1. Let $P = (1, 1, -1), Q = (1, 2, 5), R = (-1, -3, -4)$ be points in $\mathbb{R}^3$.

   (a) Find the distance between the line $\overrightarrow{PQ}$ and the point R.

$$
\begin{aligned}
\vec{v} &= Q - P = \langle 1-1, 2-1, 5-(-1)\rangle = \langle 0, 1, 6\rangle \\
\vec{w} &= R - P = \langle -1-1, -3-1, -4-(-1)\rangle = \langle -2, -4, -3\rangle \\
\vec{w}_\perp &= \vec{w} - proj_{\vec{v}}\vec{w} = \vec{w} - \frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\vec{v} \\
&= \langle -2, -4, -3\rangle - \frac{0 + (-4) + (-18)}{0 + 1 + 36}\langle 0, 1, 6\rangle \\
&= \langle \frac{-74}{37}, \frac{-148}{37}, \frac{-111}{37}\rangle - \frac{-22}{37}\langle 0, 1, 6\rangle = \frac{1}{37}\langle -74, -126, 21\rangle \\
|\vec{w}_\perp| &= \frac{1}{37}\sqrt{(-74)^2 + (-126)^2 + (21)^2} = \frac{\sqrt{21793}}{37} \\
&\approx \underline{3.989852}
\end{aligned}
$$

Is the distance between the line $\overrightarrow{PQ}$ and the point R.

   (b) Find the distance between the origin and the plane PQR.

$\vec{v} = \langle 0, 1, 6\rangle$ and $\vec{w} = \langle -2, -4, -3\rangle$ found above are two non-parallel vectors in PQR. Since we are working in $\mathbb{R}^3$, the cross product is defined and we can find a vector $\vec{n}$ that is normal to the plane:

$$
\begin{aligned}
\vec{n} &= \begin{vmatrix} i & j & k \\ 0 & 1 & 6 \\ -2 & -4 & -3 \end{vmatrix} = [(1)(-3) - (6)(-4)]i - [(0)(-3) - (6)(-2)]j + [(0)(-4) - (1)(-2)]k \\
&= \langle 21, -12, 2\rangle
\end{aligned}
$$

solving for x using $p_0 = P = (1, 1, -1)$ as an initial point we get the planar equation:

$$d = \vec{n}\cdot(p_0) = 21(1) - 12(1) + 2(-1) = 21 - 12 - 2 = 7$$

thus the equation for the plane is

$$7 = 21x_1 - 12x_2 + 2x_3$$

verify for Q, R:

$$7 = 21(1) - 12(2) + 2(5) = 21 - 24 + 10 = 7$$
$$7 = 21(-1) - 12(-3) + 2(-4) = -21 + 36 - 8 = 7$$

Find distance:

$$7 = 21(0 + 21d) - 12(0 - 12d) + 2(0 + 2d)$$
$$7 = 441d + 144d + 4d = 589d$$
$$d = \frac{7}{589}$$

Is the distance between the origin and the plane PQR.

(c) Find the point on the plane PQR closest to the point $S = (1, 1, 3)$.

Using $\vec{n} = \langle 21, -12, 2\rangle$ found above we can express the line perpendicular to PQR and thru S as

$$\overrightarrow{S} = S + c\vec{n} = (1, 1, 3) + c\langle 21, -12, 2\rangle = \langle 1 + 21c, 1 - 12c, 3 + 2c\rangle$$

for some scalar $c \in \mathbb{R}$. Our nearest point is the intersection between PQR and $\overrightarrow{S}$:

$$\begin{aligned} 7 &= 21(1 + 21c) - 12(1 - 12c) + 2(3 + 2c) \\ 7 &= 21 + 441c - 12 + 144c + 6 + 4c = 589c + 15 \\ c &= \frac{-8}{589} \end{aligned}$$

thus the nearest point S' is

$$\begin{aligned} S' &= S + \frac{-8}{589}\langle 21, -12, 2\rangle = (1, 1, 3) + \frac{1}{589}\langle -168, 96, -16\rangle = \frac{1}{589}(421, 685, 1751) \\ &\approx \underline{(0.715, 1.163, 2.973)} \end{aligned}$$

verify:

$$7 = 21(0.715) - 12(1.163) + 2(2.973) \approx 7$$

2. Web traffic can be modeled using a graph: this consists of a set of points connected by edges. We can represent the graph using an adjacency matrix A, where $a_{ij} = 1$ if there is a link from point i to point j. Note that edges are directional, and $a_{ij} = 1$ does not necessarily imply $a_{ji} = 1$. Use the figure below:

(a) Form the adjacency matrix A for the graph above.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) A useful result is that the ijth entry of the matrix A^n will be the number of ways you can get from i to j via a path of length n. Find A^4; determine the number of ways you can get from 1 to 4 using a path of length 4; then list them by describing the sequence i.e. $1 \to 4 \to 3 \to 1 \to 4$.

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3 & 5 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 5 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

There are five ways to get from 1 to 4 in four steps:
$1 \to 2 \to 4 \to 3 \to 4$
$1 \to 3 \to 1 \to 2 \to 4$
$1 \to 3 \to 1 \to 3 \to 4$
$1 \to 3 \to 4 \to 3 \to 4$
$1 \to 4 \to 3 \to 1 \to 4$

(c) Suppose B is an $N \times N$ adjacency matrix for a set of N connected web pages (where it's always possible to find a path from one page to any other). Prove, or explain why not true: There is some k for which all entries of B^k are non-zero.

Since the graph is fully connected, for each element B_{ij} of B where $i, j \in \{1, \cdots, N\}$ are the row and column indexes, there exists some natural number p_{ij} such that $B_{ij}^{p_{ij}} \neq 0$. Since a complete cycle from any B_{ij} to B_{ji} and back can be looped an arbitrary number of times, it is also the case that $(B_{ij}^{p_{ij}+p_{ji}})^m \neq 0$ for any $m \in \mathbb{N}$. Let

$$k = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p_{ij} + p_{ji}$$

For each B_{ij} we see that $k = n(p_{ij} + p_{ji})$ where $n \in \mathbb{N}$ is the product of all cycle exponents except for $p_{ij} + p_{ji}$. Thus, the kth power of the adjacency matrix provides complete cycles for all entries, and thus all values are non-zero.

3. Let $p_1(x) = x^2 - 3x - 10, p_2(x) = 5x + 7, p_3(x) = x^2 + 12x + 11$.

(a) Using standard polynomial addition, what polynomials $ax^2 + bx + c$ can be expressed as linear combinations of p_1, p_2, p_3?

$$p_n = a(p_1) + b(p_2) + c(p_3) = a(x^2 - 3x - 10) + b(5x + 7) + c(x^2 + 12x + 11)$$

for $a, b, c \in \mathbb{R}$. Simplifying:

$$p_n = (a + c)x^2 + (5b + 12c - 3a)x + (7b + 11c - 10a)$$

(b) Find a basis for the vector space spanned by p_1, p_2, p_3.

Using $\{x^2, x, 1\}$ as a basis for the set of all 2nd-degree polynomials, we can form a coefficient matrix P and row reduce:

$$P = \begin{bmatrix} 1 & -3 & -10 \\ 0 & 5 & 7 \\ 1 & 12 & 11 \end{bmatrix} \to \begin{bmatrix} 5 & -15 & -50 \\ 0 & 5 & 7 \\ 0 & 15 & 21 \end{bmatrix} \to \begin{bmatrix} 5 & 0 & 11 \\ 0 & 5 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we see p_1, p_2 are pivot columns and thus $\{x^2 - 3x - 10, 5x + 7\}$ form a basis for the vector space.

(c) Remember that a set of vectors is independent iff the only linear combination to produce $\vec{0}$ is the trivial combination. For these vectors, we could try to solve $a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = 0$. Instead, we can form new equations by differentiation. Explain how to form a system of equations with unknown constants a_1, a_2, a_3.

Our first equation are the linear combinations of the original functions set to zero

$$a_1(x^2 - 3x - 10) + a_2(5x + 7) + a_3(x^2 + 12x + 11) = 0$$

second equation is derivative of first:

$$a_1(2x - 3) + a_2(5) + a_3(2x + 12) = 0$$

third equation is derivative of second:

$$a_1(2) + a_2(0) + a_3(2) = 0$$

Which we can form into a coefficient matrix:

$$A = \begin{bmatrix} x^2 - 3x - 10 & 5x + 7 & x^2 + 12x + 11 \\ 2x - 3 & 5 & 2x + 12 \\ 2 & 0 & 2 \end{bmatrix}$$

and find the determinant:

$$\begin{aligned} \det A &= 2\left[(5x+7)(2x+12) - (x^2+12x+11)(5)\right] + 2\left[(x^2-3x-10)(5) - (5x+7)(2x-3)\right] \\ &= 2\left[10x^2 + 14x + 60x + 84 - 5x^2 - 60x - 55\right] + 2\left[5x^2 - 15x - 50 - 10x^2 - 14x + 15x + 21\right] \\ &= (10x^2 + 28x + 58) - (10x^2 + 28x + 58) \\ &= 0 \end{aligned}$$

Thus since $\det A = 0$ it follows that there are an infinite number of solutions to the equations, and thus the functions p_1, p_2, p_3 are dependent.

(d) Consider the vector space spanned by $\mathbb{V} = \{f_1(x), f_2(x), \cdots, f_n(x)\}$, where $f_i(x)$ is a smooth function of x (smooth means you can differentiate as many times as needed). Explain how you could determine if the set of vectors is linearly independent.

for any two functions f_1, f_2, we can say they are dependent if there exists some $c \in \mathbb{R}$ such that $f_1 = cf_2$, which can be written $\frac{f_1}{f_2} = c$. This implies that the ratio between the two functions is constant, and differentiating both sides results in

$$\left(\frac{f_1}{f_2}\right)' = 0$$

which evaluates to $f_1f_2' - f_1'f_2 = 0$. Note that if we form a matrix

$$A = \begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix}$$

We can see that the proceeding equation is equivalent to $\det A = 0$. This extends to n dimensions, using $n-1$ derivatives of the functions as required to form an $n \times n$ matrix to find the determinant. If this matrix A has a non-zero determinant, then only the trivial solution exists, and the functions $\{f_1, \cdots f_n\}$ are independent. If the determinant is zero, then the functions are dependent.

4. Given a set of vectors $\mathbb{V} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$, where the components of each vector are integers, a lattice consists of the set of all linear combinations of $\mathbb{V}$ whose coefficients are integers. Let $\vec{v}_1 = \langle 40, 1\rangle, \vec{v}_2 = \langle -1, 39\rangle$.

   (a) Determine which of the following vectors are in a lattice: $\langle 42, 74\rangle, \langle 198, 234\rangle, \langle 155, 199\rangle$.

Let $r, s \in \mathbb{R}$ such that

for $\langle 42, 74\rangle$:

$$\begin{aligned} 40r - s &= 42 \\ r + 39s &= 74 \\ s &= 40r - 42 \\ r + 39(40r - 42) &= 74 \\ 1561r &= 1712 \\ r &= \frac{1712}{1561} \notin \mathbb{Z} \end{aligned}$$

Thus $\langle 42, 74\rangle$ is not in a lattice of $\mathbb{V}$.

for $\langle 198, 234\rangle$:

$$\begin{aligned} 40r - s &= 198 \\ r + 39s &= 234 \\ s &= 40r - 198 \\ r + 39(40r - 198) &= 234 \\ 1561r &= 7956 \\ r &= \frac{7956}{1561} \notin \mathbb{Z} \end{aligned}$$

Thus $\langle 198, 234\rangle$ is not in a lattice of $\mathbb{V}$.

For $\langle 155, 199\rangle$:

$$\begin{aligned} 40r - s &= 155 \\ r + 39s &= 199 \\ s &= 40r - 155 \\ r + 39(40r - 155) &= 199 \\ 1561r &= 6244 \\ r &= \frac{6244}{1561} = 4 \in \mathbb{Z} \end{aligned}$$

Now solve for s:

$$s = 40(4) - 155 = 5 \in \mathbb{Z}$$

Thus $\langle 155, 199\rangle$ is in a lattice of $\mathbb{V}$.

(b) Given a lattice, the closest vector problem (CVP) requires us to find the lattice vector closest to a given vector. For the points above which are not in the lattice, solve the CVP.

For $\vec{x} = \langle 42, 74\rangle$:

$$r = \frac{1712}{1561} \approx 1.1217$$
$$s = 40\frac{1712}{1561} - 42 = \frac{68480 - 65562}{1561} \approx 1.869$$

My intuition is that the nearest integers to r, s will result in the nearest lattice point.

Try $r' = 1, s' = 2$.

$$\vec{x}' = \langle 40, 1\rangle + 2\langle -1, 39\rangle = \langle 38, 79\rangle$$
$$|\vec{x} - \vec{x}''| = \sqrt{4^2 + (-5)^2} = \sqrt{41} \approx 6.4$$

Thus $\langle 38, 79\rangle$ is the closest lattice point to $\langle 42, 74\rangle$.

For $\vec{y} = \langle 198, 234\rangle$:

$$r = \frac{7956}{1561} \approx 5.097$$
$$s = \frac{318240 - 309078}{1561} \approx 5.869$$

Try $r' = 5, s' = 6$.

$$\vec{y}' = 5\langle 40, 1\rangle + 6\langle -1, 39\rangle = \langle 200 - 6, 5 + 234\rangle = \langle 194, 239\rangle$$
$$|\vec{y} - \vec{y}'| = \sqrt{4^2 + (-4)^2} = \sqrt{32} \approx 5.657$$

Thus $\langle 194, 239\rangle$ is the closest lattice point to $\langle 198, 234\rangle$.

(c) Consider the lattice formed by $\vec{p}_1 = \langle 39, 40\rangle, \vec{p}_1 = \langle 77, 119\rangle$. Determine which of the points in problem 4a are in the lattice, then solve the CVP for the others.

Observe that $\vec{p}_1 = \vec{v}_1 + \vec{v}_2$ and $\vec{p}_2 = 2\vec{v}_1 + 3\vec{v}_2$.

For $\vec{x} = \langle 42, 74\rangle$:

$$39r + 77s = 42$$
$$40r + 119s = 74$$
$$r = \frac{42 - 77s}{39}$$
$$40(\frac{42 - 77s}{39}) + 119s = 74$$
$$1680 - 3080s + 4641s = 2886$$
$$1561s = 2886 - 1680 = 1206$$
$$s = \frac{1206}{1561} \approx 0.7725 \notin \mathbb{Z}$$
$$r = \frac{42 - 77\frac{1206}{1561}}{39} = \frac{65562 - 92862}{60879} \approx -0.448$$

Thus $\langle 42, 74\rangle$ is not a lattice point of $\mathbb{P}$.

Try $r_{\prime} = -1, s' = 1$:

$$\vec{x}' = -\langle 39, 40\rangle + \langle 77, 119\rangle = \langle 38, 79\rangle$$

This is the same closest lattice point as in $\mathbb{R}$.

For $\vec{y} = \langle 198, 234\rangle$:

$$\begin{aligned} 39r + 77s &= 198 \\ 40r + 119s &= 234 \\ r &= \frac{198 - 77s}{39} \\ 40\frac{198-77s}{39} + 119s &= 234 \\ 7920 - 3080s + 4641s &= 9126 \\ 1561s &= 9126 - 7920 = 1206 \\ s &= \frac{1206}{1561} \approx 0.7725 \notin \mathbb{Z} \\ r &= \frac{198 - 77\frac{1206}{1561}}{39} = \frac{309078 - 92862}{60879} \approx 3.552 \end{aligned}$$

Thus $\langle 198, 234\rangle$ is not a lattice point of $\mathbb{P}$.

Try $r' = 3, s' = 1$:

$$\vec{y}' = 3\langle 39, 40\rangle + \langle 77, 119\rangle = \langle 194, 239\rangle$$

This is the same closest lattice point as in $\mathbb{V}$.

For $\vec{z} = \langle 155, 199\rangle$:

$$\begin{aligned} 39r + 77s &= 155 \\ 40r + 119s &= 199 \\ r &= \frac{155 - 77s}{39} \\ 40\frac{155-77s}{39} + 119s &= 199 \\ 6200 - 3080s + 4641s &= 7761 \\ s &= \frac{1561}{1561} = 1 \in \mathbb{Z} \\ r &= \frac{155 - 77}{39} = 2 \in \mathbb{Z} \end{aligned}$$

Thus $\langle 155, 199\rangle$ is a lattice point in $\mathbb{P}$.

(d) As it turns out, the lattice spanned by the $\vec{v}_i$ś is the same lattice spanned by the $\vec{p}_i$'s. Prove this. (Suppose $\vec{x}$ can be expressed as a linear combination with integer coefficients of $\vec{v}$'s. Show that $\vec{x}$ can be expressed as a linear combination with integer coefficients of $\vec{p}$'s, then show that the converse is true.

Let $S = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ be the matrix formed with the coefficients of $\vec{p}_i = a_i\vec{v}_1 + b_i\vec{v}_2$ as columns. Then

$$S^{-1} = \frac{1}{3-2}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

Is the translation matrix between the basis $\mathbb{V}$ and $\mathbb{P}$.

Let $\vec{x}$ be a lattice point of $\mathbb{V}$ such that the coefficients $x_1, x_2 \in \mathbb{Z}$.

$$\vec{x}_{\mathbb{P}} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 \\ -2x_1 + x_2 \end{bmatrix}$$

Since $x_1, x_2 \in \mathbb{Z}$, it follows that $3x_1 - x_2, -2x_1 + x_2 \in \mathbb{Z}$ thus $\vec{x}$ is in the lattice of $\mathbb{P}$.

Let $\vec{y}$ be a lattice point of $\mathbb{P}$ such that the coefficients $y_1, y_2 \in \mathbb{Z}$.

$$S\vec{y}_{\mathbb{V}} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 + 2y_2 \\ y_1 + 3y_2 \end{bmatrix}$$

Since $y_1, y_2 \in \mathbb{Z}$, it follows that $y_1 + 2y_2, y_1 + 3y_2 \in \mathbb{Z}$ thus $\vec{y}$ is in the lattice of $\mathbb{V}$.

Therefore $\mathbb{V}, \mathbb{P}$ span the same lattice.

(e) Identify an important difference between the vectors $\vec{v}_i$ and the vectors $\vec{p}_i$. Suggestion: Given the set of vectors $\vec{v}_i$, what questions could you ask about them? Then ask the same question about the vectors $\vec{p}_i$.

Both sets of vectors are independent.
Neither set of vectors is orthogonal.
Both sets of vectors form a basis for $\mathbb{R}^2$.
Both sets of vectors have non-zero determinant.
$\mathbb{V}$ has complex eigenvalues:

$$|V - \lambda I| = \begin{vmatrix} 40-\lambda & -1 \\ 1 & 39-\lambda \end{vmatrix} = (40-\lambda)(39-\lambda) - (-1)(1) = \lambda^2 - 79\lambda + 1561 = 0$$

use quadratic equation to find roots:

$$r_{\mathbb{V}} = \frac{-(-79) \pm \sqrt{(-79)^2 - 4(1)(1561)}}{2} = \frac{79 \pm \sqrt{-3}}{2} \in \mathbb{C}$$

$\mathbb{P}$ has real eigenvalues:

$$|P - \lambda I| = \begin{vmatrix} 39-\lambda & 77 \\ 40 & 119-\lambda \end{vmatrix} = (39-\lambda)(119-\lambda) - (77)(40) = \lambda^2 - 158\lambda + 1561 = 0$$

use quadratic equation to find roots:

$$r_{\mathbb{P}} = \frac{-(-158) \pm \sqrt{(-158)^2 - 4(1)(1561)}}{2} = \frac{158 \pm \sqrt{18720}}{2} \in \mathbb{R}$$

5. Let $A = \begin{bmatrix} 6 & -5 \\ 10 & -9 \end{bmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors of A. Refer to these as $\lambda_1, \lambda_2, \vec{v}_1, \vec{v}_2$ for the following questions.

$$|(A - \lambda I)| = (6-\lambda)(-9-\lambda) - (-5)(10) = \lambda^2 + 3\lambda - 4 = (\lambda+4)(\lambda-1) = 0$$

thus $\lambda_1 = -4$ and $\lambda_2 = 1$. Now find $\vec{v}_1$ for $\lambda_1 = -4$:

$$\begin{bmatrix} 6-(-4) & -5 \\ 10 & -9-(-4) \end{bmatrix} \to \begin{bmatrix} 10 & -5 \\ 10 & -5 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

thus $\vec{v}_1 = \langle 1, 2 \rangle$. Now find $\vec{v}_2$ for $\lambda_2 = 1$:

$$\begin{bmatrix} 6-1 & -5 \\ 10 & -9-1 \end{bmatrix} \to \begin{bmatrix} 5 & -5 \\ 10 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

thus $\vec{v}_2 = \langle 1, 1 \rangle$.

(b) Prove or disprove: The set of eigenvectors you found are independent.

Suppose for the sake of contradiction that the eigenvectors are not independent. Then we can express each in terms of the other (for some $c, d \in \mathbb{R}$):

$$\begin{aligned} \vec{v}_1 &= c\vec{v}_2 \\ \langle 1,2\rangle &= c\langle 1,1\rangle \\ \vec{v}_2 &= d\vec{v}_2 \\ \langle 1,1\rangle &= d\langle 1,2\rangle \end{aligned}$$

Solving this we get $c = 1$ and also $c = 2$ and $d = 1$ and also $d = 0.5$ and thus we have reached a contradiction. Therefore the eigenvectors are independent.

(c) Prove or disprove: The set of eigenvectors you found span $\mathbb{R}^2$.

Let $\vec{x} \in \mathbb{R}^2$. We want to find $a, b \in \mathbb{R}$ such that

$$\begin{aligned} \vec{x} &= a\langle 1,2\rangle + b\langle 1,1\rangle \\ x_1 &= a + b \\ x_2 &= 2a + b \end{aligned}$$

and we define the augmented matrix X as

$$X = \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 2 & 1 & x_2 \end{array}\right] \to \left[\begin{array}{cc|r} 1 & 1 & x_1 \\ 0 & 1 & 2x_1 - x_2 \end{array}\right] \to \left[\begin{array}{cc|r} 1 & 0 & x_2 - x_1 \\ 0 & 1 & 2x_1 - x_2 \end{array}\right]$$

Thus we have found coefficients $a = x_2 - x_1$ and $b = 2x_1 - x_2$ and thus the eigenvectors span $\mathbb{R}^2$.

(d) Let $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2$. Find $A^{100}\vec{x}$.

$$\begin{aligned} A^{100}\vec{x} &= x_1\lambda_1^{100}\vec{v}_1 + x_2\lambda_2^{100}\vec{v}_2 \\ &= x_1(-4)^{100}\langle 1,2\rangle + x_2(1)^{100}\langle 1,1\rangle \\ &= x_1\langle 2^{200}, 2^{201}\rangle + x_2\langle 1,1\rangle \\ &= \langle 2^{200}x_1 + x_2, 2^{201}x_1 + x_2\rangle \end{aligned}$$

(e) If possible, set up and solve the system of equations that would allow you to express any vector $\vec{x}$ as a linear combination of the eigenvectors $\vec{v}_1, \vec{v}_2$.

This is a task analogous to assignment five problem 2c. Let $\vec{x} = \langle x_1, x_2\rangle \in \mathbb{R}^2$
Let $S = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ be the matrix formed with $\vec{v}_1, \vec{v}_2$ as columns, then

$$S^{-1} = \frac{1}{1-2}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

Is the matrix which transforms a vector from standard basis to the eigenbasis. Thus the system:

$$\begin{aligned} x_1' &= -x_1 + x_2 \\ x_2' &= 2x_1 - x_2 \end{aligned}$$

Is the system of equations to express any $\vec{x}$ as a linear combination of the eigenvectors:

$$\vec{x} = x_1'\vec{v}_1 + x_2'\vec{v}_2 = (x_2 - x_1)\vec{v}_1 + (2x_1 - x_2)\vec{v}_2$$

(f) Let $\vec{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $A^{100}\vec{y}$.

first transform into the eigen basis coordinates:

$$\vec{y}_S = S^{-1}\vec{y} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

then apply the transformation as eigenvalue multiplication

$$\begin{aligned} A^{100}\vec{y}_S &= (-2)\lambda_1^{100}\vec{v}_1 + (5)\lambda_2^{100}\vec{v}_2 \\ &= (-2)(-4)^{100}\langle 1, 2\rangle + (5)(1)^{100}\langle 1, 1\rangle \\ &= \langle 5 - 2^{201}, 5 - 2^{202}\rangle \end{aligned}$$

Now return to standard basis coordinates:

$$\begin{aligned} A^{100}\vec{y} = SA^{100}\vec{y}_S &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 - 2^{201} \\ 5 - 2^{202} \end{bmatrix} = \begin{bmatrix} (5 - 2^{201}) - (5 - 2^{202}) \\ (10 - 2^{202}) - (5 - 2^{202}) \end{bmatrix} \\ &= \begin{bmatrix} 2^{201} \\ 5 \end{bmatrix} \end{aligned}$$