Algorithms - Spring 2025

SSSPs

Recap

Computing a SSSP.

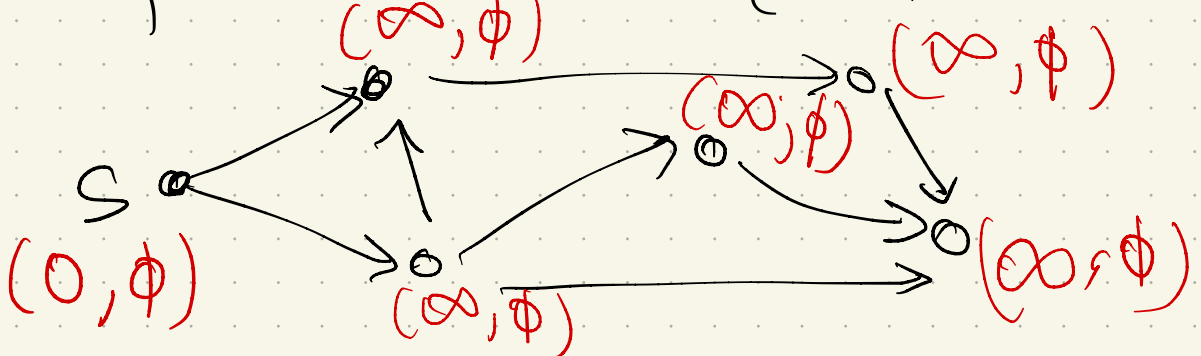
(Ford 1956 & Dantzig 1957)

Each vertex will store 2 values.
(Think of these as tentative shortest paths.)

(dist, pred)

- dist(v) is length of tentative shortest path $s \rightsquigarrow v$ (or ∞ if don't have an option yet)
- pred(v) is the predecessor of v on that tentative path $s \rightsquigarrow v$ (or NULL if none)

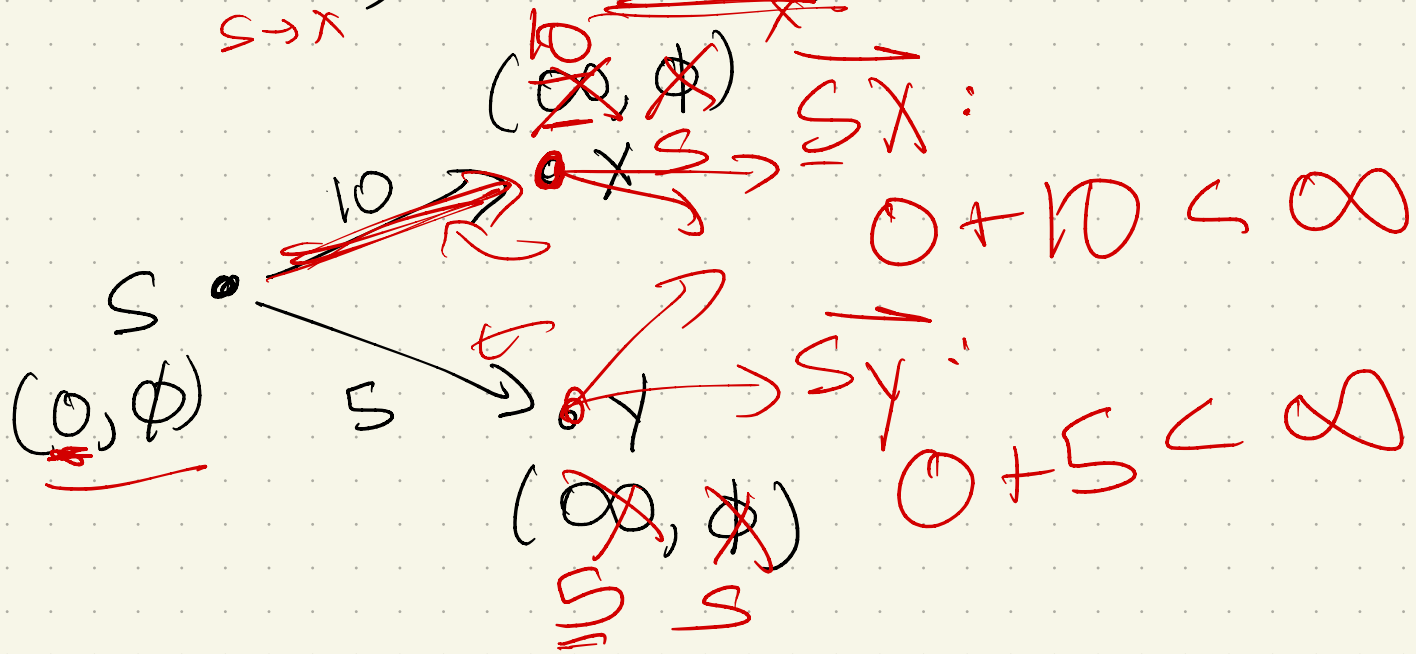
Initially:



We say an edge $\overrightarrow{uv}$ is tense if

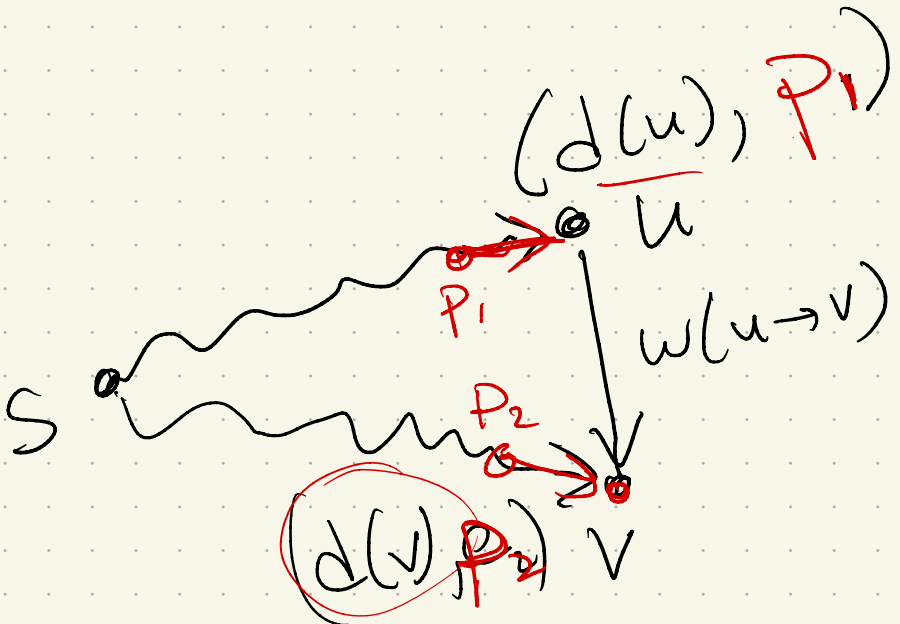
$$\text{dist}(u) + w(u \to v) < \text{dist}(v)$$

Initially:



Here:

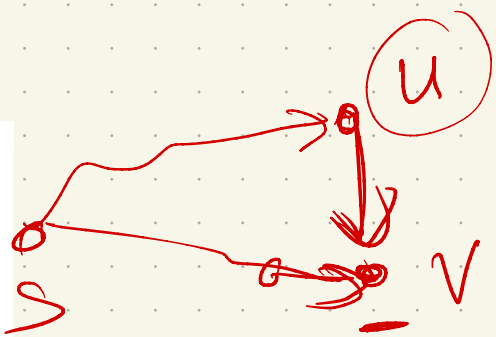
In general:



Key idea for algorithm:

Find tense edges & relax them:

```
RELAX(u→v):
    dist(v) ← dist(u) + w(u→v)
    pred(v) ← u
```



Then:

```
INITSSSP(s):
    dist(s) ← 0
    pred(s) ← NULL
    for all vertices v ≠ s
        dist(v) ← ∞
        pred(v) ← NULL
```

```
GENERICSSSP(s):
    INITSSSP(s)
    put s in the bag
    while the bag is not empty
        take u from the bag
        for all edges u→v
            if u→v is tense
                RELAX(u→v)
                put v in the bag
```

(0, ∅)

Dijkstra (59) → assume pos edges

(actually Leyzorek et al '57, Dantzig '58)

Make the bag a priority queue:

Keep "explored" part of the graph, S

Initally, $S = \{s\}$ + dist(s)=0

(all others NULL + ∞)

While $S \neq V$:

select node $v \notin S$ with one edge from S to v with:

$$\min_{e=(u,v),\ u \in S} (dist(u) + w(u \to v))$$ } tension!

Add v to S, set dist(v)+pred(v)

Let's formalize this a bit...

Correctness (w/pos edge weights!)

Thm: Consider the set S at any point in the algorithm

For each $u \in S$, the distance $dist(u)$ is the shortest path distance

(so $pred(u)$ traces a shortest path).

pf: Induction on $|S|$:

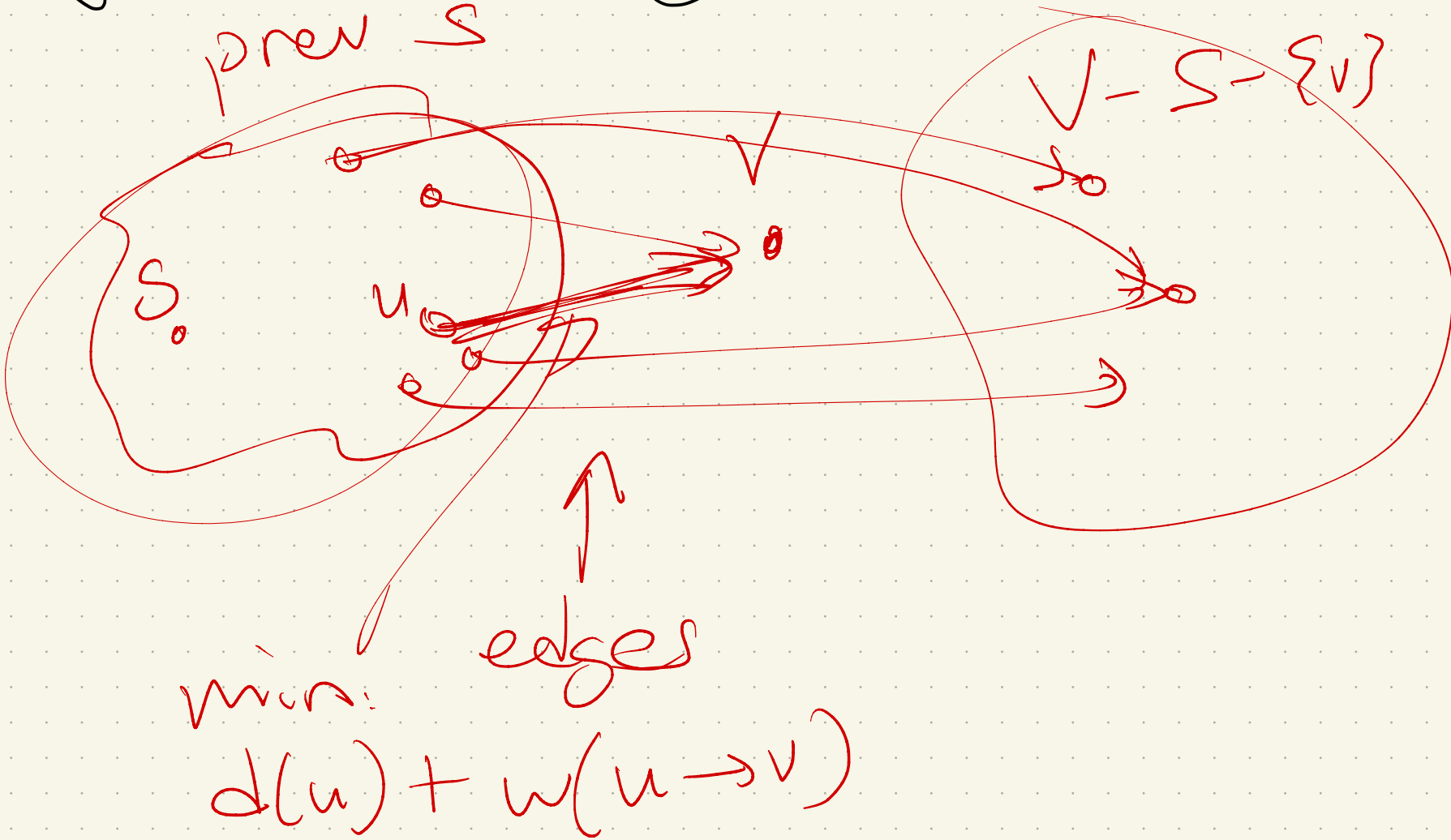
Base Case: $|S|=1$

$dist(s)=0$ ✓

IH: Spps claim holds when $|S|=k-1$.

Ind Step: Consider $|S| = k$:

algorithm is adding some v to S

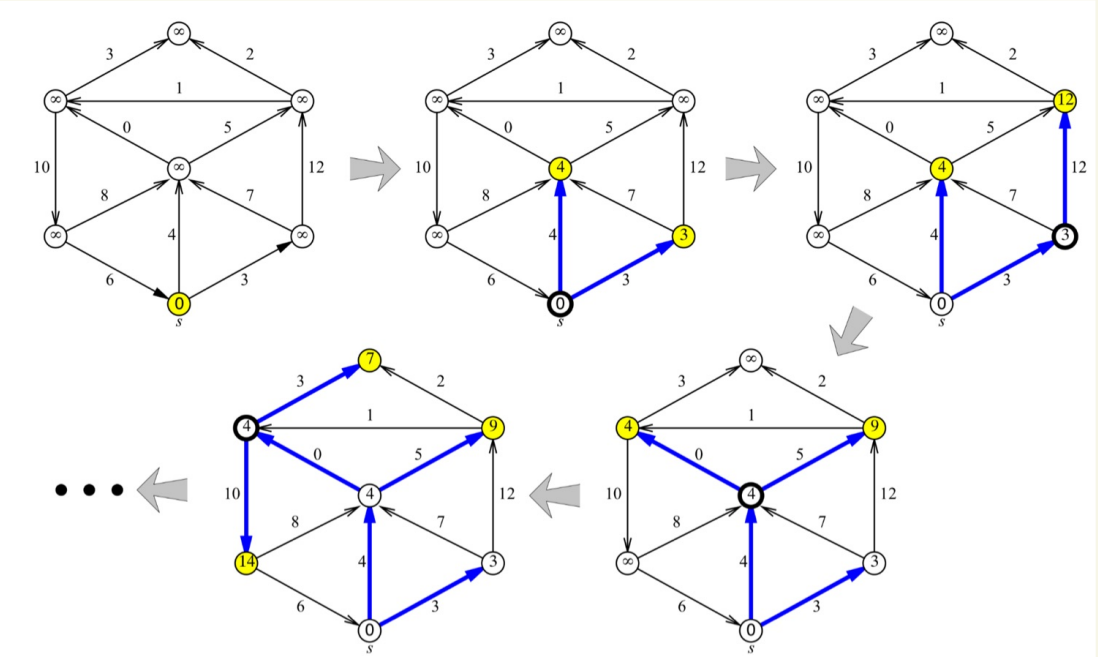


Book's implementation:

When v is added to S:

- look at v's edges and either insert w with key dist(v) + w(v→w)
- or update w's key, if dist(v) + w(v→w) beats current one

```
NonnegativeDijkstra(s):
  InitSSSP(s)
  for all vertices v
      Insert(v, dist(v))
  while the priority queue is not empty
      u ← ExtractMin( )
      for all edges u→v
          if u→v is tense
              Relax(u→v)
              DecreaseKey(v, dist(v))
```



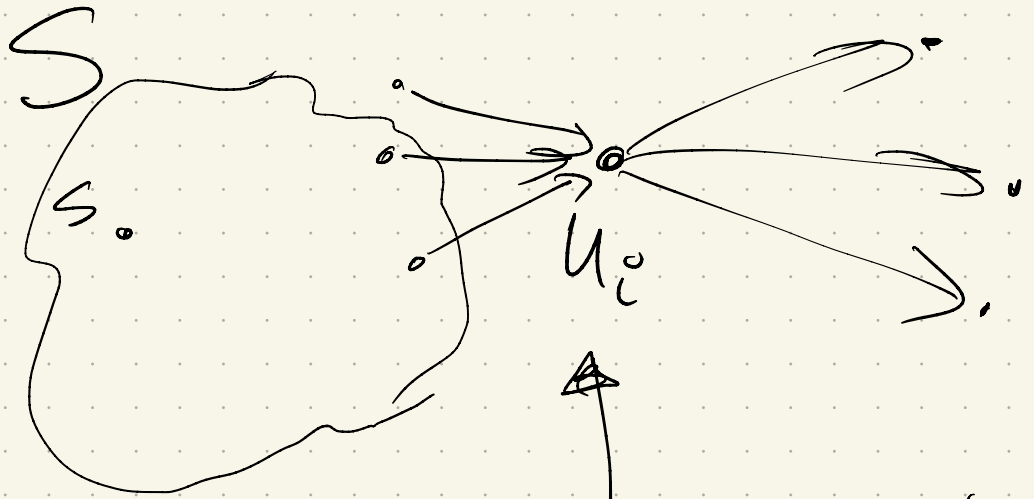
Four phases of Dijkstra's algorithm run on a graph with no negative edges.
At each phase, the shaded vertices are in the heap, and the bold vertex has just been scanned.
The bold edges describe the evolving shortest path tree.

Analysis: Let u_i be i^{th} vertex extracted from queue, & let d_i = value of $dist(u_i)$ when extracted.

Lemma: If G has no negative edges, then for all $i<j$, $d_i \leq d_j$.

Proof

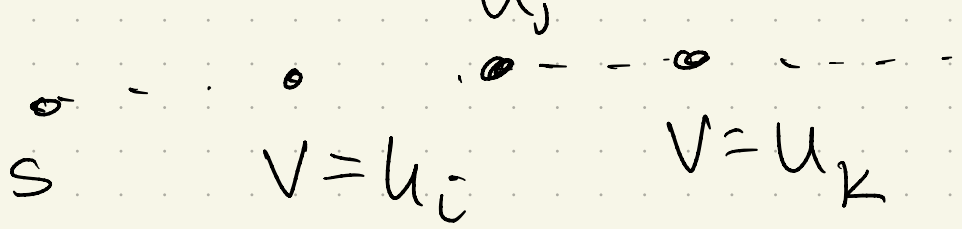
Fix an i:



current best in heap

Lemma: Each vertex is extracted from the heap once (or less)

Proof: Spps not:



prev lemma $\Rightarrow$ know $d_i \leq d_k$

But: v was readded to queue means some edge $u_j \to v$ became tense.

Runtime: In the end, runtime is $O(E \log V)$

Why?

decreasekey:

insert:

ExtractMin:

```
NonnegativeDijkstra(s):
  InitSSSP(s)
  for all vertices v
      Insert(v, dist(v))
  while the priority queue is not empty
      u ← ExtractMin( )
      for all edges u→v
          if u→v is tense
              Relax(u→v)
              DecreaseKey(v, dist(v))
```

Main downside:

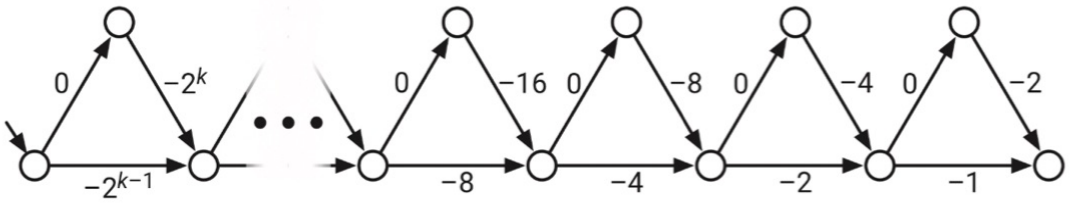


Figure 8.14. A directed graph with negative edges that forces DIJKSTRA to run in exponential time.

How to deal with negative edges?

Recall:

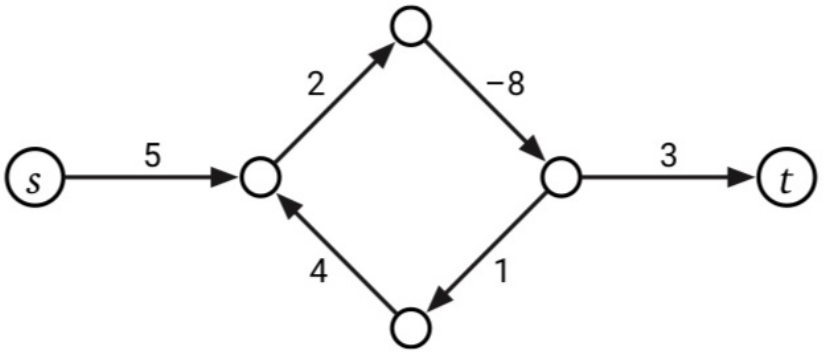


Figure 8.3. There is no shortest walk from s to t.

So two issues:

- Negative edges!
- Negative cycles:

Bellman-Ford:

Relax edges for a while.
Stop when every edge has been relaxed at least once

If any one is still tense:
you've relaxed ≥2 times!

Runtime:

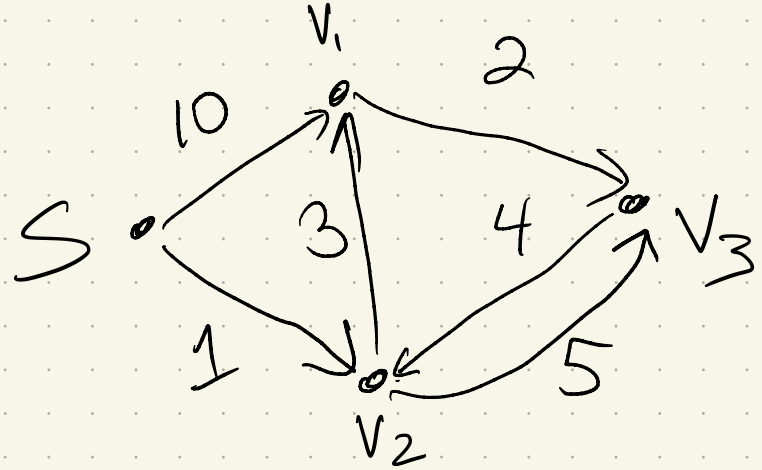
```
BellmanFord(s)
  InitSSSP(s)
  repeat V − 1 times
      for every edge u→v
          if u→v is tense
              Relax(u→v)
  for every edge u→v
      if u→v is tense
          return "Negative cycle!"
```

How to prove correctness?

Notation:

Let $dist_{\leq i}(v) :=$ length of shortest s to v path using $\leq i$ edges

Ex:



	s:	v_1:	v_2:	v_3:
≤ 1				
≤ 2				
≤ 3				
≤ 4				

Claim: $\forall v \neq i$, after i iterations of B-F,
$dist(v) \leq dist_{\leq i}(v)$

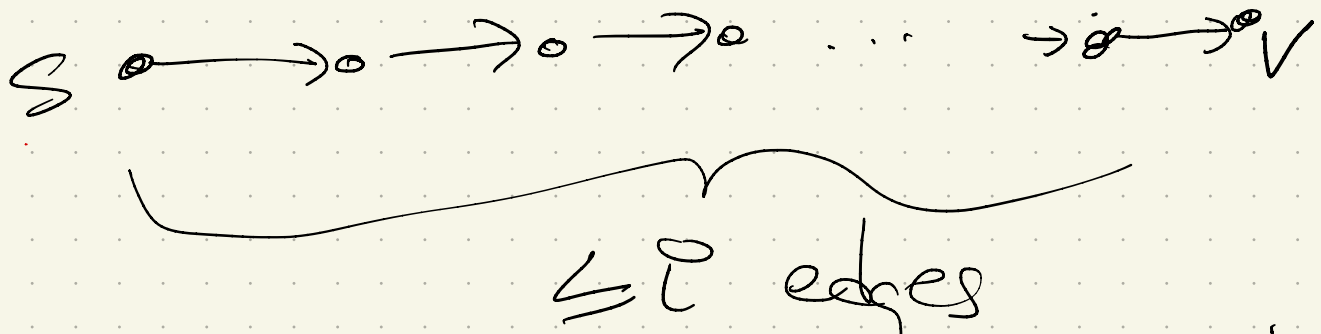
Induction on i:

BC: $i = 0$

IH: After $i-1$ iterations, all tentative guesses are $\leq d_{\leq i-1}(v)$.

IS: Now consider $d_{\leq i}(v)$:

built from a path $\rightarrow$



We know in round $i-1$, $dist(x) \leq d_{\leq i-1}(x)$ $\forall x \in V$.

Consider $u \to v$ in next round:

It was tense:

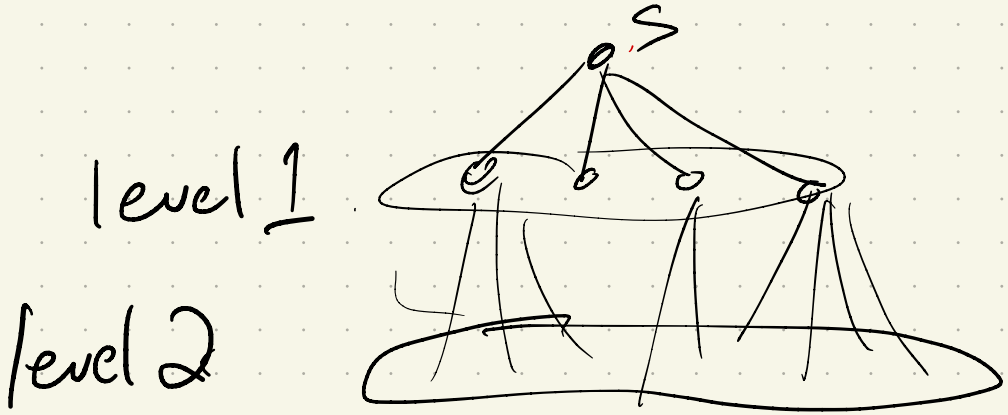
or not:

The rest: an (in practice) speed-up

BF looks at every edge.

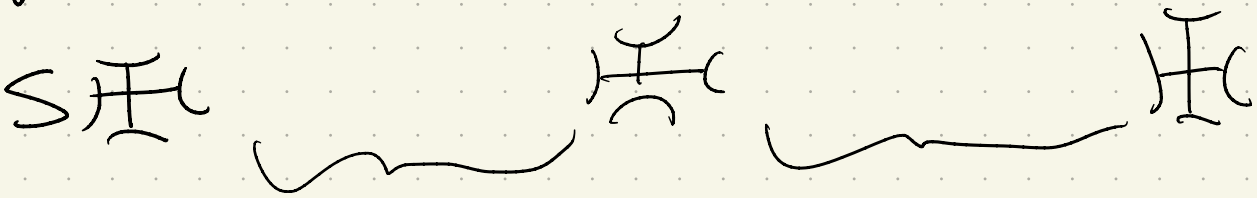
Do we need to?

Think of a BFS tree + "token"




```
MOORE(s):
  INITSSSP(s)
  PUSH(s)
  PUSH(✠)                    ⟨⟨start the first phase⟩⟩
  while the queue contains at least one vertex
      u ← PULL( )
      if u = ✠
          PUSH(✠)            ⟨⟨start the next phase⟩⟩
      else
          for all edges u→v
              if u→v is tense
                  RELAX(u→v)
                  if v is not already in the queue
                      PUSH(v)
```

queue:



Final version: Bellman's!

$$dist_{\le i}(v) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v = s \\ \infty & \text{if } i = 0 \text{ and } v \neq s \\ \min \left\{ \begin{array}{c} dist_{\le i-1}(v) \\ \min\limits_{u \to v} \left(dist_{\le i-1}(u) + w(u \to v) \right) \end{array} \right\} & \text{otherwise} \end{cases}$$

Why?? Using i again as # of edges in the path!

Since all paths are $\le V-1$,

$dist_{V-1}(v)$ is $dist(v)$

(assuming no negative cycles)

Nicer:

```
BellmanFordDP(s)
  dist[0,s] ← 0
  for every vertex v ≠ s
      dist[0,v] ← ∞
  for i ← 1 to V − 1
      for every vertex v
          dist[i,v] ← dist[i−1,v]
          for every edge u→v
              if dist[i,v] > dist[i−1,u] + w(u→v)
                  dist[i,v] ← dist[i−1,u] + w(u→v)
```

Later observation: Really don't need the i.

Just update those "tentative" distances, & trust it'll halt.

```
BellmanFordFinal(s)
  dist[s] ← 0
  for every vertex v ≠ s
      dist[v] ← ∞
  for i ← 1 to V − 1
      for every edge u→v
          if dist[v] > dist[u] + w(u→v)
              dist[v] ← dist[u] + w(u→v)
```

Runtime:

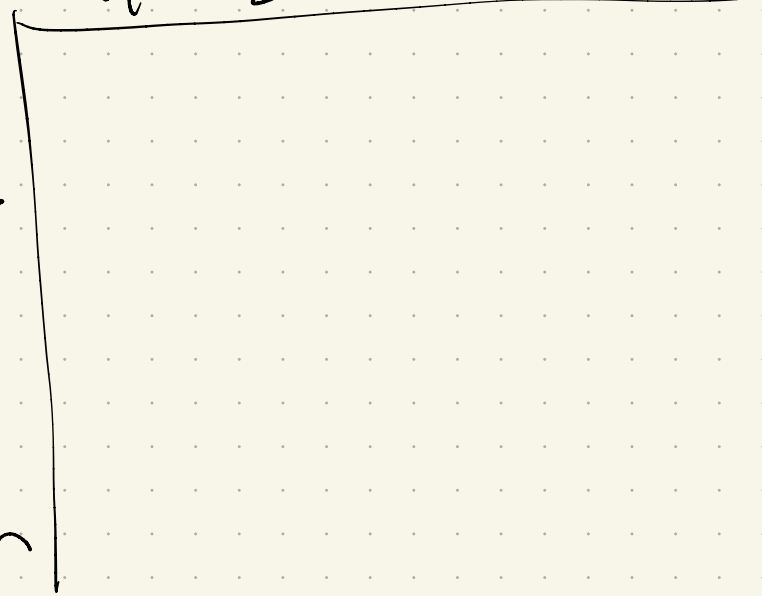
Same!

Next time:

SSSPs are nice, but:
What if we are doing lots of shortest path computations?

Goal: Precompute these, & store them!

How to store?



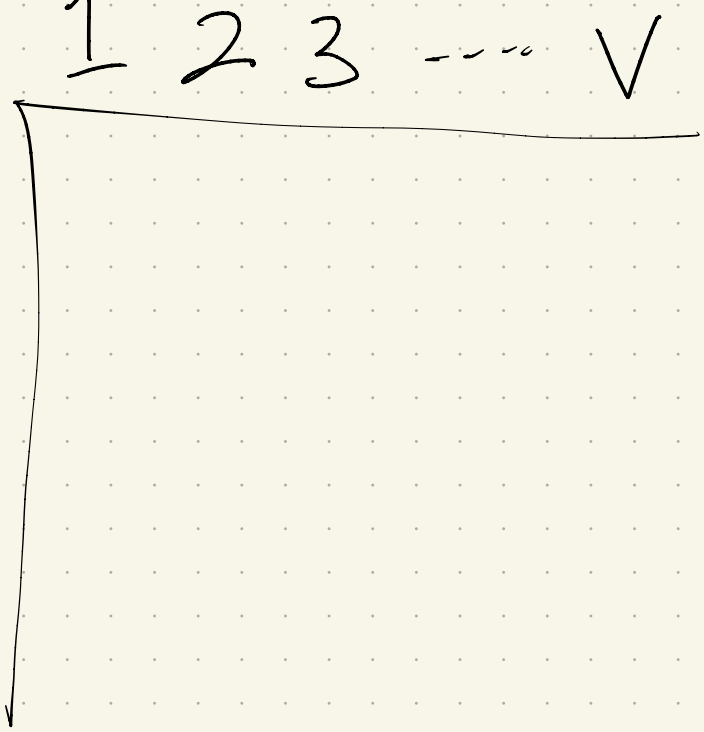
Lookup time:

But: how to compute?

Obvious answer

Well, we just designed two or three SSSP algorithms - use them!

MSSP(G):
  for each v ∈ G:
    run SSSP(v)
    store tree distances in dist[s, ·]



Can we do better?