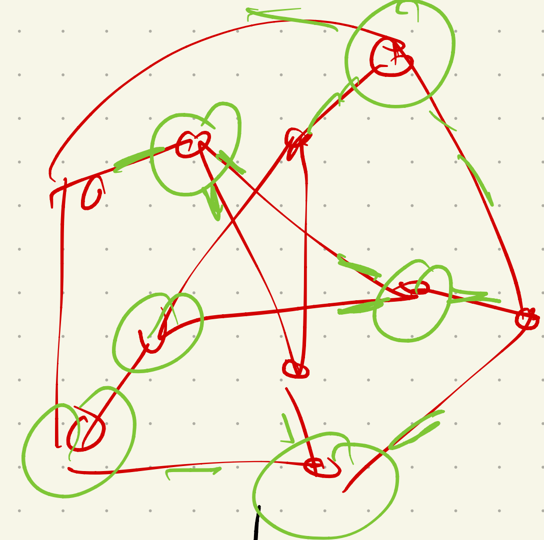
Algorithms – Spring '25

NP-Hardness
Intro to LPs

Recap

- 1 week left! Check grades, etc.
- Sign up for oral grading slot
  - ↳ HW8 group, then go to calendar
- Final: 2 weeks!
  - Thursday May 8, 10^{30}
- Practice final
  - ↳ posted later today or tomorrow
- CIFs are live!

Subset Sum is NP-Hard.



Reduction: Vertex Cover

Input: Graph G & size k

Goal: find k vertices, such that every edge in G is incident to at least one vertex in set

Challenge: Construct a set of numbers, s.t. we can hit a target value

$\iff$ G has ~~indep set~~ Vertex Cover of size k

Recall: base 4

$(32012)_4 = 3\cdot 4^4 + 2\cdot 4^3 + 0\cdot 4^2 + 1\cdot 4^1 + 2\cdot 4^0$

... 16's 4's 1's

$$\begin{array}{r} 1\,0\,3\,2 \\ +\,2\,3\,1\,2 \\ \hline 1\,0\,0\,1\,0 \end{array}$$

Idea: Use base 4:

force a target T that requires you to use only vertices, but to "cover" edges

Number edges 0...E-1, & create a number for subset sum. w/ E+1 "digits"

e_0: $b_0 = 0\,0\,\_\,\_\,\_\,\_\,\_\,0\,1$

e_1: $b_1 = 0\,0\,\_\,\_\,\_\,\_\,0\,1\,0$

⋮

e_{E-1}: $b_{E-1} = 0\,1\,0\,-\,-\,0\,1$

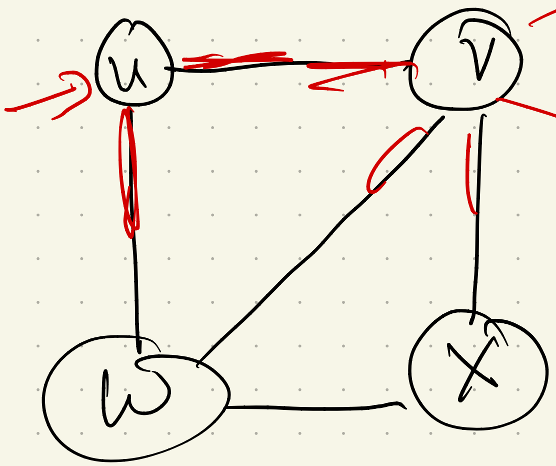
E spots → E+1

For each vertex, make another #:

$$a_v := \underbrace{1[\qquad\qquad]}_{E+1 \text{ digits}}$$

d(v) 1's, for its endpoints (all else 0)

Think of base 4 representation



{u, v}
{x, v}
{w, v}

$a_u := 111000_4 = 1344$	$b_{uv} := 010000_4 = 256$
$a_v := 110110_4 = 1300$	$b_{uw} := 001000_4 = 64$
$a_w := 101101_4 = 1105$	$b_{vw} := 000100_4 = 16$
$a_x := 100011_4 = 1029$	$b_{vx} := 000010_4 = 4$
	$b_{wx} := 000001_4 = 1$

k of these 4^E

$0\,\underbrace{1\,1 \cdots 1}_{E}$

any value > 0

→ set of #s,

Now, set $T = \underset{\text{input}}{k} \cdot 4^E + \boxed{\sum_{i=0}^{E-1} 2 \cdot 4^i}$ 2 each

Why?

Areas   k vertices

Proof: size k VC $\Longrightarrow$ sum to T

$\Rightarrow$ VC: $\exists$ k vertices $v_1, v_2, \ldots, v_k$

s.t. $\forall e \in E$, e is incident to some $v_i \in \{v_1, \ldots, v_k\}$

$\Rightarrow$ (cont)

Pick a subset:

take k vertices, ~~&~~ choose those #s.

Sum those #s: $k \cdot 4^{E}$

Each edge has 1 or 2 endpoints

$\Rightarrow$ that "digit" spot has either 1 or 2

If edge is only incident to 1 vertex in cover, add edge's # also, so now $= 2 \cdot 4^{i}$ (for edge i)

$\Leftarrow$: suppose some subset of #s sums to T. options?

Recall: $T = k \cdot 4^E + \boxed{\sum_{i=0}^{E-1} 2 \cdot 4^i}$

$+ a_v = 1\underline{\qquad\qquad\qquad} +$

$+ b_e = 0\underline{\qquad\qquad\qquad}$

Plus:
Each digit position has only 3 1's across all #s:

1 for edge #
2 for edge's endpoint vertex #s

So!

Must have exactly k of the vertex #s.

(only way to get $k \cdot 4^E$)

These also get some # of "edge bit" spots $0 \ldots E-1$

2 in each edge spot either comes from 2 vertices or 1 vertex plus one edge' #.

Partition:

Given $X = \{x_1, \ldots, x_n\}$, can we partition X into A & B (so $A \cup B = X$, $A \cap B = \emptyset$, & $A, B \neq \emptyset$)

s.t.

$$\sum_{x_i \in A} x_i = \sum_{x_j \in B} x_j \ ?$$

Reduction? try reducing from subset sum: $\{a_1, \ldots a_n\}$ plus target t:

Possible idea:

Let $X = \{a_1, \ldots, a_n, t\}$

subset sum soln $\subseteq A$

rest of $A \setminus \{\text{subset}\}$ $= \left(\sum a_i\right) - t$

t

Try again: $X = \{a_1, \ldots, a_n, t, \left(\sum_{i=1}^{n} a_i\right) - t\}$

Proof:

Some fun examples

arXiv.org > cs > arXiv:1203.1895

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Computer Science > Computational Complexity

Classic Nintendo Games are (Computationally) Hard

Greg Aloupis, Erik D. Demaine, Alan Guo, Giovanni Viglietta

(Submitted on 8 Mar 2012 (v1), last revised 8 Feb 2015 (this version, v3))

We prove NP-hardness results for five of Nintendo's largest video game franchises: Mario, Donkey Kong, Legend of Zelda, Metroid, and Pokemon. Our results apply to generalized versions of Super Mario Bros. 1-3, The Lost Levels, and Super Mario World; Donkey Kong Country 1-3; all Legend of Zelda games; all Metroid games; and all Pokemon role-playing games. In addition, we prove PSPACE-completeness of the Donkey Kong Country games and several Legend of Zelda games.

Comments: 36 pages, 36 figures. Fixed some typos. Added NP-hardness results (with proofs and figures) for American SMB2 and Zelda 2

Subjects: **Computational Complexity (cs.CC)**; Computer Science and Game Theory (cs.GT)

Cite as: arXiv:1203.1895 **[cs.CC]**

(or arXiv:1203.1895v3 **[cs.CC]** for this version)

Submission history

From: Alan Guo [view email]

[v1] Thu, 8 Mar 2012 19:37:20 GMT (627kb,D)

[v2] Thu, 6 Feb 2014 18:24:15 GMT (3330kb,D)

[v3] Sun, 8 Feb 2015 19:45:26 GMT (3425kb,D)

Which authors of this paper are endorsers? | Disable MathJax (What is MathJax?)

Link back to: arXiv, form interface, contact.

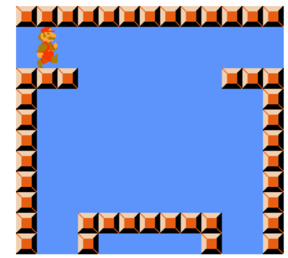


Figure 10: Variable gadget for Super Mario Bros.

shes until it is collected by Mario.

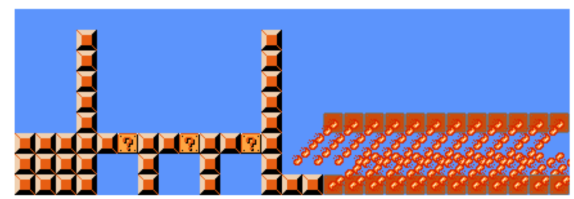


Figure 11: Clause gadget for Super Mario Bros.

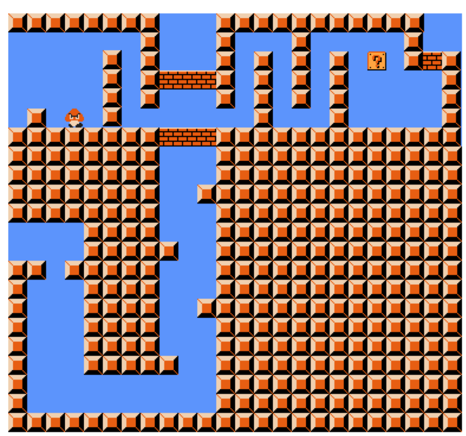
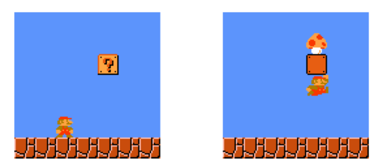


Figure 12: Crossover gadget for Super Mario Bros.



Left: Start gadget for Super Mario Bros. Right: The item block contains a

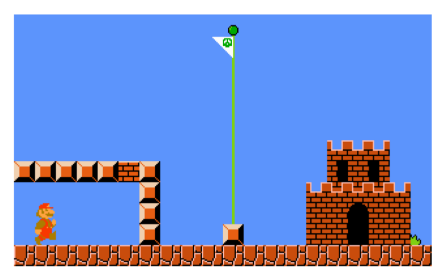
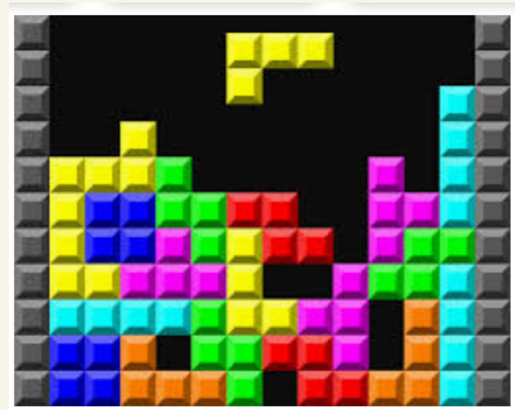


Figure 9: Finish gadget for Super Mario Bros.

Another: Tetris



NP-Hard: Reduce 3-partition

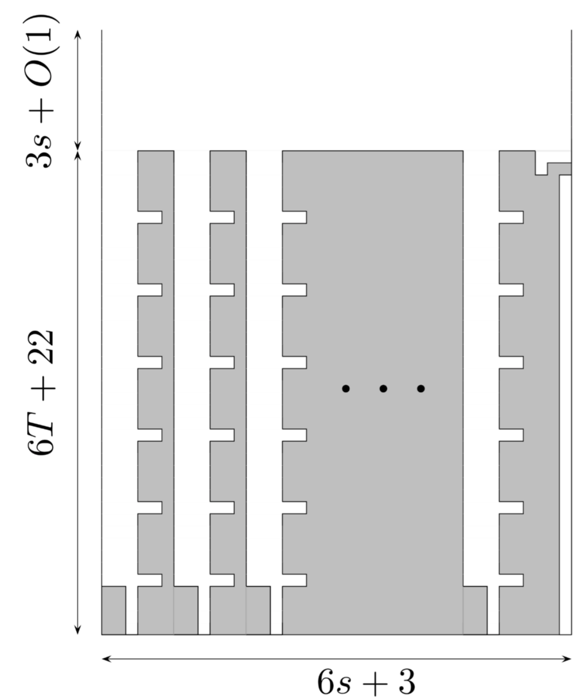


Fig. 2. *The initial gameboard for a Tetris game mapped from an instance of* 3-PARTITION.

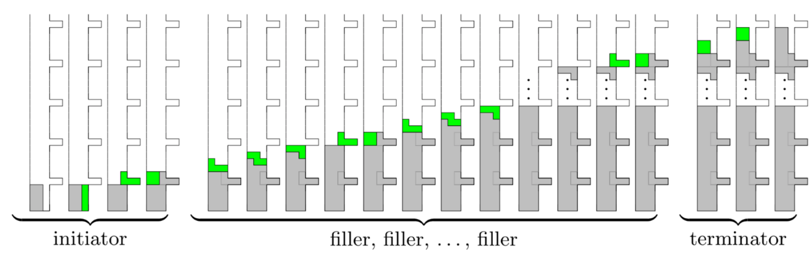


Fig. 3. *A valid sequence of moves within a bucket.*

Again: An active area of research!

arXiv.org > cs > arXiv:1711.00788

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Computer Science > Computational Geometry

On the complexity of optimal homotopies

Erin Wolf Chambers, Arnaud de Mesmay, Tim Ophelders

(Submitted on 2 Nov 2017)

In this article, we provide new structural results and algorithms for the Homotopy Height problem. In broad terms, this problem quantifies how much a curve on a surface needs to be stretched to sweep continuously between two positions. More precisely, given two homotopic curves γ_1 and γ_2 on a combinatorial (say, triangulated) surface, we investigate the problem of computing a homotopy between γ_1 and γ_2 where the length of the longest intermediate curve is minimized. Such optimal homotopies are relevant for a wide range of purposes, from very theoretical questions in quantitative homotopy theory to more practical applications such as similarity measures on meshes and graph searching problems.

We prove that Homotopy Height is in the complexity class NP, and the corresponding exponential algorithm is the best one known for this problem. This result builds on a structural theorem on monotonicity of optimal homotopies, which is proved in a companion paper. Then we show that this problem encompasses the Homotopic Fréchet distance problem which we therefore also establish to be in NP, answering a question which has previously been considered in several different settings. We also provide an O(log n)-approximation algorithm for Homotopy Height on surfaces by adapting an earlier algorithm of Har-Peled, Nayyeri, Salvatipour and Sidiropoulos in the planar setting.

Almost any problem in AI is NP-Hard.

Not impossible! Just exponential to solve perfectly:

- heuristics
- approximation
- pruning strategies

Linear program

In a linear program, we are given a set of variables

The goal is to give these real values so that:

1. We satisfy some set of linear equations or inequalities

2. We maximize or minimize some linear objective function

↳ often, profit!

Example LP: Cargo plane:

- can carry 100 tons, + volume of 60 cubic meters
- 3 materials:
  - x_1 1st: 2 tons/cubic meter, 40 cubic meters available, worth \$1000 per c.m
  - x_2 2nd: 1 ton/c.m, 30 cm total available, and worth \$1200 per cm
  - x_3 3rd: 3 tons/cm, 20 cms total, & \$12,000 per c.m

Profit!

maximize $1000x_1 + 1200x_2 + 12000x_3$

dollars

such that:

$x_1 \le 40$ | $x_1 \ge 0$

$x_2 \le 30$ | $x_2 \ge 0$

$x_3 \le 20$ | $x_3 \ge 0$

weight eqn

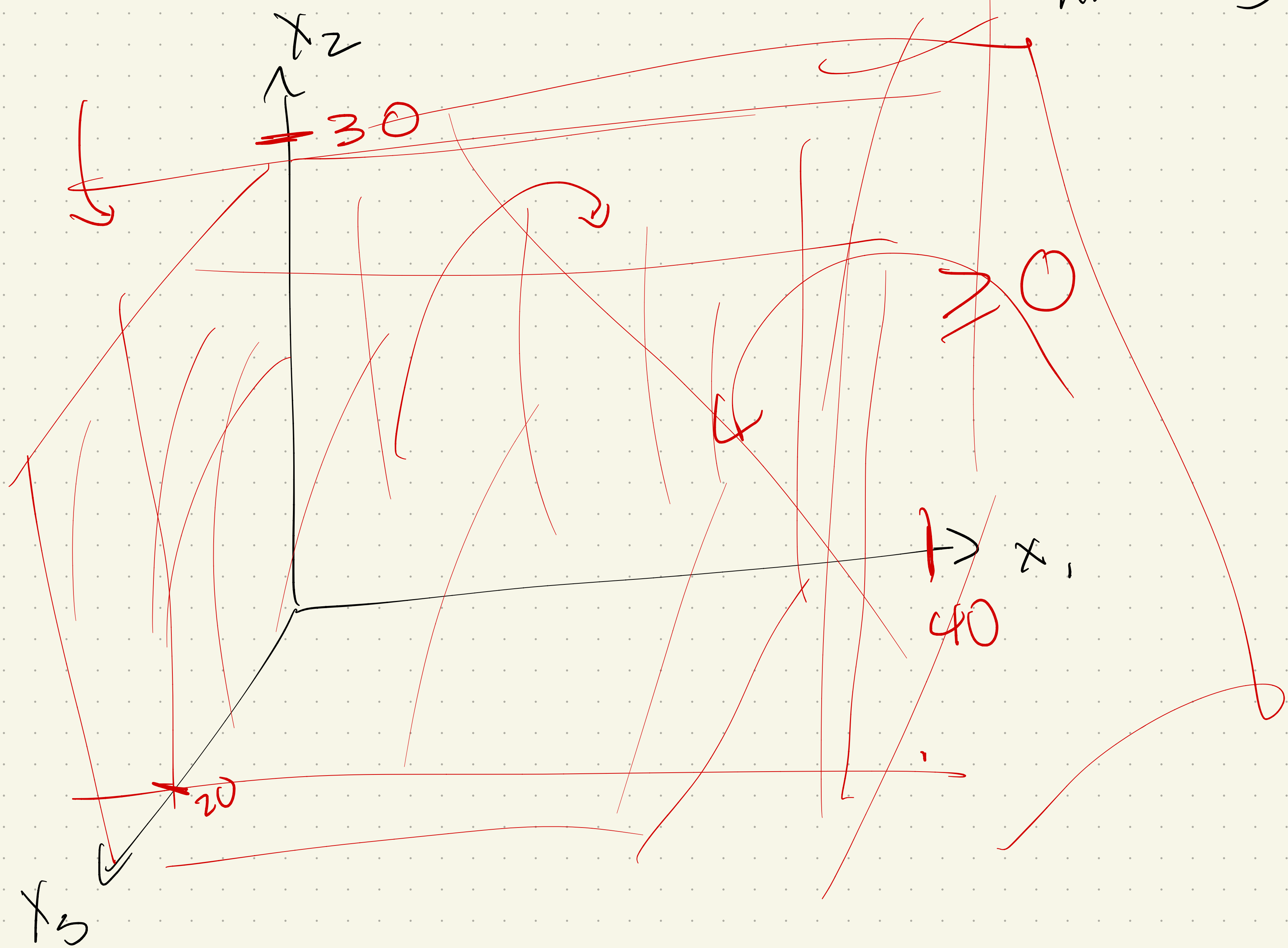
$2x_1 + 1 \cdot x_2 + 3x_3 \le 100$

size:

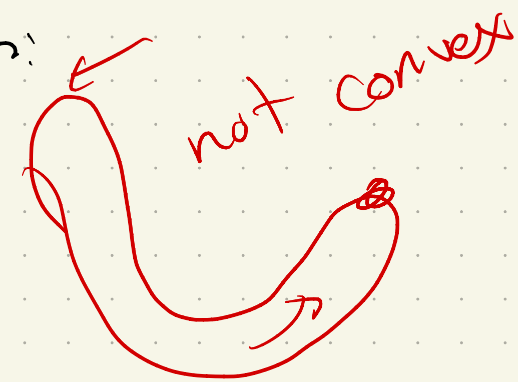
$\underline{x_1} + \underline{x_2} + \underline{x_3} \le 60$

plane in 3d

Geometry: Each equation makes a plane (since linear!):



Each variable adds a dimension:



Maximize $x_1 + 6x_2 + 13x_3$

s.t.

$x_1 \le 200$ 1

$x_2 \le 300$ 2

$x_1 + x_2 + x_3 \le 400$

$x_2 + 3x_3 \le 600$

and

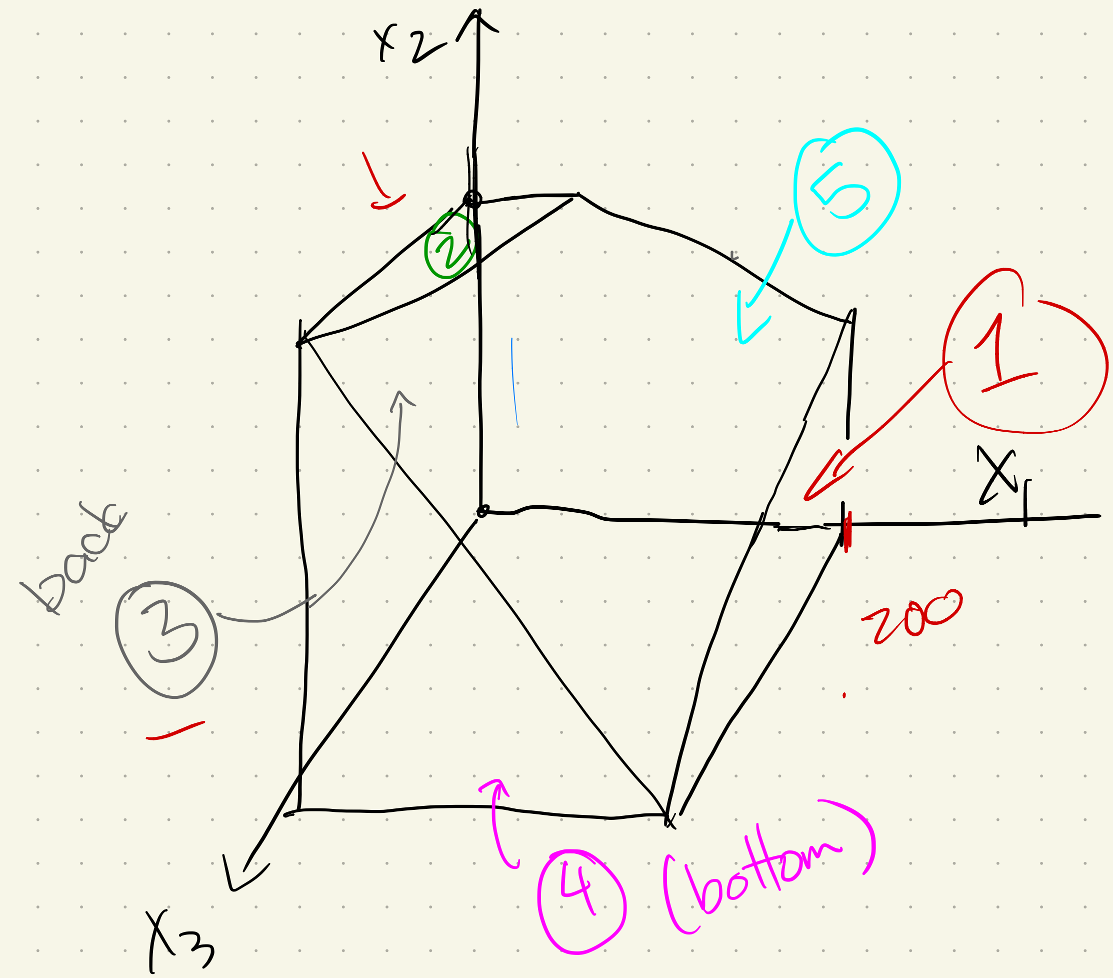
$x_1 \ge 0$ 3

$x_2 \ge 0$ 4

$x_3 \ge 0$ 5

$\mathbb{R}^3$:

And each eqn adds a face to polyhedron:



Another (more general)

n foods, m nutrients

Let a_{ij} = amount of nutrient i in food j

r_i = requirement of nutrient i

x_j = amount of food j purchased

c_j = cost of food j

Goal: Buy food so you satisfy nutrients while minimizing cost

Can view as matrix $\longrightarrow$

$$A = \begin{array}{cc} & j \\ i & \begin{bmatrix} & & & \\ & & a_{ij} & \\ & & & \end{bmatrix} \end{array}$$

$$\vec{r} = (r_1, r_2, \ldots r_m)$$

$$\vec{x} = (x_1, \ldots, x_n)$$

$$\vec{c} = (c_1, \ldots, c_n)$$

So: minimize

s.t.

In general, get systems like this:

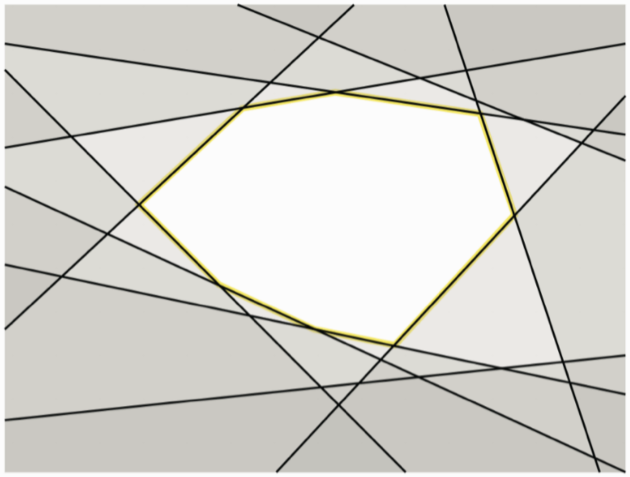
$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \le b_i \quad \text{for each } i = 1 \mathinner{..} p$$

$$\sum_{j=1}^{d} a_{ij} x_j = b_i \quad \text{for each } i = p+1 \mathinner{..} p+q$$

$$\sum_{j=1}^{d} a_{ij} x_j \ge b_i \quad \text{for each } i = p+q+1 \mathinner{..} n$$

Geometric picture:



A two-dimensional polyhedron (white) defined by 10 linear inequalities.

Canonical form:

Avoid having both $\leq$ and $\geq$.

Why?

So get something more like our first example:

$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for each } i = 1..n$$

$$x_j \geq 0 \quad \text{for each } j = 1..d$$

Or, given a vector $\vec{c}$, matrix A + vector $\vec{b}$:

Anything can be put into canonical form:

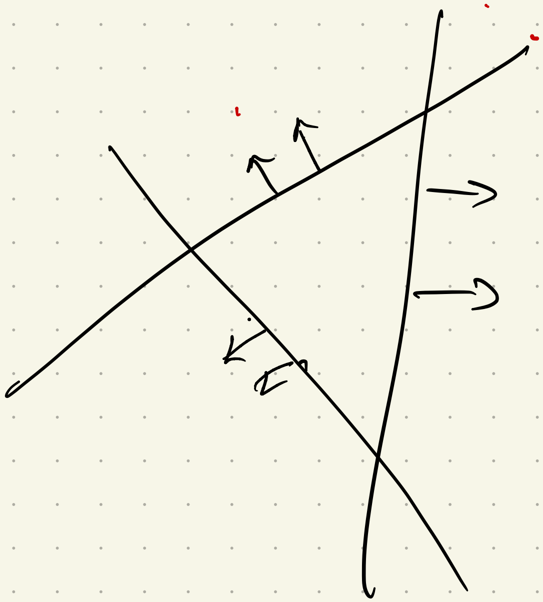
The reduction:

1. Avoid $=$:

2. Avoid $\geq$:

How could these not have a solution?

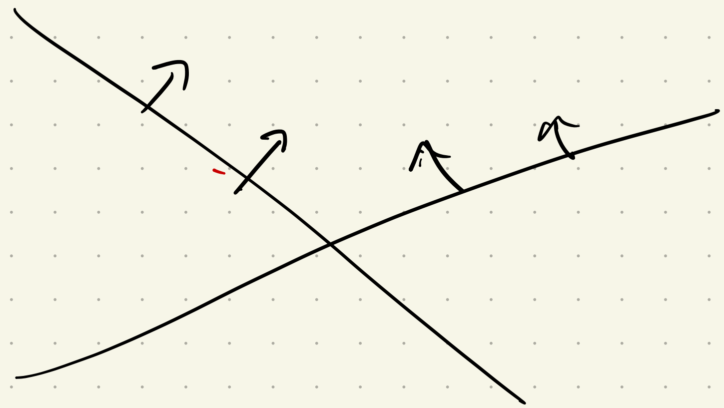
2 ways:



or

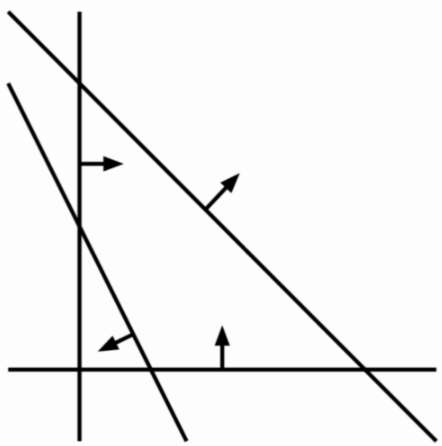
maximize $x_1 + x_2$

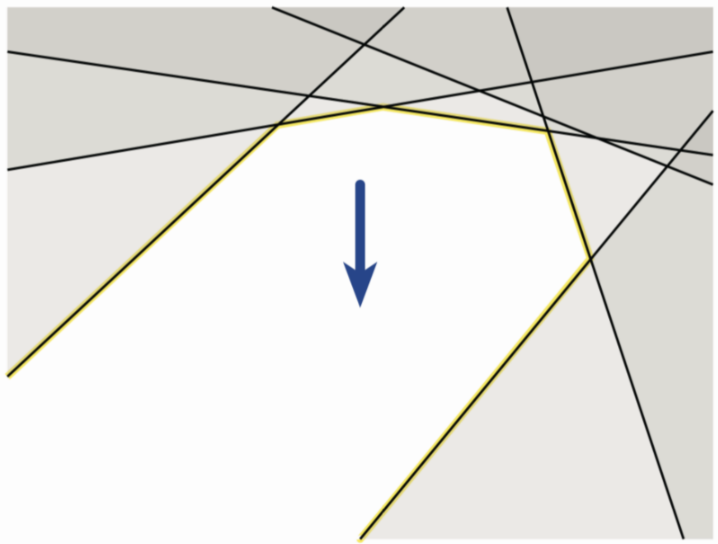
s.t.



Better pictures (still 2-d):

$$\begin{aligned} \text{maximize } & x - y \\ \text{subject to } & 2x + y \leq 1 \\ & x + y \geq 2 \\ & x, y \geq 0 \end{aligned}$$





Note:

1. Multiplying by -1 turns any maximization problem into a minimization one:

why?

2. Can turn inequalities into equalities via slack variables:

$$\sum_{i=0}^{n} a_i x_i \leq b \implies$$

(3) Can change equalities into inequalities, also!

$$\sum_{i=1}^{n} a_i x_i = b$$

$$\Downarrow$$

The algorithm: Simplex

Assumes canonical form:

$$\begin{aligned} &\text{maximize} \sum_{j=1}^{d} c_j x_j \\ &\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \le b_i \quad \text{for each } i = 1 \mathinner{..} n \\ &\qquad\qquad\qquad x_j \ge 0 \quad \text{for each } j = 1 \mathinner{..} d \end{aligned}$$

So:

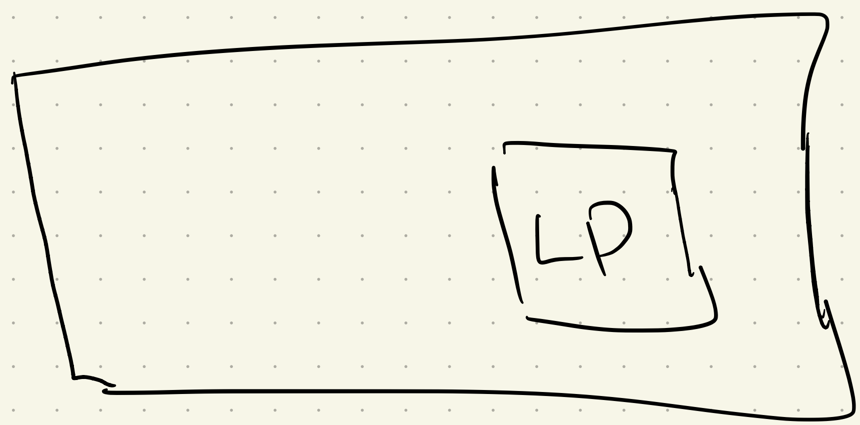
- no min
- only $\le$
- $x \ge 0$ for all variables

How to implement, plus runtime:

Connections to other problems:

It turns out that LPs are powerful enough to express many types of problems.

In a sense, we solve many problems by reducing them to an LP:



Ex: Flows & Cuts

Input: directed G w/ edge capacities $c(e)$ & $s, t \in V$

Goal: Compute flow $f: E \to \mathbb{R}$ s.t.

1. $0 \le f(e) \le c(e)$
2. $\forall v \ne s, t$,

$$\sum_u f(u \to v) = \sum_w f(v \to w)$$

Make an LP: Maximize

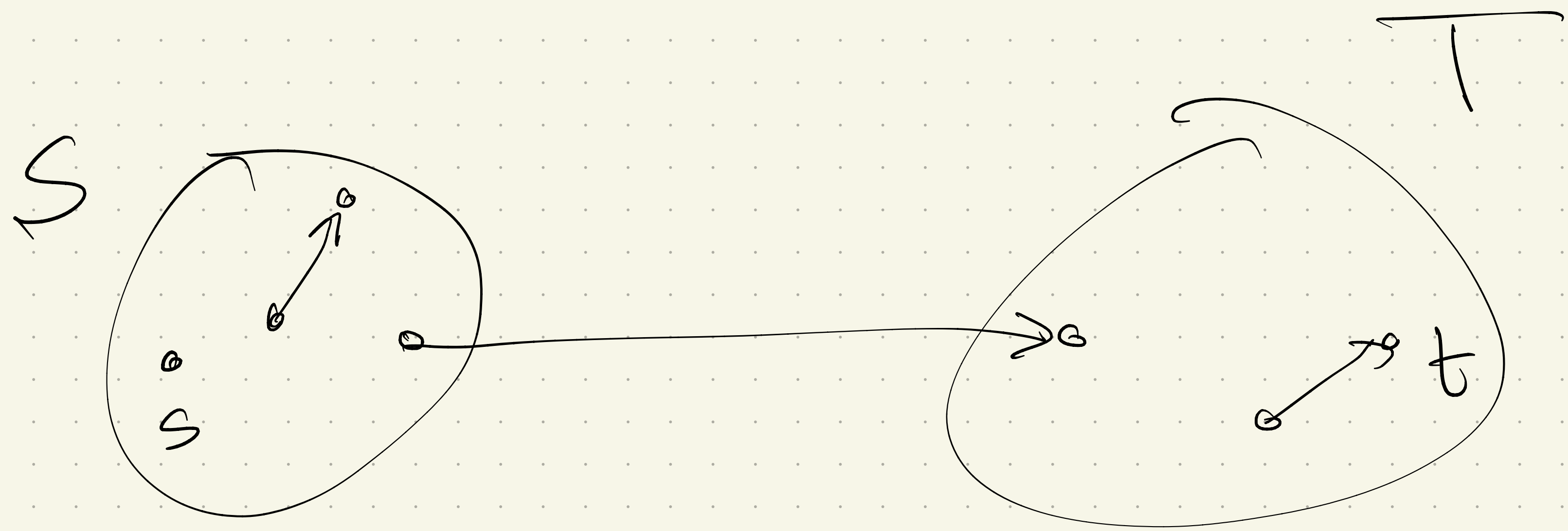
s.t.

Related: Min cuts (S,T)

Use indicator variables:

$$s_v = 0 \text{ or } 1$$

$$x_e = x_{(u\to v)} = 1 \quad \text{if } u \in S \text{ and } v \in T$$



The LP:

Minimize $\displaystyle\sum_{u\to v} c_{u\to v}\cdot x_{u\to v}$ ← want few edges

s.t.

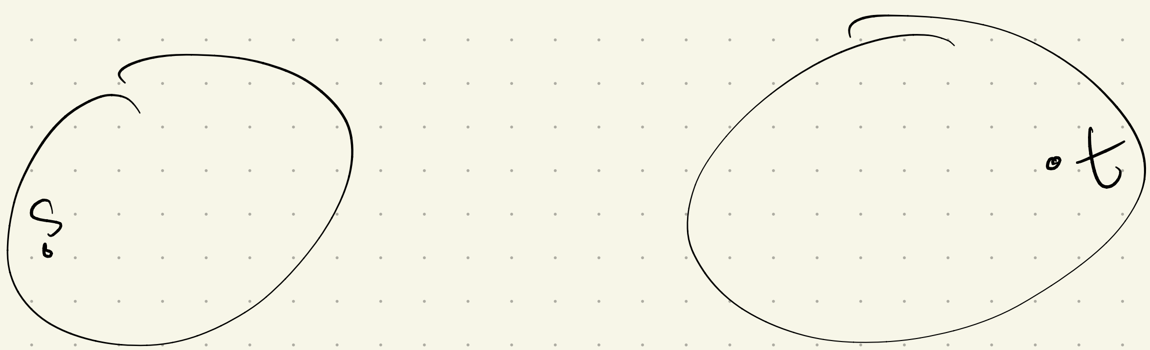
$$x_{u\to v} + s_v - s_u \geq 0 \quad \forall u\to v$$

Which are forced?

$$x_{u\to v} \geq 0 \quad \forall u,v$$

$$s_s = 1$$

$$s_t = 0$$



Note:

For flow/cuts, a solution would yield optimal LP solution.

The reverse is not obvious!

LP might have strange fractional answer which doesn't describe a cut.

It can be shown that this won't happen

↳ but not obvious...