1. For each of the following, find a basis for Row(A), Col(A), and Null(A); also identify the dimension of each. If the vectors are not independent, express the dependent vector(s) as linear combinations of the others.

(a) $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 5 \\ 0 & 3 & 1 \end{bmatrix}$

First, we find rref(A) to find bases for Col(A) and Null(A)

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 5 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -4 \\ 1 & 2 & 5 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & 9 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & 9 \\ 0 & 0 & -26 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since each column in rref(A) has a pivot we can see a basis for Col(A) is $\{\langle 2,1,0\rangle^T, \langle 3,2,3\rangle^T, \langle 1,5,1\rangle^T\}$ with dimension 3, and the basis for Null(A) is $\{\langle 0,0,0\rangle^T\}$ with dimension 0. All columns are independent.

Now we find rref(A^T) to find a basis for Row(A)

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 3 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & -1 \\ 0 & -13 & 0 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & -1 \\ 0 & 1 & 0 \\ 0 & 9 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So we can see a basis for Row(A) is $\{\langle 2,3,1\rangle, \langle 1,2,5\rangle, \langle 0,3,1\rangle\}$ with dimension 3.

(b) $A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 3 & 1 & 0 \\ 4 & -1 & -2 & 1 \end{bmatrix}$

First, we find rref(A) to find bases for Col(A) and Null(A):

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 3 & 1 & 0 \\ 4 & -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & -3 & -1 & -4 \\ 0 & -13 & -6 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 39 & 13 & 52 \\ 0 & -39 & -18 & -21 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & -5 & 31 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & -5 & 31 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 15 & 5 & 20 \\ 0 & 0 & -5 & 31 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 15 & 0 & 51 \\ 0 & 0 & -5 & 31 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 51/15 \\ 0 & 0 & 1 & -31/5 \end{bmatrix}$$

Since columns 1, 2, 3 have pivots we can see a basis for Col(A) is $\{\langle 1,2,4\rangle^T, \langle 3,3,-1\rangle^T, \langle 1,1,-2\rangle^T\}$ with dimension 3.

and a basis for Null(A) is $\{\langle 30,-51,93,15\rangle^T\}$ with dimension 1.

column 4 is dependent on the others, and can be expressed as $-2\langle 1,2,4\rangle^T + \frac{51}{15}\langle 3,3,-1\rangle^T - \frac{31}{5}\langle 1,1,-2\rangle^T$

Now, we find rref(A^T) to find basis for Row(A):

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & -1 \\ 1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -13 \\ 0 & -1 & -6 \\ 0 & -4 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -6 \\ 0 & -1 & -6 \\ 0 & -4 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -12 \\ 0 & 0 & -31 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since rows 1, 2, 3 have pivots, we can see a basis for Row(A) is $\{\langle 1,3,1,2\rangle, \langle 2,3,1,0\rangle, \langle 4,-1,-2,1\rangle\}$ with dimension 3.

(c) $A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & -4 & -5 \end{bmatrix}$

First, we find rref(A) to find bases for Col(A) and Null(A):

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & -4 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 9 \\ 1 & 5 & 9 \\ 2 & -4 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 9 \\ 0 & -14 & -23 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 9 \\ 0 & 1 & 23/14 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 11/14 \\ 0 & 1 & 23/14 \\ 0 & 0 & 0 \end{bmatrix}$$

Since columns 1, 2 have pivots we can see a basis for Col(A) is $\{\langle 3, 1, 2\rangle^T, \langle 1, 5, -4\rangle^T\}$ with dimension 2.

and a basis for Null(A) is $\{\langle -11, -23, 14\rangle^T\}$ with dimension 1.

column 3 is dependent on the others, and can be expressed as $\frac{11}{14}\langle 3, 1, 2\rangle^T + \frac{23}{14}\langle 1, 5, -4\rangle^T$

2. Let $\mathbb{V} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for a 2-dimensional vector space, and let

$$\vec{w}_1 = a_{11}\vec{v}_1 + a_{12}\vec{v}_2$$
$$\vec{w}_2 = a_{21}\vec{v}_1 + a_{22}\vec{v}_2$$

where all $a_{ij} \in \mathbb{R}$.

(a) Under what conditions will $\mathbb{W} = \{\vec{w}_1, \vec{w}_2\}$ be the basis of a 2-dimensional vector space?

$\vec{w}_1$ and $\vec{w}_2$ will be a basis if they do not point in the same direction. Since $\vec{v}_1$ and $\vec{v}_2$ form a basis, we know we can meet this condition by making sure the ratio between components of $\vec{v}_i$ are not the same in $\vec{w}_i$.

$$\frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}}$$

since we do not know that $a_{21} \neq 0, a_{22} \neq 0$ it is better to say

$$a_{11}a_{22} \neq a_{12}a_{21}$$
$$a_{11}a_{22} - a_{12}a_{21} \neq 0$$

And we see that this is analogous to the conditions for matrix invertibility in assignment 4 question 2. Thus we can deduce that if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is an invertible matrix corresponding to a change of basis transformation performed on a set of basis vectors then the outcome is a set of basis vectors.

(b) Prove or disprove: If $\mathbb{W}$ is the basis for a 2-dimensional vector space, it will be the same as the vector space spanned by $\mathbb{V}$.

Suppose $\mathbb{W}$ is a basis for a 2d vector space. Let $\vec{x} \in Span(\mathbb{W})$ such that $\vec{x} = x_1\vec{w}_1 + x_2\vec{w}_2$ for some $x_1, x_2 \in \mathbb{R}$ not both zero.

$$\begin{aligned}\vec{w} &= x_1\vec{w}_1 + x_2\vec{w}_2 \\ &= x_1(a_{11}\vec{v}_1 + a_{12}\vec{v}_2) + x_2(a_{21}\vec{v}_1 + a_{22}\vec{v}_2) \\ &= (x_1a_{11} + x_2a_{21})\vec{v}_1 + (x_1a_{12} + x_2a_{22})\vec{v}_2\end{aligned}$$

thus $\vec{x} \in Span(\mathbb{V})$ and $Span(\mathbb{W}) \subseteq Span(\mathbb{V})$.
Let $\vec{y} \in Span(\mathbb{V})$ such that $\vec{y} = y_1\vec{v}_1 + y_2\vec{v}_2$ for some $y_1, y_2 \in \mathbb{R}$ not both zero.

$$\vec{y} = y_1\vec{v}_1 + y_2\vec{y}_2$$

since we know $a_{11}a_{22} - a_{12}a_{21} \neq 0$ we can substitute $y_1 = x', y_2 = y'$ found in part c below:

$$\begin{aligned}\vec{v} &= x'\vec{w}_1 + y'\vec{w}_2 \\ &= \frac{y_1 \cdot a_{22} - y_2 \cdot a_{21}}{a_{11}a_{22} - a_{12}a_{21}}\vec{w}_1 + \frac{y_2 \cdot a_{11} - y_1 \cdot a_{12}}{a_{11}a_{22} - a_{12}a_{21}}\vec{w}_2\end{aligned}$$

By the answer to 2(c) below. Thus $\vec{y} \in Span(\mathbb{W}), Span(\mathbb{V}) \subseteq Span(\mathbb{W})$, and subsequently $Span(\mathbb{V}) = Span(\mathbb{W})$.

(c) Suppose $\vec{x} = a\vec{v}_1 + b\vec{v}_2$. Find a', b' so that $\vec{x} = a'\vec{w}_1 + b'\vec{w}_2$.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. By the definition above $A\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \end{bmatrix}$. Multiply both sides by A^{-1}:

$$A^{-1}A\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = A^{-1}\begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \end{bmatrix}$$

thus

$$\vec{x} = \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}\frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \end{bmatrix}$$

$$\begin{bmatrix} a' & b' \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \underline{\begin{bmatrix} \frac{a \cdot a_{22} - b \cdot a_{21}}{a_{11}a_{22} - a_{12}a_{21}} & \frac{b \cdot a_{11} - a \cdot a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \end{bmatrix}}$$

3. The following will prove a useful theorem about independence, and motivate why we care about it. Suppose $\mathbb{V} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a set of n vectors, none of which is the zero vector. Let $x = \sum_{i=1}^n a_i \vec{v}_i$.

   (a) Suppose $x = (a_1, a_2, \cdots, a_n)$ and also $x = (b_1, b_2, \cdots, b_n)$, where $a_i \neq b_i$ for at least one i. Show that this means $\vec{0} = \sum_{i=1}^n c_i \vec{v}_i$, where at least one c_i is non-zero.

   Let $j \in \{1, 2, \cdots, n\}$ such that $a_j \neq b_j$. It follows that $b_j - a_j \neq 0$. Since $x = \sum_{i=1}^n a_i \vec{v}_i = \sum_{i=1}^n b_i \vec{v}_i$

$$x = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} v_1^1 b_1 + v_1^2 b_2 + \cdots + v_1^j b_j + \cdots + v_1^n b_n \\ v_2^1 b_1 + v_2^2 b_2 + \cdots + v_2^j b_j + \cdots + v_2^n b_n \\ \vdots \\ v_n^1 b_1 + v_n^2 b_2 . \cdots + v_j^n b_j + \cdots + v_n^n b_n \end{bmatrix} - \begin{bmatrix} v_1^1 a_1 + v_1^2 a_2 + \cdots + v_1^j a_j + \cdots + v_1^n a_n \\ v_2^1 a_1 + v_2^2 a_2 + \cdots + v_2^j a_j + \cdots + v_2^n a_n \\ \vdots \\ v_n^1 a_1 + v_n^2 a_2 . \cdots + v_j^n a_j + \cdots + v_n^n a_n \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} (b_1 - a_1) v_1^1 + (b_2 - a_2) v_1^2 + \cdots + (b_j - a_j) v_1^j + \cdots + (b_n - a_n) v_1^n \\ (b_1 - a_1) v_2^1 + (b_2 - a_2) v_2^2 + \cdots + (b_j - a_j) v_2^j + \cdots + (b_n - a_n) v_2^n \\ \vdots \\ (b_1 - a_1) v_n^1 + (b_2 - a_2) v_n^2 + \cdots + (b_j - a_j) v_n^j + \cdots + (b_n - a_n) v_n^n \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_j & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_j - a_j \\ \vdots \\ b_n - a_n \end{bmatrix} = \sum_{i=1}^n (b_i - a_i) \vec{v}_i$$

   Let $\vec{c} = \vec{b} - \vec{a}$. Since $b_j - a_j \neq 0$ it follows that there is at least one c_i that is non-zero.

   (b) Suppose $x = (a_1, a_2, \cdots, a_n)$ and also $x = (b_1, b_2, \cdots, b_n)$ as above. Show that if $y = (p_1, p_2, \cdots, p_n)$, then $y = (q_1, q_2, \cdots, q_n)$ where $p_i \neq q_i$ for at least one i.

   Let $j \in \{1, 2, \cdots, n\}$ such that $a_j \neq b_j$. It follows that $a_j - b_j \neq 0$. Suppose $y = (p_1, p_2, \cdots, p_n)$.

$$\begin{aligned} x + y &= (a_1 \vec{v}_1 + \cdots + a_j \vec{v}_j + \cdots + a_n \vec{v}_n) + (p_1 \vec{v}_1 + \cdots + p_j \vec{v}_j + \cdots + p_n \vec{v}_n) \\ x - x + y &= (a_1 \vec{v}_1 + \cdots + a_j \vec{v}_j + \cdots + a_n \vec{v}_n) - (b_1 \vec{v}_1 + \cdots + b_j \vec{v}_j + \cdots + b_n \vec{v}_n) \\ &\quad + (p_1 \vec{v}_1 + \cdots + p_j \vec{v}_j + \cdots + p_n \vec{v}_n) \\ y &= (a_1 - b_1 + p_1) \vec{v}_1 + \cdots + (a_j - b_j + p_j) \vec{v}_j + \cdots + (a_n - b_n + p_n) \vec{v}_n \end{aligned}$$

   Let $q_i = a_i - b_i + p_1$ for all $i \in \{1 \cdots n\}$.

$$y = q_1 \vec{v}_1 + \cdots + q_j \vec{v}_j + \cdots + q_n \vec{v}_n$$

   Since $a_j - b_j \neq 0$ it follows that $q_j \neq p_j$ thus $y = (q_1, q_2, \cdots, q_n)$ where $p_i \neq q_i$ for at least one i.

(c) Suppose the zero vector can be expressed as a non-trivial linear combination of the vectors in $\mathbb{V}$. Show that this means that the vectors of $\mathbb{V}$ are not independent.

Suppose $\vec{0}$ is a non-trivial linear combination of vectors in $\mathbb{V}$ with at least one $c_i \neq 0$. Let $j \in \{1, 2, \cdots, n\}$ such that $c_j \neq 0$.

$$\begin{aligned}
\vec{0} &= c_1\vec{v}_1 + \cdots + c_j\vec{v}_j + \cdots + c_n\vec{v}_n \\
-c_j\vec{v}_j &= c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1} + c_{j+1}\vec{v}_{j+1} + \cdots + c_n\vec{v}_n \\
\vec{v}_j &= \frac{c_1}{-c_j}\vec{v}_1 + \cdots + \frac{c_{j-1}}{-c_j}\vec{v}_{j-1} + \frac{c_{j+1}}{-c_j}\vec{v}_{j+1} + \cdots + \frac{c_n}{-c_j}\vec{v}_n
\end{aligned}$$

Thus we have shown that $\vec{v}_j$ is a linear combination of the other vectors, therefore the vectors of $\mathbb{V}$ are not independent.

(d) Suppose the vectors of $\mathbb{V}$ are independent. Show this implies $\vec{0} = \sum_{i=1}^n a_i\vec{v}_i$ has a unique solution.

Let $\vec{a} = \langle a_1, a_2, \cdots, a_n\rangle$ and $\vec{b} = \langle b_1, b_2, \cdots, b_n\rangle$ such that $\sum_{i=1}^n a_i\vec{v}_i = \sum_{i=1}^n b_i\vec{v}_i$. Let $j \in \{1 \cdots n\}$.

$$\begin{aligned}
a_1\vec{v}_1 + \cdots + a_n\vec{v}_n &= b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n \\
\vec{0} &= b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n - a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \\
&= (b_1 - a_1)\vec{v}_1 + (b_2 - a_2)\vec{v}_2 + \cdots + (b_j - a_j)\vec{v}_j + \cdots + (b_n - a_n)\vec{v}_n
\end{aligned}$$

subtracting the scaled multiple of $\vec{v}_j$ from both sides results in

$$\begin{aligned}
(a_j - b_j)\vec{v}_j &= (b_1 - a_1)\vec{v}_1 + \cdots + (b_{j-1} - a_{j-1})\vec{v}_{j-1} + (b_{j+1} - a_{j+1})\vec{v}_{j+1} + \cdots + (b_n - a_n)\vec{v}_n \\
\vec{v}_j &= \frac{b_1 - a_1}{a_j - b_j}\vec{v}_1 + \cdots + \frac{b_{j-1} - a_{j-1}}{a_j - b_j}\vec{v}_{j-1} + \frac{b_{j+1} - a_{j+1}}{a_j - b_j}\vec{v}_{j+1} + \cdots + \frac{b_n - a_n}{a_j - b_j}\vec{v}_n
\end{aligned}$$

since the vectors of $\mathbb{V}$ are independent, it cannot be the case that $\vec{v}_j$ is a linear combination of the other vectors, therefore $a_k - b_k = 0$ and subsequently $a_k = b_k$ must be true for all $k \in \{1 \cdots n\}$. Returning to the step

$$\begin{aligned}
\vec{0} &= (b_1 - a_1)\vec{v}_1 + (b_2 - a_2)\vec{v}_2 + \cdots + (b_j - a_j)\vec{v}_j + \cdots + (b_n - a_n)\vec{v}_n \\
&= \sum_{i=1}^n (b_i - a_i)\vec{v}_i = \sum_{i=1}^n 0\vec{v}_i = 0\sum_{i=1}^n \vec{v}_i
\end{aligned}$$

And we have thus shown there is a singular solution $x = \langle 0, \cdots, 0\rangle$ for $\vec{0} = \sum_{i=1}^n x_i\vec{v}_i$.

(e) Show that if $\vec{0} = \sum_{i=1}^n a_i\vec{v}_i$ has a unique solution, then the vectors of $\mathbb{V}$ are independent.

Suppose that there exists $j \in \{1, \cdots, n\}$ such that $b_j\vec{v}_j = b_1\vec{v}_1 + \cdots + b_{j-1}\vec{v}_{j-1} + b_{j+1}\vec{v}_{j+1} + \cdots + b_n\vec{v}_n$ for some $b_1, \cdots, b_n \in \mathbb{R}$, i.e. the vectors are not independent.

$$\begin{aligned}
b_j\vec{v}_j &= b_1\vec{v}_1 + \cdots + b_{j-1}\vec{v}_{j-1} + b_{j+1}\vec{v}_{j+1} + \cdots + b_n\vec{v}_n \\
\vec{0} &= b_1\vec{v}_1 + \cdots + (-b_j)\vec{v}_j + \cdots + b_n\vec{v}_n \\
(-1)\vec{0} = \vec{0} &= (-b_1)\vec{v}_1 + \cdots + b_j\vec{v}_j + \cdots + (-b_n)\vec{v}_n
\end{aligned}$$

Thus we have found two solutions for $\vec{0} = \sum_{i=1}^n a_i\vec{v}_i$:

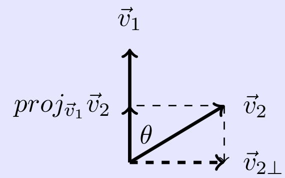
$$\begin{aligned}
a &= \langle b_1, \cdots, -b_j, \cdots, b_n\rangle \\
a' &= \langle -b_1, \cdots, b_j, \cdots, -b_n\rangle
\end{aligned}$$

Thus if the vectors are not independent then there is not a unique solution to $\vec{0} = \sum_{i=1}^n a_i\vec{v}_i$, and by contrapositive if there is a unique solution to $\vec{0} = \sum_{i=1}^n a_i\vec{v}_i$ then the vectors are independent.

4. A set of vectors $\mathbb{V}$ is said to be orthogonal if any two vectors are perpendicular.

(a) Let $\mathbb{V} = \{\vec{v}_1, \vec{v}_2\}$, and assume these form a basis. Find a vector $\vec{v}_{2\perp}$ that is perpendicular to $\vec{v}_1$.

$$\begin{aligned}
\vec{v}_{2\perp} &= \vec{v}_2 - proj_{\vec{v}_1}\vec{v}_2 \\
&= \vec{v}_2 - (|\vec{v}_2|\cos\theta)\frac{\vec{v}_1}{|\vec{v}_1|} \\
&= \vec{v}_2 - \left(\frac{|\vec{v}_2|}{|\vec{v}_1|}\right)\left(\frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|}\right)\vec{v}_1 \\
&= \vec{v}_2 - \left(\frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_1|}\right)\vec{v}_1 \\
&= \vec{v}_2 - \frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1
\end{aligned}$$



(b) Show that $\mathbb{V}_\perp = \{\vec{v}_1, \vec{v}_{2\perp}\}$ consists of a set of independent vectors.

Suppose $\mathbb{V}_\perp$ is not independent, thus there exists $x_1, x_2 \in \mathbb{R}$ such that $\vec{0} = x_1\vec{v}_1 + x_2\vec{v}_{2\perp}$ and at least one x_i is non-zero.

$$\begin{aligned}
\vec{0} &= x_1\vec{v}_1 + x_2\vec{v}_{2\perp} \\
\vec{0} &= x_1\vec{v}_1 + x_2\left(\vec{v}_2 - \frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1\right) = \left(x_1 - x_2\frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + x_2\vec{v}_2
\end{aligned}$$

since we know that $\vec{v}_1, \vec{v}_2$ form a basis, we know that only the trivial solution to $\vec{0} = x_1'\vec{v}_1 + x_1'\vec{v}_2$ exists, thus $x_1' = 0$ and $x_2' = 0$ and

$$\begin{aligned}
x_2\prime &= x_2 = 0 \\
x_1\prime &= x_1 - x_2\frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1} = 0 \\
x_1 &= x_2\frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1} + x_1' = 0\frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1} + 0 = 0
\end{aligned}$$

And we have reached a contradiction since $x_1 = 0$ and $x_2 = 0$ contradicts our initial claim that at least one is non-zero. Therefore $\mathbb{V}_\perp$ is a set of independent vectors.

(c) Show that the span of $\mathbb{V}$ is the same as the span of $\mathbb{V}_\perp$. We say $\mathbb{V}_\perp$ forms an orthogonal basis for $\mathbb{V}$.

Let $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2$ be a non-zero vector. We want to find x_1', x_2' such that $\vec{x} = x_1'\vec{v}_1 + x_2'\vec{v}_{2\perp}$.

$$\begin{aligned}
\vec{x} &= x_1\vec{v}_1 + x_2\vec{v}_2 \\
\vec{x} - x_2 proj_{\vec{v}_1}\vec{v}_2 &= x_1\vec{v}_1 + x_2\vec{v}_2 - x_2 proj_{\vec{v}_1}\vec{v}_2 \\
&= x_1\vec{v}_1 + x_2\left(\vec{v}_2 - proj_{\vec{v}_1}\vec{v}_2\right) \\
&= x_1\vec{v}_1 + x_2\vec{v}_{2\perp} \\
\vec{x} &= x_1\vec{v}_1 + x_2\vec{v}_{2\perp} + x_2 proj_{\vec{v}_1}\vec{v}_2 \\
&= x_1\vec{v}_1 + x_2\vec{v}_{2\perp} + x_2\left(\frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1\right) \\
&= \left(x_1 + x_2\frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + x_2\vec{v}_{2\perp}
\end{aligned}$$

Thus we have found $x_1' = x_1 + x_2\frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1}$ and $x_2' = x_2$ and therefore $\mathbb{V}_\perp$ is an orthogonal basis for $\mathbb{V}$.

(d) Find an orthogonal basis for $\mathbb{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, assuming the vectors are independent.

We will try to form $\mathbb{U}_\perp = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Let

$$\begin{aligned}
\vec{v}_1 &= \vec{u}_1 \\
\vec{v}_2 &= \vec{u}_2 - proj_{\vec{v}_1}\vec{u}_2 \\
&= \vec{u}_2 - \frac{\vec{v}_1 \cdot \vec{u}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\
\vec{v}_3 &= \vec{u}_3 - proj_{\vec{v}_1}\vec{u}_3 - proj_{\vec{v}_2}\vec{u}_3 \\
&= \vec{u}_3 - \frac{\vec{v}_1 \cdot \vec{u}_3}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{u}_3}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2
\end{aligned}$$

(e) Suppose vectors of $\mathbb{W} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ are not independent. What happens when you try to form $\mathbb{W}_\perp$?

Let $\vec{w}_3 = a_1\vec{w}_1 + a_2\vec{w}_2$. We will try to form $\mathbb{W}_\perp = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$:

$$\begin{aligned}
\vec{v}_1 &= \vec{w}_1 \\
\vec{v}_2 &= \vec{w}_2 - \frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 \\
\vec{v}_3 &= \vec{w}_3 - \frac{\vec{w}_1 \cdot \vec{w}_3}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - \frac{\vec{w}_2 \cdot \vec{w}_3}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2 \\
&= (a_1\vec{w}_1 + a_2\vec{w}_2) - \left(\frac{a_1(\vec{w}_1 \cdot \vec{w}_1) + a_2(\vec{w}_1 \cdot \vec{w}_2)}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1\right) - \left(\frac{a_1(\vec{w}_1 \cdot \vec{w}_2) + a_2(\vec{w}_2 \cdot \vec{w}_2)}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2\right) \\
&= (a_1\vec{w}_1 + a_2\vec{w}_2) - \left(a_1 + a_2\frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_1 \cdot \vec{w}_1}\right)\vec{w}_1 - \left(a_2 + a_1\frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\right)\vec{w}_2 \\
&= -a_2\frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - a_1\frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2 \\
&= -a_2\frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_1 \cdot \vec{w}_1}\vec{v}_1 - \left(a_1\frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\right)\left(\vec{v}_2 + \frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_1 \cdot \vec{w}_1}\vec{v}_1\right) \\
&= \frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_1 \cdot \vec{w}_1}\left(a_1\frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2} - a_2\right)\vec{v}_1 + a_1\frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\vec{v}_2
\end{aligned}$$

Thus we see that $\vec{v}_3$ is a linear combination of $\vec{v}_1, \vec{v}_2$ and thus $\mathbb{W}_\perp$ is not an orthogonal basis.

5. Answer the following questions.

(a) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. If possible, find $(AB)^2$ and $(B^TB)^{-1}$.

Since A is 2×2 and B is 2×3 it follows that AB is 2×3, thus $(AB)^2$ is undefined.

$$B^TB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

to find the inverse we row reduce:

$$\left[\begin{array}{ccc|ccc} 5 & -1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 5 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 2 & 1 & 5 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 5 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 2 & 1 & 5 & 0 \\ 0 & 2 & 1 & -2 & 0 & 5 \end{array}\right] \rightarrow$$
$$\left[\begin{array}{ccc|ccc} 5 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 2 & 1 & 5 & 0 \\ 0 & 0 & 0 & -5 & -5 & 10 \end{array}\right]$$

and we have reached a situation in which B^TB is not invertible, thus $(B^TB)^{-1}$ is undefined.

(b) Find the angle between $\vec{v} = \langle 1, 3, -1, -1 \rangle$ and $\vec{u} = \langle 1, 1, -1, 4 \rangle$.

$$\begin{aligned} |\vec{v}| &= \sqrt{1^2 + 3^2 + (-1)^2 + (-1)^2} = \sqrt{12} = 2\sqrt{3} \\ |\vec{u}| &= \sqrt{1^2 + 1^2 + (-1)^2 + 4^2} = \sqrt{19} \\ \vec{v} \cdot \vec{u} &= (1 \cdot 1) + (3 \cdot 1) + (-1 \cdot -1) + (-1 \cdot 4) = 1 + 3 + 1 - 4 = 1 \\ \cos\theta &= \frac{\vec{v} \cdot \vec{u}}{|\vec{v}||\vec{u}|} = \frac{1}{2\sqrt{57}} = \frac{\sqrt{57}}{114} \\ \theta &= \cos^{-1} \frac{\sqrt{57}}{114} \approx 1.504 \end{aligned}$$

(c) Prove or disprove: if $\mathbb{V}$ is a vector space, then Null($\mathbb{V}$) is a vector space.

Let V be the matrix whose columns are the vectors spanning $\mathbb{V}$, and let $\vec{x}_1$ and $\vec{x}_2$ be vectors in Null($\mathbb{V}$), and $c \in \mathbb{R}$. We can see from the definition of Nullspace that $V\vec{x}_1 = \vec{0}$ and $V\vec{x}_2 = \vec{0}$. Since $\mathbb{V}$ is a vector space we can say

$$V(\vec{x}_1 + \vec{x}_2) = V\vec{x}_1 + V\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$$

thus Null($\mathbb{V}$) is closed under vector addition

$$V(c\vec{x}_1) = c(V\vec{x}_1) = c\vec{0} = \vec{0}$$

thus Null($\mathbb{V}$) is closed under scalar multiplication

$$V(\vec{x}_1 + \vec{0}) = V\vec{x}_1 + V\vec{0} = \vec{0} + \vec{0} = \vec{0}$$

thus Null($\mathbb{V}$) contains a zero vector

$$V(\vec{x}_1 + (-1)\vec{x}_1) = V((1-1)\vec{x}_1) = V(0\vec{x}_1) = 0V(\vec{x}_1) = \vec{0}$$

thus Null($\mathbb{V}$) contains an additive inverse for each element

Since Null($\mathbb{V}$) $\subseteq$ Col($\mathbb{V}$) which is a vectorspace, the other six properties are inherited and we can conclude from these four properties that Null($\mathbb{V}$) is a vector space if $\mathbb{V}$ is a vector space.