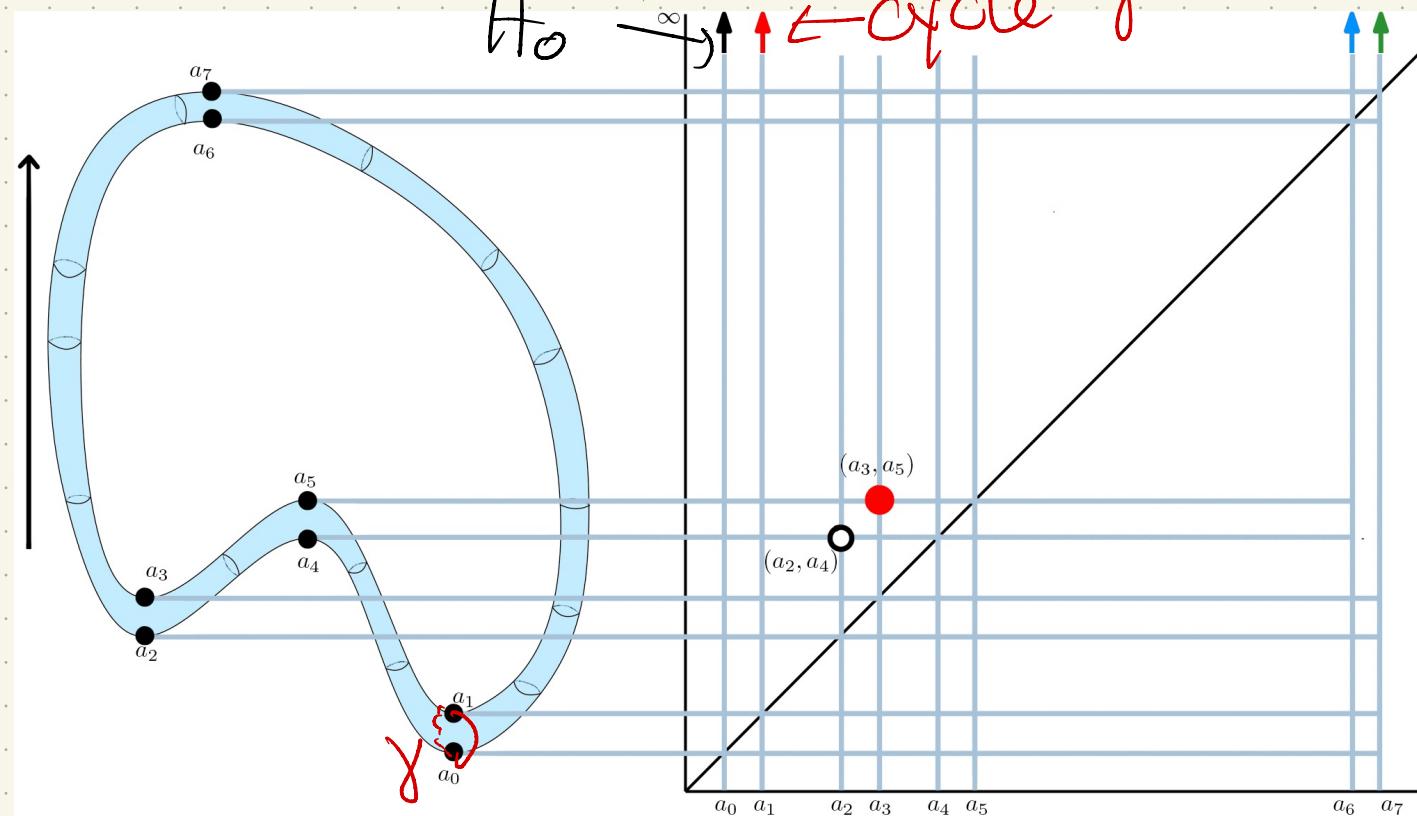
- Paper due: due October 17  
(sorry for typo on webpage!)
- Next assignment will be final  
project proposal
- Shorten office hours today

# Extended Persistence

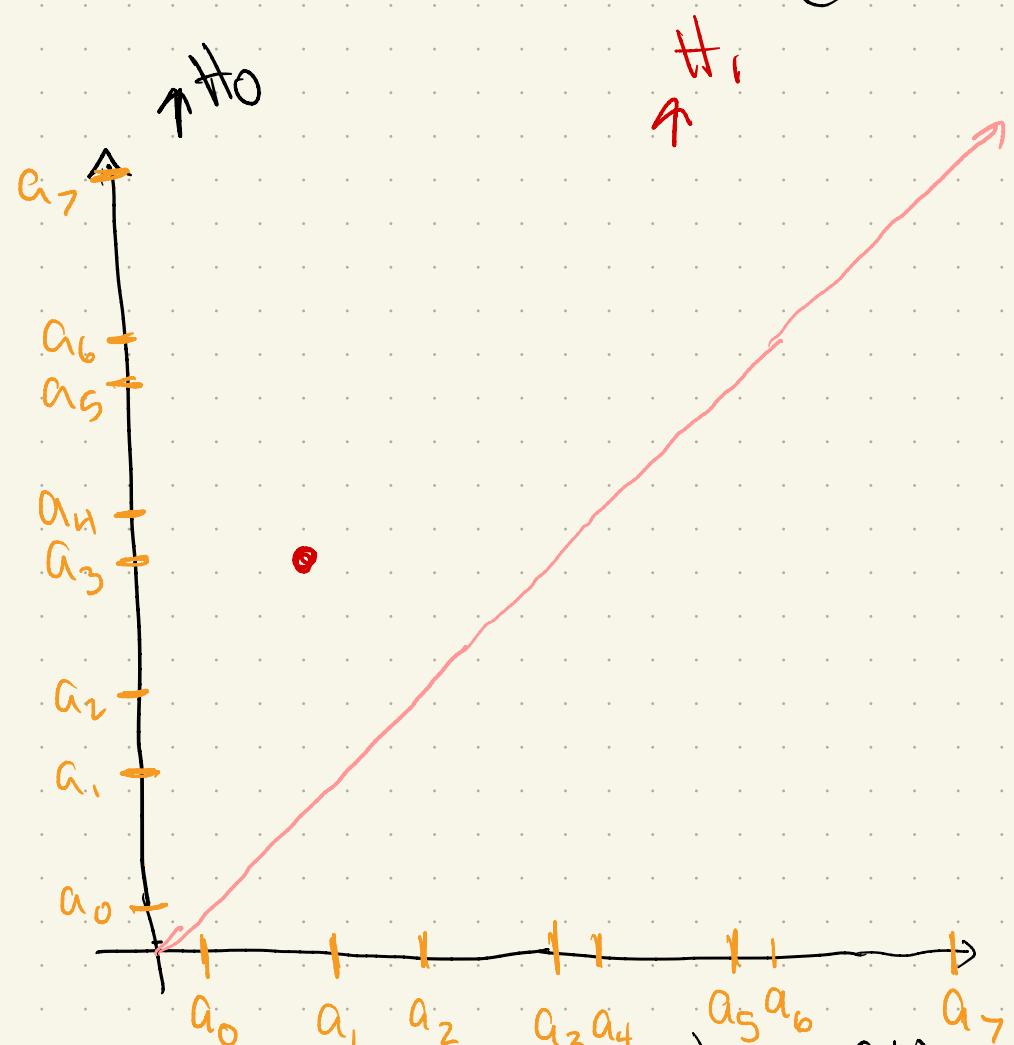
Odd parts of persistence:

- points at infinity

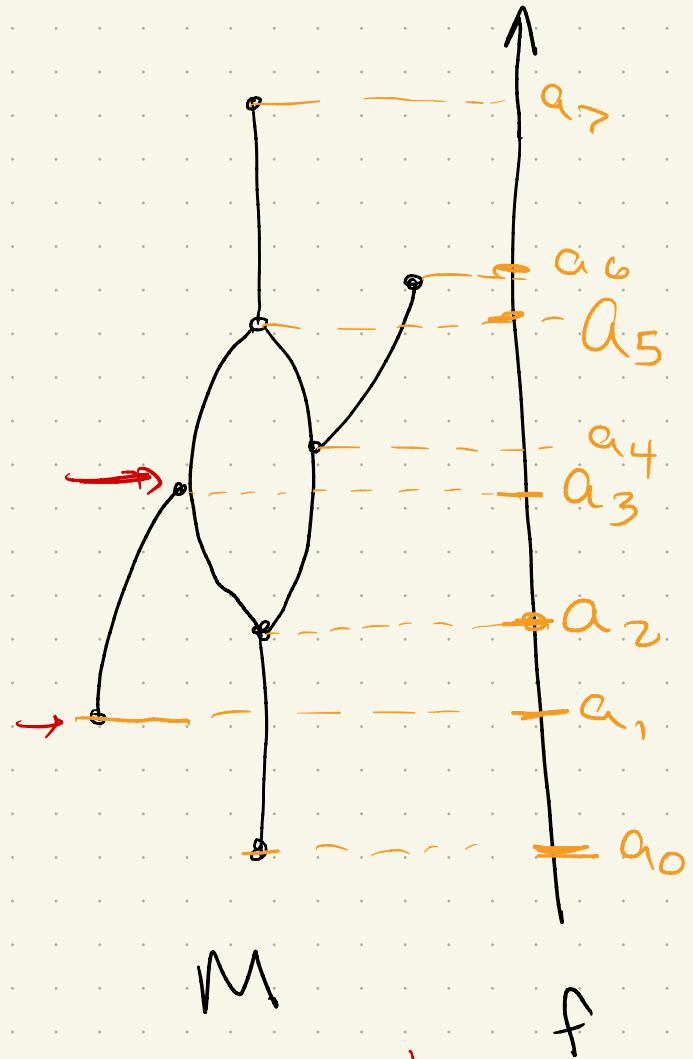
- some Morse critical points don't seem to matter



Really becomes obvious on Reeb graphs  
(more next week)



Persistence diagram



what about the rest!

Agarwala-Edelsbrunner-Harer-Wang 2006

⇒ Cohen-Steiner, Edelsbrunner, Harer 2009

Use relative homology to find other critical points, & get better pairings.

Relative homology Fix  $L \subseteq K$

Define  $C_p(K, L) := C_p(K) / C_p(L)$

$$+ [\alpha] = \{ \gamma \in C_p(K) \mid \alpha + \gamma \in C_p(L) \}$$

Then maps extend to homology-

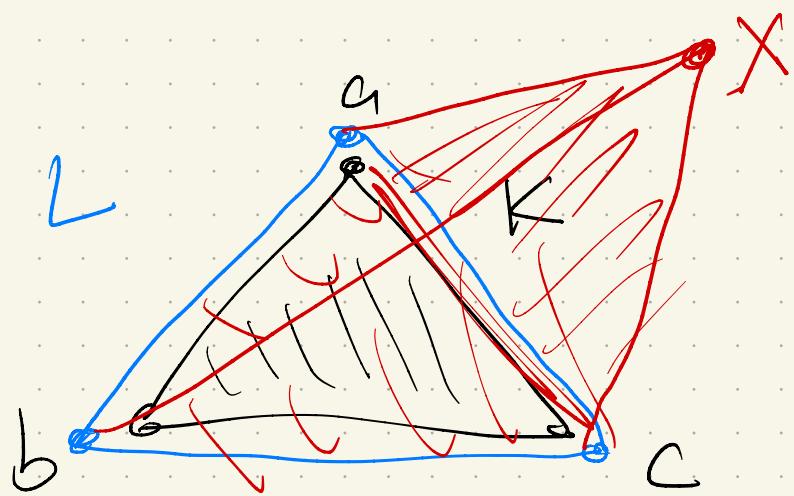
Remember a month ago? →

## Fun fact

Let  $K^* = K \cup \{x\} \cup \{\underline{\delta} \cup \{x\} \mid \delta \in L\}$

"coned off"

tetrahedra



## Theorem:

$$H_p(K, L) = H_p(K^*) \text{ for } p > 0$$

$$\& \quad \beta_0(K, L) = \beta_0(K^*) - 1$$

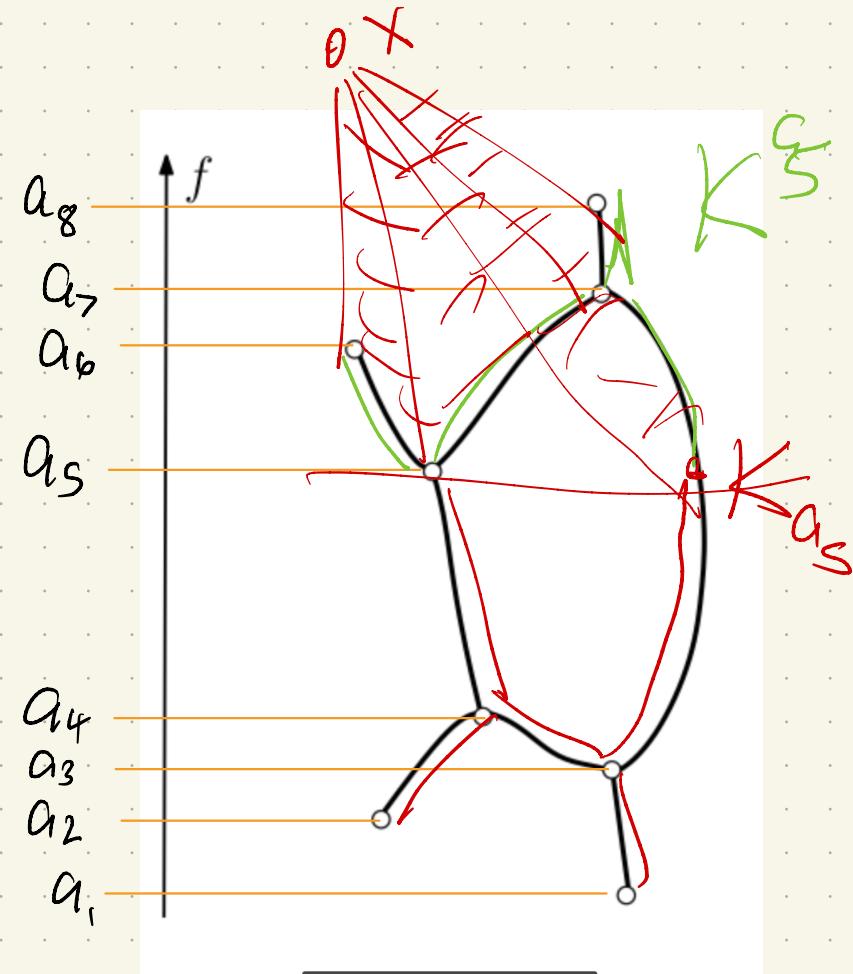
Here, we want to look at relative homology of the superlevel sets!

Given  $f: K \rightarrow \mathbb{R}$

$$K_a = \{ \sigma \in K \mid f(\sigma) \leq a \}$$

$$K^a = \{ \sigma \in K \mid f(\sigma) \geq a \}$$

& study  $H_p(K, K^a)$   
 (as well as  $H_p(K_a)$ )



What are important bits? ("cone off"  $K^{q_0}$ )  
 (revisited)

$H_0(K^*)$  minus 1



$H_0(K, K^{a_8})$

$$1 - 1 = 0$$

$H_1(K, K^{a_8})$

$\langle \alpha \rangle$

$H_0(K, K^{a_6})$

$$1 - 1 = 0$$

$H_1(K, K^{a_6})$

$\langle \alpha \rangle$

$H_0(K, K^{a_5})$

$$\langle \alpha \rangle$$

$H_1(K, K^{a_5})$

$\langle \alpha \rangle$

$H_0(K, K^{a_3})$

$$0$$

$H_1(K, K^{a_3})$

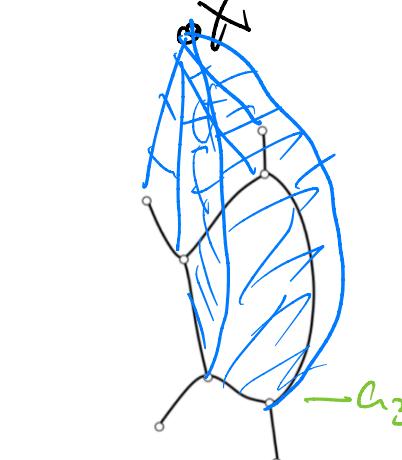
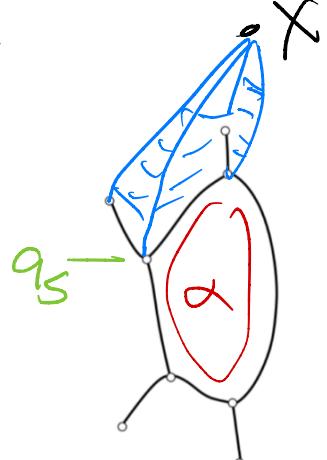
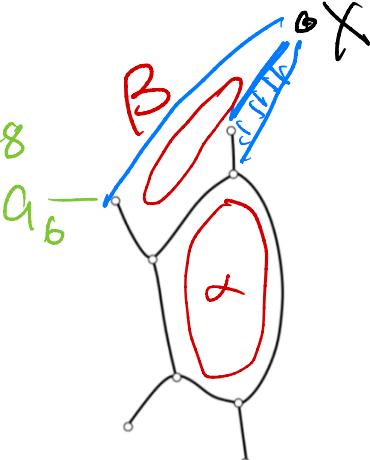
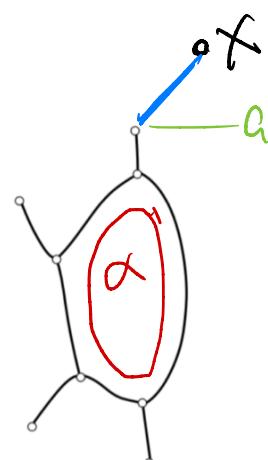
$0$

$H_0(K, K^{a_1})$

$$0$$

$H_1(K, K^{a_1})$

$0$



# Extended persistence Module

regular  
persistence

$$H_p(K_{a_1}) \rightarrow H_p(K_{a_2}) \rightarrow \dots \rightarrow H_p(K_{a_n})$$

$$H_p(K, K^{a_n}) \rightarrow \dots \rightarrow H_p(K, K^{a_1})$$

relative  
superlevel  
set homology

3 kinds of points:

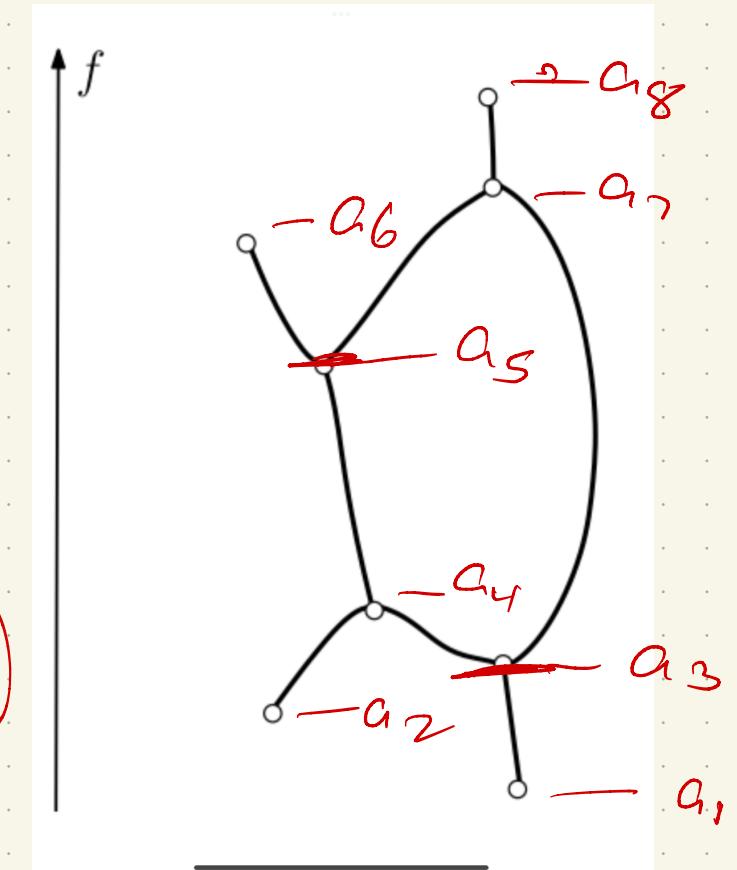
- Ordinary : usual in PD
- Relative : born & die on way down
- Extended : use both

$$H_p(K_{a_1}) \rightarrow H_p(K_{a_2}) \rightarrow \dots \rightarrow H_p(K_{a_8})$$

$$\hookrightarrow H_p(K, K^{a_8}) \rightarrow H_p(K, K^a)$$

$$\hookrightarrow H_p(K, K^{a_6}) \xrightarrow{\langle \beta \rangle} H_p(K, K^{a_5}) \rightarrow 0$$

$$\hookrightarrow H_p(K, K^{a_3})$$



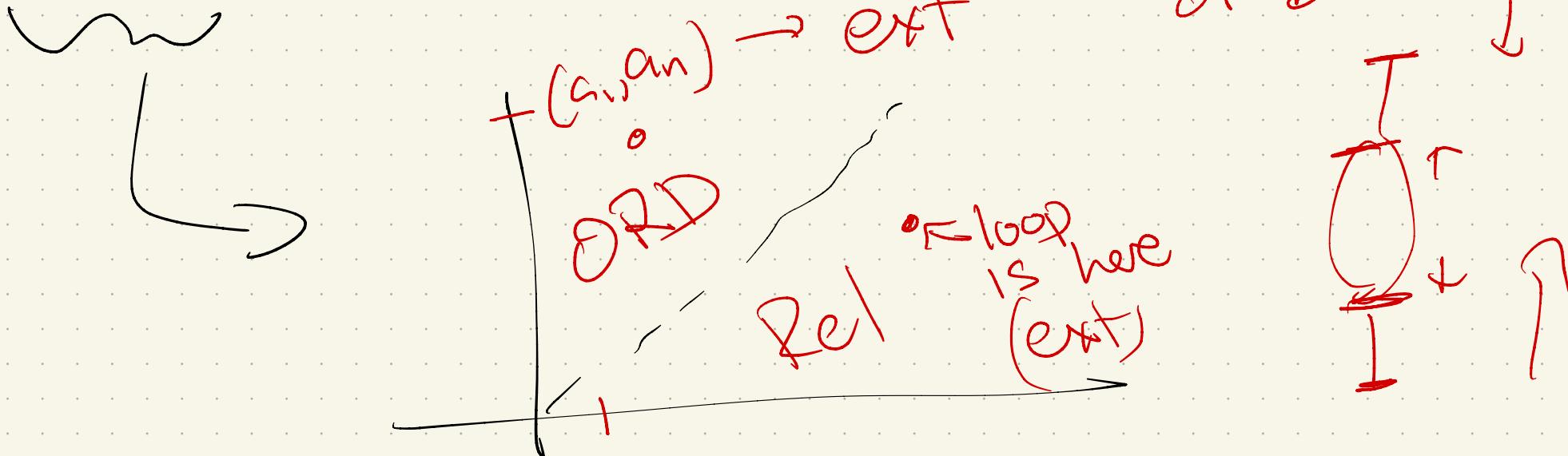
Points in diagram:

$$H_p(K_{a_1}) \rightarrow \dots \rightarrow H_p(K_{a_n}) \rightarrow H_p(K, K^{a_n}) \rightarrow \dots \rightarrow H_p(K, K^{a_1})$$

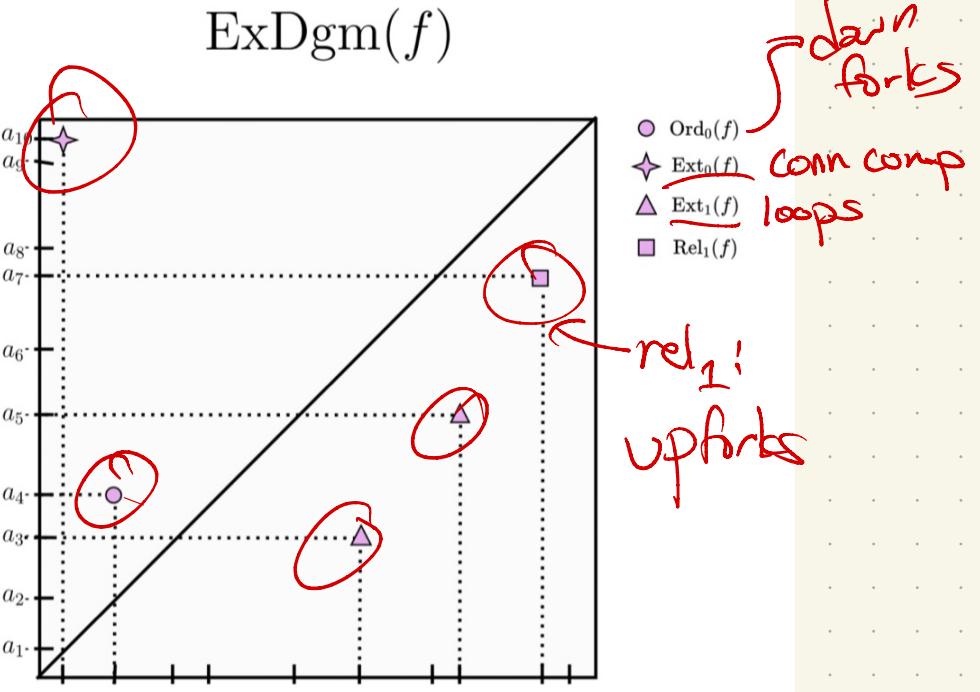
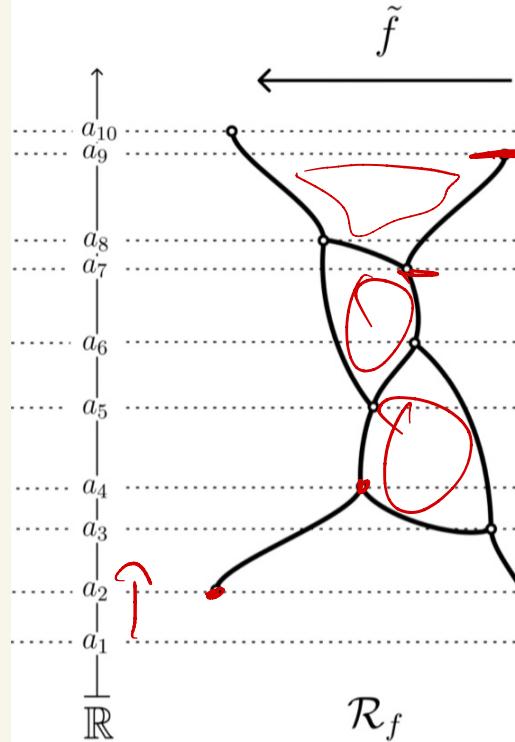
Ord: Starts & ends in 1<sup>st</sup> half, so  $(a, b) \Rightarrow a < b$

Rel: Starts & ends in 2<sup>nd</sup> half,  $(a, b) \Rightarrow b < a$

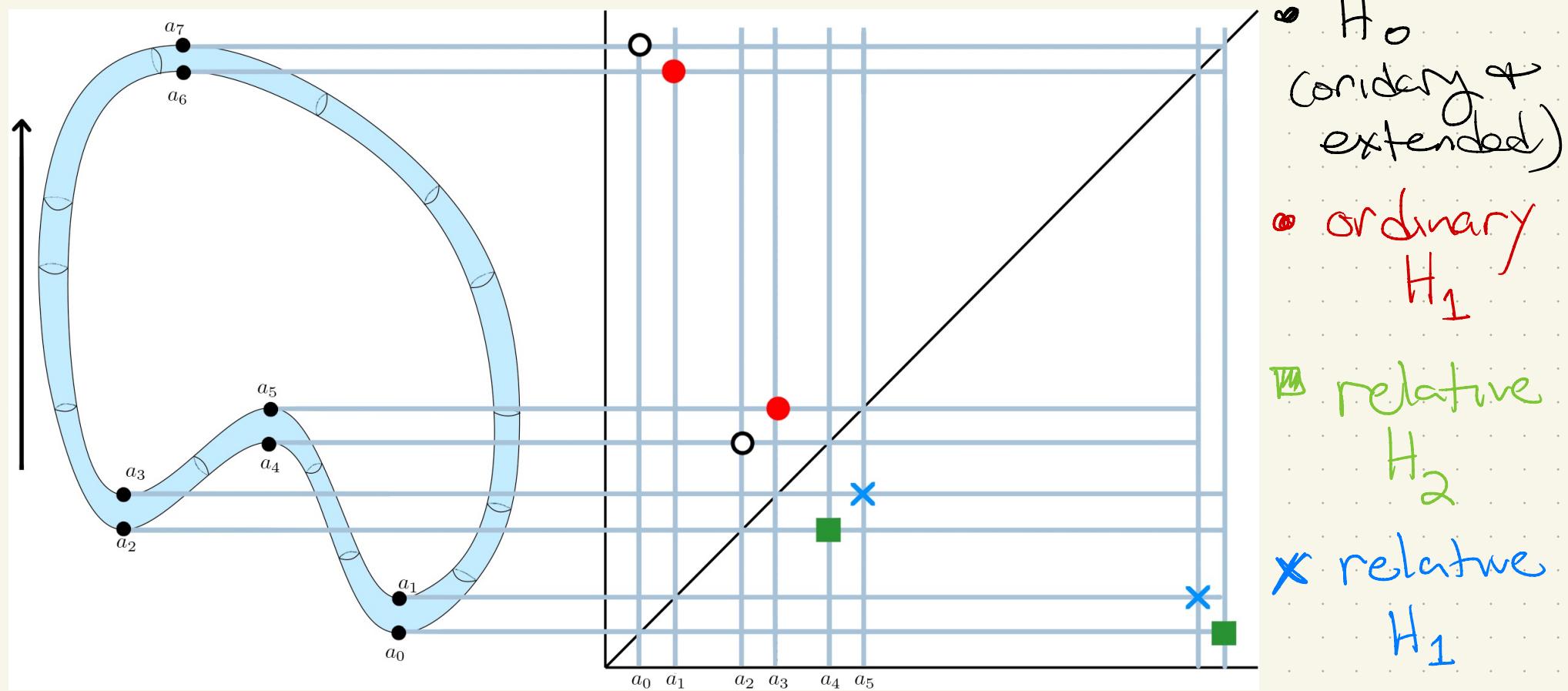
Ext: starts in 1<sup>st</sup> half  
ends in 2<sup>nd</sup> half  $\Rightarrow (a, b)$  either  
 $a < b$   
or  $b < a$



# On graphs



More generally: on manifolds get symmetry



Under the hood:

- Very beautiful combination of Lefschetz & Poincaré duality (that 2010 paper)
- Some more algebraic connections

Turner-Robins - Morgan  
2022

# Reeb Graphs

(named for Georges Reeb, 1946)  
by René Thom)

Recall: level sets

Given (PL)  $f: |K| \rightarrow \mathbb{R}$

a level set is  $f^{-1}(c)$

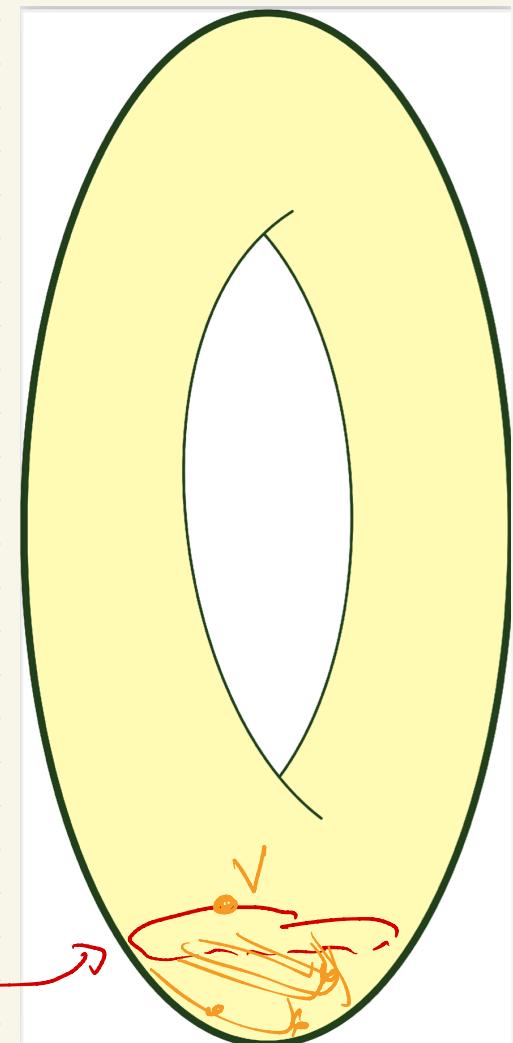
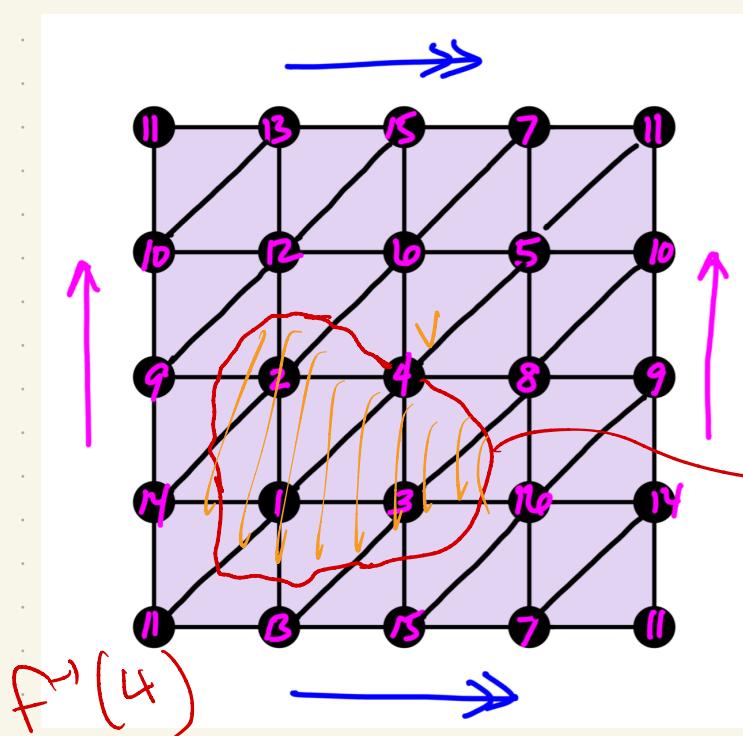
for  $c \in \mathbb{R}$

Ex:

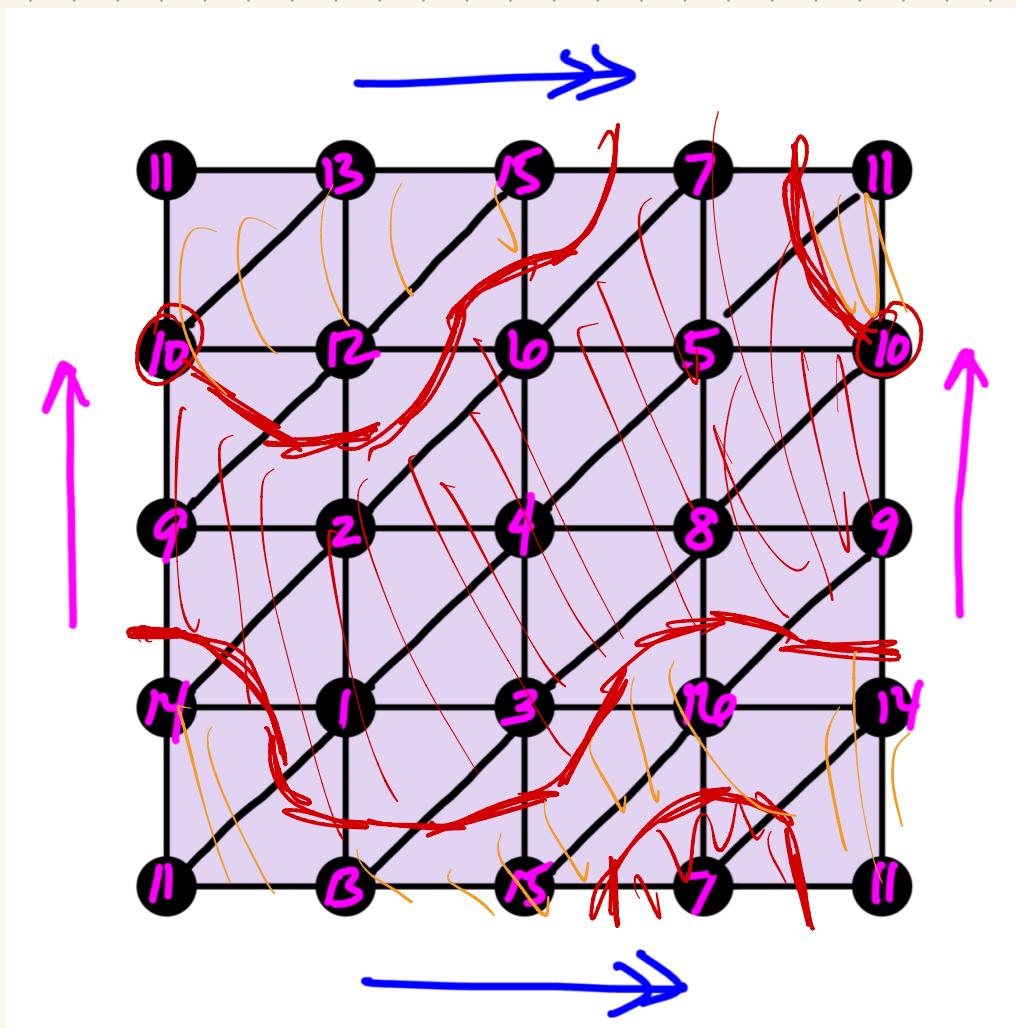
Consider

$f^{-1}(-\infty, 4]$

&  $f^{-1}(4)$



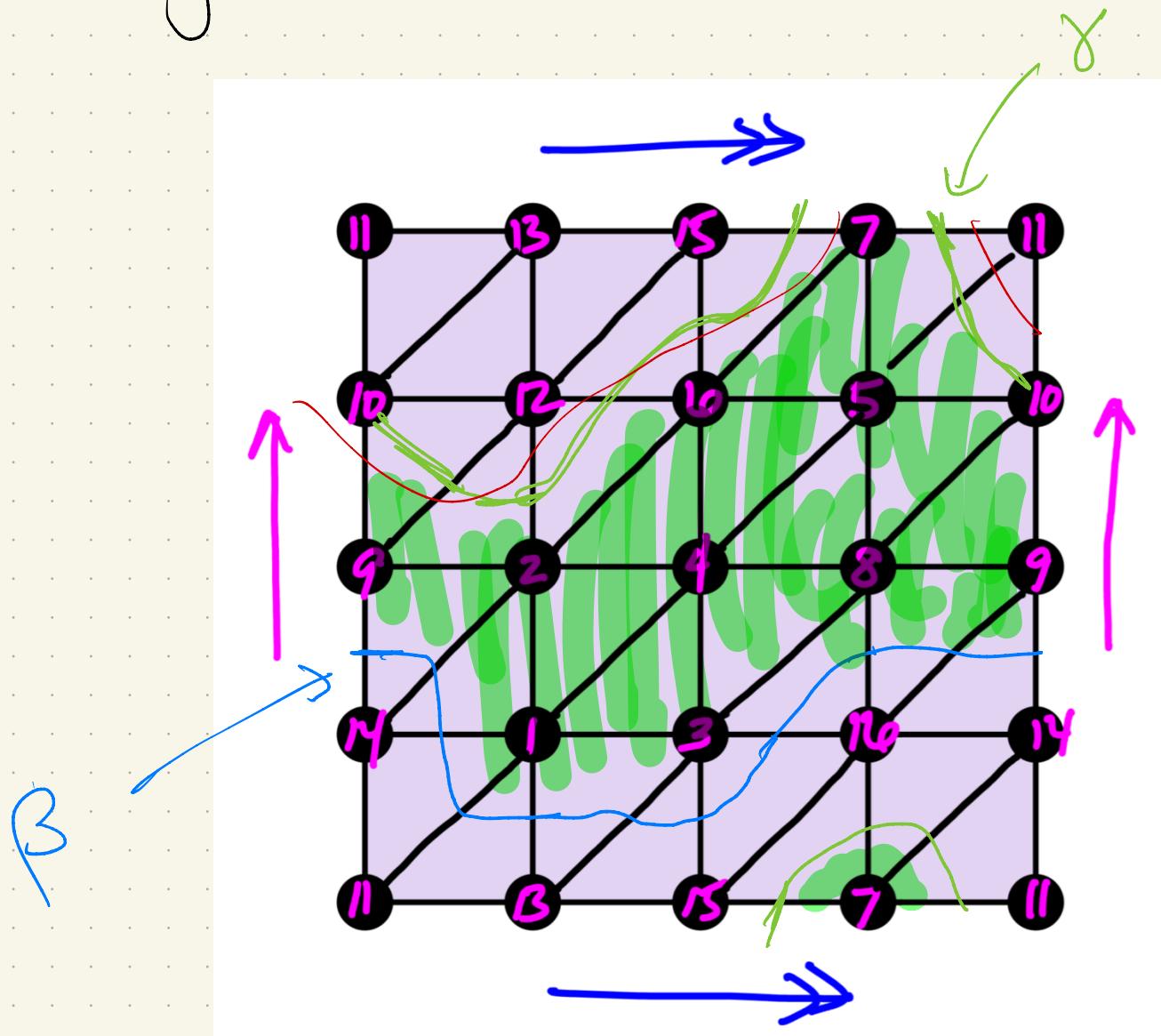
Practice: What is  $f^{-1}(10)$ ?



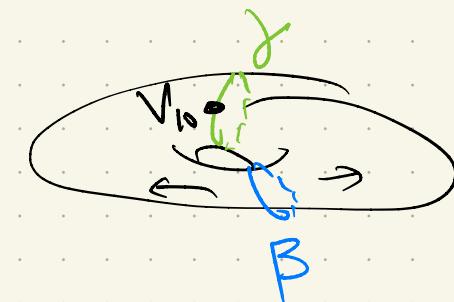
$$f^{-1}(-\infty, 10)$$

$$f^{-1}([10, \infty))$$

Answer: just in case! :)



# On Torus:



Sublevel set  $f^{-1}(-\infty, 10])$

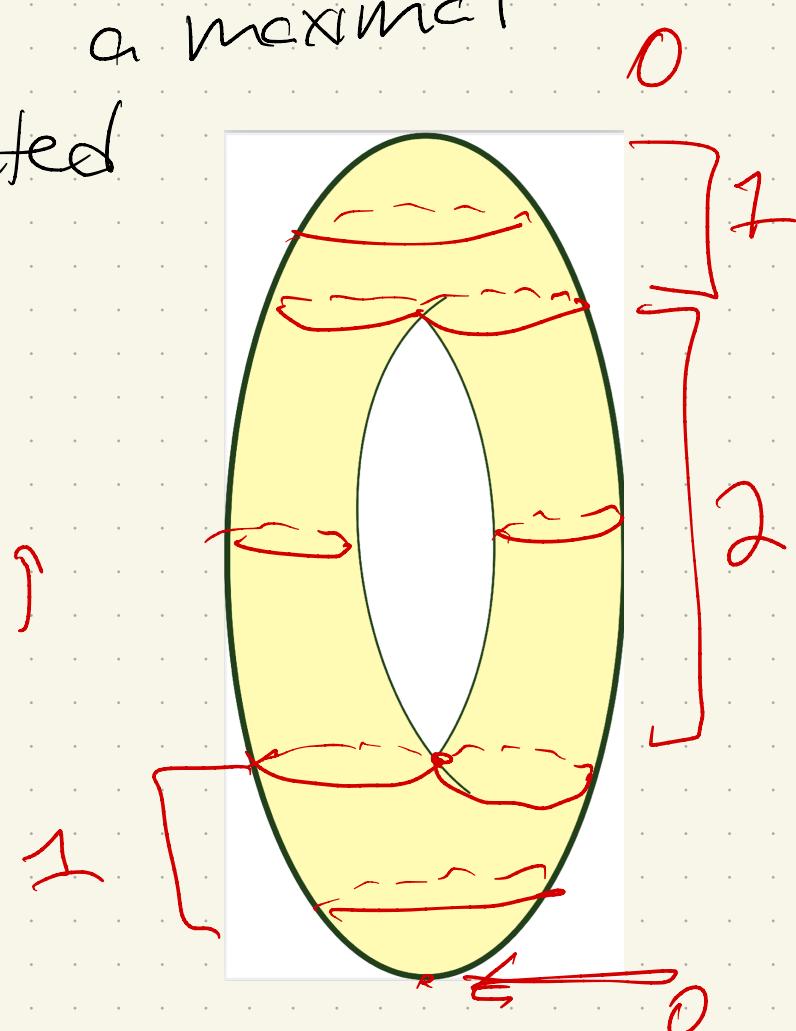
Recall:

A space is disconnected if  $\exists$  disjoint open sets  $U, V$  with  $X = U \cup V$   
↳ otherwise, connected.

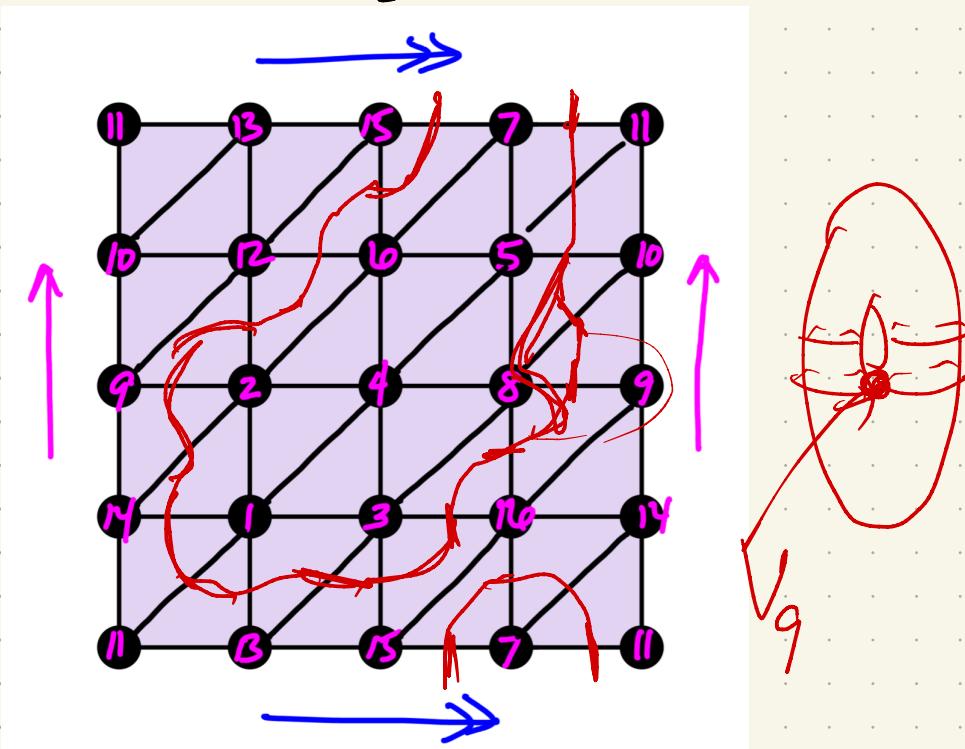
A **connected component** is a maximal  
subset that is connected

A **contour** is a connected  
component of a level set

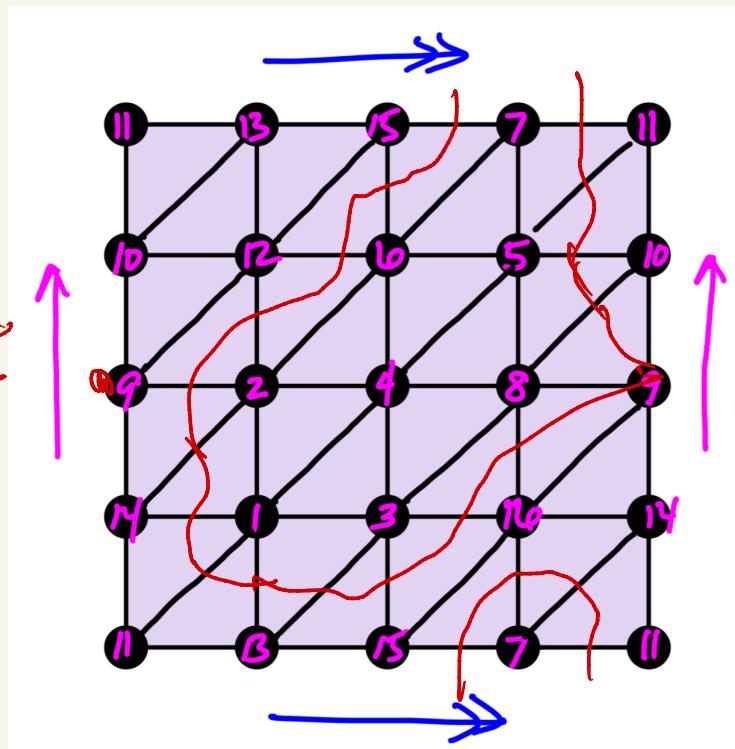
Q: How do contours change  
a level set changes?



Back to triangulation:  
 $f^{-1}(8)$



$f^{-1}(9)$



What shape, & how many components?

(+ recall  $f^{-1}(10)$  a few slides back had  
2 contours)

## Equivalence relations & quotients

A binary relation  $\sim$  on a set  $T$  is an equivalence relation if  $\forall a, b, c \in T$ ,

- $a \sim a$  [reflexive]

- $a \sim b \Leftrightarrow b \sim a$  [symmetric]

- If  $a \sim b \wedge b \sim c \Rightarrow a \sim c$  [transitive]

Equivalence class of  $a$  under  $\sim$

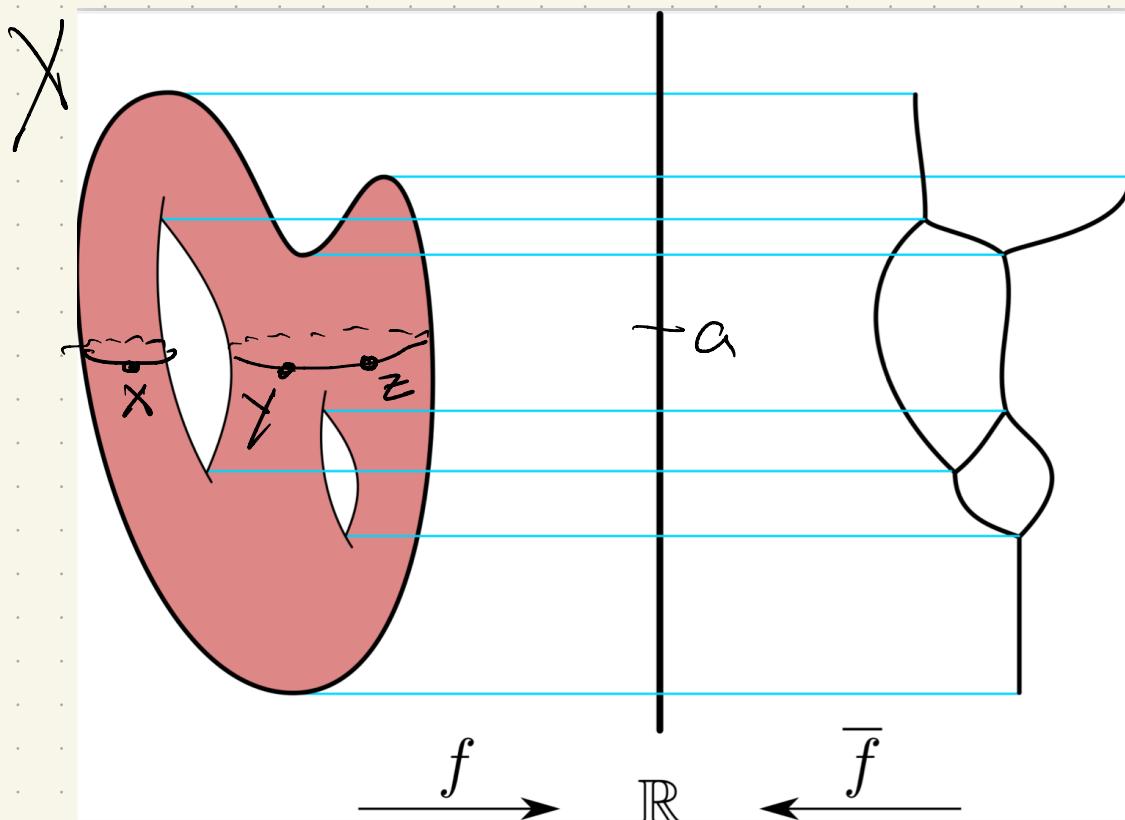
$$[a] = \{x \in T \mid x \sim a\}$$

On a topological space with an equivalence relation, get quotient map  $q(a) = [a]$

(and  $U$  is open  $\Leftrightarrow q^{-1}(U)$  is open)

# Reeb graph equivalence relation

Given function  $f: X \rightarrow \mathbb{R}$ .

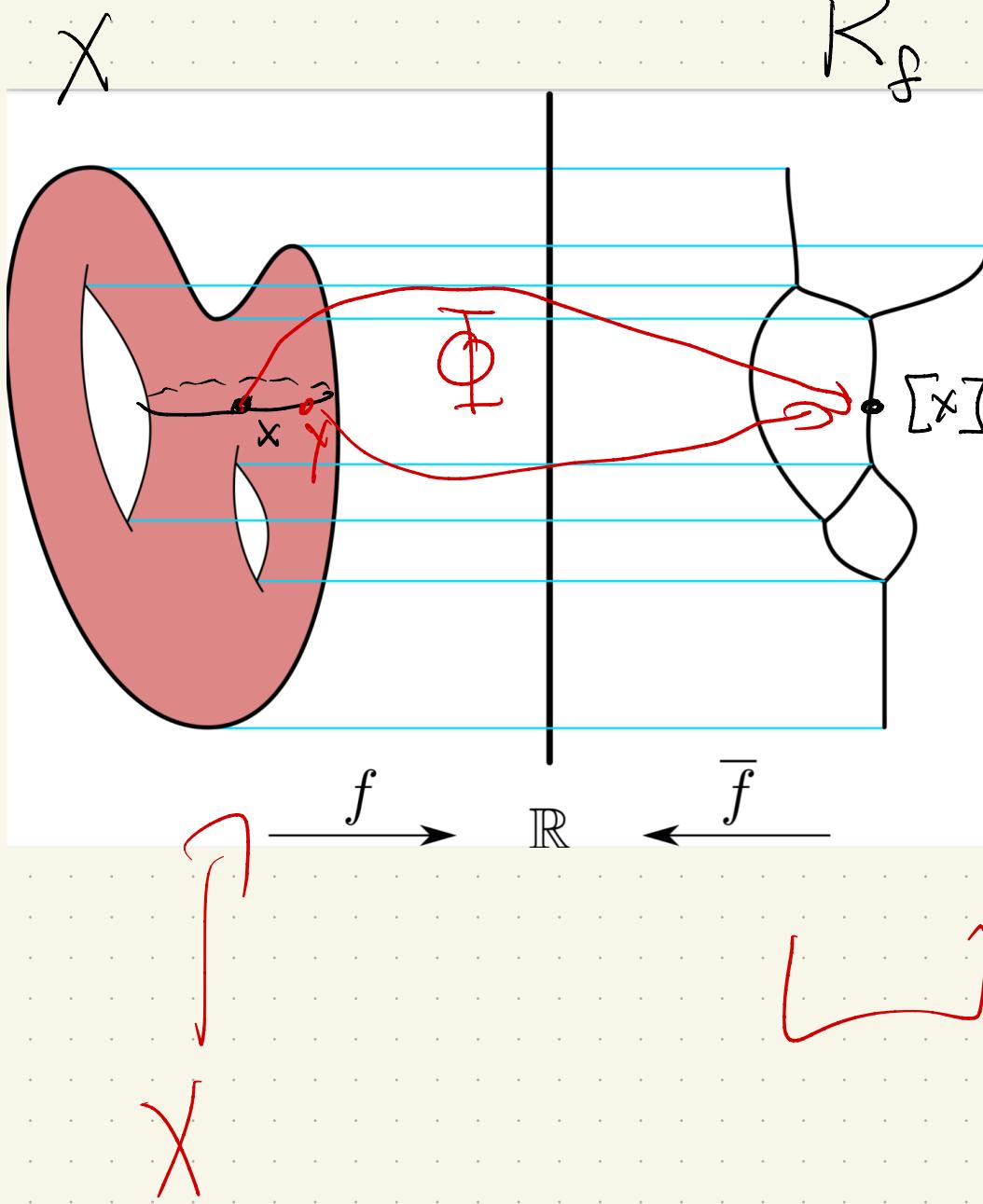


Define equiv. rel.  $\sim$   
by  $x \sim y$  iff:

- $f(x) = f(y) = a$
- $x$  &  $y$  are in same connected component of level set  $f^{-1}(a)$ .

$$\begin{aligned} x &\sim y, z \\ y &\sim z \end{aligned}$$

# Reeb graphs



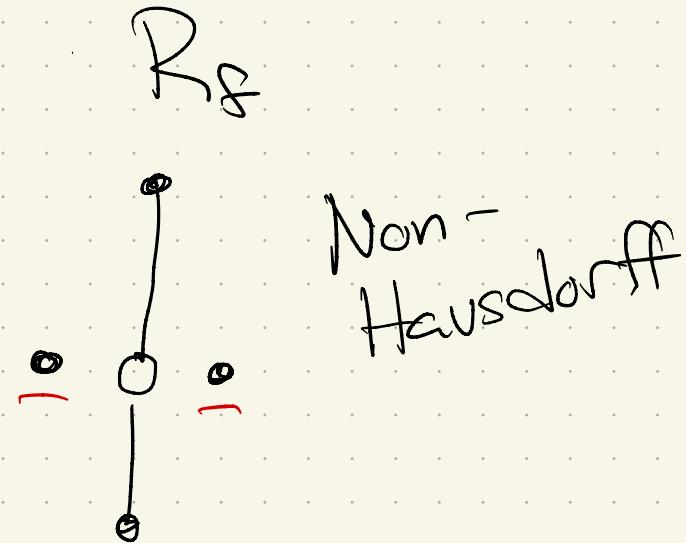
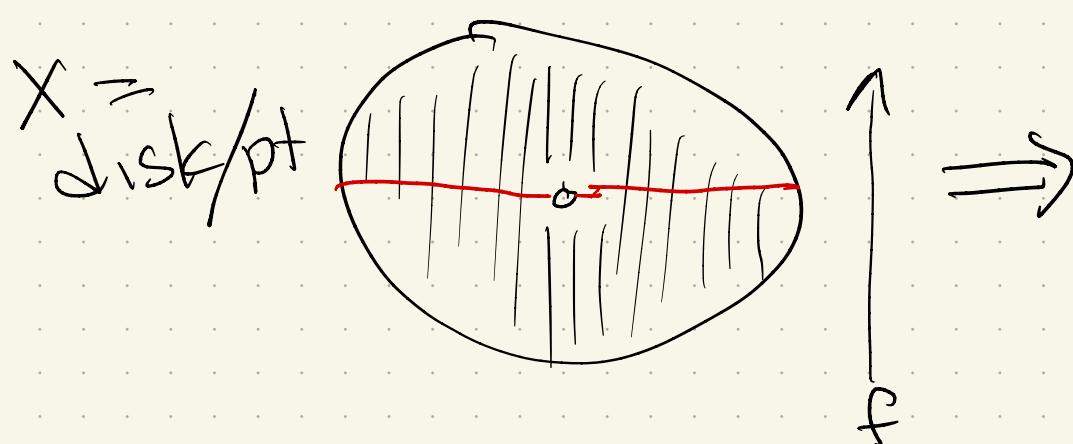
- Let  $[x]$  denote the equivalence class of  $x \in X$ .
- The Reeb graph  $R_f = R(X, f)$  is the quotient space  $X / \sim$
- Let  $\Phi: X \rightarrow R_f$ ,  $x \mapsto [x]$  be the quotient map

## Niceness assumptions

If input is "nice" (both  $X$  &  $f$ ), then  
 $R_f$  is a graph.

- Morse function on compact manifold
- PL function of compact polyhedron
- Others ...

Bad example:



# Applications: a topological signature

Pascucci et al

no 3d

$B_1=5$



Note: a bit of a lie in this figure:

no embedding!

R-values

$\mathbb{Z}_4$