

TDA - Fall 2025

Maps  
Morse theory



# Recap

- Overview of class
  - ↳ questions?
- At some point, check HW page for overview of the class project
- Office hours: Monday after class  
Tues or Thurs.

# Question from last time

→ Metric Space:

a pair  $(T, d)$ , where  $T$  is a set and  
 $d: T \times T \rightarrow \mathbb{R}$  satisfies other:  $d(p, q) \geq 0$

$$\bullet d(p, q) = 0 \Leftrightarrow p = q$$

$$\bullet d(p, q) = d(q, p) \quad \forall p, q \in T$$

$$\text{triangle inequality: } \bullet d(p, q) \leq d(p, r) + d(r, q) \quad \forall p, q, r \in T$$

Sometimes a 4<sup>th</sup>:  $\forall p, q \quad d(p, q) \geq 0$  ↪  
But: the first 3 imply the 4<sup>th</sup>.

Why? (exercise)

A topological space is **disconnected**  
if  $\exists$  2 disjoint nonempty open sets  
 $U, V \in T$  s.t.  $T = U \cup V$ .

(The space is **connected** if it is  
not disconnected.)

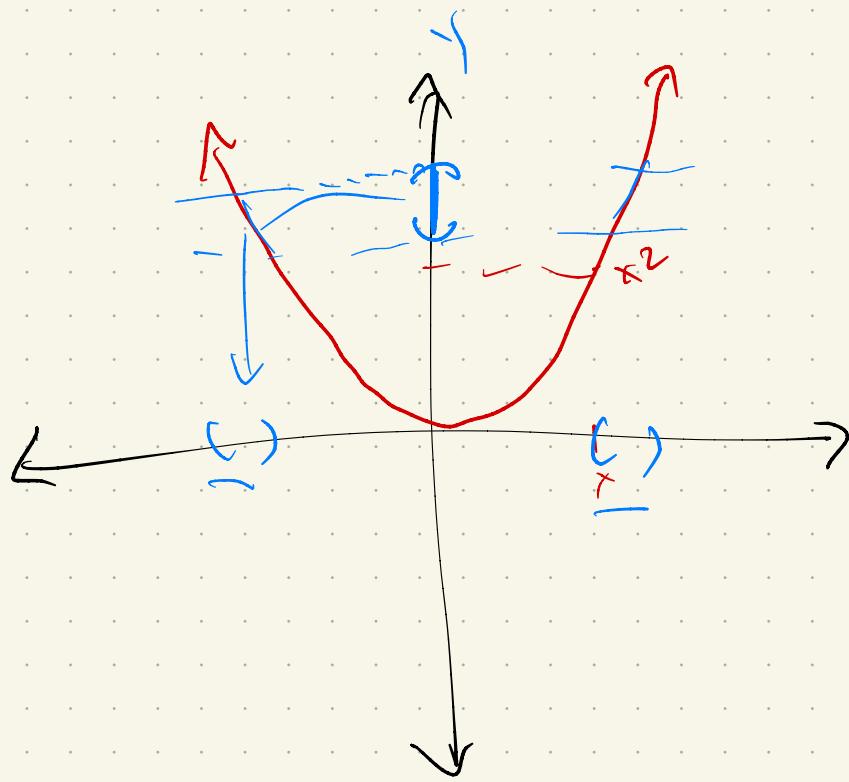
Ex:  $A = (1, 2) \cup (3, 4) \subset \mathbb{R}$

Note: **Subspace topology**: Given  $U \subseteq T$ ,  
 $U$  can inherit topology from  $T$  via  
 $\{x \cap U \mid x \in T\}$

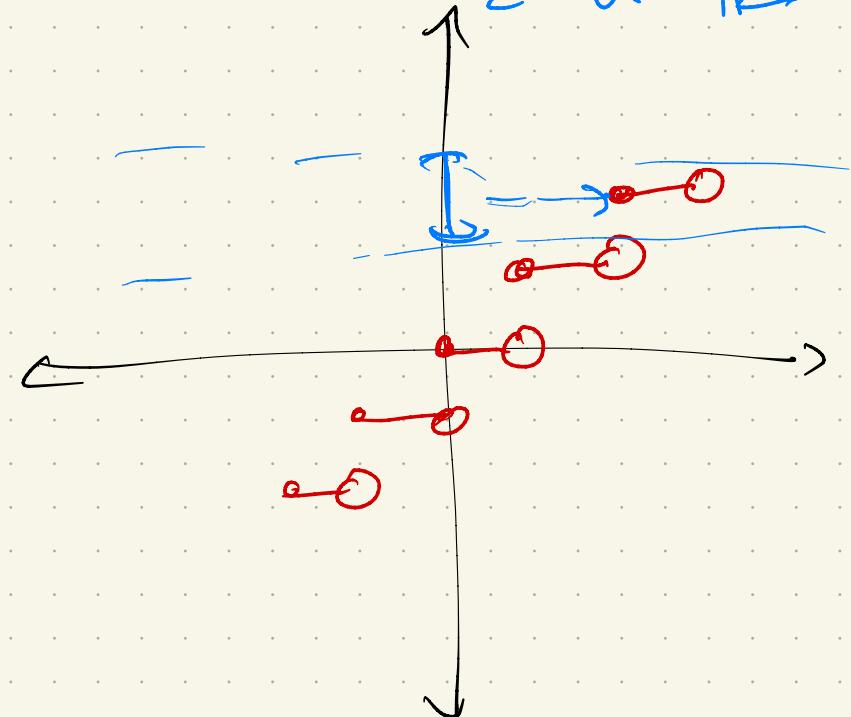
# Maps

A function  $f: T \rightarrow U$  is continuous if for every open set  $Q \subseteq U$ ,  $f^{-1}(Q)$  is open. (These are also called maps.)

Example:  $f: \mathbb{R} \rightarrow \mathbb{R}$   
 $f(x) = x^2$



Example:  $g: \mathbb{R} \rightarrow \mathbb{R}$   
 $g(x) = \lfloor x \rfloor$   
 $\leftarrow U = \mathbb{R}$



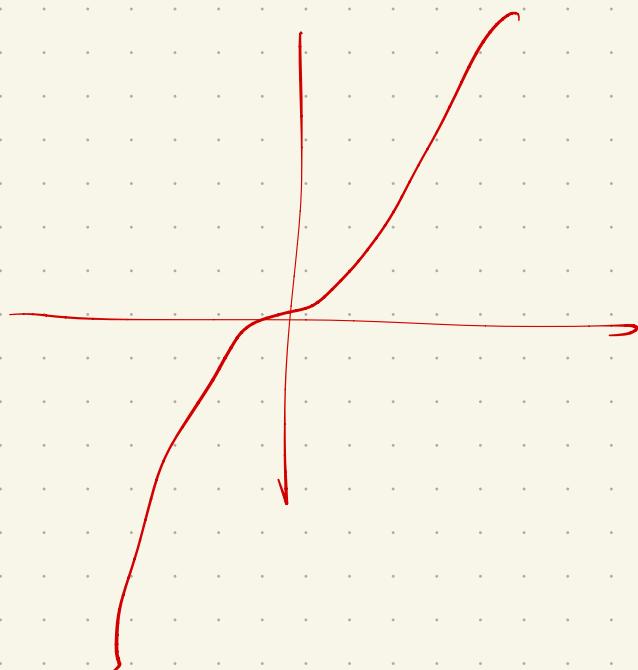
A map  $f: T \rightarrow U$  is an embedding  
of  $T$  into  $U$  if  $f$  is injective.

injective, or 1-1:

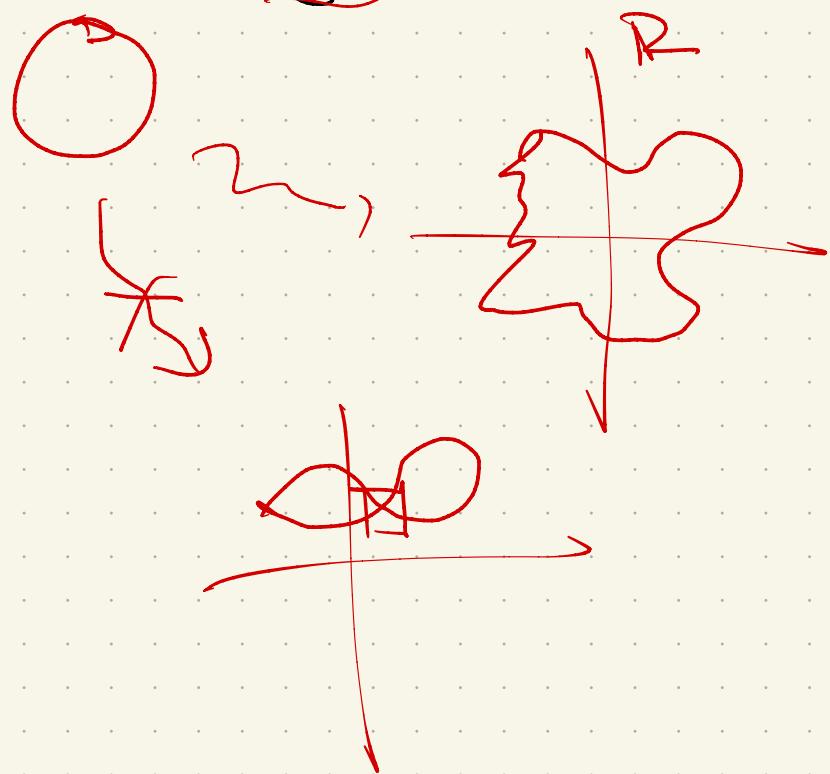
$$f(a) = f(b) \Leftrightarrow a = b$$

Example:  $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^3$$

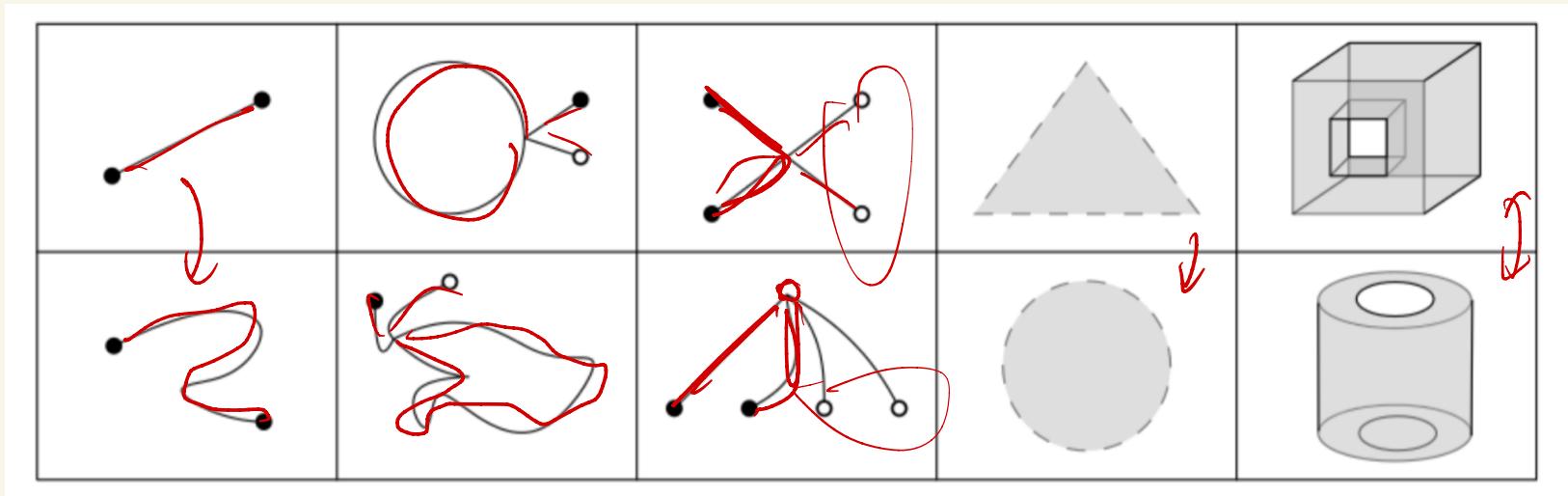


Example:  $g: S^1 \rightarrow \mathbb{R}^2$



Let  $T$  &  $U$  be topological spaces.  
A **homeomorphism**  $h: T \rightarrow U$  is a bijective  
map whose inverse is also continuous.  
(We say  $T$  &  $U$  are **homeomorphic** if  
such an  $h$  exists.)

Examples:

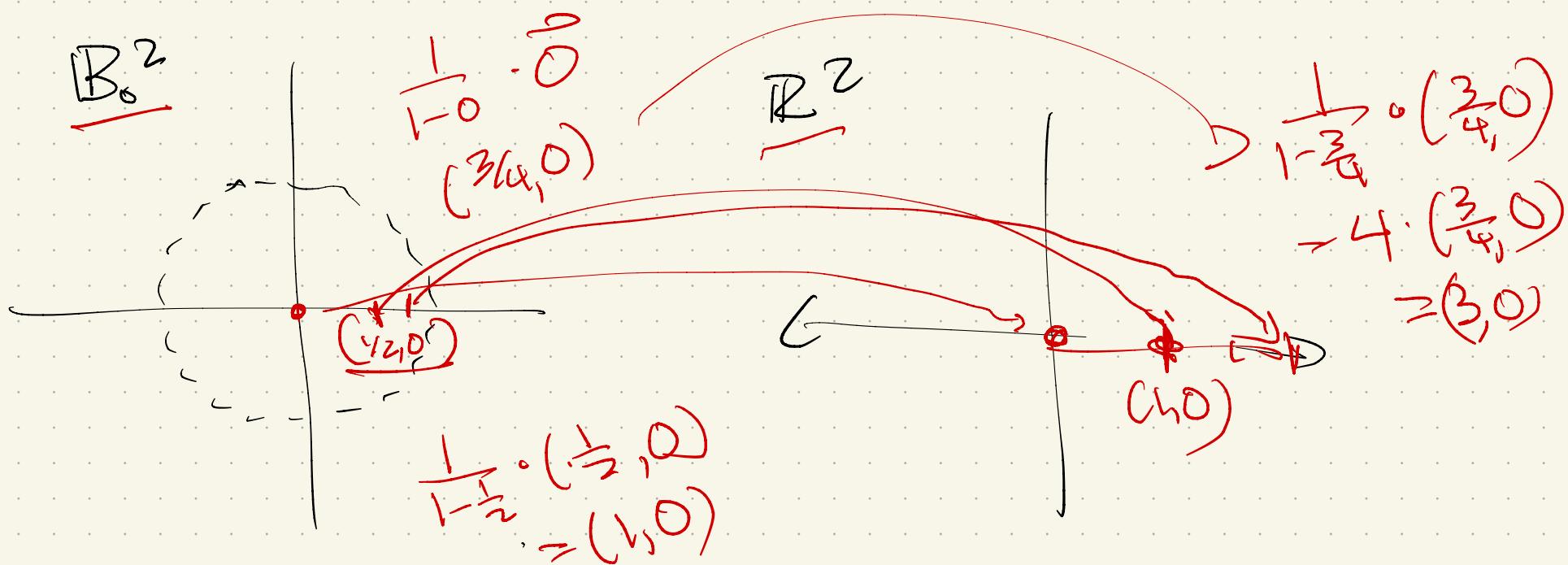


Note: requires construction of a function!

Example: open d-ball  $B_0^d$  and  $\mathbb{R}^d$

$$h(x) = \frac{1}{1-\|x\|} \cdot x$$

$$(so \ h^{-1}(y) = \frac{\sqrt{1+4\|y\|^2}-1}{2\|y\|^2} \cdot y \text{ if } y \neq 0, \text{ and } = 0 \text{ if } y = 0)$$



For nice enough spaces, a "cheap trick":

### Proposition

If  $T$  &  $U$  are compact metric spaces,  
every bijective map  $T \rightarrow U$  has  
a continuous inverse.

# Isotopy

When  $T, U$  are subspaces of a common topological space, can study something stronger:

An isotopy connecting  $T \subseteq \mathbb{R}^d + U \subseteq \mathbb{R}^d$  is a map  $\xi: T \times [0, 1] \rightarrow \mathbb{R}^d$  where

- $\xi(T, 0) = T$

- $\xi(T, 1) = U$

- $\forall t \in [0, 1]$ ,  $\xi(\cdot, t)$  is a homeomorphism from  $T$  to its image

Ambient isotopy: map  $\xi: \mathbb{R}^d \times [0, 1] \rightarrow \mathbb{R}^d$

Examples: For open d-ball again:

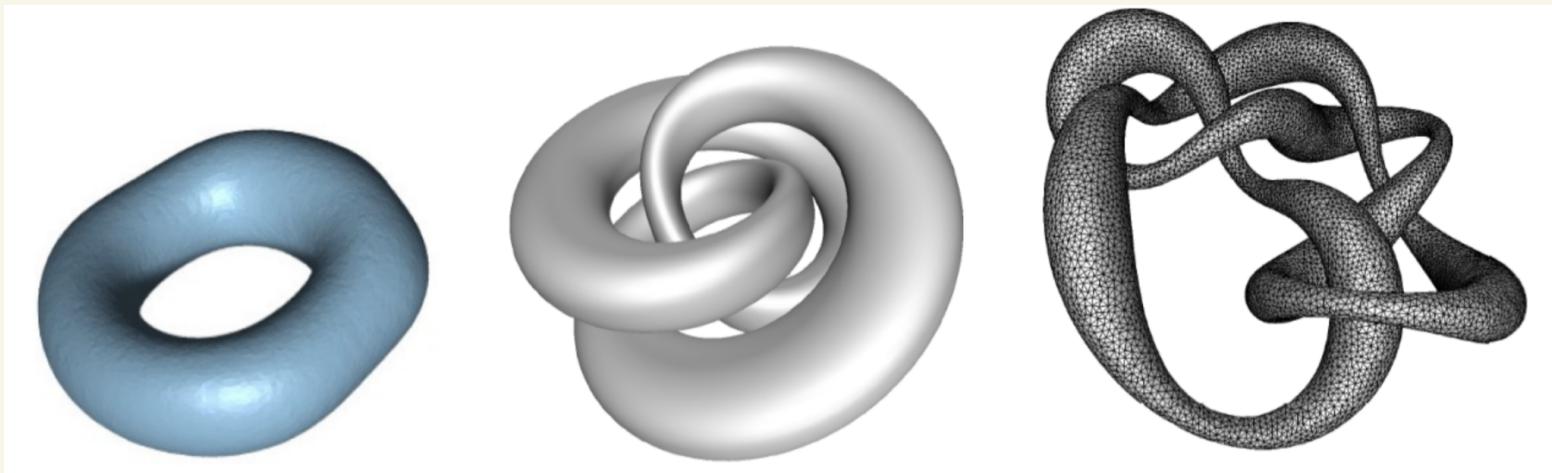
Consider  $\{f(x,t) = \frac{1 - (1-t)\|x\|^2}{1 - \|x\|^2} \cdot \vec{x}\}$

If  $t=0$ :  $\frac{1 - (1+0)^2\|x\|^2}{1 - \|x\|^2} \cdot \vec{x} = \vec{x}$

If  $t=1$ :  $\frac{1 - (1-1)\cdot\|x\|^2}{1 - \|x\|^2} \cdot \vec{x} = \frac{1}{1 - \|x\|^2} \cdot \vec{x}$

So  $B^d_0$  &  $\mathbb{R}^d$  are isotopic.

Homeomorphism  $\ll$  Isotopy  $\ll$  ambient  
isotopy:



Obstruction comes from the ambient  
space:  $\mathbb{R}^3 \setminus$  knot here

# Homotopy

Consider maps  $g: X \rightarrow U$  and  $h: X \rightarrow U$ .

A homotopy is a map  $H: X \times [0, 1] \rightarrow U$

such that  $H(\cdot, 0) = g$  and  $H(\cdot, 1) = h$

Example:

$$g: \overline{B}_0^3 \rightarrow \overline{\mathbb{R}}^3$$

$$g: \overline{B}_0^3 \hookrightarrow \overline{\mathbb{R}}^3$$

Inclusion map  $h(\vec{x}) = \vec{x}$

$$h: \overline{B}_0^3 \rightarrow \overline{\mathbb{R}}^3, h(\vec{x}) = \vec{0}$$

homotopy:  $H(x, t) = \underbrace{(1-t) \cdot \vec{x}}$

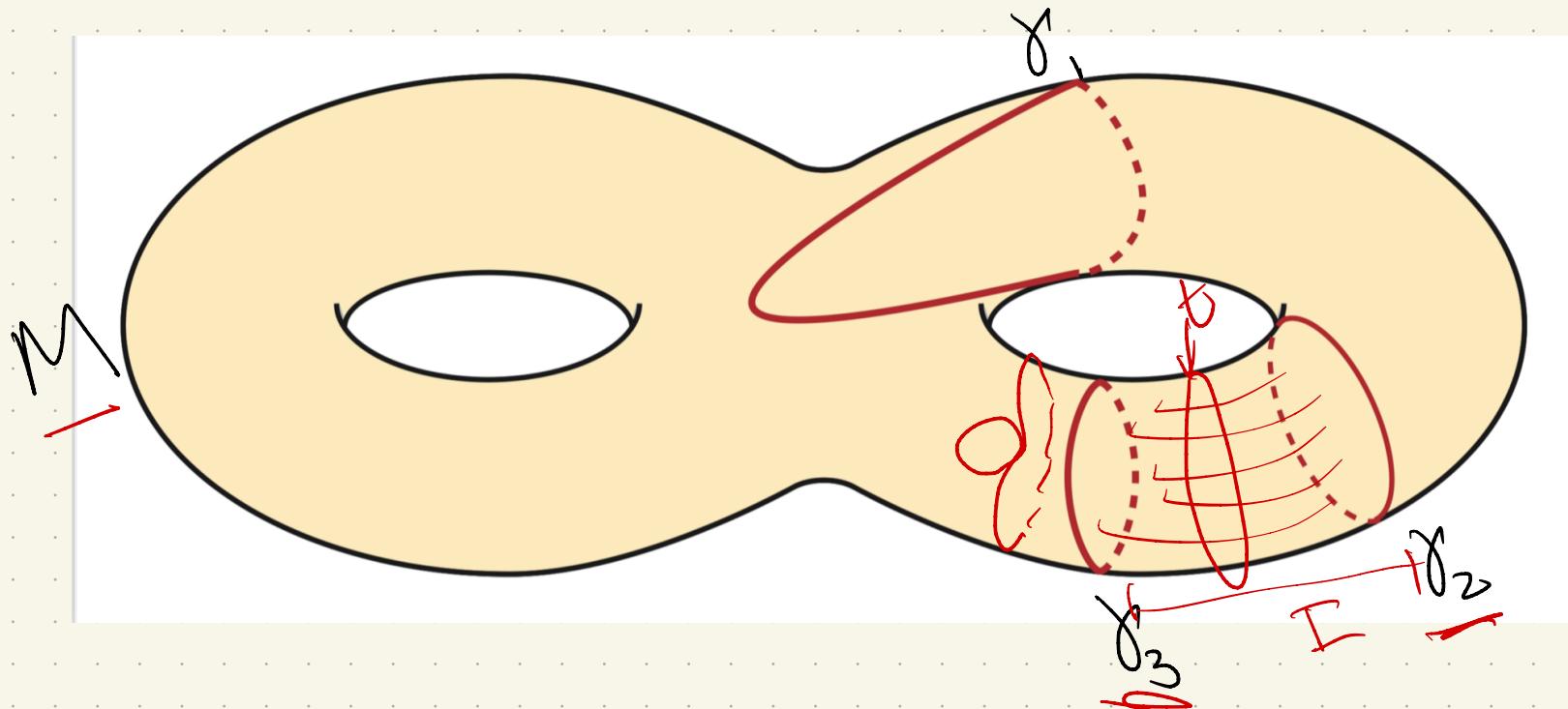
check:  $H(\cdot, 0) = (1-0) \cdot \vec{x} = \vec{x}$

$$H(\cdot, 1) = (1-1) \cdot \vec{x} = \vec{0}$$

\* between:

Another: Curves on Surfaces

$$\gamma_1, \gamma_2, \gamma_3: S^1 \rightarrow M$$



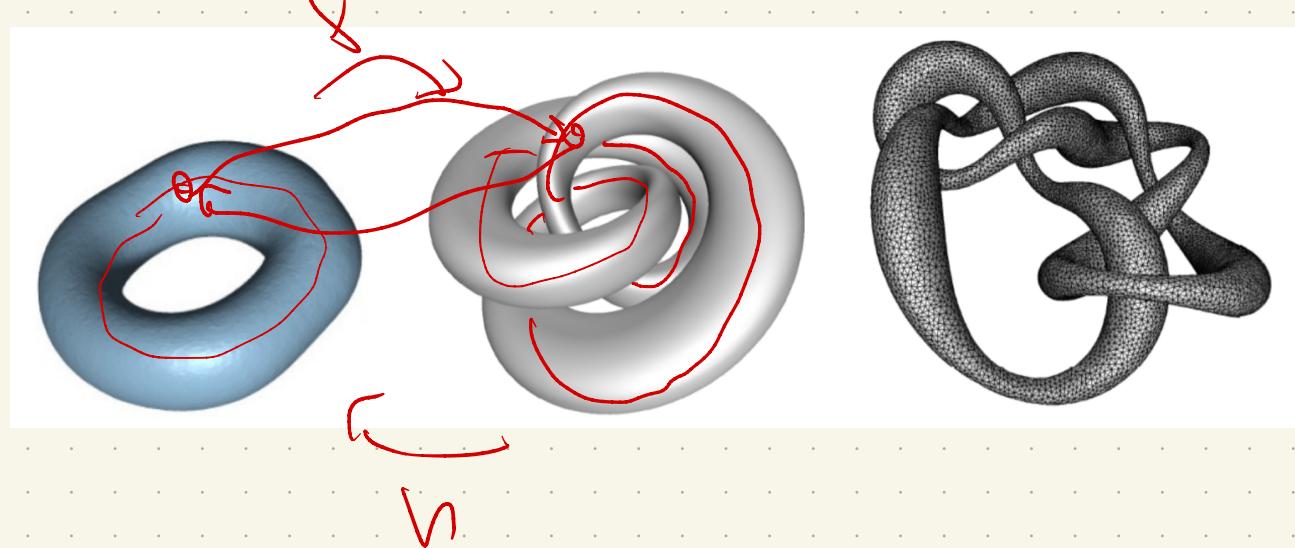
Here, homotopy  $H: S^1 \times [0,1] \rightarrow M$

$$H(\cdot, 0) = \overline{\gamma_3} \quad H(\cdot, 1) = \overline{\gamma_2}$$

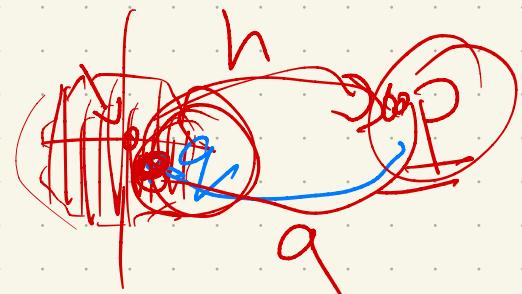
## Homotopy equivalence

Two topological spaces  $T$  &  $M$  are homotopy equivalent if  $\exists g: T \rightarrow M$  and  $h: M \rightarrow T$  such that  $h \circ g$  and  $g \circ h$  are homotopic to identity maps

Example:



Another:  $B_0^2$  and any point  $P$ .



$$h: B_0^2 \rightarrow \{P\}, \quad h(x) = P$$

$$g: \{P\} \rightarrow B_0^2, \text{ with } g(P) = q \leftarrow$$

(an arbitrary point in  $B_0^2$ )

$h \circ g: \{P\} \rightarrow \{P\}$  ✓

$g \circ h: B_0^2 \rightarrow B_0^2$

sends every  $x \in B_0^2$  to  $q$

Homotopy:  $\underline{H}(x, t) = \underline{(1-t) \cdot g + t \cdot x}$

at  $t=0$ :  $q$

at  $t=1$ :  $t \cdot x = x$

## Retracts

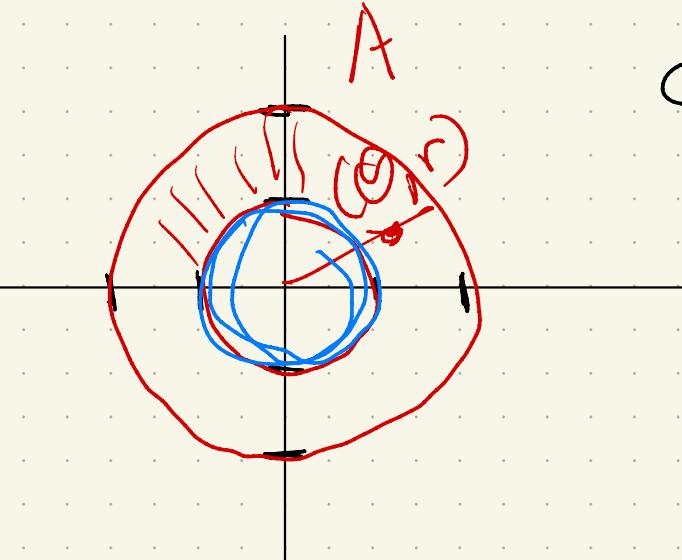
Consider  $T$  a topological space, &  
 $U \subseteq T$  a subspace.

A retraction  $r$  of  $T$  to  $U$  is a

map  $\underline{f: T \rightarrow U}$  s.t.  $\underline{f(x) = x} \quad \forall x \in U$ .

Example: annulus  $\rightarrow A = \{(0, r) \mid 1 \leq r \leq 2\}$   
and  $0 \in [0, 2\pi]\}$

circle  $S^1 = \{(0, 1) \mid 0 \in [0, 2\pi]\}$



How to make  $f$ ?

$$f(\theta, r) = (\theta, 1)$$

## Deformation Retract

$U \subseteq T$  is a deformation retract if the identity map on  $T$  can be continuously deformed to a retraction with no change to points in  $U$ . More precisely:

$\exists$  homotopy

$$\textcircled{R}: T \times [0, 1] \rightarrow T \text{ s.t.}$$

- $R(\cdot, 0) = \text{Id}_T$  ↪
- $\text{R}(\cdot, 1)$  is a retraction  $T \rightarrow U$
- $R(x, t) = \underline{x}$  for every  $x \in U$  and  $t \in [0, 1]$ .

Try previous example:

annulus  $A = \{(0, r) \mid 1 \leq r \leq 2$   
and  $\theta \in [0, 2\pi]\}$

circle  $S^1 = \{(0, 1) \mid \theta \in [0, 2\pi]\}$

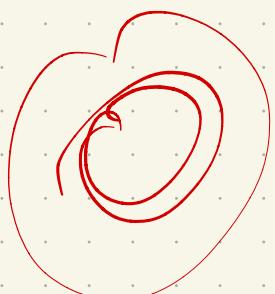
Set  $R((0, r), t) = (0, \underline{(1-t) \cdot r + t})$

Check 3 things:

If  $t=0$ :  $(0, r+t)$

If  $t=1$   $(0, t)$

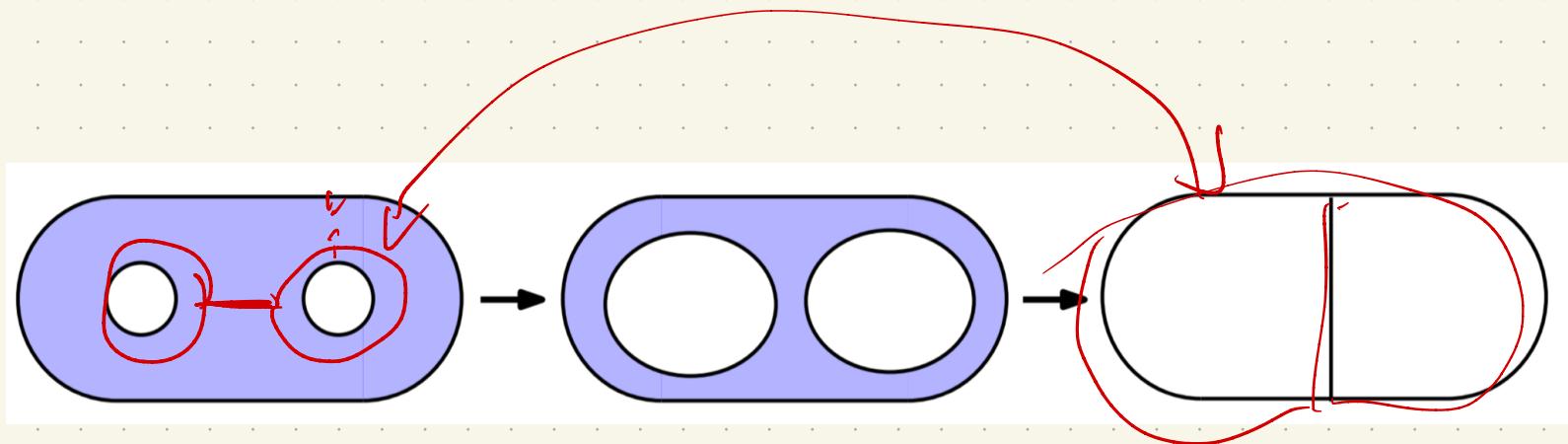
$$R((0, 1), t) = (0, (1-t) \cdot 1 + t) \\ = (0, 1)$$



Why care??

Theorem If  $U$  is a deformation retract of  $\Pi$ , then  $\Pi + U$  are homotopy equivalent.

Ex:



(Note: and are homotopy equivalent, but no deformation retract.)

# Manifolds

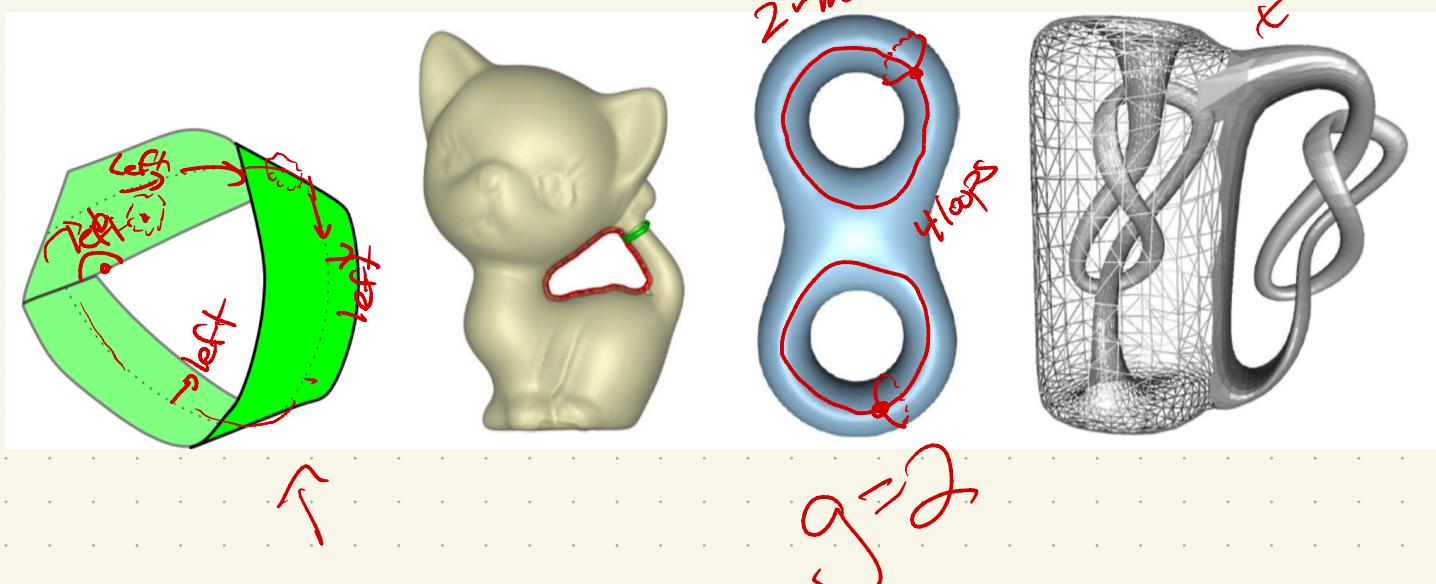
A topological space is an  $m$ -manifold if every  $x \in M$  has a point homeomorphic to the  $m$ -ball  $B_o^m$  or the  $m$ -halfspace  $H^m$ :

$$B_o^m = \{y \in \mathbb{R}^m \mid \|y\| < 1\}$$

$$H^m = \{y \in \mathbb{R}^m \mid \|y\| < 1 \text{ and } y_m \geq 0\}$$

$$\mathbb{B}_o^2$$

$$H^2$$



## Notation / terminology

- Boundary : look like  $H^d$
- Surface : 2-manifold
- Non-orientable : walk along a curve starting on one side.  
If you could end up on other side when you return  $\rightarrow$  non-orientable
- Loop : 1-manifold, no boundary  $R$  
- Genus  $g$  :  $\exists$  a set of  $2g$  loops which can be removed without disconnecting it.

# Smooth

Topological manifolds are spaces  
But usually, consider an embedding  
into Euclidean space  $\Rightarrow$  geometry.

Given a smooth function  $f: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  
the gradient vector field  $\nabla f: \mathbb{R}^d \rightarrow \mathbb{R}^d$   
at a point  $x$  is:  
$$\nabla f = \left[ \underbrace{\frac{\partial f}{\partial x_1}(x)}, \dots, \underbrace{\frac{\partial f}{\partial x_d}(x)} \right]$$

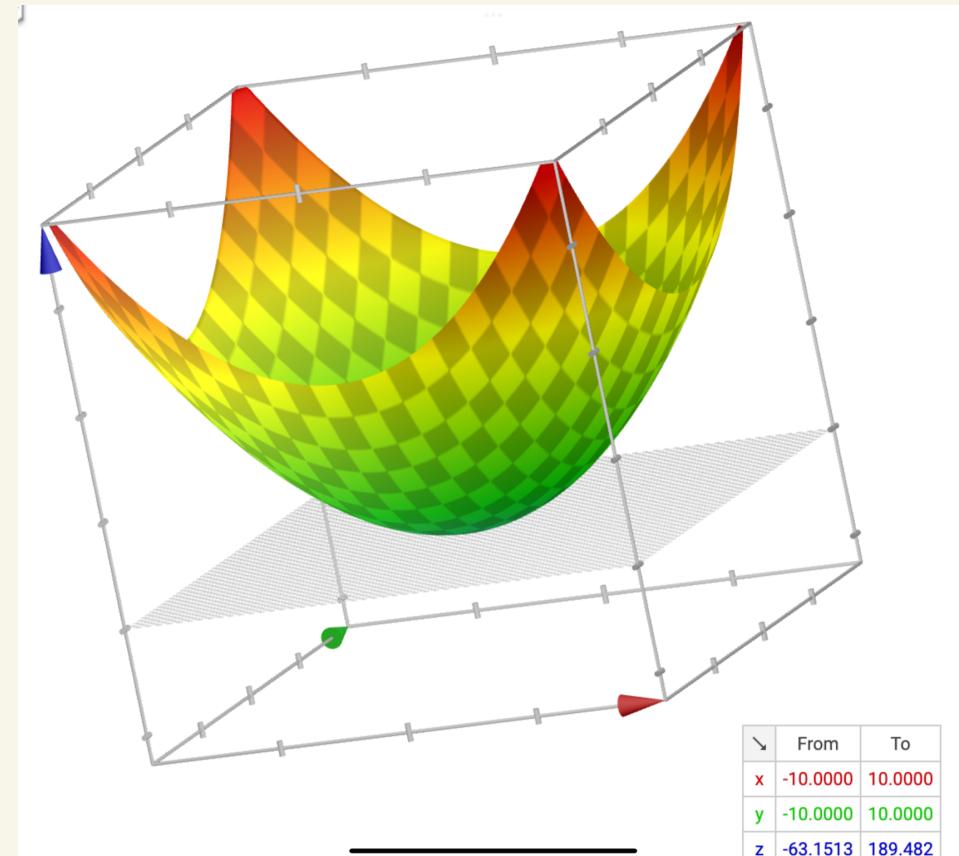
Ex:  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$\nabla f =$$

$$\left[ \frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f \right]$$

$$\Rightarrow [2x_1, 2x_2]$$



Then  $\nabla f(0,0) = [0,0]$

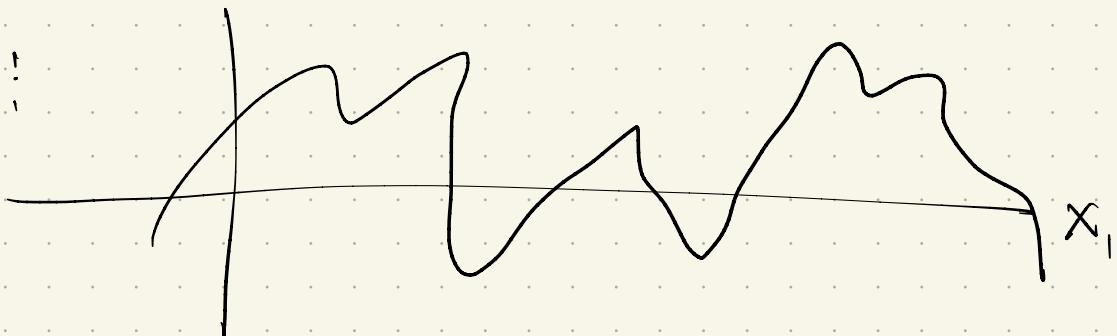
$$\nabla f(1,0) = [2,0]$$

## Critical point

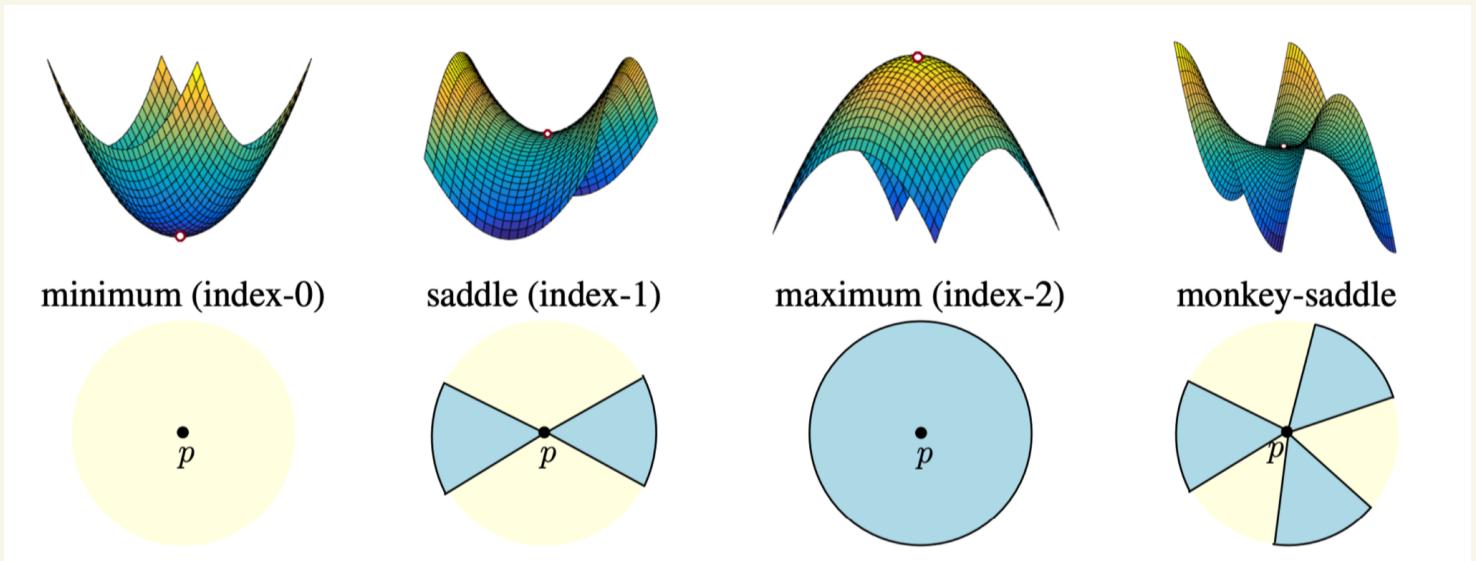
Any  $p \in \mathbb{R}^d$  where  $\nabla f(p) = \vec{0}$   
(Otherwise we say  $p$  is regular)

On 1 manifolds:

$$\frac{\partial f}{\partial x} \cdot x = 0$$



On 2 manifolds:



Extending to manifolds:

Given  $\phi: U \rightarrow W$ ,  $U \subseteq \mathbb{R}^k$  &  $W \subseteq \mathbb{R}^d$   
open sets, where

$$\phi(x) = (\phi_1(x), \dots, \phi_d(x))$$

The Jacobian of  $\phi$  is a  $d \times k$   
matrix of partial derivatives:

$$\begin{bmatrix} \frac{\partial \phi_1(x)}{\partial x_1} & \cdots & \frac{\partial \phi_1(x)}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_d(x)}{\partial x_1} & \cdots & \frac{\partial \phi_d(x)}{\partial x_k} \end{bmatrix}$$

## Types of critical points

For a smooth  $m$ -manifold, the Hessian matrix of  $f: M \rightarrow \mathbb{R}$  is the matrix of 2nd order partial derivatives:

$$\text{Hessian}(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix}$$

A critical point is non-degenerate if Hessian is nonsingular ( $\det \neq 0$ ); otherwise degenerate.

An example:  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x_1, x_2) = x_1^3 - 3x_1x_2^2$$

$$\nabla f =$$

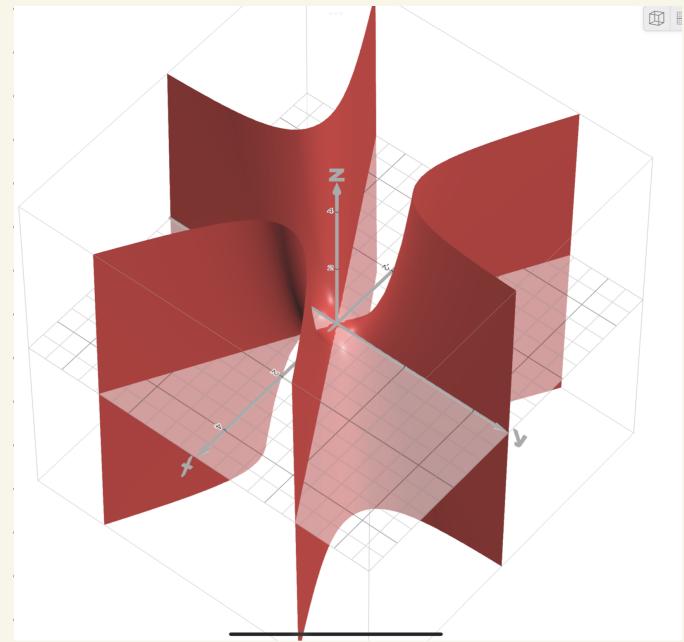


Is it degenerate?

Hessian:

$$\begin{pmatrix} \frac{\partial}{\partial x_1 \partial x_1} & \frac{\partial}{\partial x_1 \partial x_2} \\ \frac{\partial}{\partial x_2 \partial x_1} & \frac{\partial}{\partial x_2 \partial x_2} \end{pmatrix} =$$

So at  $(0,0)$ ,  $\det =$



## Morse Lemma

Given a smooth function  $f: M \rightarrow \mathbb{R}$  defined on a smooth manifold  $M$ , let  $p$  be a non-degenerate critical point of  $f$ . Then  $\exists$  a local coordinate system in a neighborhood  $U(p)$  s.t.

- $p$ 's coordinate is  $\overset{\rightharpoonup}{0}$

- locally, any  $x$  is in the form

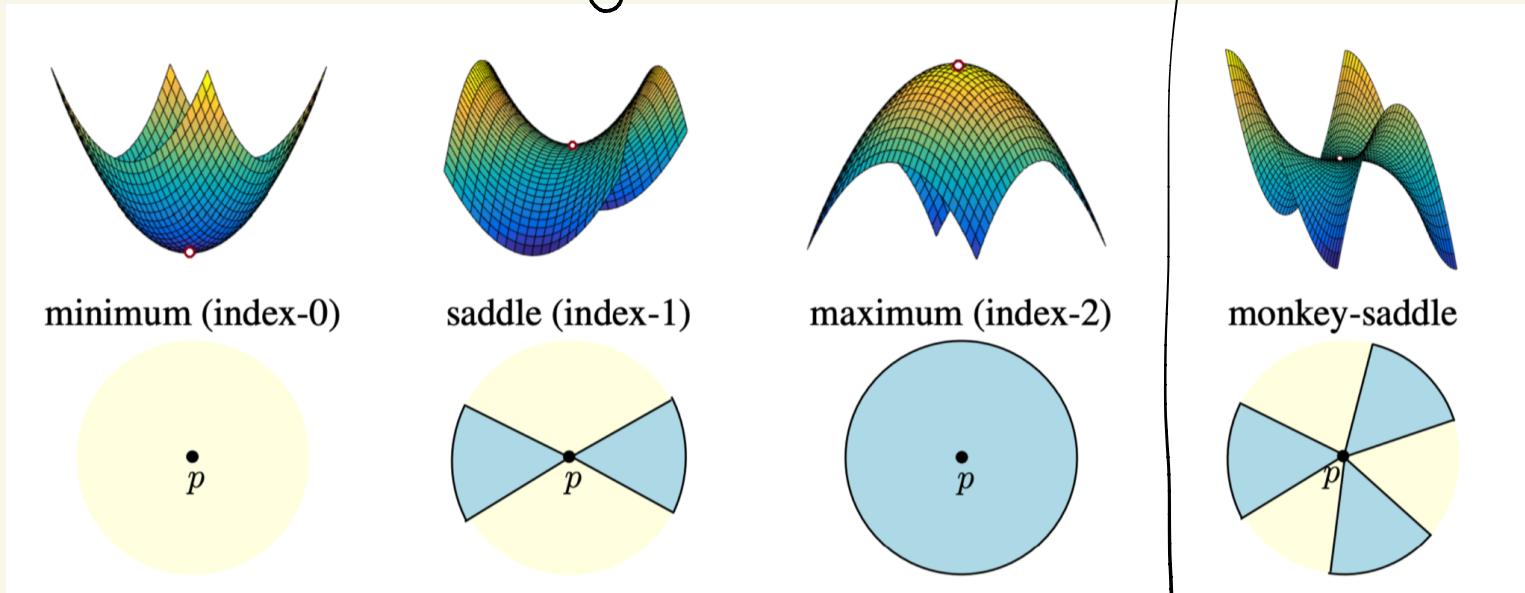
$$f(x) = f(p) - x_1^2 - \dots - x_s^2 + x_{s+1}^2 + \dots + x_m^2$$

for some  $s \in [0, m]$

$s$  is called the **index** of  $p$ .

Back to that picture...  
non-degenerate

degenerate



↑  
everything is  
bigger around p

↑  
everything is  
smaller around p  
One coordinate bigger,  
one smaller

Next time:

Why we care about Morse  
theory...