Recap

- HW2 - due in 1 week
- Next week: special assignment
  Intro: the AARTN
- Next week:

And back to persistence...

Induced maps on homology

Each $K_i \hookrightarrow K_{i+1}$, so we get induced maps $H_p(K_i) \to H_p(K_{i+1})$.

Homology module (simplicial case):

$H_p(\mathcal{F}(K))$:

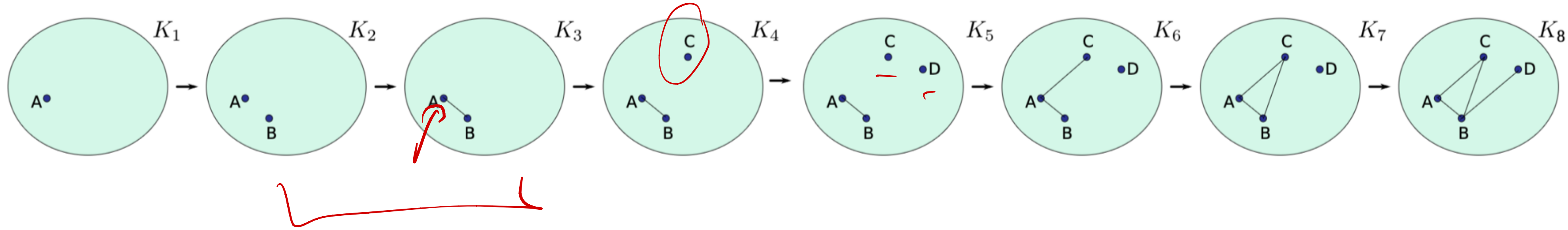
$$\emptyset = H_p(K_0) \to H_p(K_1) \to \cdots \to H_p(K_n) = H_p(K)$$

inclusion

$i, j \to f_p^{i,j} : H_p(K_i) \to H_p(K_j)$

What do these capture?

Now let's try tracking generators of homology!

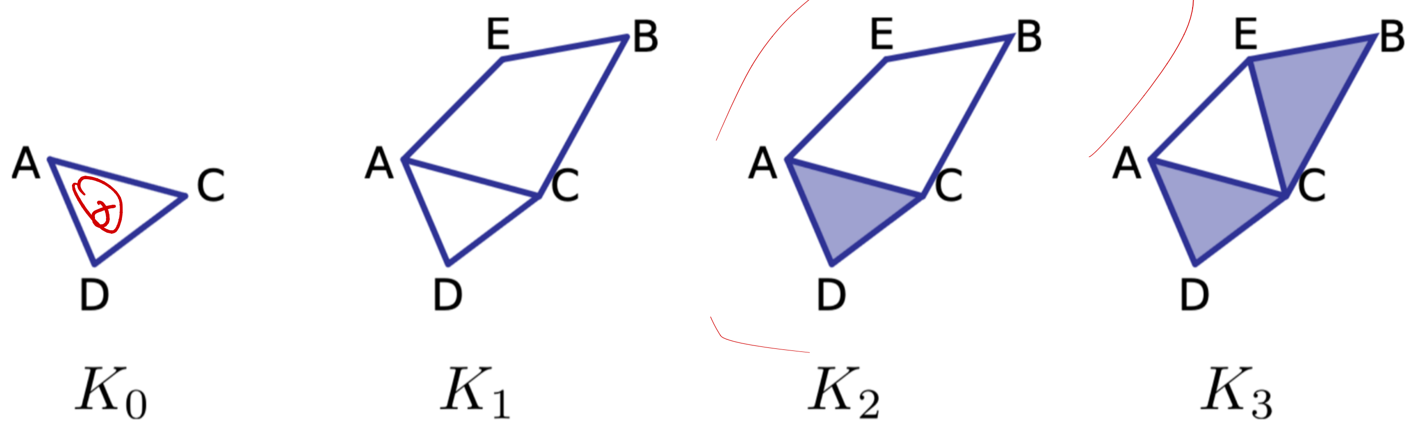


$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$$\begin{tikzcd}
[A] \arrow[r] & [A] \arrow[r] & [A] \arrow[r] & [A] \arrow[r] & [A] \arrow[r] & [A] \arrow[r] & [A] \arrow[r] & [A] \\
 & [B] \arrow[ur] & & [C] \arrow[r] & [C] \arrow[ur] & [D] \arrow[r] & [D] \arrow[ur] & \\
 & & & & [D] \arrow[ur] & & &
\end{tikzcd}$$

$$\begin{array}{cc} & [A] \\ \begin{array}{c}[A]\\ [B]\end{array} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}$$

Another:



$$H_1(K_0) \xrightarrow{f_*} H_1(K_1) \xrightarrow{g_*} H_1(K_2) \xrightarrow{h_*} H_1(K_2)$$

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ [ACD] \to & [ACD] & [ABCE] \to & [ACE] \\ & [ABCE] & & \end{array}$$

The p^{th}-persistent homology groups are the images induced by inclusion:

$$H_p^{i,j} = \mathrm{Im}\left(H_p(K_i) \longrightarrow H_p(K_j)\right)$$

$K_i \subseteq K_j \quad i \leq j$

The p^{th}-persistent Betti numbers are

$$\beta_p^{i,j} = \mathrm{rank}\left(H_p^{i,j}\right)$$

for a persistence module

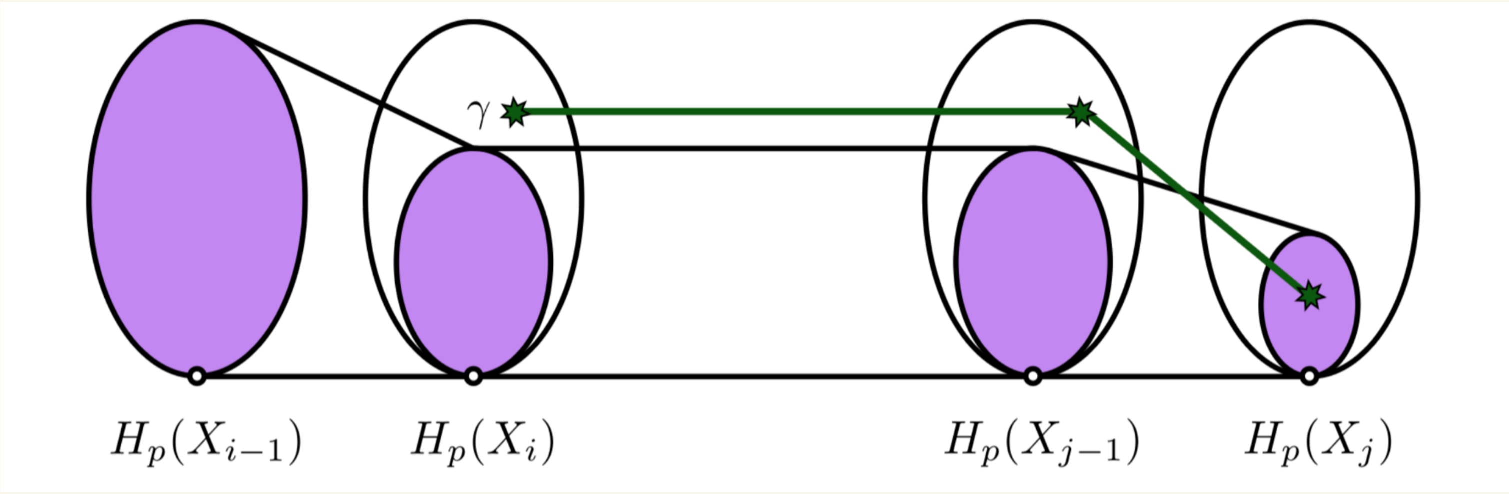
$$H_p(K_0) \longrightarrow H_p(K_1) \longrightarrow \cdots H_p(K_i) \rightarrow \cdots H_p(K_j) \rightarrow \cdots H_p(K_n)$$

Birth & death

We say a homology class $\gamma \in H_p(K_i)$ is **born** at K_i if it is not in $H_p^{i-1,i}$,

& γ dies entering K_j if it merges with an older class, ie if $f_p^{i,j-1}(\gamma) \notin H_p^{i-1,j-1}$ but $f_p^{i,j}(\gamma) \in H_p^{i-1,j}$

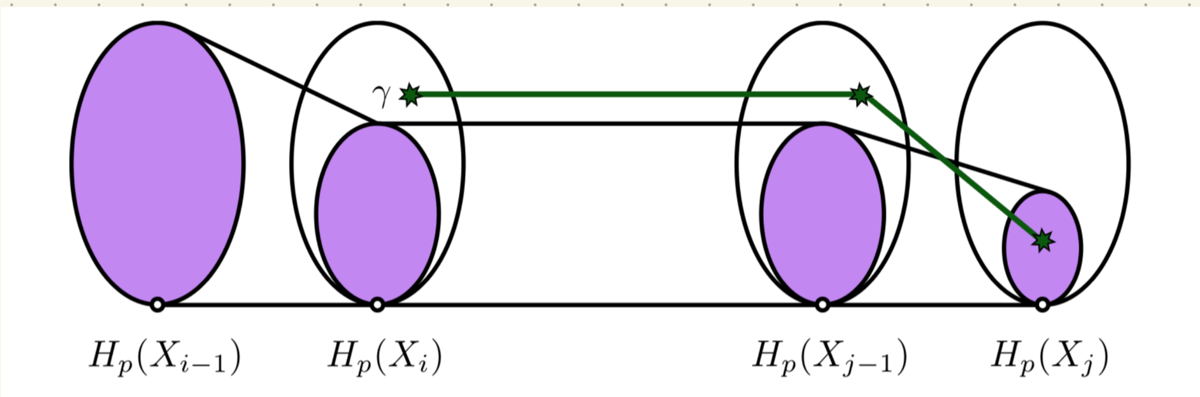
Warning: not book's version!



Book's version of death:

γ dies entering K_j if

- $\gamma \in H_p(X_{j-1})$ is not trivial
- but $h_p^{j-1,j}(\gamma) = 0$



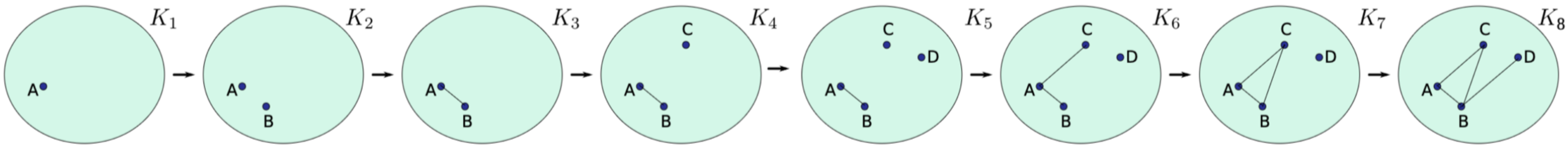
Only issue: no birth/death pairs in this definition

Pairing (book dfn)

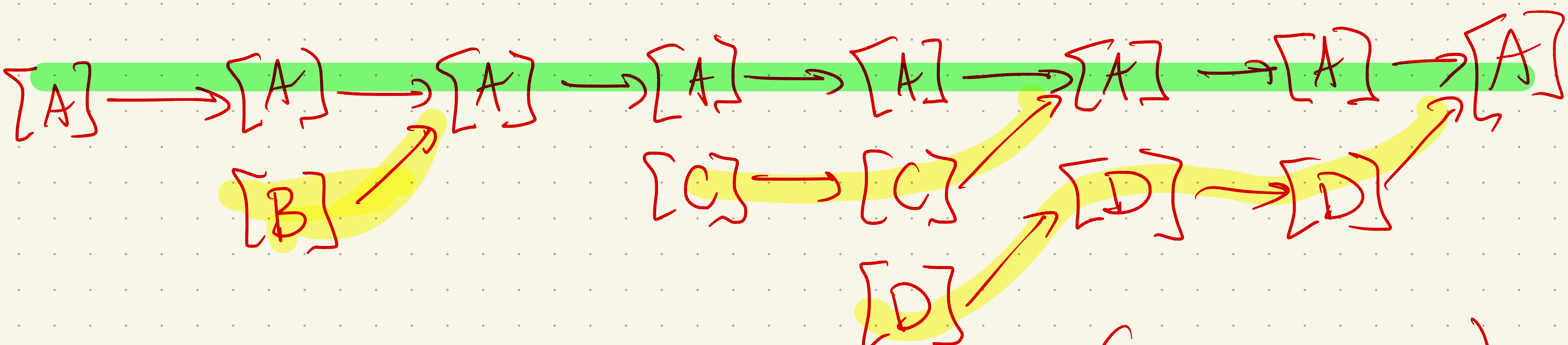
Let $[c]$ be a p^{th} homology class that is born at X_i and dies entering X_j. Then, it is born at X_i if and only if $\exists\ i_1 \le i_2 \le \cdots \le i_k = i$ (with $k \ge 1$) s.t. $\forall$

- $[c_{i_\ell}]$ is born at X_{i_ℓ} ($\ell \in [1..k]$)
- $[c] = f_p^{i_1, j-1}([c_{i_1}]) + \cdots + f_p^{i_k, j-1}([c_{i_k}])$
- $i_k = i$ is smallest possible choice

Revisiting: when are births & deaths?



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$



b is born

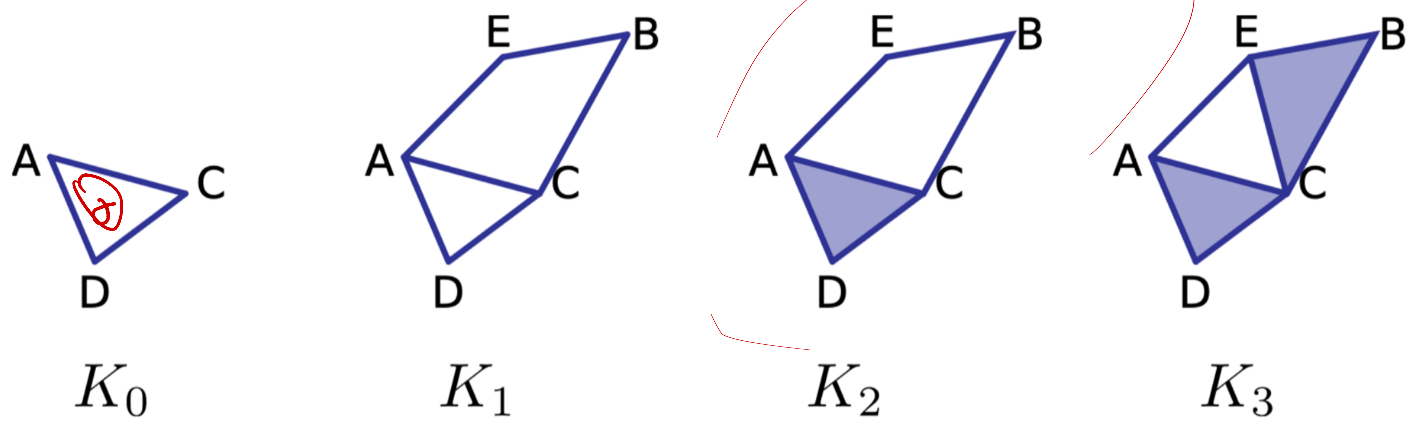
b dies

c is born

d is born

c dies

d dies



$$H_1(K_0) \xrightarrow{f_*} H_1(K_1) \xrightarrow{g_*} H_1(K_2) \xrightarrow{h_*} H_1(K_2)$$

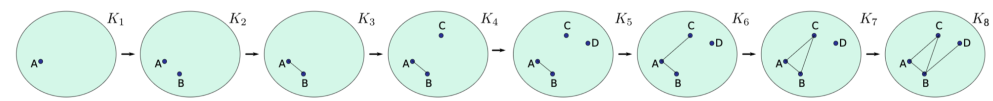
$$\begin{tikzcd}
0 & 0 & 0 & 0 \\
{[ACD]} \arrow[r] & {[ACD]} \arrow[ur] & {[ABCE]} \arrow[r] & {[ACE]} \\
 & {[ABCE]} \arrow[u, start anchor=east, end anchor=south west] & &
\end{tikzcd}$$

Note: the maps $f_p^{i,j}$ change if basis changes or reorders

↳ but <u>times</u> are the same.

Result:

Given filtration:



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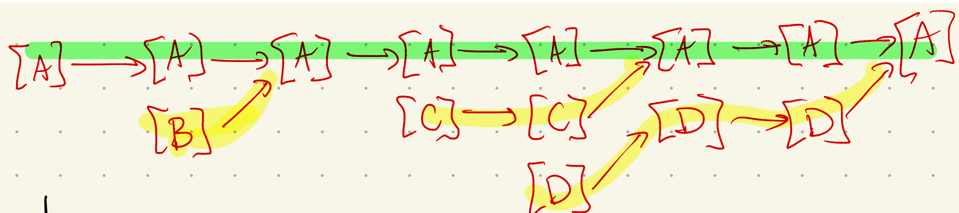
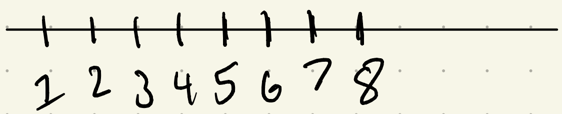
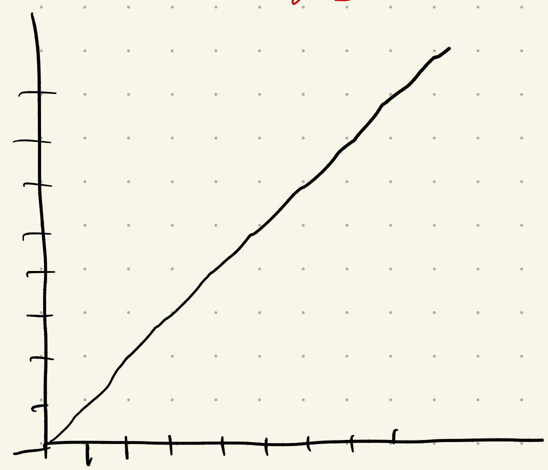






Barcodes

Persistence diagrams

More formally: Counting classes

$$0 \to H_p(K_1) \to H_p(K_2) \to \cdots \to H_p(K_n)$$

- Attach 0 vector space at end
- Associate $n+1$ to $a_{n+1} = \infty$
- Then $\beta_p^{i,j}$ counts classes born <u>before</u> i which die <u>after</u> j

How can we get # of classes born <u>at</u> i which die at j?

Pairing function

for $0 < i < j \leq n+1$, define

$$\mu_p^{i,j} = \left(\beta_p^{i,j-1} - \beta_p^{i,j}\right) - \left(\beta_p^{i-1,j-1} - \beta_p^{i-1,j}\right)$$

Why?

$$H_p(X_{i-1}) \xrightarrow{f_p^{i-1,i}} H_p(X_i) \xrightarrow{\quad f_p^{i,j-1} \quad} H_p(X_{j-1}) \xrightarrow{f_p^{j-1,j}} H_p(X_j)$$

When $\mu_p^{i,j} \neq 0$, the persitence of a class $[c]$, $Per([c])$, which is born at X_i + dies at X_j is defined as $a_j - a_i$.

↳ length of barcode

"lifetime"

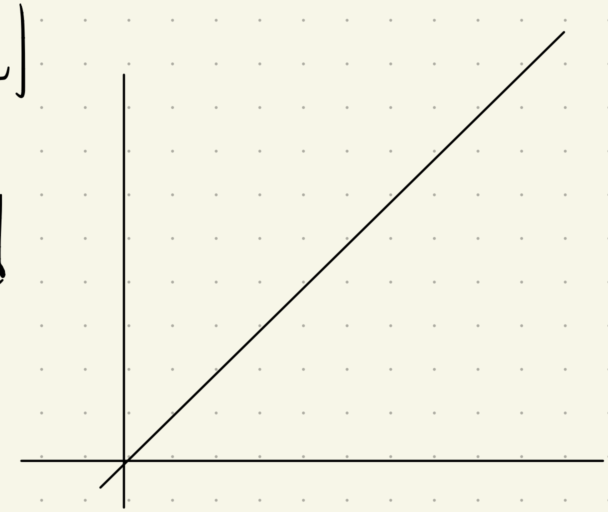
[If $j = n+1$ with $a_{n+1} = \infty$, $Pers([c]) = \infty$].

Persistence diagram $Dgm_p(F)$

(also written $Dgm_p(f)$)

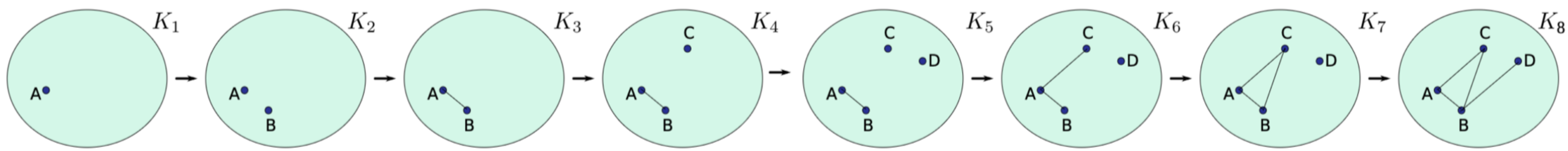
Filtration F on K induced by f.

$Dgm_p(F)$ is obtained by drawing a point (a_i, a_j) with non-zero multiplicity $\mu_p^{i,j}$ $(i<j)$ on extended plane, where points on the diagonal $\Delta = \{(x,x) \in \mathbb{R}^2\}$ are added with infinite multiplicity



Let's try! First, calculate $\beta^{i,j}$

↳ then $\mu^{i,j}$



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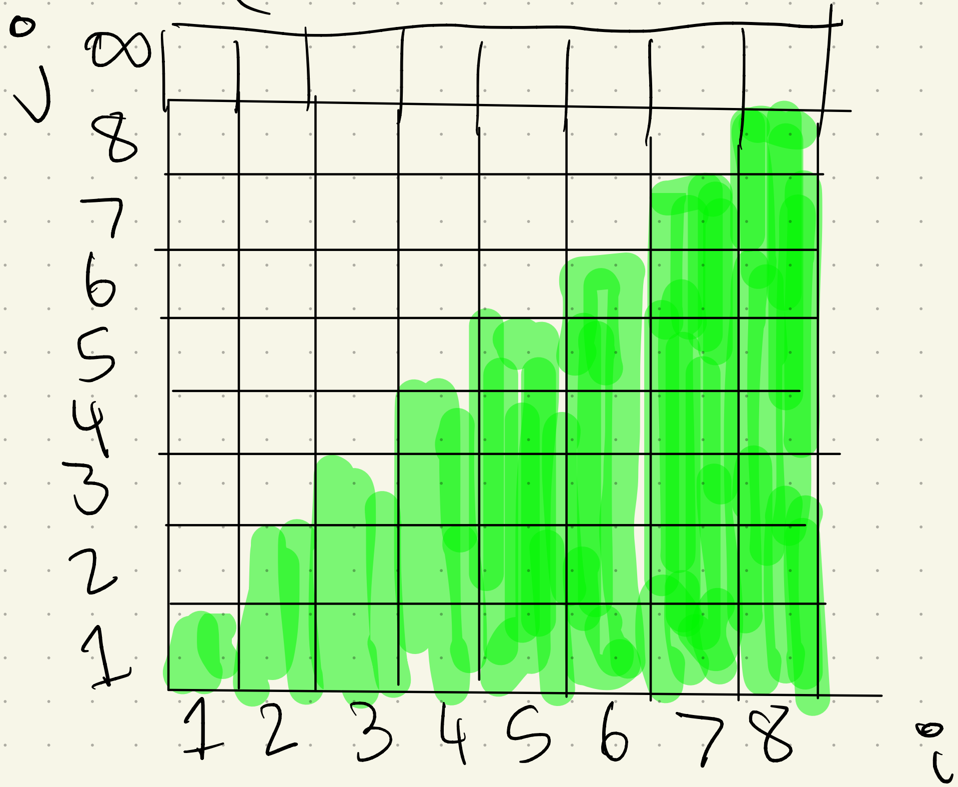
$\beta_0^{i,j}$

j \ i	1	2	3	4	5	6	7	8
∞	0	0	0	0	0	0	0	0
8								
7								
6								
5								
4								
3								
2								
1								

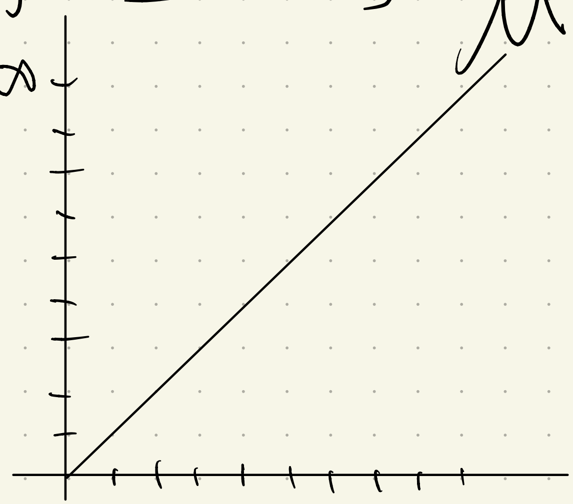
$$\mu^{i,j} = \left(\beta^{i,j-1} - \beta^{i,j}\right) - \left(\beta^{i-1,j-1} - \beta^{i-1,j}\right)$$

$j \backslash i$	1	2	3	4	5	6	7	8
∞	0	0	0	0	0	0	0	0
8	1	1	1	1	1	1	1	1
7	1	1	1	1	2	2	2	
6	1	1	1	1	2	2		
5	1	1	1	1	3			
4	1	1	1	2				
3	1	1	1					
2	1	2						
1	1							



$\beta^{i,j}_{\infty} \longrightarrow \mu^{i,j} \quad (i<j)$

Then $\mathrm{Dgm}(F) =$



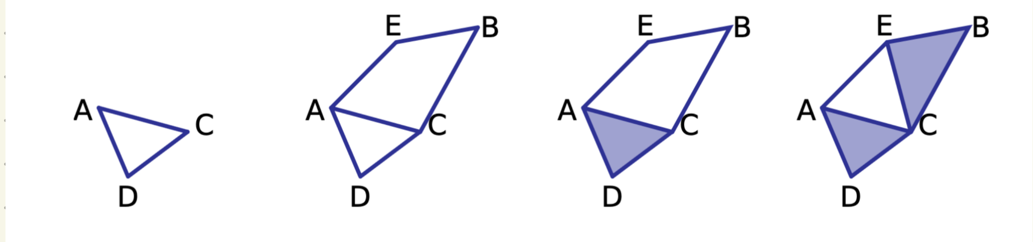
OK, let's avoid ever doing this by hand again...

Let $f: K \to \mathbb{N}$ give the index where a simplex σ appears in filtration.

A compatible ordering of the simplices is a sequence $\sigma_1, \sigma_2, \dots, \sigma_m$ s.t.

- $f(\sigma_i) < f(\sigma_j) \implies i < j$
- $\sigma_i \leq \sigma_j \implies i < j$

Ex:



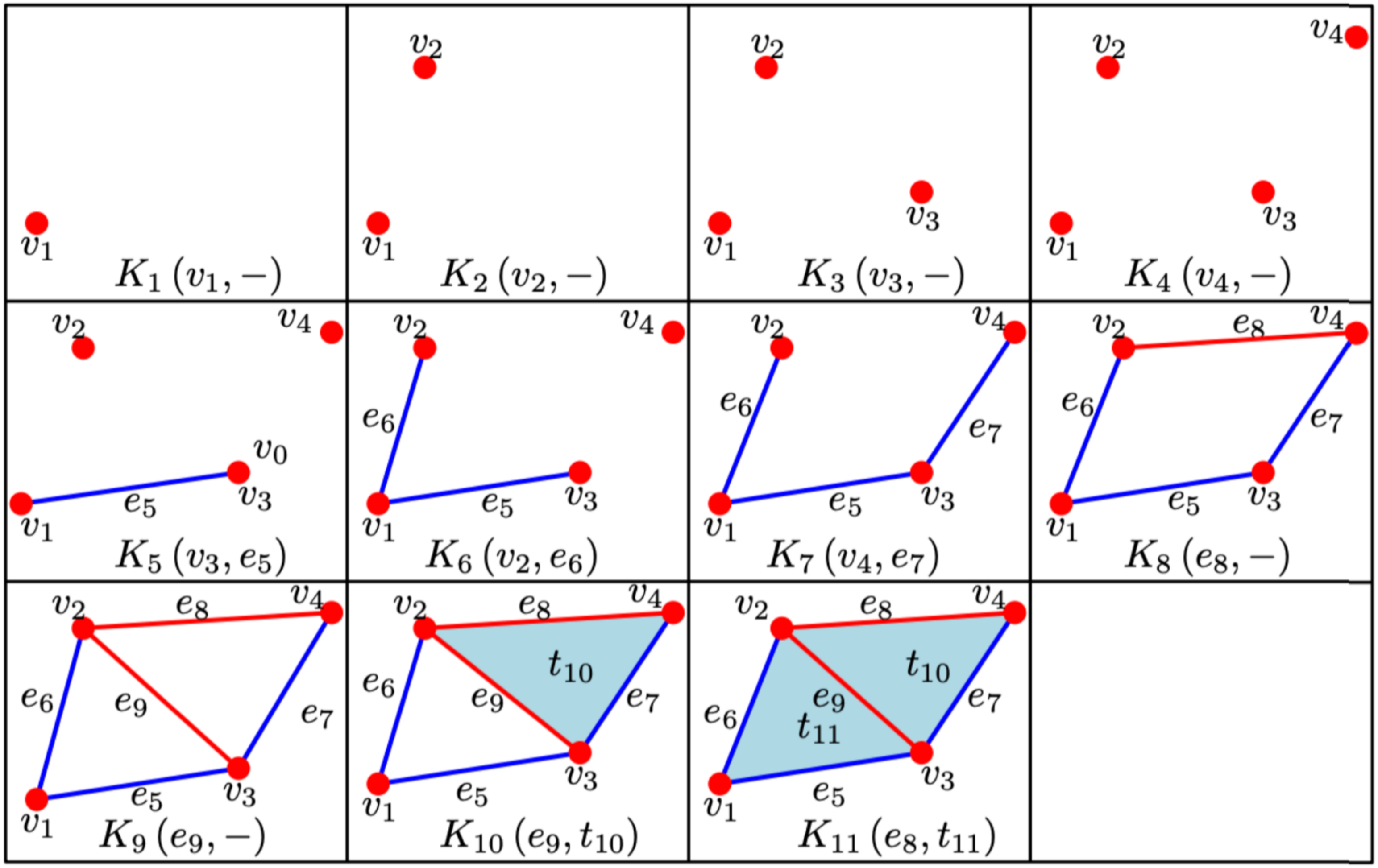
K_0 K_1 K_2 K_3

Essentially, we now have a simplex-wise filtration: assume $K_j / K_{j-1} = \sigma_j$ is a single simplex.

When p-simplex σ_j is added, two possibilities:

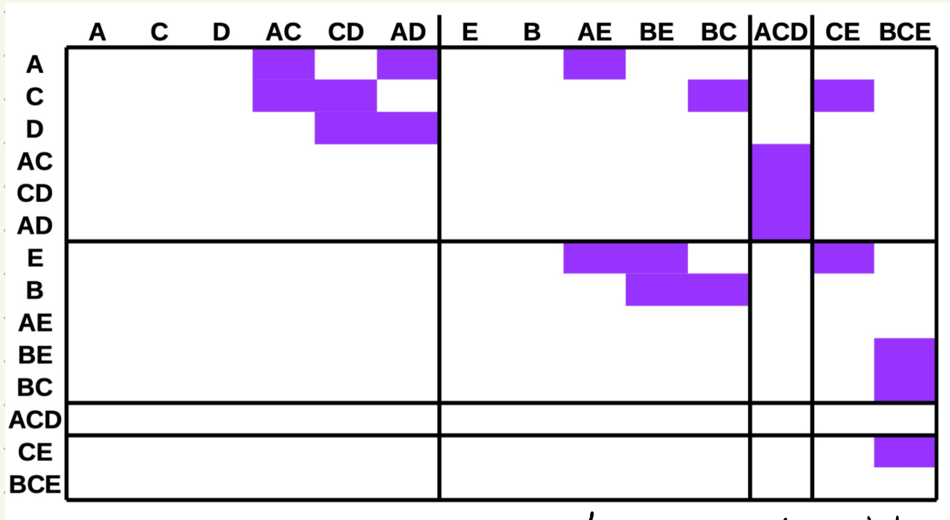
1. A non-boundary p-cycle c along with its classes $[c]+h$ for $h \in H_p(K_{j-1})$ are born. Call σ_j positive (or a creator).
2. An existing $(p-1)$-cycle c along with its class $[c]$ dies. Call σ_j negative (or a destroyer).

Examples

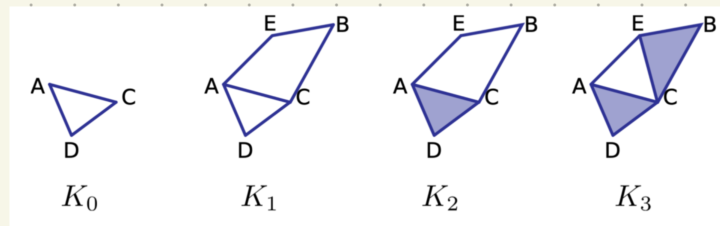

$K_1\,(v_1, -)$
$K_2\,(v_2, -)$
$K_3\,(v_3, -)$
$K_4\,(v_4, -)$
$K_5\,(v_3, e_5)$
$K_6\,(v_2, e_6)$
$K_7\,(v_4, e_7)$
$K_8\,(e_8, -)$
$K_9\,(e_9, -)$
$K_{10}\,(e_9, t_{10})$
$K_{11}\,(e_8, t_{11})$
v_0
v_1
v_2
v_3
v_4
e_5
e_6
e_7
e_8
e_9
t_{10}
t_{11}

An algorithm

Take boundary matrix, with rows & columns in simplex-wise order:



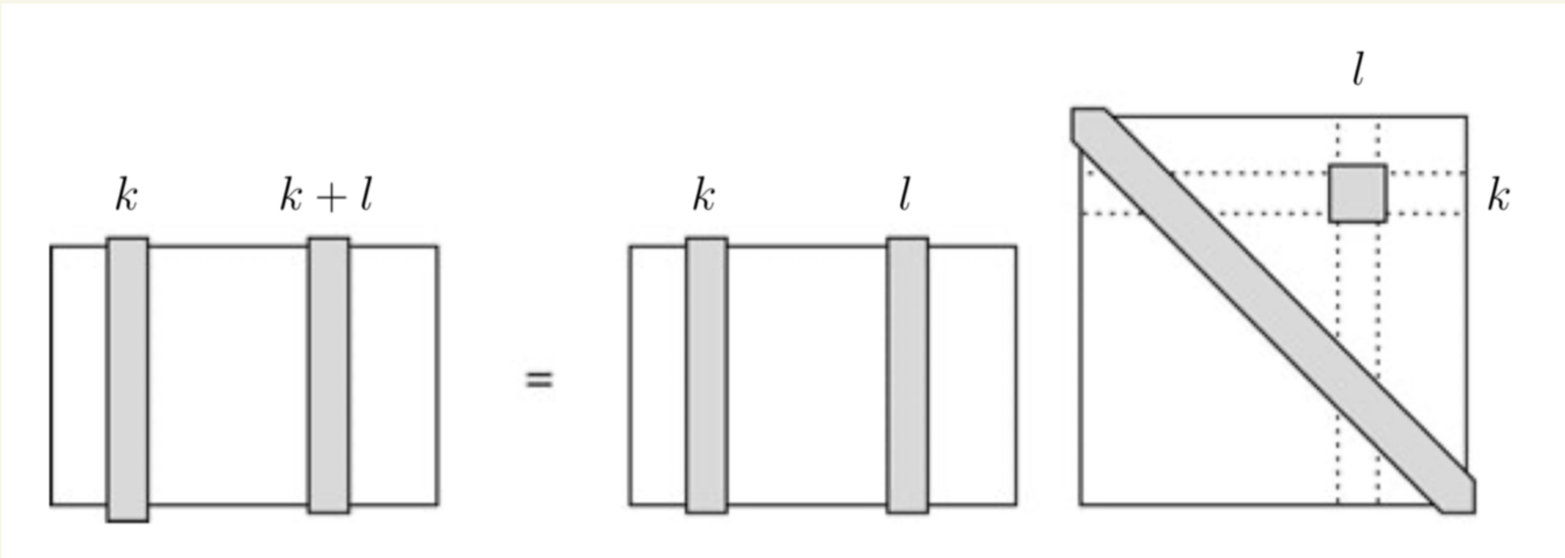
K_0 K_1 K_2 K_3



- Let low(j) = row of lowest 1 in column j (& if all 0's, low(j) = NaN)
- R is reduced if low(j) ≠ low(j') for any j ≠ j'

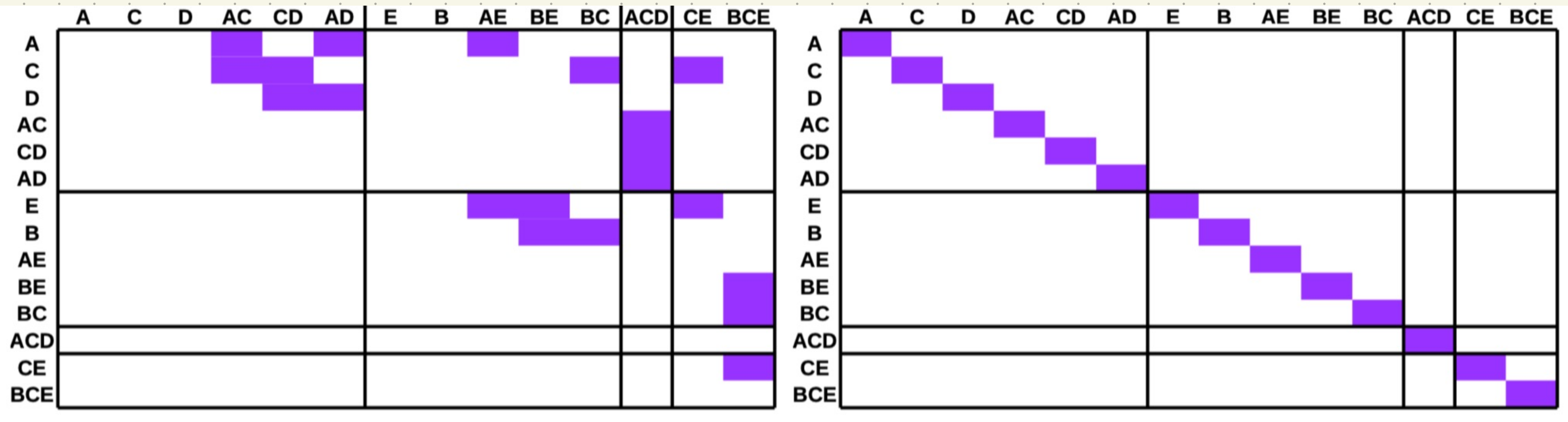
Matrix operations

To add row k to row l, can create matrix with 1 in l, k:



Here!

$R = B$
for $j = 1 \cdots m$ **do**
  while $\exists\, j' < j$ with $low(j') = low(j)$ **do**
    add column j' to column j
  end while
end for

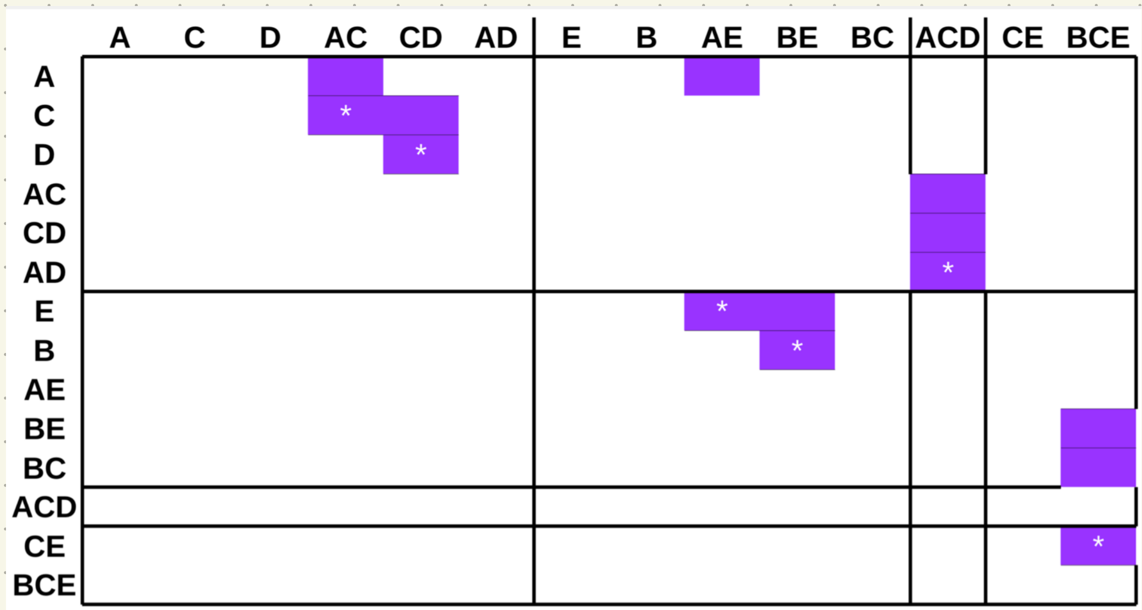


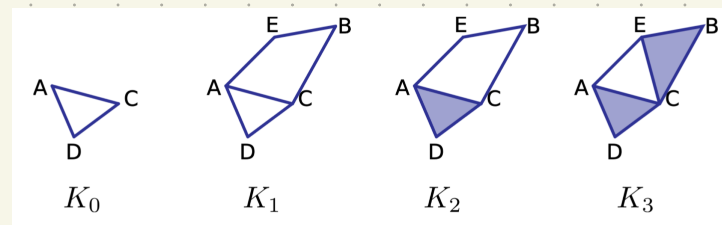
Idea

- B is upper triangular, & if we add from left it stays that way
- If a column is entirely 0, that simplex created a homology class (so it is positive)
- If a column has a lowest 1, then this simplex killed a class from the previous step.

Pairing

- Every negative simplex must be paired with a previous positive (birth/death)
  - → pair with its lowest 1



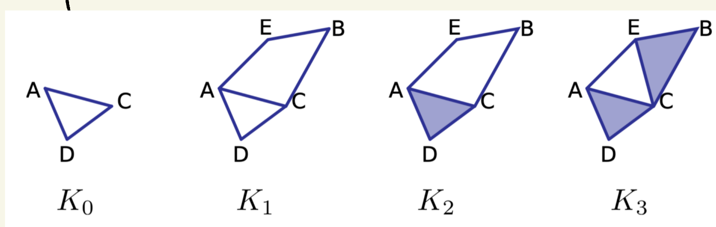


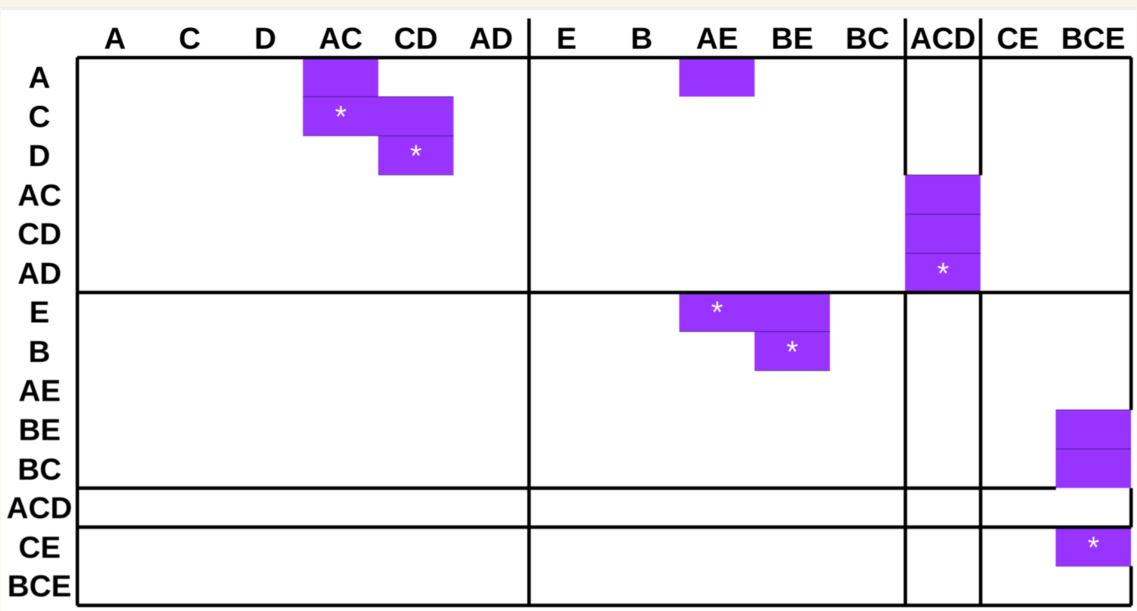
Pairs:

Fact

The number of unpaired p-simplices in a simplex-wise filtration of K is its p^{th} Betti number.

So: use pairs to build persistence diagram.





History

Matrix algorithm is from

Edelsbrunner-Letscher-Zomorodian 2000

Algebraic formulation given in

Carlsson + Zomorodian 2004

Independent formulations

Frosni 1990

Robbins 1999