1. Find the determinant of each matrix.

 (a) $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

 $(2 \cdot -1) - (3 \cdot 1) = -2 - 3 = \underline{-5}$

 (b) $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$

 $(2 \cdot -6) - (-3 \cdot 4) = -12 + 12 = \underline{0}$

 (c) $\begin{bmatrix} 3 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$

 $(1)(-1-1)(-1)^3 + 0 + (-1)(3 - -2)(-1)^5 = 2 + 5 = \underline{7}$

2. Every mathematical formula is a summary of all the steps of an algorithm. Cramer's Rule is a formula that summarizes all steps necessary to solve a system of n equations with n unknowns.

 (a) Consider a system of two equations with two unknowns:

$$\begin{aligned} a_{11}x + a_{12}y &= c_1 \\ a_{21}x + a_{22}y &= c_2 \end{aligned}$$

 Solve this system to produce formulas for the values x and y.

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \end{array}\right] \to \left[\begin{array}{cc|c} a_{11} & a_{12} & c_1 \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}} & c_2 - \frac{c_1 a_{21}}{a_{11}} \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} a_{11} & 0 & c_1 - \left(\frac{a_{11}a_{12}}{a_{11}a_{22}-a_{12}a_{21}}\right)\left(\frac{c_2a_{11}-c_1a_{21}}{a_{11}}\right) \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}} & \frac{c_2a_{11}-c_1a_{21}}{a_{11}} \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} a_{11} & 0 & \frac{c_1a_{11}a_{22}-c_1a_{12}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} - \frac{c_2a_{11}a_{12}-c_1a_{12}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} \\ 0 & 1 & \left(\frac{c_2a_{11}-c_1a_{21}}{a_{11}}\right)\left(\frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}}\right) \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & \frac{c_1a_{22}-c_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ 0 & 1 & \frac{c_2a_{11}-c_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}} \end{array}\right]$$

$$x = \frac{c_1a_{22} - c_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$y = \frac{c_2a_{11} - c_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

(b) Let A be the coefficient matrix; A_x be the matrix produced by replacing first column of the coefficient matrix (corresponding to x) with the constant vector; A_y be the matrix produced by replacing the second column of the coefficient matrix (corresponding to y) with the constant vector. Find $\det A$, $\det A_x$, and $\det A_y$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \det A = a_{11}a_{22} - a_{12}a_{21}$$
$$A_x = \begin{bmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{bmatrix} \qquad \det A_x = c_1a_{22} - c_2a_{12}$$
$$A_y = \begin{bmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{bmatrix} \qquad \det A_y = c_2a_{11} - c_1a_{21}$$

(c) Express the values of x and y in terms of these determinants.

$$x = \frac{\det A_x}{\det A} \qquad y = \frac{\det A_y}{\det A}$$

(d) Use Cramer's Rule to solve:

$$\begin{aligned} 3x + 4y &= 9 \\ 7x - 27 &= 8 \end{aligned}$$

$$x = \frac{(9 \cdot -27) - (4 \cdot 8)}{(3 \cdot 8) - (4 \cdot 7)} = \frac{-243 - 32}{-4} \qquad = 68\frac{3}{4}$$
$$y = \frac{(3 \cdot 8) - (9 \cdot 7)}{(3 \cdot 8) - (4 \cdot 7)} = \frac{32 - 63}{-4} \qquad = 7\frac{3}{4}$$

(e) Cramer's Rule generalizes (with A_z defined as you'd expect it to be). Use it to solve:

$$\begin{aligned} x + 3y - 4z &= 8 \\ 2x - 4y - 7z &= 1 \\ x - 2y + 8 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & -4 & -7 \\ 1 & -2 & 0 \end{bmatrix} \quad A_x = \begin{bmatrix} 8 & 3 & -4 \\ 1 & -4 & -7 \\ -8 & -2 & 0 \end{bmatrix} \quad A_y = \begin{bmatrix} 1 & 8 & -4 \\ 2 & 1 & -7 \\ 1 & -8 & 0 \end{bmatrix} \quad A_z = \begin{bmatrix} 1 & 3 & 8 \\ 2 & -4 & 1 \\ 1 & -2 & -8 \end{bmatrix}$$

$$\det A = (-4)(-4 - -4)(-1)^4 + (-7)(-2 - 3)(-1)^5 + 0 = -35$$
$$\det A_x = (-4)(-2 - 32)(-1)^4 + (-7)(-16 - -24)(-1)^5 + 0 = 136 + 56 = 192$$
$$\det A_y = (-4)(-16 - 1)(-1)^4 + (-7)(-8 - 8)(-1)^5 = 68 - 112 = -44$$
$$\det A_z = (1)(32 - -2)(-1)^2 + (2)(-24 - -16)(-1)^3 + (1)(3 - -32)(-1)^4 = 34 + 16 + 35 = 85$$
$$x = \frac{192}{-35} \quad y = \frac{-44}{-35} \quad z = \frac{85}{-35}$$

(f) Cramer's Rule is a good example of why it's the journey, not the destination: It's a terrible way to solve linear systems. However, it's important because on the way, you find a way of determining when a system of n equations of n unknowns does not have a unique solution. How can you use Cramer's Rule to predict whether a system of n equations of n unknowns has a unique solution?

If $\det A \neq 0$ where A is the coefficient matrix, then $A \mid \vec{c}$ has a unique solution (where $\vec{c}$ are non-homogeneous constants).

3. Find the eigenvalues and associated eigenvectors for the following matrices.

(a) $\begin{bmatrix} 4 & -15 \\ 2 & -7 \end{bmatrix}$

$$\det(A - I\lambda) = \begin{vmatrix} 4-\lambda & -15 \\ 2 & -7-\lambda \end{vmatrix} = (4-\lambda)(-7-\lambda) - (-15)(2) = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1) = 0$$

Thus $\lambda_1 = -1$ and $\lambda_2 = -2$.

Find eigenvectors using row reduction:

$$(A - \lambda_1 I)\vec{v}_1 = \begin{bmatrix} 5 & -15 \\ 2 & -6 \end{bmatrix} \to \begin{bmatrix} 5 & -15 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \to \vec{v}_1 = \langle 3, 1 \rangle$$

$$(A - \lambda_2 I)\vec{v}_2 = \begin{bmatrix} 6 & -15 \\ 2 & -5 \end{bmatrix} \to \begin{bmatrix} 6 & -15 \\ 0 & 0 \end{bmatrix} \to \vec{v}_2 = \langle 15, 6 \rangle$$

Thus $\vec{v}_1 = \langle 3, 1 \rangle$ and $\vec{v}_2 = \langle 15, 6 \rangle$.

verify:

$$A\vec{v}_1 = \begin{bmatrix} 4 & -15 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12-15 \\ 6-7 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \lambda_1 \vec{v}_1$$

$$A\vec{v}_2 = \begin{bmatrix} 4 & -15 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} 15 \\ 6 \end{bmatrix} = \begin{bmatrix} 60-90 \\ 30-42 \end{bmatrix} = \begin{bmatrix} -30 \\ -12 \end{bmatrix} = (-2) \begin{bmatrix} 15 \\ 6 \end{bmatrix} = \lambda_2 \vec{v}_2$$

(b) $\begin{bmatrix} 18 & -20 \\ 15 & -17 \end{bmatrix}$

$$\det(A - I\lambda) = \begin{vmatrix} 18-\lambda & -20 \\ 15 & -17-\lambda \end{vmatrix} = (18-\lambda)(-17-\lambda) - (-20)(15) = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2) = 0$$

Thus $\lambda_1 = -2$ and $\lambda_2 = 3$.

Find eigenvectors using row reduction:

$$(A - \lambda_1 I)\vec{v}_1 = \begin{bmatrix} 20 & -20 \\ 15 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \to \vec{v}_1 = \langle 1, 1 \rangle$$

$$(A - \lambda_2 I)\vec{v}_2 = \begin{bmatrix} 15 & -20 \\ 15 & -20 \end{bmatrix} \to \begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix} \to \vec{v}_2 = \langle 4, 3 \rangle$$

Thus $\vec{v}_1 = \langle 1, 1 \rangle$ and $\vec{v}_2 = \langle 4, 3 \rangle$.

Verify:

$$A\vec{v}_1 = \begin{bmatrix} 18 & -20 \\ 15 & -17 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 18-20 \\ 15-17 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \vec{v}_1$$

$$A\vec{v}_2 = \begin{bmatrix} 18 & -20 \\ 15 & -17 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 72-60 \\ 60-51 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \end{bmatrix} = (3) \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \lambda_2 \vec{v}_2$$

(c) $\begin{bmatrix} -2 & 5 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 4 \end{bmatrix}$

$$\begin{aligned}\det(A-\lambda I) &= (-1-\lambda)\begin{vmatrix} -2-\lambda & 4 \\ -2 & 4-\lambda \end{vmatrix}(-1)^4 = (-1-\lambda)\left[(-2-\lambda)(4-\lambda)-(4)(-2)\right] \\ &= (-1-\lambda)(\lambda^2-2\lambda) = -\lambda^2+2\lambda-\lambda^3+2\lambda^2 = -\lambda(\lambda^2-\lambda-2) \\ &= -\lambda(\lambda+1)(\lambda-2) = 0\end{aligned}$$

Thus $\lambda_1=-1$, $\lambda_2=0$, and $\lambda_3=2$.

Find eigenvectors using row reduction:

$$(A-\lambda_1 I)\vec{v}_1 = \begin{bmatrix} -1 & 5 & 4 \\ 0 & 0 & 0 \\ -2 & 7 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & -4 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -15 & -12 \\ 0 & 15 & 15 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \langle 1, 1, -1\rangle$$

$$(A-\lambda_2 I)\vec{v}_2 = \begin{bmatrix} -2 & 5 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 4 \end{bmatrix} \to \begin{bmatrix} -2 & 5 & 4 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_2 = \langle 2, 0, 1\rangle$$

$$(A-\lambda_3 I)\vec{v}_3 = \begin{bmatrix} -4 & 5 & 4 \\ 0 & -3 & 0 \\ -2 & 7 & 2 \end{bmatrix} \to \begin{bmatrix} -4 & 5 & 4 \\ 0 & -3 & 0 \\ 0 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_3 = \langle 1, 0, 1\rangle$$

Verify:

$$A\vec{v}_1 = \begin{bmatrix} -2 & 5 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \lambda_1\vec{v}_1$$

$$A\vec{v}_2 = \begin{bmatrix} -2 & 5 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (0)\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \lambda_2\vec{v}_2$$

$$A\vec{v}_3 = \begin{bmatrix} -2 & 5 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = (2)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda_3\vec{v}_3$$

4. Let x_n and y_n be the number of immature and mature pairs of rabbits at the end of month n. If the rabbits breed according to the model of Leonardo of Pisa, we can find x_{n+1} and y_{n+1} via

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}$$

(a) Find the eigenvalues and corresponding eigenvectors for the transition matrix.

$$\det(A-\lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (-\lambda)(1-\lambda) - 1 = \lambda^2 - \lambda - 1 = 0$$

Use quadratic equation to find roots:

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2-(4)(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1-\sqrt{5}}{2} = 1-\phi \qquad \lambda_2 = \frac{1+\sqrt{5}}{2} = \phi$$

Find eigenvectors using row reduction:

$$(A-\lambda_1 I)\vec{v}_1 = \begin{bmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ 1 & 1-\frac{1-\sqrt{5}}{2} \end{bmatrix} \to \begin{bmatrix} 0 & 1+\left(1-\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) \\ 1 & 1-\frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1-\frac{1-\sqrt{5}}{2} \\ 0 & 1+\frac{1-\sqrt{5}}{2}-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^2 \end{bmatrix} \to \begin{bmatrix} 1 & 1-\frac{1-\sqrt{5}}{2} \\ 0 & 1+\frac{1}{2}-\frac{\sqrt{5}}{2}-\left(\frac{1}{4}-\frac{\sqrt{5}}{2}+\frac{5}{4}\right) \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1-\frac{1-\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix} \to \vec{v}_1 = \langle \frac{1-\sqrt{5}}{2} - 1, 1\rangle = \langle -\frac{1+\sqrt{5}}{2}, 1\rangle$$

$$(A-\lambda_2 I)\vec{v}_2 = \begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & 1-\frac{1+\sqrt{5}}{2} \end{bmatrix} \to \begin{bmatrix} 0 & 1+\left(1-\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) \\ 1 & 1-\frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1-\frac{1+\sqrt{5}}{2} \\ 0 & 1+\frac{1+\sqrt{5}}{2}-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^2 \end{bmatrix} \to \begin{bmatrix} 1 & 1-\frac{1+\sqrt{5}}{2} \\ 0 & \frac{3+\sqrt{5}}{2}-\left(\frac{1}{4}+\frac{\sqrt{5}}{2}+\frac{5}{4}\right) \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1-\frac{1+\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix} \to \vec{v}_2 = \langle \frac{1+\sqrt{5}}{2} - 1, 1\rangle = \langle -\frac{1-\sqrt{5}}{2}, 1\rangle$$

Thus $\vec{v}_1 = \langle -\phi, 1\rangle$ and $\vec{v}_2 = \langle \phi - 1, 1\rangle$.

(b) Suppose $x_0 = 1$ and $y_0 = 0$. Express $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of the eigenvectors you found.

Row reduce:

$$\left[\begin{array}{cc|c} -\frac{1+\sqrt{5}}{2} & -\frac{1-\sqrt{5}}{2} & 1 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & \frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & \sqrt{5} & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{\sqrt{5}} \\ 0 & 1 & \frac{1}{\sqrt{5}} \end{array}\right]$$

Verify:

$$-\frac{1}{\sqrt{5}}\begin{bmatrix} -\frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + \frac{1}{\sqrt{5}}\begin{bmatrix} -\frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{10}+\frac{1}{2}-\frac{\sqrt{5}}{10}+\frac{1}{2} \\ \frac{\sqrt{5}}{5}-\frac{\sqrt{5}}{5} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(c) Determine the exact number of immature and mature rabbits at the end of the 12th month (x_{12}, y_{12}).

$$\vec{x}_{12} = F^{12}\vec{x}_0$$
$$F^{12} = (F^4)(F^4)(F^4)$$
$$F^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad F^4 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$
$$F^8 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix} \quad F^{12} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix} = \begin{bmatrix} 89 & 144 \\ 144 & 233 \end{bmatrix}$$
$$\vec{x}_{12} = \begin{bmatrix} 89 & 144 \\ 144 & 233 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 89 \\ 144 \end{bmatrix}$$

Thus at the end of 12 months there are 89 pairs of immature and 144 pairs of mature rabbits.

(d) Use the eigenvalues and eigenvectors to approximate the number of immature and mature at the end of the 12th month.

Let $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, let λ_1, λ_2 be the eigenvalues and $\vec{v}_1, \vec{v}_2$ be the associated eigenvectors. Expressing $\vec{x} = \langle 1, 0\rangle$ in terms of the eigenbasis we get $\vec{x}' = \langle -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$.

$$\vec{x}_{12} = F^{12}\vec{x} \approx x_1'\lambda_1^{12}\vec{v}_1 + x_2'\lambda_2^{12}\vec{v}_2 = -\frac{1}{\sqrt{5}}(1-\phi)^{12}\begin{bmatrix} -\phi \\ 1 \end{bmatrix} + \frac{1}{\sqrt{5}}\phi^{12}\begin{bmatrix} \phi - 1 \\ 1 \end{bmatrix}$$
$$\approx (-0.447)(0.00311)\begin{bmatrix} -1.618 \\ 1 \end{bmatrix} + (0.447)(321.997)\begin{bmatrix} 0.618 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 89 \\ 144 \end{bmatrix}$$

Thus at the end of 12 months there are approximately 89 pairs of immature and 144 pairs of mature rabbits.

5. Answer the following questions.

(a) Let $A = \begin{bmatrix} 2 & 1 & 5 & 1 \\ 0 & 2 & 3 & -1 \\ 1 & -1 & 2 & 1 \end{bmatrix}$. Find a basis for $\mathrm{Col}(A)$ and a basis for $\mathrm{Null}(A)$.

Row reduce:
$$\begin{bmatrix} 2 & 1 & 5 & 1 \\ 0 & 2 & 3 & -1 \\ 1 & -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & -3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 7 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 7 & -1 \end{bmatrix}$$

$\mathrm{Col}(A) = \mathrm{Span}(\{\langle 2,0,1\rangle, \langle 1,2,-1\rangle, \langle 5,3,2\rangle\})$

Parameterize: Let $7s = w$.

$x = -7s$

$y = 2s$

$z = -s$

$\mathrm{Null}(A) = \mathrm{Span}(\langle -7, 2, -1, 7\rangle)$

(b) Mathematicians like to recycle concepts, so the same basic idea will occur in many different places. Remember the range of a function is the set of all possible outputs; Thus we define the range of a linear transformation T to be the set of all vectors $\vec{y}$ for which $T\vec{x} = \vec{y}$ for some $\vec{x}$. Let $T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 5 & 3 \end{bmatrix}$.

Find the range of T.

Row reduce:

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & y_1 \\ -1 & 2 & 1 & y_2 \\ 1 & 5 & 3 & y_3 \end{array}\right] \to \left[\begin{array}{ccc|r} 1 & 5 & 3 & y_1 + 2y_2 \\ 0 & 7 & 4 & y_2 + y_3 \\ 0 & 0 & 0 & y_3 - y_1 - 2y_2 \end{array}\right] \to \left[\begin{array}{ccc|r} 7 & 0 & 1 & 7y_1 + 9y_2 - 5y_3 \\ 0 & 7 & 4 & y_2 + y_3 \\ 0 & 0 & 0 & y_3 - y_1 - 2y_2 \end{array}\right] \to$$

Parameterize: Let $x_3 = 7s$. Then $x_2 = -4s$, and $x_1 = -s$. Thus $\vec{x} = s\langle -1, -4, 7\rangle$ for $s \in \mathbb{R}$.
And thus we see the range of the transformation T satisfies the equation $-y_1 - 2y_2 - y_3 = 0$.
Let $y_1 = 0$, then $y_3 = -2y_2$, and $\vec{y}_1 = \langle 0, 1, -2\rangle$.
Let $y_2 = 0$, then $y_1 = -y_3$, and $\vec{y}_2 = \langle -1, 0, 1\rangle$.
Let $y_3 = 0$, then $y_1 = -2y_2$, and $\vec{y}_3 = \langle -2, 1, 0\rangle$.
Since $y_3 = y_1 + 2y_2$ we can see that $\{y_1, y_2\}$ is independent but $\{y_1, y_2, y_3\}$ is not.
Thus the range of $T = \text{Span}\{\langle 0, 1, -2\rangle, \langle -1, 0, 1\rangle\}$ which is a plane in $\mathbb{R}^3$.