

TDA - Fall 2025

Computing
Persistence



History Persistence actually came up often!

Matrix algorithm is from

Edelsbrunner-Letscher-Zomorodian 2008

Algebraic formulation given in

Carlsson + Zomorodian 2004

Independent formulations

Frosini 1990

Robbins 1999

- manifold comparison
in Euclidean space
- crystaline structures
& periodicity
 $\hookrightarrow H_0$

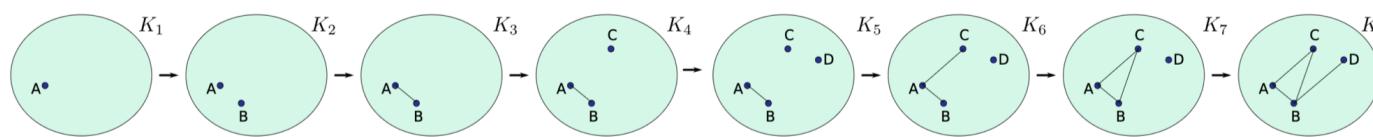
So far: Persistence

Chain complexes + filtrations $K_1 \subseteq K_2 \subseteq \dots \subseteq K_n$

\downarrow pass to homology

$$0 \rightarrow H_p(K_1) \xrightarrow{f^{1,2}} H_p(K_2) \xrightarrow{f^{2,3}} \dots \xrightarrow{f^{n-1,n}} H_p(K_n) \rightarrow 0$$

where f_{ij} is induced by inclusion



$$H_0(K_1) \rightarrow H_0(K_2) \rightarrow H_0(K_3) \rightarrow H_0(K_4) \rightarrow H_0(K_5) \rightarrow H_0(K_6) \rightarrow H_0(K_7) \rightarrow H_0(K_8)$$

and $H_P^{i,j} = \text{Im} (H_p(K_i) \rightarrow H_p(K_j))$

↳ homology classes in K_i but checked in bigger complex

Aside: $H_p(K_i)$ is a homology group still!

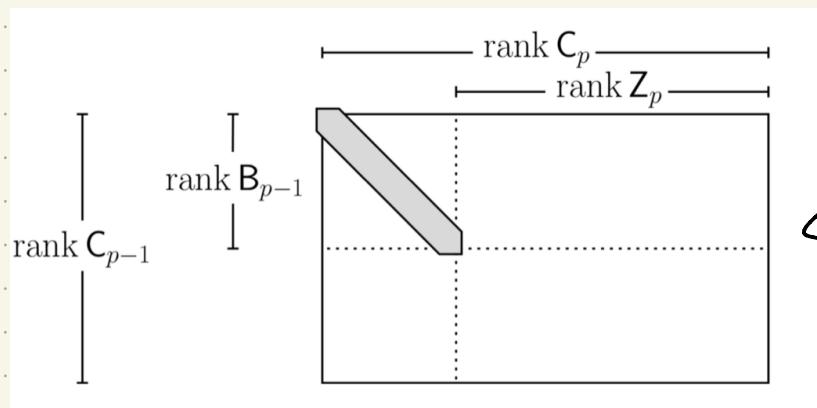
So calculated as we saw before:

$$C_d(K_i) \xrightarrow{\partial_d} C_{d-1}(K_i) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_{p+1}} C_p(K_i) \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_1} C_0(K_i)$$

then $H_p(K_i) = Z_p / B_p$

$$= \text{Ker } \partial_p / \text{im } \partial_{p-1}$$

Calculated via
boundary matrix
func.



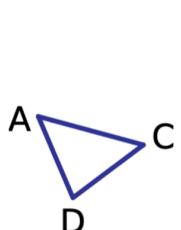
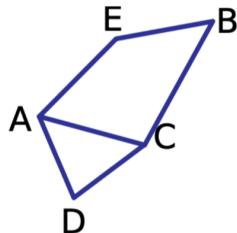
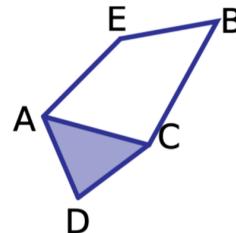
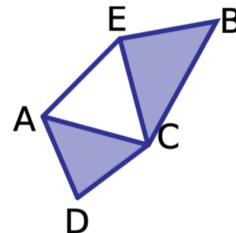
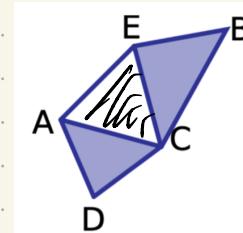
Smith
Normal
Form

Pairing (book defn from last class - revisit)

Let $[c]$ be a p^{th} homology class that dies entering x_j . Then, it is born at x_i if and only if $\exists i_1 \leq i_2 \leq \dots \leq i_k = i$ (with $k \geq 1$) s.t.

- $[c_{i_e}]$ is born at x_{i_e} ($e \in [1..k]$)
- $[c] = f_p^{i_1, j-1}([c_{i_1}]) + \dots + f_p^{i_k, j-1}([c_{i_k}])$
- $i_k = i$ is smallest possible choice

Why?? See 2 filtrations

 K_0  K_1  K_2  K_3  K_4

$$H_1(K_0) \xrightarrow{f_*} H_1(K_1) \xrightarrow{g_*} H_1(K_2) \xrightarrow{h_*} H_1(K_3) \rightarrow H_1(K_4)$$

[ACE] exists in $H_1(K_3)$, not in $H_1(K_4)$

Birth?

Counting classes
+ Persistence

Set $B_P^{i,j} = \text{rank}(H_P^{i,j})$

$$0 \rightarrow H_p(K_1) \rightarrow H_p(K_2) \rightarrow \dots \rightarrow H_p(K_n) \rightarrow 0$$

K_{n+1}

• Attach 0 vector space at end

• Associate $n+1$ to $a_{n+1} = \infty$

• Then $B_P^{i,j}$ counts classes born

before i which die after j
are active in $[i, j]$

How can we get # of classes

born at i which die at j ?

$$H_P^{i-1} \rightarrow H_P^i \rightarrow \dots \rightarrow H_P^{j-1} \rightarrow H_P^j$$

Pairing function

for $0 < i < j \leq n+1$, define

$$\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$$

~~A~~ ~~the~~ \curvearrowright # of classes born at i that die at j

Why?

$$H_p(X_{i-1}) \xrightarrow{f_p^{i,j-1}} H_p(X_i) \xrightarrow{f_p^{i,j-1}} H_p(X_{j-1}) \xrightarrow{f_p^{j-1,j}} H_p(X_j)$$

$\beta_p^{i-1,j-1} - \beta_p^{i-1,j}$ $\beta_p^{i,j-1} - \beta_p^{i,j}$

When $\mu_p^{i,j} \neq 0$, the persistence of a class $[c]$, $\text{Per}([c])$, which is born at x_i + dies at x_j is defined

as $a_j - a_i$.

→ length of barcode
"Lifetime"

[If $j = n+1$ with $a_{n+1} = \infty$, $\text{Per}([c]) = \infty$].

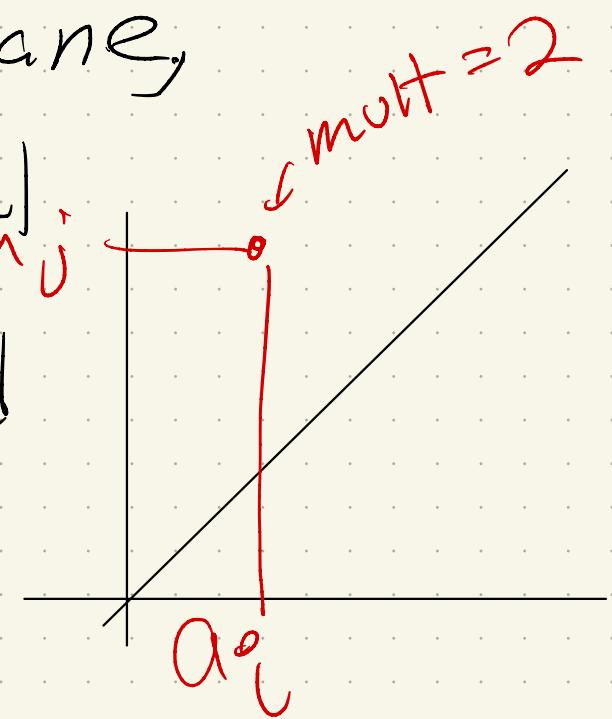
Persistence diagram $Dg_{mp}(F)$

(also written $Dg_m(F)$)

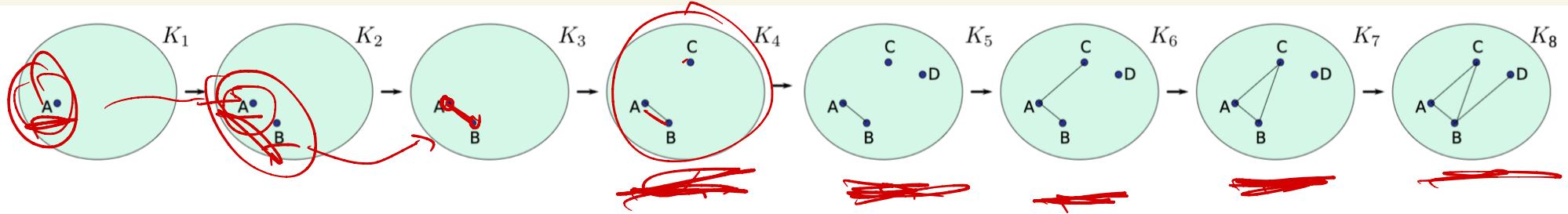
Filtration F on K induced by \mathbb{P} .
 $Dg_{mp}(F)$ is obtained by drawing a point (a_{ij}, a_j) with non-zero multiplicity m_{ij}^{mp} ($i < j$) on extended plane, where points on the diagonal $a_i = a_j$ are added

$$\Delta = \{(x, x) \in \mathbb{R}^2\}$$

with infinite multiplicity



Let's try! First calculate B^{ij}
 Then m^{ij}

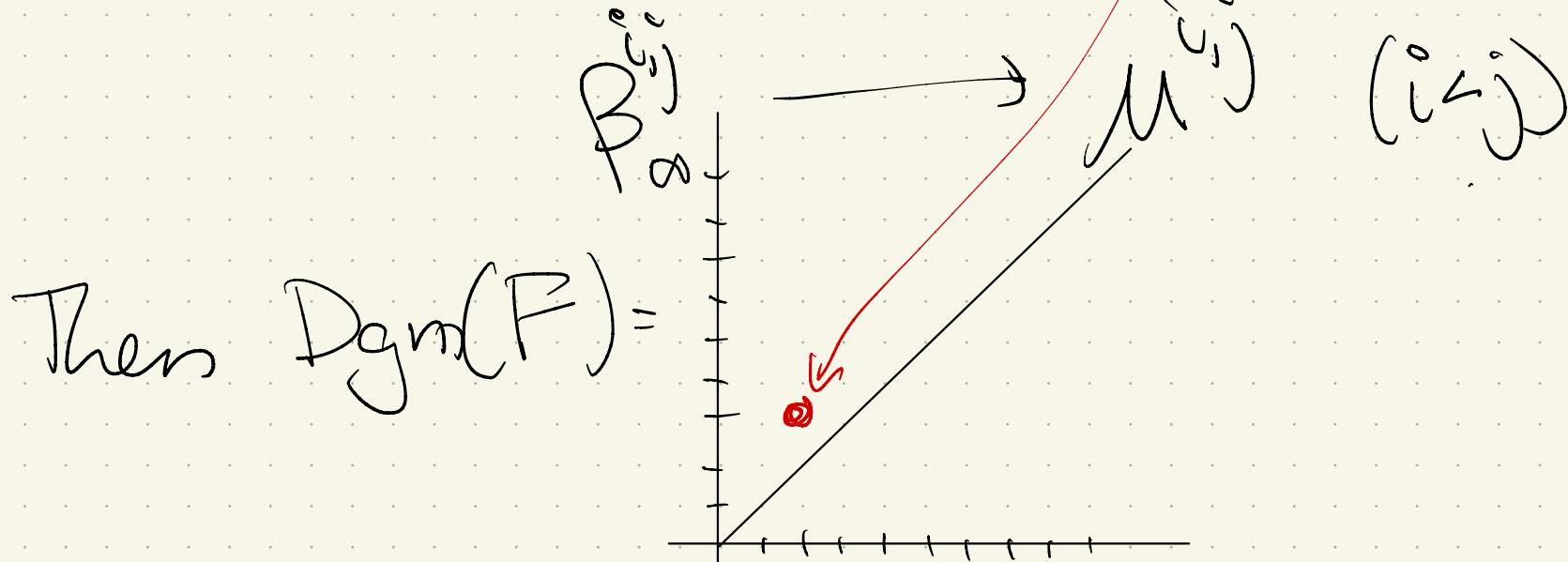
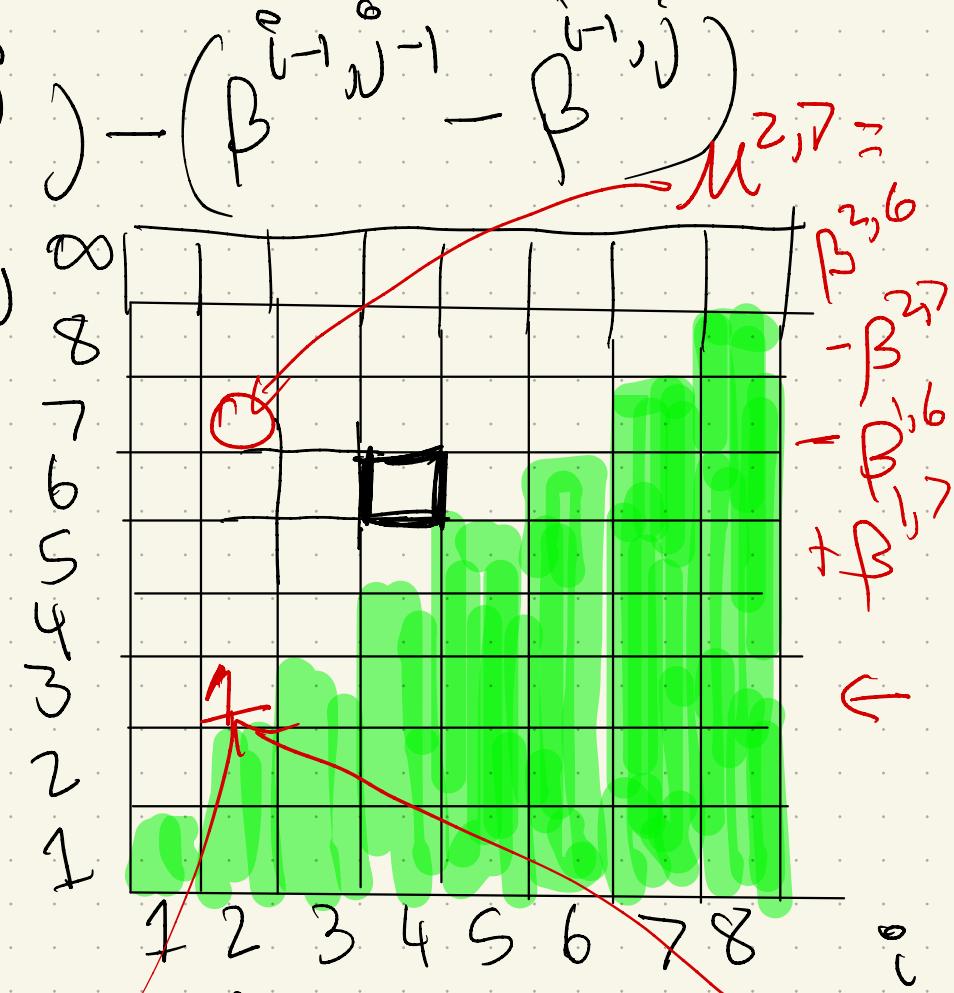
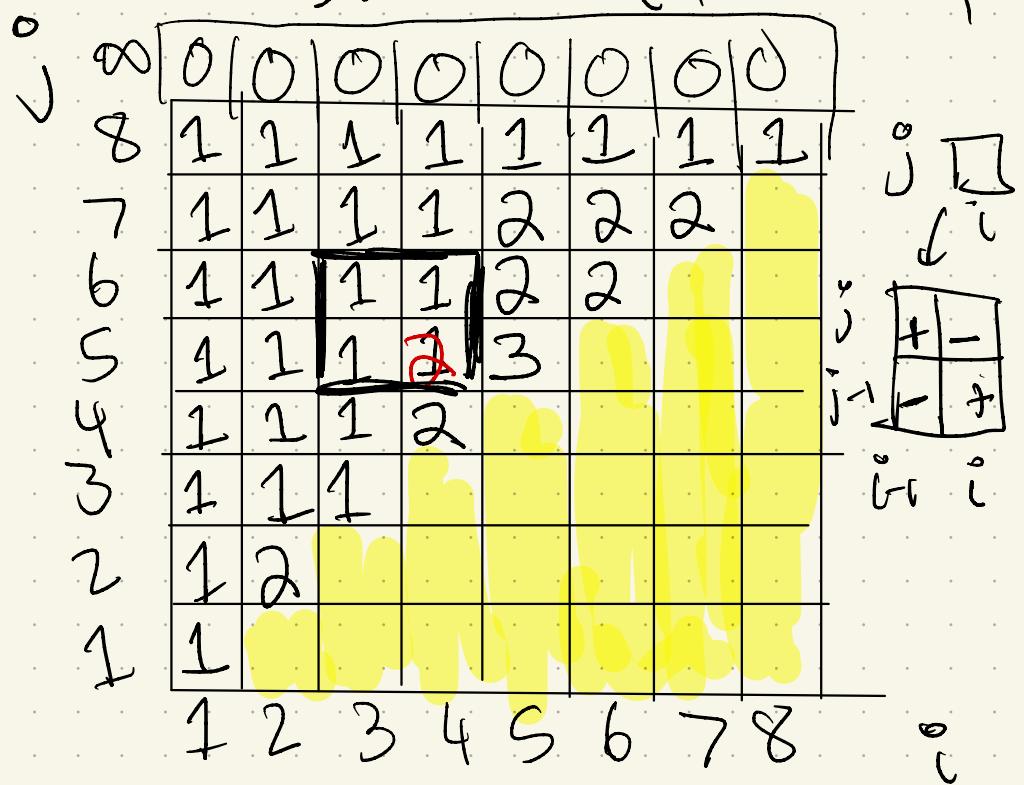


$$H_0(K_1) \rightarrow H_0(K_2) \rightarrow H_0(K_3) \rightarrow H_0(K_4) \rightarrow H_0(K_5) \rightarrow H_0(K_6) \rightarrow H_0(K_7) \rightarrow H_0(K_8)$$

β_{ij}^0

j	8	7	6	5	4	3	2	1
i	1	2	3	4	5	6	7	8
8	0	0	0	0	0	0	0	0
7	1	1	1	1	1	1	1	1
6	1	1	1	1	2	2	2	
5	1	2	1	1	2	2		
4	1	1	1	2	3			
3	2	1	2					
2	1	1	1					
1	1	2						

$$M^{i,j} = (B^{i,j})^{-1} - B^{i,j}$$



Then $Dgm(F) =$

$$\begin{aligned} M^{2,3} &= B^{2,2} \\ &- B_{2,2} \\ &- B_{2,3} \\ &+ B_{3,2} \end{aligned}$$

Taking stock:

Can compute $H_p(K_i)$.
How to get $H_p^{i,j}$?

Really, want $B_p^{i,j}$, so can calculate
 $\mu^{i,j} \rightarrow$ then $Dgm_p(F)$.

So: need to adapt matrix
algorithm somehow, to get
ranks of induced homology.

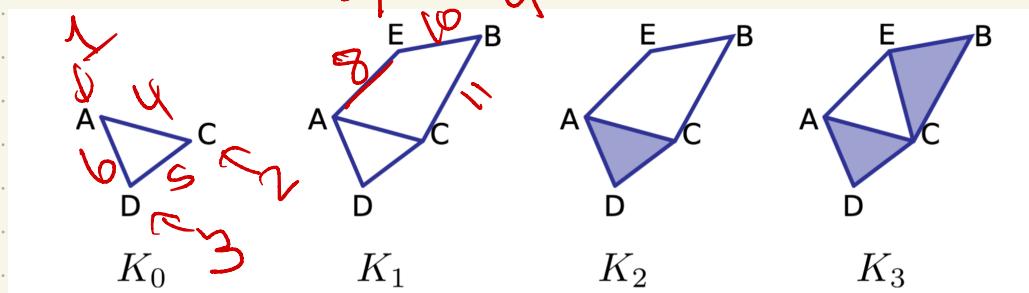
Some Preliminaries

Let $f: K \rightarrow \mathbb{N}$ give the index where a simplex σ appears in filtration. A compatible ordering of the simplices is a sequence $\sigma_1, \sigma_2, \dots, \sigma_m$ s.t.

$$\cdot f(\sigma_i) < f(\sigma_j) \Rightarrow i < j$$

$$\rightarrow \cdot \sigma_i \subseteq \sigma_j \Rightarrow i < j$$

Ex:



Essentially, we now have a simplex-wise filtration: assume $K_j / K_{j-1} = \sigma_j$ is a single simplex.

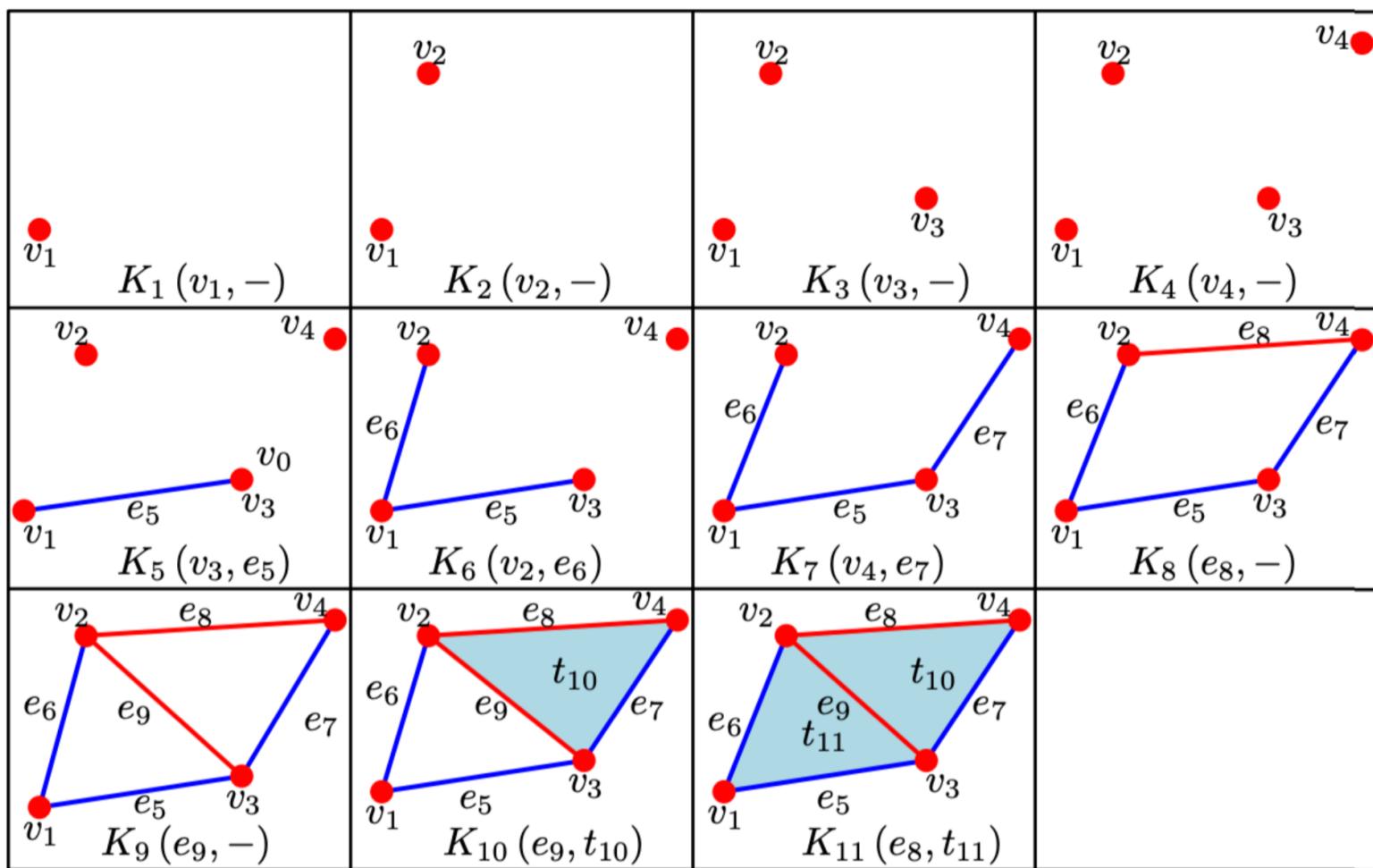
When p-simplex σ_j is added, two possibilities:

- ① A non-boundary p-cycle c along with its classes $[c]_h$ for $h \in H_p(K_{j-1})$ are born. Call σ_j positive (or a creator).
- ② An existing $(p-1)$ -cycle c along with its class $[c]$ dies. Call σ_j negative (or a destroyer).



Examples

H_0 birth/death
 H_1 birth/death (no H_2 here)



An algorithm

Take boundary matrix, with rows & columns in simplex-wise order:

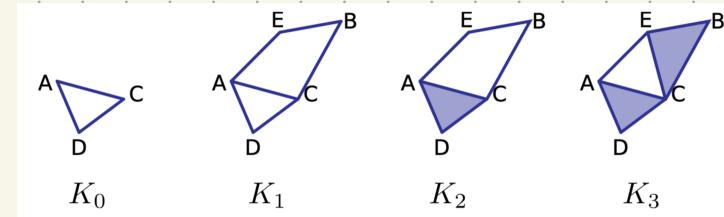
	A	C	D	AC	CD	AD	E	B	AE	BE	BC	ACD	CE	BCE
A														
C														
D														
AC														
CD														
AD														
E														
B														
AE														
BE														
BC														
ACD														
CE														
BCE														

K_0

K_1

K_2

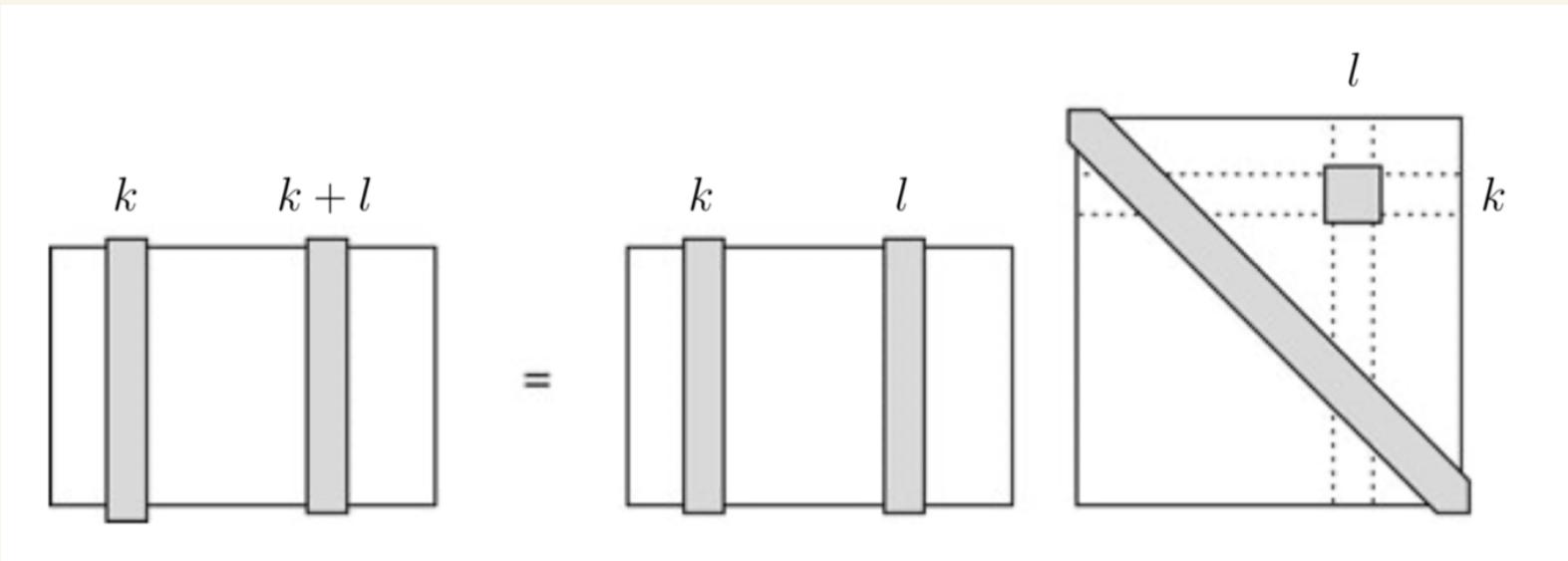
K_3



- Let $low(j) = \text{row of lowest 1 in column } j$
(+ if all 0's, $low(j) = N+1$)
- R is reduced if $low(j) \neq low(j')$ for any $j \neq j'$

Matrix operations

To add row k to row l , can
create matrix with 1 in l, k° :



Here!

$$R = B$$

for $j = 1 \dots m$ **do**

while $\exists j' < j$ with $low(j') = low(j)$ **do**

add column j' to column j

end while

end for

Idea

- B is upper triangular & if we add from left it stays that way
- If a column is entirely 0, that simplex created a homology class (so it is positive) \nwarrow unique (indep of operations)
- If a column has a lowest 1, then this simplex killed a class from the previous step.

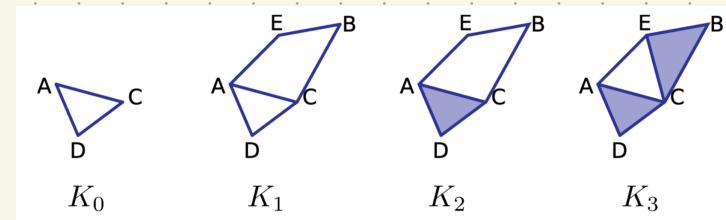
[Proof: essentially, 1 in that spot means $M_{i,j}^{i,j}=1$]

Pairing

Every negative simplex must be paired with a previous positive
 (birth/death)

pair with its lowest 1

	A	C	D	AC	CD	AD	E	B	AE	BE	BC	ACD	CE	BCE
A														
C				*										
D					*									
AC														
CD														
AD												*		
E							*							
B								*						
AE									*					
BE										*				
BC											*			
ACD														
CE													*	
BCE														



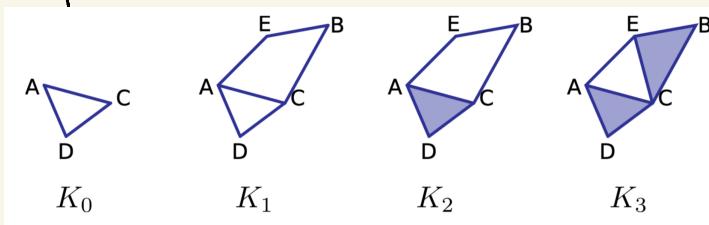
Pairs

Fact

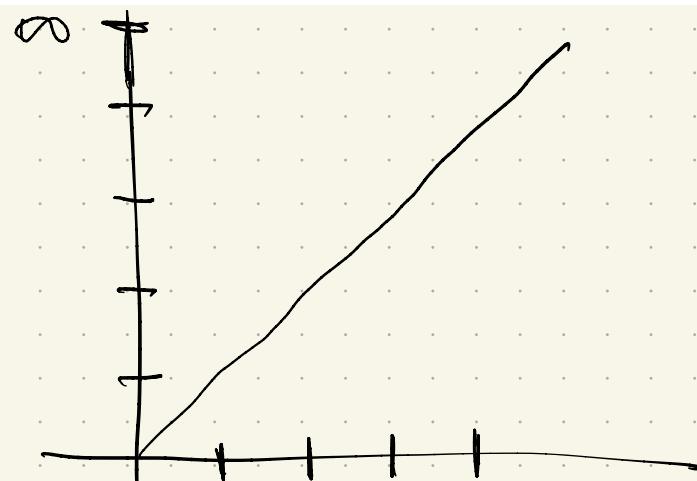
The number of unpaired p -simplices
in a simplex-wise filtration of K
is its p^{th} Betti number.

Why?

So! Use pairs to build persistence diagram.



A	C	D	AC	CD	AD	E	B	AE	BE	BC	ACD	CE	BCE
A			*										
C					*								
D													
AC													
CD													
AD													
E						*							
B							*						
AE								*					
BE									*				
BC										*			
ACD											*		
CE												*	
BCE													



Unpaired:

K_0 : A and AD
 $B_0(K_0) = + B_1(K_0) =$

K_1 : still A + AD,

plus BC
 $B_0 = B_1 =$

K_2 : AD now paired!

$B_0 = B_1 =$

K_3 : no change

Next time!

Some code discussion

Stability + distance metrics
for persistence

Longer term!

- Statistics + ML
- Reeb graphs + Mapper graphs
- Extensions of persistence