Introduction

This handout includes all assignments for Math 2101: Linear Algebra (Summer I 2016). It's conceivable you can finish all these assignments ahead of time; go ahead.

There are two things that may make this course different from others you've taken: Proofs, and problem solving.

Proofs

Linear algebra is a transition course to higher mathematics (in some sense, it may be the first "real" mathematics course you've taken), and one of the components of such a course is that you'll be asked to do some real proofs. We'll introduce some proof strategies along the way, but it's important to understand why proof is so important.

By far the *least* important reason to do proof is that it provides a guarantee that a result is true. While this is important, it's more important in the abstract: We want to know that there *is* a proof, but might not care about the details.

More importantly, proof requires you to ask repeatedly "What do I mean by that?" This is important in mathematics, because mathematical concepts are very carefully constructed to mean very specific things. From an academic standpoint, the value of proof is that you have to go back through things that you've learned—or go over things you've forgotten. In other words: Proof is a study technique.

The most important aspect of proof is that it highlights the relationships between concepts—and raises new questions about those relationships. As a result, a good proof raises more questions than it answers, and suggests new problems to solve and new directions of investigation to undertake. Proof *creates* mathematics.

We'll leave you with an initial strategy for proofs: *Definitions are the whole of mathematics; all else is commentary.* You'll run into a lot of theorems, propositions, lemmas, and other results, but they can all be ignored, because every single one of them can be derived from the definitions.[1] However, the definitions themselves cannot be derived from anything: you either

[1] This doesn't mean you shouldn't know the theorems: just because you *can* derive them from the definitions doesn't mean you'll want to do it every time.

know what a nonsingular $n \times n$ matrix is, or you don't, and if you don't, there's no way of figuring it out.

Problem Solving

A recent trend in mathematics education is an emphasis of **problem solving**. This may cause some confusion: Isn't that what you've been doing every time you do homework? The quick answer is: Probably not. To illustrate this, consider the following question: Find $23 + 158$. Since you (hopefully!) already know how to add multidigit numbers, this is not a problem; rather, it's a task to be completed by following an algorithm invented by someone else.

On the other hand, suppose you had never learned the algorithm for adding multidigit numbers. Then finding $23 + 158$ *is* a problem. However, the problem isn't to calculate a value; the problem is *creating* the algorithm to add multidigit numbers. In general, when we speak of problem solving, we mean *creating* an algorithm or formula to answer a question, and **not** following step-by-step directions produced by somebody else.

This last distinction is important, because machines are very good at following step-by-step directions. Every question in linear algebra that can be answered following step-by-step directions can be answered using Wolfram Alpha, which is available free online ($2.99 for a smartphone installation).[2] So if all you learn in a linear algebra course is how to follow step-by-step directions, then what you've learned is worth less than three dollars (significantly less, because Wolfram Alpha also does calculus, trigonometry, and algebra).

On the other hand, machines are terrible at creating algorithms (and there's some reason to believe they will *never* be able to do so, though that's a topic for philosophers). By learning problem solving, you're learning something that no machine can do. Equivalently: It's something that is more valuable than *every computer in existence put together.*

Learning problem solving requires a class that is, in all likelihood, very different from any you've taken before. In particular:

- **DON'T** expect to be given a step-by-step sequence of directions for solving a problem. Following instructions is good for machines, not for

[2] Disclaimer: I'm not a paid sponsor of Wolfram Alpha. But I *highly recommend* that you learn to use it.

people.

- **DO** expect to be given key concepts and problems to solve based on those concepts.
- **DON'T** expect "one-line" problems. It often takes several pages to *state* a real world problem, and solutions must satisfy dozens of requirements. Most of our problems will take at least a paragraph to lay out, and two or three requirements, all of which must be met in order for the problem to be considered solved.

This may seem daunting—and it is! Fortunately, you won't have to do it alone. In general, expect class time to alternate between short lectures, introducing basic concepts, and time to work on problems, based on that concept. During the work time, you'll have the opportunity to seek guidance and to have your work evaluated.

One more note: This *is* the 21st century, and if you can't Google the answer to a question, the answer probably doesn't exist. There will be a great temptation to look up solutions. *Resist this temptation!*

Problem solving is like everthing else: the more you practice, the better you get. But the only way to practice problem solving is to solve problems, and opportunities for real problem solving are *extremely* rare.

How rare are they? Remember problem solving is creating algorithms or formulas, so every algorithm you've learned corresponds to one problem you could have solved. Most courses can be reduced to between ten and twenty algorithms and/or formulas.

When you look up how to solve a problem, you lose the opportunity to solve it. Moreover, *this opportunity never comes again*: You will never get a second chance to solve a problem for the first time.

Suggestions for Presenting Your Work

Remember that in mathematics, it's the journey, not the destination. *How* you answer a question is often as important, if not more important, than whehter you got "the right answer." Thus, some questions might ask you to use a particular method (or not use a particular method). Here are some other guidelines to keep in mind as you write up your work.

First: In all your previous math courses, you have been asked to "show your work." This requirement still holds! However, we'll qualify it: Show the work based on things you've learned *in this class*. For example, if a question requires you solve the equation $3x+5=7$, which is something from elementary algebra, you can simply write down the solution $x=\frac{2}{3}$, and I don't need to see all the steps.

Second: This is an advanced math class, which means we should spend our time, energy, and effort on advanced math, not elementary math. In particular, I don't care about simplifying answers. You *should* perform the obvious reductions, like $(-1)^2$ or $\frac{3}{1}$, but anything more complicated, like $\frac{5}{25}$, can be left in that form. **Caution**: Simplification is often useful as an *intermediate* step, and if you fail to do so in the middle of a problem, you might have a hard time completing the problem.

Third: One of the things you'll be asked to do is to **verify**. Verification *always* means to solve the problem in a *different* way, and confirm that the two different methods give you the same answer. This is particularly important when someone suggests a new method of solving a problem: If the method doesn't give you "the right answer" when you know what the right answer is supposed to be, why should you trust it? For example, suppose you've just learned how to multiply multidigit numbers, and determined that $3\times 15=45$. To verify this, you might say "Since 3×15 is the sum of three 15s, and $15+15+15=45$, this verifies our product $3\times 15=45$."

Fourth: Another thing you'll be asked to do is to **explain**. An explanation describes your thought process: *why* you undertook the steps you did. For example, our verification in the preceding paragraph included an explanation, namely "Since 3×15 is the sum of three 15s." The explanation tells the reader *why* the work is relevant.

Math 2101: Assignment 1

1. In the following, let $\vec{v}_1 = (3, -2, -1)$, $\vec{v}_2 = (1, 1, -1)$, and $\vec{v}_3 = (-3, 2, 6)$. Assume $\cdot$ always refers to the dot product. Find the following, if possible; if not possible, explain why not.

   (a) $3\vec{v}_1 - 2\vec{v}_2 + 4\vec{v}_3$
   (b) $\vec{v}_1 \cdot \vec{v}_2$
   (c) $2\vec{v}_1 \cdot \vec{v}_2 + 2\vec{v}_3$
   (d) $\vec{v}_1 \cdot \vec{v}_2 \cdot \vec{v}_3$
   (e) $\vec{v}_1 \cdot (3\vec{v}_2 - 2\vec{v}_3)$
   (f) $|\vec{v}_1|$
   (g) $|5\vec{v}_1|$

2. In the following, assume $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, and θ is the angle between $\vec{u}$ and $\vec{v}$. You might want to review your trigonometry.

   (a) Prove: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$. Suggestion: Use the Law of Cosines.
   (b) Let $\vec{u}$ and $\vec{v}$ be two sides of a triangle. Find the area of the triangle.

3. Let $P = (1, 4, -3)$, $Q = (5, 1, -3)$, and $R = (-1, 1, 2)$. Note these are points in $\mathbb{R}^3$, not vectors!

   (a) Explain how you would write the parametric equation of the line through X, where $\vec{v}$ gives the direction of a line. (You may want to draw a picture to organize your thoughts) Suggestion: How would you go from X to any other point on the line?
   (b) Find the parametric equation of the lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$, and $\overleftrightarrow{QR}$.
   (c) Explain why any linear combination of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ must be in the same plane as P, Q, R. (You may use a picture)
   (d) Use the preceding idea to write the parametric equation of the plane PQR.
   (e) Find the area of the triangle ΔPQR.

(f) Find the distance between the point R and the line $\overleftrightarrow{PQ}$.

4. Suppose M is the coefficient matrix for a system of equations augmented by the constants of the equations. We'll say that an operation is *allowable* if we can apply it to M and produce the augmented coefficient matrix for a system of equations with the *same* solution as the original system of equations: these correspond to the steps you use to solve a system of equations. For **each** of the following, identify the corresponding operation on a system of equations; then state whether the operation is allowable or forbidden. Assume $c \neq 0$.

   (a) Multiplying every term of a row by c.

   (b) Switching two columns.

   (c) Switching two rows.

   (d) Adding c to each term in a row.

   (e) Multiplying every term in a row by c, then adding the corresponding terms to another row.

   (f) Suppose that, after you've applied a sequence of allowable row operations to M, you end up with a row consisting of all zeroes except the last entry, which is a non-zero number. What does this say about the original system of equations?

5. Find a parametrization for solutions to the following systems of equations.

   (a) $3x + 2y = 5$.

   (b)

$$\begin{aligned} 3w - 2x - 4y + 2z &= 0 \\ 3x + 2y - 5z &= 0 \\ 2y - z &= 0 \end{aligned}$$

   (c)

$$\begin{aligned} 2w + 3x - 5y - 2z &= 0 \\ x + 2y - z &= 0 \end{aligned}$$

Math 2101: Assignment 2

1. Produce the row echelon form of the augmented matrix for the given system of equations. **Specify** the elementary row operations used to produce the matrix. Then give a (possibly parametrized) expression for all solutions in vector form, and identify whether the solution(s) correspond to a point, line, or plane. If a solution does not exist, say so. Finally, give the rank of each coefficient matrix.

   (a) The system of equations

   $$\begin{aligned} 8x + 5y &= 25 \\ 3x - 7y &= 10 \end{aligned}$$

   (b) The system of equations

   $$\begin{aligned} 2x + 3y - 4z &= 11 \\ x + 5y - 2z &= 10 \\ 4x - 3y - z &= 25 \end{aligned}$$

   (c) The system of equations

   $$\begin{aligned} 7x + 5y &= 11 \\ 2x - 4y &= 3 \\ 3x - 2y &= 15 \end{aligned}$$

   (d) The system of equations

   $$\begin{aligned} x + 3y - 2z &= 25 \\ 2x - 5y + 4z &= 10 \end{aligned}$$

   (e) The system of equations

   $$\begin{aligned} 2x - 4y + z &= 10 \\ 4x - 8y + 2z &= 25 \end{aligned}$$

   (f) The system of equations

   $$\begin{aligned} 2x - 4y + z &= 5 \\ 4x - 8y + 2z &= 30 \end{aligned}$$

2. Let $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1, v_2, v_3)$. We want to find a vector perpendicular to both $\overrightarrow{u}$ and $\overrightarrow{v}$.

   (a) Let the components of this vector be (x_1, x_2, x_3). Write down a system of equations corresponding to the requirement that this vector is perpendicular to $\overrightarrow{u}$ and to $\overrightarrow{v}$.

   (b) What is the rank of the corresponding coefficient matrix; what does this say about the number of vectors that are perpendicular to both $\overrightarrow{u}$ and $\overrightarrow{v}$; and why is this surprising (alternatively, why *isn't* this surprising)?

   (c) Find a vector perpendicular to $\overrightarrow{u}$ and to $\overrightarrow{v}$, whose components are *not* fractional expressions of the components of $\overrightarrow{u}$ and $\overrightarrow{v}$. (It's possible the components are fractions or even real numbers; what you want to avoid is having a component like $\frac{u_1}{v_3}$)

   (d) Find a vector perpendicular to $(1, 1, -1)$ and $(2, 1, -3)$. (This is called a **normal vector**)

   (e) Let $X = (x, y, z)$ be a point in the plane containing the points $P = (1, 2, 4)$, $Q = (-3, -1, 2)$, and $R = (1, 1, 0)$. Write the equation of the plane. Suggestion: The vector perpendicular to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ will also be perpendicular to $\overrightarrow{PX}$.

   (f) Find the distance between the point $(2, 1, 0)$ and the plane PQR.

3. An important use of matrices is **stochastic modeling**. An example is the following: Imagine a park with three locations: a lake; a picnic area; and a playground. Every hour, on the hour, the parkgoers move according to the following rules:

   - Half of those at the lake move to the picnic area, and one-quarter of those at the lake move to the playground.
   - Half of those at the picnic area go to the lake, and the other half go to the playground.
   - Half of those at the playground go to the picnic area.

   Once they arrive, the parkgoers stay until the next hour, at which point they move again, according to the same rules.

(a) Suppose there are 100 persons at each location. How many persons are there at each of the locations one hour later? Suggestion: In life, it's often easier to determine where you've come *from* than it is to determine where you're going *to*.

(b) Let p_n, l_n, g_n be variables representing the number of persons at the picnic area, lake, and playground at hour n. Write down equations for $p_{n+1}, l_{n+1}, g_{n+1}$, giving the number of persons at the picnic area, lake, and playground at hour $n+1$.

(c) Write down the coefficient matrix T for the system of equations you just wrote. (The first, second, and third columns should correspond to the variables p_n, l_n, and g_n, respectively; and the first, second, and third rows should correspond to the equations giving p_{n+1}, l_{n+1}, and g_{n+1}, respectively).

(d) The matrix you've produced is called a **transition matrix** or (since it is for a stochastic model) a **stochastic matrix**. One of the following matrices could be a stochastic matrix; the other cannot be. Identify the matrix that cannot be a stochastic matrix, and explain why it can't be one. *Also* identify the model that matrix that can be a stochastic matrix, and give the corresponding movement rules.

$$A = \begin{pmatrix} 1/2 & 1/4 & 1/2 \\ 1/2 & 1/4 & 1/2 \\ 0 & 1/2 & 1/4 \end{pmatrix}, B = \begin{pmatrix} 1/4 & 0 & 1/3 \\ 1/2 & 1/2 & 1/6 \\ 1/4 & 1/2 & 1/2 \end{pmatrix}$$

(e) Generalize your observation: A matrix M can be the stochastic matrix for a model provided ______________________. (Give the properties of a stochastic matrix, including its possible dimensions and entries. **Defend** your conclusion by connecting the requirement to a feature of the stochastic model)

4. Another use of matrices is for **discrete time modeling**. Consider the following problem, made famous in the 13th century by Leonardo of Pisa: Suppose you have a pair of rabbits that breed according to the following rules:

- Rabbits mature after two months, and will produce a pair of rabbits every month after.

- The pairs of rabbits are always male/female, and mature rabbits will always find a mate.
- No rabbits die.

Let x_n be the number of pairs of immature rabbits at the end of month n, and y_n be the number of pairs of mature rabbits at the end of month n.

(a) What is x_0 and y_0? (Note that the end of month 0 is the start of month 1)

(b) Find x_1 and y_1.

(c) Suppose you know x_n and y_n. How would you find x_{n+1} and y_{n+1}?

(d) Treating x_n and y_n as our two variables, write down the coefficient matrix for the system of equations that gives the values of x_{n+1} and y_{n+1}. As in a stochastic model, let the the first and second columns correspond to the variables x_n and y_n, and let the first and second rows correspond to the equations giving x_{n+1} and y_{n+1}.

(e) Note that this transition matrix **cannot** be a stochastic matrix. What essential difference exists between the rabbit problem and the park problem? ("Essential" = can't be resolved by changing the names. The fact that one problem is about parkgoers and the other is about rabbits is non-essential, because we can replace the word "parkgoer" with the word "rabbit" and eliminate the discrepancy)

5. Answer the following questions. Let $\overrightarrow{v} = (2, 1, 3, 4)$, $\overrightarrow{u} = (1, 1, -1, 2)$.

(a) Find the angle θ between $\overrightarrow{v}$ and $\overrightarrow{u}$.

(b) Find a unit vector (a vector of length 1) that goes in the same direction as $\overrightarrow{v}$.

(c) Find $3\overrightarrow{v} - 2\overrightarrow{u}$.

(d) Find a, b so that $a\overrightarrow{v} + b\overrightarrow{u}$ is perpendicular to $3\overrightarrow{v} - 2\overrightarrow{u}$.

Math 2101: Assignment 3

1. Given any point $P = (x, y)$, we can define a **linear transformation** $(x, y) \to (x', y')$, where

$$\begin{aligned} ax + by &= x' \\ cx + dy &= y' \end{aligned}$$

for some real numbers a, b, c, d. We write $T : P \to P'$ to indicate P' is the point (x', y') produced by applying the transformation T to the point P; we also write $TP = P'$. In the following, you don't have to draw a picture ... but it will probably help.

   (a) Write down the linear transformation corresponding to the geometric transformation of reflecting a point across the x-axis. (In other words: Find a, b, c, d so that (x', y') is the reflection of (x, y) across the x-axis) Then write corresponding coefficient matrix (call this matrix M_x).

   (b) Write down the linear transformation corresponding to the geometric transformation of rotating the point (x, y) $90°$ counterclockwise around the origin. Then write the corresponding coefficient matrix (call this matrix $R_{90°}$).

   (c) Write down the linear transformation corresponding to the geometric transformation of reflecting the point (x, y) across the x-axis, followed by rotating the point $90°$ counterclockwise around the origin. Then write the corresponding coefficient matrix (call this T).

   (d) Suppose S, T are two linear transformation, and P a point. Define $M = S + T$ and remember **we haven't yet defined matrix addition** (so even if you know how to add two matrices, you may not use this knowledge). Show that if we want the distributive law $(S+T)P = SP+TP$ to hold, we must define $m_{ij} = s_{ij}+t_{ij}$. **Note**: There are only so many symbols: the + on the right should be viewed as vector addition, while the + on the left is the addition of two matrices. You *do* know how to add two vectors, which means you know the value of the right hand side.

2. If A, B are the matrices corresponding to a geometric transformation, we interpret the **matrix product** AB as the geometric transformation produced by applying B, *then* A (note that this is the "reverse" of their written order). From the previous problem, we defined R_{90° to correspond to the geometric transformation of rotating a point 90° counterclockwise rotation about the origin, and M_x to correspond to the geometric transformation of reflecting a point across the x-axis.

   (a) Is matrix multiplication commutative? (In other words, given two matrices A, B, will $AB = BA$?) Explain your conclusion (see the introduction for how you should explain a conclusion). Your answer **MUST** be based on the geometric interpretation of the matrix product.

   (b) Find $R_{90^\circ}M_x$. (**Don't** try to multiply the two matrices. Instead, find the product by identifying the geometric transformation it corresponds to.)

   (c) It's tempting to define the product of two matrices componentwise:
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} am & bn \\ cp & dq \end{pmatrix}$$
   Show that this definition does *not* calculate $R_{90^\circ}M_x$ correctly. (You already determined what $R_{90^\circ}M_x$ should be; show that the componentwise multiplication does *not* produce the desired product)

   (d) We actually find the product of 2×2 matrices as
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} am + bp & an + bq \\ cm + dp & cn + dq \end{pmatrix}$$
   Show that using this definition gives us $T = R_{90^\circ}M_x$. **Note**: A general rule of life is when someone offers you a new way of solving a problem, test it on a problem whose solution you already know.

   (e) Use matrix multiplication as defined above to find $T' = M_x R_{90^\circ}$. Then verify that this is correct, by identifying the geometric transformations corresponding to the product $M_x R_{90^\circ}$ and finding the corresponding linear transformation. **NOTE**: See the introduction for how to verify a conclusion.

3. Let $M = \begin{pmatrix} 1/2 & 1/4 & 1/2 \\ 0 & 1/4 & 1/2 \\ m_{31} & 1/2 & m_{33} \end{pmatrix}$ be a stochastic matrix.

   (a) Interpret this as a set of movement rules between three locations by identifying what fraction of those at each location at $t = k$ will move to the other locations at $t = k + 1$.

   (b) Find m_{31} and m_{33}.

   (c) Find M^2, where m_{ij} corresponds to the fraction of those in location i at $t = k$ who will be in location j at $t = k + 2$. **DO NOT** use matrix multiplication. You may want to use a table like the following to organize your thoughts:

Location	1	2	3
Number at $t = k$	x_k	y_k	z_k
Number at $t = k + 1$			
Number at $t = k + 2$			

   (d) Explain how you found the number of people in location 3 at $t = k + 2$. **Note**: See the introduction for how to explain an answer.

   (e) The preceding problem suggests that if A, B are 2×2 matrices, the product AB is the matrix whose entries correspond to the dot products of the rows of A with the columns of B. Verify that this works (or show that it fails to work) on 3×3 matrices? In other words: You know what M^2 is supposed to be; does the rule give you the correct value of M^2?

4. The **identity transformation** is simply $(x, y) \to (x, y)$. We'll continue to use M_x and R_{90°, from above.

   (a) Write down the coefficient matrix corresponding to the identity transformation. (You've just written down the **identity matrix**, designated I)

(b) Given a matrix B, the inverse matrix B^{-1} satisfies $B^{-1}B = I$. First, describe the *geometric* transformation corresponding to the matrix M_x^{-1}. Then find M_x^{-1}.

(c) Explain, **without** computing or otherwise referring to the values of the matrices involved, why $R_{90°}^3 = R_{90°}^{-1}$.

(d) Find $R_{90°}^{-1}$. **Note**: You could use the preceding. Or you could find it from the geometric transformation.

(e) Suppose A, B are matrices corresponding to some geometric transformations. What is $(AB)^{-1}$? Explain. (Remember AB corresponds to the transformation produced by applying B, then applying A)

5. Answer the following questions.

(a) Reduce the following matrix to row echelon form:

$$\begin{pmatrix} 3 & 1 & 5 & 1 \\ 2 & -1 & 3 & -2 \\ 1 & 4 & 0 & 3 \end{pmatrix}$$

(b) Let $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1, v_2, v_3)$. Prove or disprove: $|\overrightarrow{u} + \overrightarrow{v}| \leq |\overrightarrow{u}| + |\overrightarrow{v}|$.

(c) Let $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1, v_2, v_3)$. Prove or disprove: $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u}$.

(d) Let $\overrightarrow{p} = (3, -1, 1)$. Find *all* vectors (parametrized, as necessary) that are perpendicular to $\overrightarrow{p}$.

Math 2101: Assignment 4

1. Let A, B, C be the following matrices:

$$A = \begin{pmatrix} 3 & 5 \\ 2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 \\ 2 & -1 \\ -3 & 1 \end{pmatrix}$$

If possible, find the following. If not possible, explain why not.

(a) $A + B$

(b) $-2A$

(c) $A^2 + A$

(d) AB

(e) B^T

(f) $A(B + C)$

(g) ABC

2. Given a matrix M, the *right* inverse of M is a matrix D where $MD = I$ (where I is of the appropriate size); the *left* inverse is a matrix S where $SM = I$.

(a) Suppose M is a $m \times n$ matrix. What are the dimensions of D and I?

(b) Suppose $D = S$ (in other words, the right and left inverses are the same). What must be true about M?

(c) Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let $D = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ be a right inverse of M. Write down and solve the system of equations needed to solve for x_1, x_2, x_3, x_4. Also, indicate what must be true in order for the system to be solvable.

(d) Show that the matrix D you found is also a left inverse. (Consequently, we'll just say it's the inverse of M, and write it M^{-1}).

(e) If possible, find A^{-1}, for the matrix A above.

(f) If possible, find the right inverse of B, for the matrix B above.

(g) If possible, find $(CB)^{-1}$.

3. Determine whether the statement is true. If the statement is true, prove it. If the statement is false, find the correct expression for the right side and prove your result. You may assume P, Q are invertible matrices of the appropriate size for addition and/or multiplication.

   (a) $(P^{-1})^{-1} = P$

   (b) $(PQ)^{-1} = P^{-1}Q^{-1}$

   (c) $aP + bP = (a+b)P$ (where $a, b \in \mathbb{R}$).

   (d) $(P^T)^T = P$

   (e) If $PP^{-1} = I$, then $P^{-1}P = I$.

4. Verify that each is a vector space, **OR** explain why it is not.

   (a) The set of polynomials with integer coefficients, under the standard addition of polynomials.

   (b) The set of 3×3 matrices, under matrix addition.

   (c) The set of invertible 4×4 matrices, under matrix addition.

   (d) The set of polynomials with integer coefficients, where, given two polynomials f, g, we define $+$ as $f + g = \frac{d}{dx}(fg)$ (the derivative of the product).

   (e) The set of polynomials with integer coefficients, where, given two polynomials f, g, we define $+$ as $f + g = f'g'$ (the product of the derivatives).

5. Let $\overrightarrow{v}_1 = (2, 1, 0)$, $\overrightarrow{v}_2 = (1, 0, -1)$, and $\overrightarrow{v_3} = (3, 1, -1)$.

   (a) Prove or disprove: The set of linear combinations of $\overrightarrow{v}_1$, $\overrightarrow{v}_2$, $\overrightarrow{v}_3$ forms a vector space, using the ordinary definitions for vector addition and scalar multiplication.

   (b) Find the vectors (x, y, z) that can be written as a linear combination of $\overrightarrow{v}_1$, $\overrightarrow{v}_2$, $\overrightarrow{v}_3$.

Math 2101: Assignment 5

1. For each of the following, find a basis for Row(A), Col(A), and Null(A); also identify the dimension of each. If the vectors are not independent, express the dependent vector(s) as linear combinations of the others.

   (a) $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 5 \\ 0 & 3 & 1 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 3 & 1 & 0 \\ 4 & -1 & -2 & 1 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & -4 & -5 \end{pmatrix}$

2. Let $\mathcal{V} = \{\overrightarrow{v}_1, \overrightarrow{v}_2\}$ be a basis for a 2-dimensional vector space, and let

$$\begin{aligned} \overrightarrow{w}_1 &= a_{11}\overrightarrow{v}_1 + a_{12}\overrightarrow{v}_2 \\ \overrightarrow{w}_2 &= a_{21}\overrightarrow{v}_1 + a_{22}\overrightarrow{v}_2 \end{aligned}$$

   where and $a_{ij} \in \mathbb{R}$.

   (a) Under what conditions will $\mathcal{W} = \{\overrightarrow{w}_1, \overrightarrow{w}_2\}$ be the basis for a 2-dimensional vector space? Suggestion: A useful proof strategy is that if you want to *avoid* something (in this case, $\mathcal{W}$ not being a basis), you determine what will make it happen, then forbid this occurrence.

   (b) Prove or disprove: If $\mathcal{W}$ is the basis for a 2-dimensional vector space, it will be the same as the vector space spanned by $\mathcal{V}$. Suggestion: First try to prove that every vector in the span of $\mathcal{W}$ is in the span of $\mathcal{V}$. Then prove that every vector in the span of $\mathcal{V}$ is in the span of $\mathcal{W}$.

   (c) Suppose $\overrightarrow{x} = a\overrightarrow{v}_1 + b\overrightarrow{v}_2$. Find a', b' so that $\overrightarrow{x} = a'\overrightarrow{w}_1 + b'\overrightarrow{w}_2$. (It helps to think of a', b' as the variables).

3. The following will prove a useful theorem about independence, and motivate why we care about it. Suppose $\mathcal{V} = \{\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n\}$ is a

set of n vectors, none of which are the zero vector $\overrightarrow{0}$. Let $x = \sum_{i=1}^{n} a_i \overrightarrow{v}_i$. We say the ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the **set of coordinates of x relative to $\mathcal{V}$** (with the individual terms called the ordinates), and we might write $x = (a_1, a_2, \ldots, a_n)$. We *don't* want x to have two different sets of coordinates, so a standard strategy is to suppose that it does, and see what that would cause; we can then require that this doesn't happen.

(a) Suppose $x = (a_1, a_2, \ldots, a_n)$ and also $x = (b_1, b_2, \ldots, b_n)$, where $a_i \neq b_i$ for at least one i. Show that this means $\overrightarrow{0} = \sum_{i=1}^{n} c_i \overrightarrow{v}_i$, where at least one of the c_is are non-zero. (We say that the zero vector can be expressed as a nontrivial linear combination of the vectors in $\mathcal{V}$)

(b) Suppose $x = (a_1, a_2, \ldots, a_n)$ and also $x = (b_1, b_2, \ldots, b_n)$ as above. Show that if $y = (p_1, p_2, \ldots, p_n)$, then $y = (q_1, q_2, \ldots, q_n)$ where $p_i \neq q_i$ for at least one i. (This means that if a *single* point has more than one set of coordinates, *every* point has more than one set of coordinates) Suggestion: Why is $\overrightarrow{0}$ the zero vector?

(c) Suppose the zero vector can be expressed as a nontrivial linear combination of the vectors in $\mathcal{V}$. Show that this means the vectors of $\mathcal{V}$ are *not* independent.

The preceding problems show that *if* x has more than one set of coordinates, *then* the vectors of $\mathcal{V}$ are not independent. Since we *don't* want x to have more than one set of coordinates, we need to ensure that the vectors of $\mathcal{V}$ are independent. We'll use the following strategy, suitable for catching elephants: Figure out a property *of* elephants, then see if that property *identifies* elephants. In this case, we'll find a property *of* sets of independent vectors, then see if that property *identifies* sets of independent vectors.

(d) Suppose the vectors of $\mathcal{V}$ are independent. Show that this means $\overrightarrow{0} = \sum_{i=1}^{n} a_i \overrightarrow{v}_i$ has a unique solution. **Do not** use proof by contrapositive.[3] Suggestion: A common way to prove something is

[3] If you don't know what this is, obviously you shouldn't use it. If you do know what

unique is to assume there is another one, and see where that takes you.

(e) Show that if $\overrightarrow{0} = \sum_{i=1}^{n} a_i \overrightarrow{v}_i$ has a unique solution, then the vectors of $\mathcal{V}$ are independent.

4. A set of vectors $\mathcal{V}$ is said to be **orthogonal** if any two vectors are perpendicular.

   (a) Let $\mathcal{V} = \{\overrightarrow{v}_1, \overrightarrow{v}_2\}$, and assume these form a basis. Find a vector $\overrightarrow{v}_{2\perp}$ that is perpendicular to $\overrightarrow{v}_1$. (In other words: Find a formula that will produce such a perpendicular vector).

   (b) Show that $\mathcal{V}_\perp = \{\overrightarrow{v}_1, \overrightarrow{v}_{2\perp}\}$ consists of a set of independent vectors. Suggestion: The previous problem gives you a way to check whether a set of vectors is independent, and by construction, $\overrightarrow{v}'_2$ is perpendicular to $\overrightarrow{v}_1$.

   (c) Show that the span of $\mathcal{V}$ is the same as the span of $\mathcal{V}_\perp$. (We say that $\mathcal{V}_\perp$ forms an orthogonal basis for $\mathcal{V}$)

   (d) Find an orthogonal basis for $\mathcal{U} = \{\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3\}$, assuming the vectors are independent. (In particular, find $\mathcal{U}_\perp = \{\overrightarrow{u}_1, \overrightarrow{u}_{2\perp}, \overrightarrow{u}_{3\perp}\}$, where all vectors are perpendicular. Note that the first vector of $\mathcal{U}_\perp$ is the same as the first vector of $\mathcal{U}$.

   (e) Suppose the vectors of $\mathcal{W} = \{\overrightarrow{w}_1, \overrightarrow{w}_2, \dots, \overrightarrow{w}_n\}$ are *not* independent. What will happen when you try to form $\mathcal{W}$?

5. Answer the following questions.

   (a) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$. If possible, find $(AB)^2$ and $(B^T B)^{-1}$.

   (b) Find the angle between $\overrightarrow{v} = (1, 3, -1, -1)$ and $\overrightarrow{u} = (1, 1, -1, 4)$.

   (c) Prove or disprove: If $\mathbf{V}$ is a vector space, then $\mathrm{Null}(V)$ is a vector space.

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this is, then the reason you shouldn't use it is because proof is about the journey, not the destination.

Math 2101: Assignment 6

1. Determine whether the mapping is linear. Show your work (give me a reason to believe your conclusion!).

    (a) The derivative mapping $D : f \to f'$.

    (b) The scalar mapping $S : x \to cx$. (Assume $c \in \mathbb{R}$)

    (c) The inverse mapping: $I : x \to \frac{1}{x}$, where $x \neq 0$.

    (d) The log mapping: $L : x \to \log(x)$, where $x > 0$.

    (e) Matrix multiplication: $M : x \to Mx$, where M is a specific matrix, and x has the right dimension to be multiplied by M.

2. A **permutation of** n objects is a rearrangement of the n objects. A **permutation matrix** is a matrix whose entries are either 0 or 1, with the added restriction that each column where each column *and* each row consists of a single 1, with all other entries in the row or column 0.

    (a) Let P be a $n \times m$ matrix. What restrictions, if any, must we impose on n, m if we want P to be a permutation matrix? Prove your answer. Suggestion: Pick different values for n, m and see if you can construct a $n \times m$ permutation matrix.

    (b) Let P be a permutation matrix and $\overrightarrow{x}$ a vector of the appropriate size. Explain why $P\overrightarrow{x}$ is a permutation (rearrangement) of the components of $\overrightarrow{x}$.

    (c) Explain why $P^k = I$ for some k. Suggestion: Remember P^k corresponds to applying P k times.

    Permutations show up in many applications. One of the more exotic involves a *sestina*, a poetic form produced as follows: First, choose six words A, B, C, D, E, F. Each of these will be the terminal (last) word of a line; the six lines produce the first stanza. The next stanza has six lines, which have terminal words F, A, E, B, D, C (so the first line of the second stanza ends with the word F, the second line of the second stanza ends with the word A, and so on).

(d) Find a permutation matrix for the sestina. In other words, find P so that

$$P\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} F \\ A \\ E \\ B \\ D \\ C \end{pmatrix}$$

(e) The sestina has a total of six stanzas; you've just determined the terminal words for the first two stanzas. The remaining stanzas are found by applying P repeatedly. Find the rhyme scheme for the remaining stanzas. Can you think of a good reason why the sestina has six stanzas, and not five or seven?

(f) Suppose you want to invent a new poetic form, similar to a sestina, based on five terminal words A, B, C, D, E. If possible, construct a permutation matrix that will give this form exactly two stanzas of five lines.

(g) Suppose you decide two stanzas isn't interesting enough. If possible, construct a permutation matrix that will give this form *six* stanzas of five lines.

3. Remember we were able to express rotations and reflections, which are *geometric* transformations, using a linear transformation T, the coefficient matrix corresponding to the geometric transformation $(x, y) \to (x', y')$.

(a) What problem do you encounter with translations $(x, y) \to (x + h, y + k)$?

To handle this problem, we introduce **homogeneous coordinates**. We let the vector (x_1, y_1) in $\mathbb{R}^2$ correspond to the vector $(x_1, y_1, 1)$, and conversely. (In effect, we're projecting the xy-plane onto the plane $z = 1$)

(b) Find M, the matrix corresponding to the translation $(x, y, 1) \to (x + h, y + k, 1)$. In other words, find M so that

$$M\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + h \\ y + k \\ 1 \end{pmatrix}$$

(c) Let $T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$ Consider the square with opposite vertices at $(0,0)$ and $(1,1)$. Find where each of the *four* vertices of the square ends up when the linear transformation M is applied to each of them.

(d) Find the area of the resulting quadrilateral.

(e) Suppose that when T is applied to a unit square, the transformed square has area 0. Does $T = 0$ (the 3×3 matrix of all zeroes)? Prove, or give a counter-example.

4. Given a matrix M, an **eigenvalue-eigenvector pair** is a number λ and vector $\overrightarrow{v}$ where $M\overrightarrow{v} = \lambda\overrightarrow{v}$. $\overrightarrow{v}$ must be a non-zero vector; however, λ could be zero. In the following, assume M is a 2×2 matrix, with distinct real eigenvalues λ_1, λ_2 and corresponding eigenvectors $\overrightarrow{v}_1$, $\overrightarrow{v}_2$. Assume $0 < |\lambda_1| < 1 < \lambda_2$, and that $\overrightarrow{v}_1$, $\overrightarrow{v}_2$ span $\mathbb{R}^2$.

(a) Since λ_1, $\overrightarrow{v}_1$ is an eigenvalue-eigenvector pair, we have $M\overrightarrow{v}_1 = \lambda_1\overrightarrow{v}_1$. Describe what this means *geometrically*.

(b) Prove or disprove: $c\overrightarrow{v}_1$ is an eigenvector for any real number c.

(c) Evaluate: $M^n\overrightarrow{v}_1$. Also evaluate: $M^n\overrightarrow{v}_2$.

(d) Let $\overrightarrow{x}$ be any vector in $\mathbb{R}^2$. Evaluate: $M^n\overrightarrow{x}$.

(e) One way to find the largest eigenvalue (λ_2 in our case) and its corresponding eigenvector is the following: Pick any non-zero vector $\overrightarrow{x}$, and find $M^{n+1}\overrightarrow{x}$ and $M^n\overrightarrow{x}$ for a sufficiently large value of n. Explain why (provided n is "large enough") $\lambda_2 \approx \frac{|M^{n+1}\overrightarrow{x}|}{|M^n\overrightarrow{x}|}$.

(f) Let $M = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Find M^8, M^9, an approximation for λ_2 and a corresponding eigenvector $\overrightarrow{v}_2$. (Suggestion: Rather than multiplying M by itself repeatedly, note that $M^2 = M \times M$, $M^4 = M^2 \times M^2$, and $M^8 = M^4M^4$).

5. Answer the following questions.

(a) Solve the following system, parametrizing as necessary.

$$\begin{aligned} 3x + 2y - 4z + w &= 0 \\ x - 4y + z + 2w &= 0 \\ x + 10y - 6z - 3w &= 0 \\ 2x + y - 4z - 5w &= 0 \end{aligned}$$

(b) Let $\mathcal{A}$ be the set of polynomials of the form $ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$, where we define addition and scalar multiplication using the standard rules for polynomial arithmetic. Prove or disprove: $\mathcal{A}$ is a vector space.

Math 2101: Assignment 7

1. Find the determinant of each matrix.

   (a) $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

   (b) $\begin{pmatrix} 2-3 \\ 4-6 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix}$

2. Every mathematical formula is a summary of all the steps of an algorithm. **Cramer's Rule** is a formula that summarizes all steps necessary to solve a system of n equations in n unknowns.

   (a) Consider a system of two equations in two unknowns:

   $$\begin{aligned} a_{11}x + a_{12}y &= c_1 \\ a_{12}x + a_{22}y &= c_2 \end{aligned}$$

   Solve this system to produce formulas for the value of x and y.

   (b) Let A be the coefficient matrix; A_x be the matrix produced by replacing first column of the coefficient matrix (corresponding to the variable x) with the constant vector; A_y be the matrix produced by replacing the second column of the coefficient matrix (corresponding to the variable y) with the constant vector. Find $\det A$, $\det A_x$, and $\det A_y$.

   (c) Express the values of x and y in terms of these determinants.

   (d) Use Cramer's Rule to solve:

   $$\begin{aligned} 3x + 4y &= 9 \\ 7x - 2y &= 8 \end{aligned}$$

   (e) Cramer's rule generalizes (with A_z defined as you'd expect it to be). Use it to solve:

   $$\begin{aligned} x + 3y - 4z &= 8 \\ 2x - 4y - 7z &= 1 \\ x - 2y + 8 &= 0 \end{aligned}$$

(f) Cramer's rule is a good example of why it's the journey, not the destination: It's a *terrible* way to solve linear systems. However, it's important because on the way, you find a way of determining when a system of n equations in n unknowns does *not* have a unique solution. How can you use Cramer's rule to predict whether a system of n equations in n unknowns has a unique solution?

3. Find the eigenvalues and associated eigenvectors for the following matrices.

   (a) $\begin{pmatrix} 4 & -15 \\ 2 & -7 \end{pmatrix}$

   (b) $\begin{pmatrix} 18 & -20 \\ 15 & -17 \end{pmatrix}$

   (c) $\begin{pmatrix} -2 & 5 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 4 \end{pmatrix}$

4. Let x_n and y_n be the number of immature and mature pairs of rabbits at the end of month n. If the rabbits breed according to the model of Leonardo of Pisa, we can find x_{n+1}, y_{n+1} via

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$$

   (a) Find the eigenvalues and corresponding eigenvectors for the transition matrix.

   (b) Suppose $x_0 = 1$ and $y_0 = 0$. Express $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a linear combination of the eigenvectors you found.

   (c) Determine the *exact* number of immature and mature rabbits at the end of the 12th month (in other words, find x_{12}, y_{12}).

   (d) Use the eigenvalues and eigenvectors to approximate the number of immature and mature rabbits at the end of the 12th month.

5. Answer the following questions.

(a) Let $A = \begin{pmatrix} 2 & 1 & 5 & 1 \\ 0 & 2 & 3 & -1 \\ 1 & -1 & 2 & 1 \end{pmatrix}$. Find a basis for $\mathrm{Col}(A)$, and a basis for $\mathrm{Null}(A)$.

(b) Mathematicians like to recycle concepts, so the same basic idea will occur in many different places. Remember the range of a function is the set of all possible outputs; thus we define the **range of a linear transformation** T to be the set of all vectors $\overrightarrow{y}$ for which $T\overrightarrow{x} = \overrightarrow{y}$ for some $\overrightarrow{x}$. Let $T = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 5 & 3 \end{pmatrix}$. Find the range of T.

Math 2101: Assignment 8

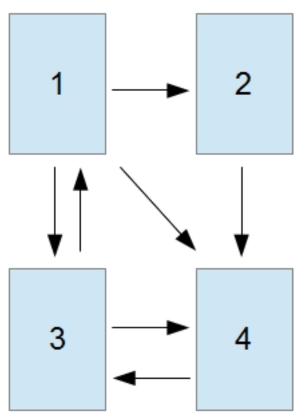
1. Let $P = (1, 1, -1)$, $Q = (1, 2, 5)$, $R = (-1, -3, -4)$ be **points** (not vectors!) in $\mathbb{R}^3$.

   (a) Find the distance between the line $\overleftrightarrow{PQ}$ and the point R.

   (b) Find the distance between the origin and the plane PQR. Suggestion: Remember that you can find distance by dropping a perpendicular, then determining the length of the perpendicular.

   (c) Find the point on the plane closest to the point $S = (1, 1, 3)$. Suggestion: If T is the point on the plane closest to S, then $|ST|$ must be the distance from the point to the plane.

2. Web traffic (and many, many other things) can be modeled using a **graph**: this consists of a set of points (the web pages) connected by edges (the links). We can represent the graph using an **adjacency matrix** A, where $a_{ij} = 1$ if there is a link *from* point i *to* point j. (Note that since hyperlinks are directional, a link from i to j does not necessarily imply a link from j to i) The figure shows a set of four linked web pages.



   (a) Form the adjacency matrix A for the graph shown.

   (b) A useful result (from combinatorics) is that the ijth entry of the matrix A^n will be the number of ways you can get from i to j via a path of length n (which might revisit web pages). Find A^4; determine the number of ways you can get from page 1 to page 4 using a path of length 4; then list them (by describing the click-thru sequence: e.g., one such path is $1 \to 4 \to 3 \to 1 \to 3$).

(c) Suppose B is the adjacency matrix for a set of connected web pages (where it's always possible to find a path from one web page to another). Prove, or explain why this might not be true: There is some n for which all entries of B^n are non-zero. Suggestion: What does it mean when an entry of B^n is non-zero?

3. Let $p_1(x) = x^2 - 3x - 10$, $p_2(x) = 5x + 7$, and $p_3(x) = x^2 + 12x + 11$.

(a) Using standard polynomial addition, what polynomials ax^2+bx+c can be expressed as linear combinations of $p_1(x)$, $p_2(x)$, $p_3(x)$?

(b) Find a basis for the vector space spanned by $p_1(x)$, $p_2(x)$, $p_3(x)$.

(c) Remember that a set of vectors is independent if and only if the only linear combination to produce $\overrightarrow{0}$ is the trivial linear combination (all coefficients 0). For these vectors, we could try to solve $a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = 0$. Instead, we can form new equations by *differentiation*. Explain how to produce a system of equations with unknown constants a_1, a_2, a_3. (There are three unknowns, so you'll need three equations: find them!)

(d) Consider the vector space spanned by $\overrightarrow{V} = \{f_1(x), f_2(x), \ldots, f_n(x)\}$, where $f_i(x)$ is a smooth function of x (smooth = you can differentiate it as many times as you need to). Explain how you could determine if the set of vectors is linearly independent.

4. Given a set of vectors $\mathcal{V} = \{\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n\}$, where the components of each vector are integers, a **lattice** consists of the set of all linear combinations of $\mathcal{V}$ whose coefficients are integers. Let $\overrightarrow{v}_1 = (40, 1)$ and $\overrightarrow{v}_2 = (-1, 39)$.

(a) Determine which of the following vectors are in lattice: $(42, 74)$, $(198, 234)$, $(155, 199)$.

(b) Given a lattice, **closest vector problem** (CVP) requires us to find the lattice vector closest to a given vector. For the points above which are not in the lattice, solve the CVP.

(c) Consider the lattice formed by $\overrightarrow{p}_1 = (39, 40)$, $\overrightarrow{p}_2 = (77, 119)$. Determine which of the points in Problem 4a are in the lattice; then solve the CVP for the remaining points.

(d) As it turns out, the lattice spanned by the $\overrightarrow{v}_i$s is the same lattice spanned by the $\overrightarrow{p}_i$s (surprise!). Prove this. (In particular: Suppose $\overrightarrow{x}$ can be expressed as a linear combination with integer coefficients of the $\overrightarrow{v}_i$s. Show that $\overrightarrow{x}$ can be expressed as a linear combination with integer coefficients of the $\overrightarrow{p}_i$s. Then show that the converse is also true)

(e) Identify an important difference between the vectors $\overrightarrow{v}_i$ and the vectors $\overrightarrow{p}_i$. Suggestion: Given the set of vectors $\overrightarrow{v}_i$, what questions could you ask about them? Then ask the same questions about the vectors $\overrightarrow{p}_i$, and compare the answers.

5. Let $A = \begin{pmatrix} 6 & -5 \\ 10 & -9 \end{pmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors of A. (In the following, we'll refer to these as λ_1, λ_2, with corresponding $\overrightarrow{v}_1$, $\overrightarrow{v}_2$)

(b) Prove or disprove: The set of eigenvectors you found are independent.

(c) Prove or disprove: The set of eigenvectors you found span $\mathbb{R}^2$.

(d) Let $\overrightarrow{x} = x_1\overrightarrow{v}_1 + x_2\overrightarrow{v}_2$. Find $A^{100}\overrightarrow{x}$.

(e) If possible, set up and solve the system of equations that would allow you to express any vector $\overrightarrow{x}$ as a linear combination of the eigenvectors $\overrightarrow{v}_1$, $\overrightarrow{v}_2$.

(f) Let $\overrightarrow{y} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Find $A^{100}\overrightarrow{y}$.