Algorithms - Spring '25

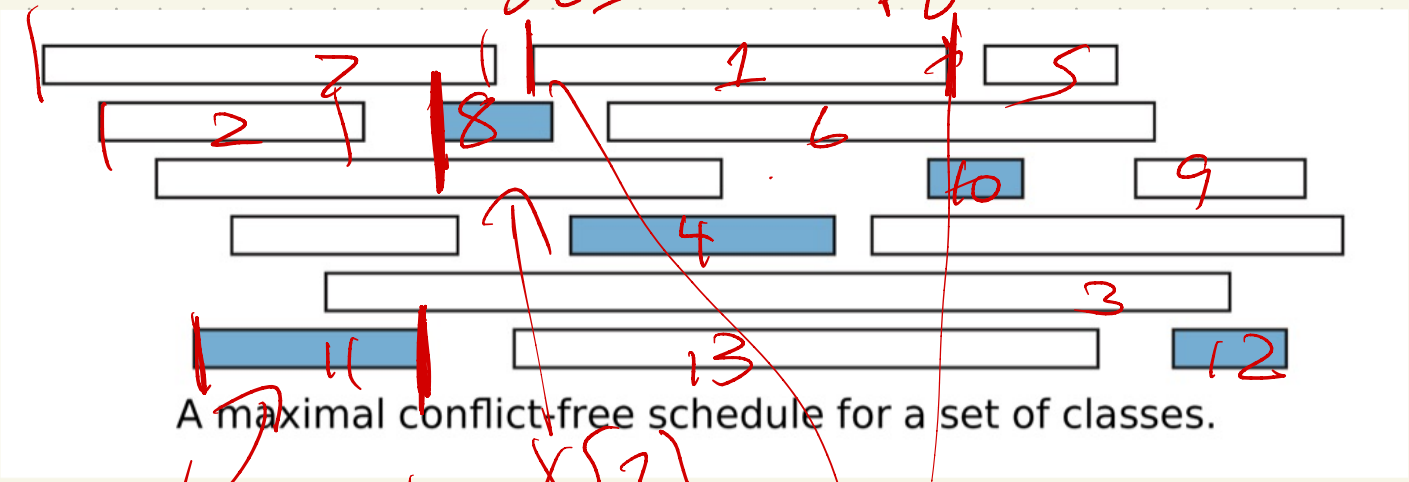
Greed:
Intervals
Huffman codes

Recap

- Office hours: today at 2pm (me) & 3:15 pm (TA)
- HW3 due
- HW4 — oral grading next Thurs/Friday
- Midterm: March 4, 8am

Problem: Interval Scheduling

Given a set of events (ie intervals, with a start and end time), select as many as possible so that no 2 overlap.



A maximal conflict-free schedule for a set of classes.

X[1] = 11

X[2]

More formally:

Two arrays

S[1..n]

F[1..n]:

X[1] interval

↳ S[X[1]]

↑ of F[X[1]]

Goal: A subset $X \subseteq \{1..n\}$ as big as possible s.t. $\forall i$,

$F[i] \leq S[i+1]$

How would we formalize a dynamic programing approach?

Recursive structure:

Consider job 1:

take it
↳ add to X
recurse on 2..n

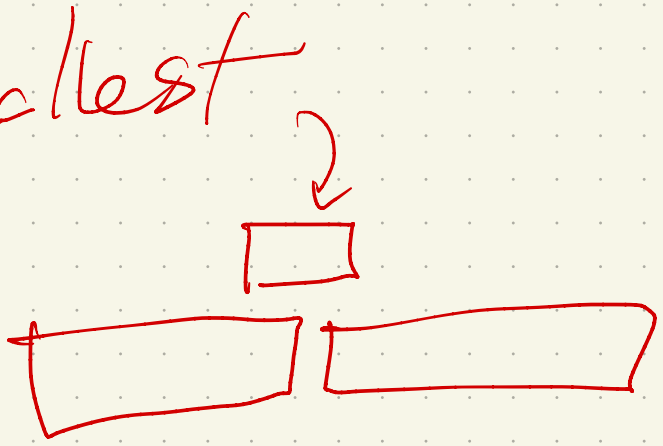
don't
recurse on 2..n

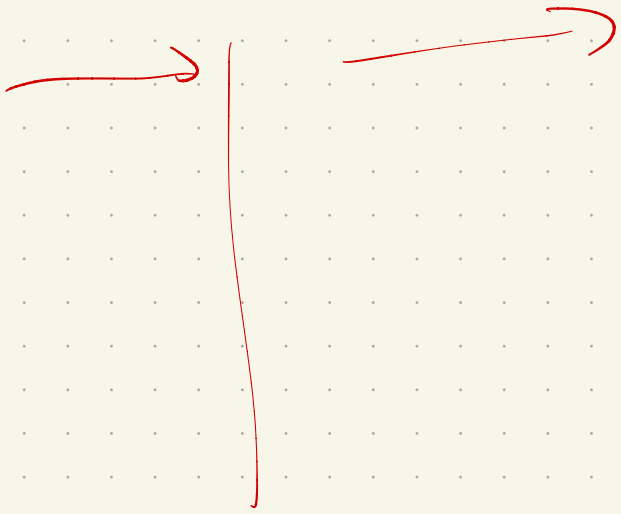
Intuition for greedy:

Consider what might be a good first one to choose.

Ideas?

Smallest





Key intuition:

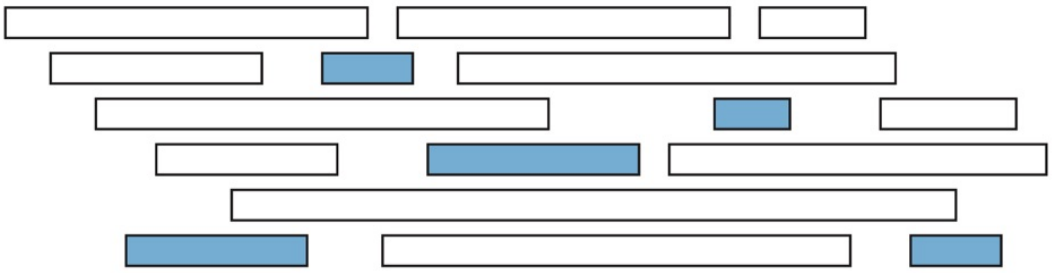
If it finishes as early as possible, we can fit more things in!

So - strategy:

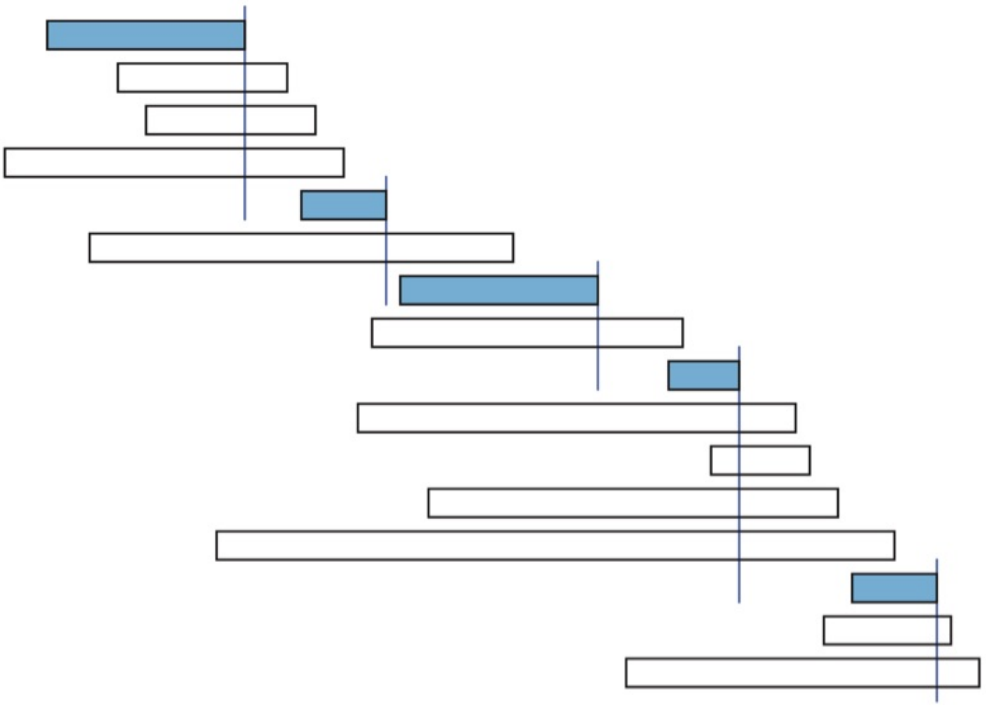
The code:

```
GreedySchedule(S[1..n], F[1..n]):
  sort F and permute S to match
  count ← 1
  X[count] ← 1
  for i ← 2 to n
      if S[i] > F[X[count]]
          count ← count + 1
          X[count] ← i
  return X[1..count]
```

Picture:



A maximal conflict-free schedule for a set of classes.



The same classes sorted by finish times and the greedy schedule.

Correctness:

Why does this work?

Note: No longer trying all possibilities or relying on optimal substructure!

So we need to be very careful on our proofs.

(Clearly, intuition can be wrong!)

Lemma: We may assume the optimal schedule includes the class that finishes first.

pf:

Thm: The greedy schedule is optimal.

Pf: Suppose not.

Then ∃ an optimal schedule that has more intervals than the greedy one.

Consider first time they differ:

Greedy: g_1 g_2 ... g_i ... g_k

OPT: o_1 o_2 ... o_i ... o_ℓ

Example: Huffman trees

Many of you saw this in data structures.

Why?

- cool use of trees
- non-trival use of other data structure

Really - it's greedy!

Goal: Minimize cost

↳ here, minimize total length of encoded message:

Input: frequency counts $f[1..n]$

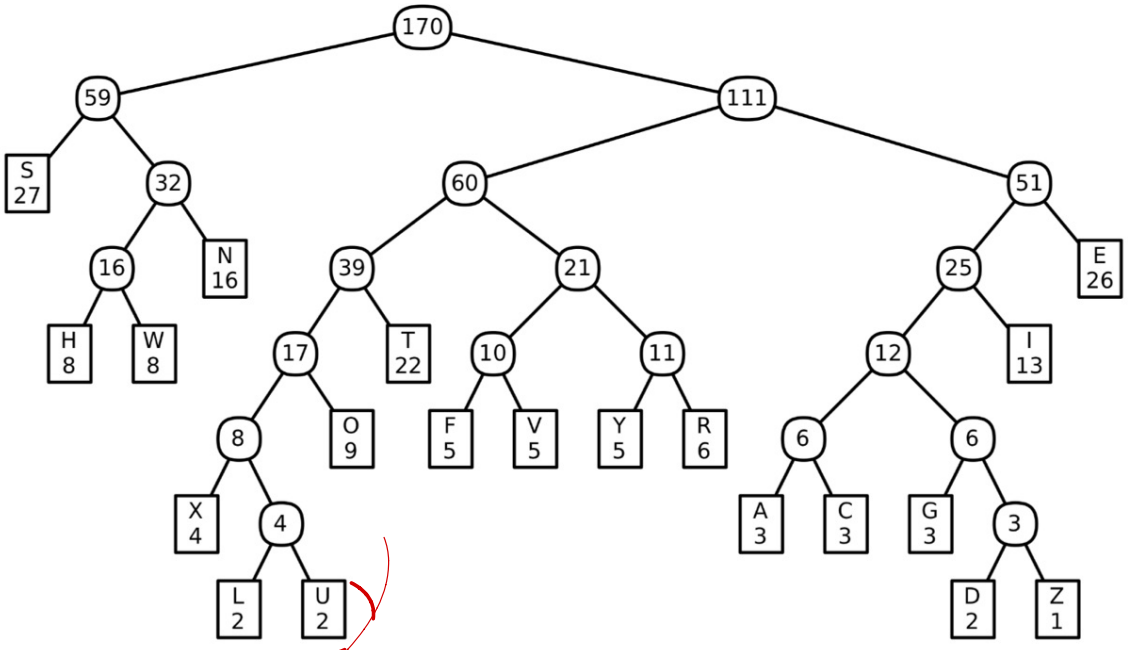
Compute:

$$cost(T) = \sum_{i=1}^{n} f[i] \cdot depth(i)$$

Strategy:
- Pick 2 least common letters & make them leaves
- "Merge them": remove letters, & add a new letter with sum of their frequencies
- Recurse!

↑ well, a bit imprecise..

In the end, get a tree with letters at the leaves:



A Huffman code for Lee Sallows' self-descriptive sentence; the numbers are frequencies for merged characters

A	C	D	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	Z
3	3	2	26	5	3	8	13	2	16	9	6	27	22	2	5	8	4	5	1

If we use this code, the encoded message starts like this:

1001	0100	1101	00	00	111	011	1001	111	011	110001	111	110001	10001	011	1001	110000	...
T	H	I	S	S	E	N	T	E	N	C	E	C	O	N	T	A	

Implementation: use priority queue

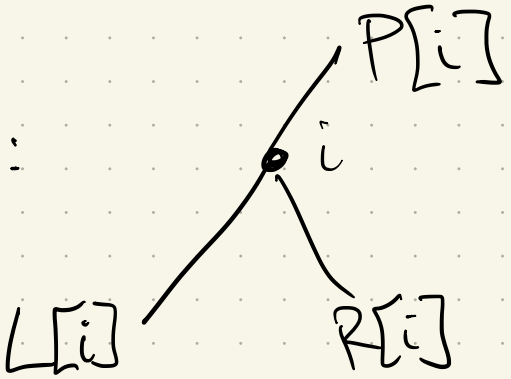
```
BUILDHUFFMAN(f[1..n]):
  for i ← 1 to n
      L[i] ← 0;  R[i] ← 0
      INSERT(i, f[i])

  for i ← n to 2n − 1
      x ← EXTRACTMIN( )
      y ← EXTRACTMIN( )
      f[i] ← f[x] + f[y]
      L[i] ← x;  R[i] ← y
      P[x] ← i;  P[y] ← i
      INSERT(i, f[i])

  P[2n − 1] ← 0
```

3 arrays: L, R, P to encode the tree

node i :



So: BANANA

index: 1 2 3 4 (n)

letters: B A N EM

freq: f:

```
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```

L: 1 2 3 4 5 6 7

R:

P:

Runtime?

```
BUILDHUFFMAN(f[1..n]):
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      P[x] ← i;  P[y] ← i
      INSERT(i, f[i])

  P[2n − 1] ← 0
```
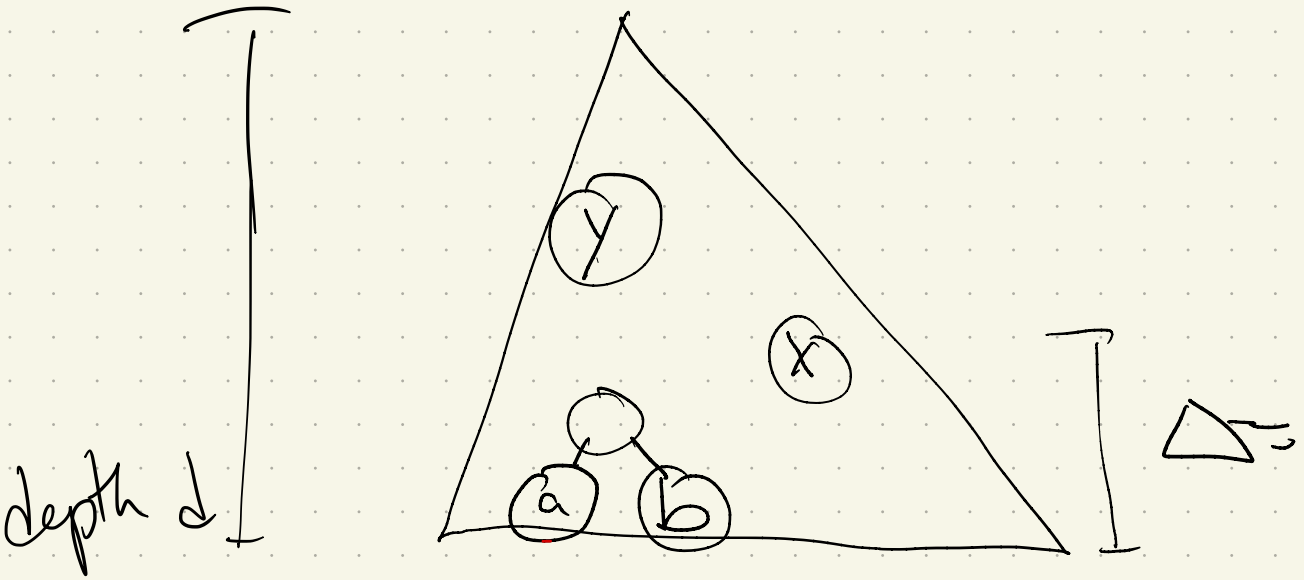

Correctness:

1st Lemma: There is an optimal prefix tree where the two least common letters are siblings at the largest depth.

pf: Spps not. Then optimal tree T has some depth d, but 2 least common letters x & y are not at that depth.



Note some other letters a & b are deepest

pf cont:

Know $f[x] \leq f[a]$,

but $depth(a) = depth(x) + \Delta$

↳ recall that:

$$cost(T) = \sum_{i=1}^{n} f[i] \cdot depth(i)$$

Build T':

Thm: Huffman trees are optimal.

pf:

Use induction (+ swap).

BC: For $n = 1, 2,$ or 3, Huffman works

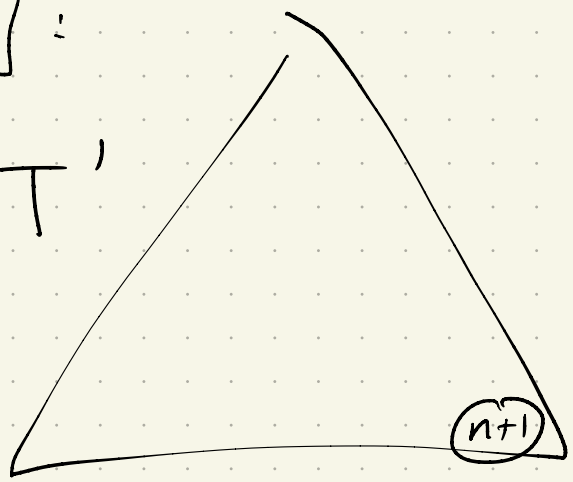
Why?

IH: Assume Huffman works on $\leq n-1$ characters

IS: Input $F[1..n]$, + spps $F[1]$ + $F[2]$ are min freq.

↳ Create a smaller array

IS: optimal tree T' of $F[3..n+1]$:



Note: $n+1$ is in tree

Build a tree T for $F[1..n]$:

Claim: T is optimal.

Why?

Why is T optimal??
(we know T' is optimal → IH!)

$$cost(T) = \sum_{i=1}^{n} F[i] \cdot depth[i]$$

$$= cost(T') + \text{changes we made}$$

↳

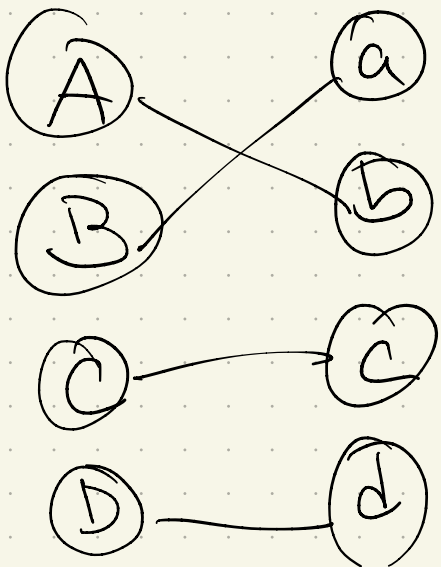
Stable matching

Really useful! Many variants:

- ties
- incomplete preference lists
- one side picks many from the other
- "egalitarian" matchings
- minimizing "regret"

Really a lot of choices to be made.

First: "unstable":



(A, a) is unstable
- if A prefers a to current match
- and a prefers A to current match

In a sense: if put together & realizing they both prefer each other, would (A, a) leave current matches?

↳ unstable!

History: used to be "stable marriage"

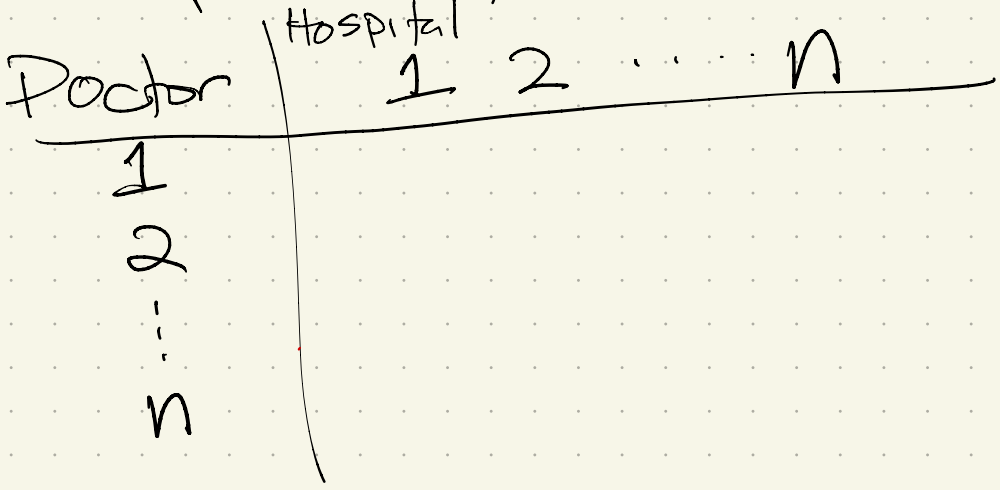
(long history of strange papers & variants.)

Algorithm: (Wikipedia)

Algorithm [edit]

```
algorithm stable_matching is
    Initialize all m ∈ M and w ∈ W to free
    while ∃ free man m who still has a woman w to propose to do
        w := first woman on m's list to whom m has not yet proposed
        if w is free then
            (m, w) become engaged
        else some pair (m', w) already exists
            if w prefers m to m' then
                m' becomes free
                (m, w) become engaged
            else
                (m', w) remain engaged
            end if
        end if
    repeat
```

In book, data structures matter for runtime!



Not obvious why it works.
(or even how to be greedy!)
Good example of why the proof matters.

Nice example of fairness:
This algorithm sucks for one side.
(Not all solutions are equal!)
How to even define "fair"?