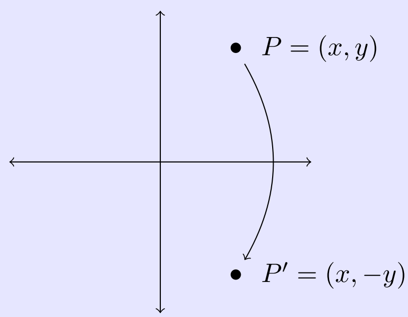
Math 2101 — Robert Wagner
Assignment 3 — June 11, 2016

1. Given any point $P = (x, y)$, we can define a **transformation** $(x, y) \to (x', y')$, where

$$\begin{aligned} ax + by &= x' \\ cx + dy &= y' \end{aligned}$$

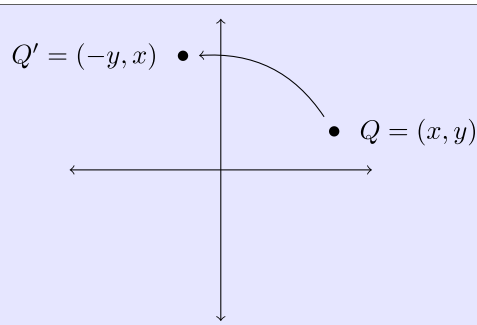
for some real numbers a, b, c, d. We write $T : P \to P'$ to indicate P' is the point (x', y') produced by applying the transformation T to the point P; We also write $TP = P$. In the following, you don't have to draw a picture ... but it will probably help.

(a) Write down the linear transformation corresponding to the geometric transformation of reflecting a point across the x-axis. (In other words, find a, b, c, d so that (x', y') is the reflection of (x, y) across the x-axis) Then write the corresponding coefficient matrix (call this matrix M_x).



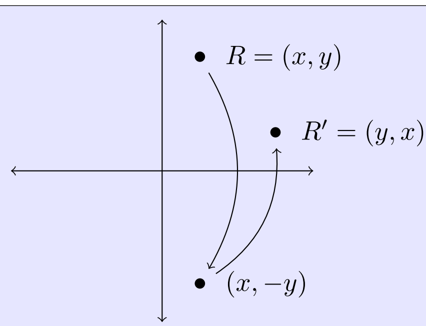
$$\begin{aligned} x' &= 1x + 0y \\ y' &= 0x - 1y \\ M_x &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ M_x P &= P' \end{aligned}$$

(b) Write down the linear transformation corresponding to the geometric transformation of rotating the point (x, y) 90° counter-clockwise around the origin. Then write the corresponding coefficient matrix (call this matrix R_{90}).



$$\begin{aligned} x' &= 0x - 1y \\ y' &= 1x + 0y \\ R_{90} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ R_{90} Q &= Q' \end{aligned}$$

(c) Write down the linear transformation corresponding to the geometric transformation of reflecting the point (x, y) across the x-axis, followed by rotating the point 90° counter-clockwise around the origin. Then write the corresponding coefficient matrix (call this matrix T).



$$\begin{aligned} x' &= 0x + 1y \\ y' &= 1x + 0y \\ T &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ R_{90} M_x R &= TR = R' \end{aligned}$$

(d) Suppose S, T are two linear transformations, and P is a point. Define $M = S + T$ and remember **we haven't yet defined matrix addition** (so even if you know how to add two matrices, you may not use this knowledge). Show that if we want the distributive law $(S+T)P = SP + TP$ to hold, we must define $m_{ij} = s_{ij} + t_{ij}$.

> Let $P = (x, y)$, $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $T = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then
>
> $$\begin{aligned} SP + TP &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \langle x, y \rangle + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \langle x, y \rangle \\ &= \langle ax + by, cx + dy \rangle + \langle ex + fy, gx + hy \rangle \\ &= \langle (a+e)x + (b+f)y, (c+g)x + (d+h)y \rangle \\ &= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \langle x, y \rangle = (S+T)P \end{aligned}$$
>
> Therefore it follows in the two dimensional case that for $M = S + T$, matrix addition defined as $m_{ij} = s_{ij} + t_{ij}$ will hold the distributive property.

2. If A, B are the matrices corresponding to a geometric transformation, we interpret the matrix product AB as the geometric transformation produced by applying B first, then applying A.

(a) Is matrix multiplication commutative? (In other words, given two matrices A, B, will $AB = BA$?) Explain your conclusion.

> No, since we can see from the geometric transformation example that $R_{90}M_x \neq M_x R_{90}$, and in general it is not necessarily true for any matrices A, B since A is treated as a set of row vectors, and B is treated as a set of column vectors such that for $M = AB$,
>
> $$m_{ij} = \vec{\boldsymbol{a}}_i \cdot \vec{\boldsymbol{b}}_j$$
>
> for all rows $i \in A$ and columns $j \in B$.
> In general, if $A = A^T$ and $B = B^T$ then $AB = BA$.

(b) Find $R_{90}M_x$.

> From item 1(c) above, we see that
>
> $$R_{90}M_x = T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) It's tempting to define the product of two matrices componentwise:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} m & n \\ p & q \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} am & bn \\ cp & dq \end{bmatrix}$$

Show that this definition does not calculate $R_{90}M_x$ correctly.

> $$R_{90}M_x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 & -1 \cdot 0 \\ 1 \cdot 0 & -1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq T$$

(d) We actually find the product of 2×2 matrices as

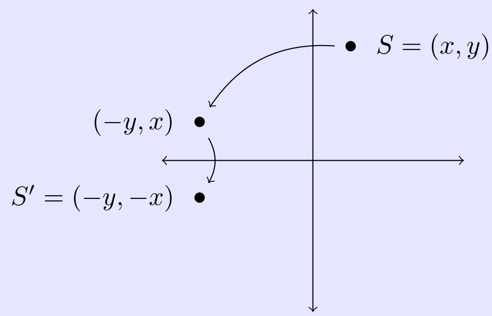
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} m & n \\ p & q \end{bmatrix} = \begin{bmatrix} am + bp & ab + bq \\ cm + dp & cn + dq \end{bmatrix}$$

Show that this definition gives us $T = R_{90} M_x$.

$$R_{90} M_x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} (0 \cdot 1) + (-1 \cdot 0) & (0 \cdot 0) + (-1 \cdot -1) \\ (1 \cdot 1) + (0 \cdot 0) & (1 \cdot 0) + (0 \cdot -1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T$$

(e) Use matrix multiplication as defined above to find $T' = M_x R_{90}$. Then verify that this is correct, by identifying the geometric transformation to the product $M_x R_{90}$ and finding the corresponding linear transformation.

$$M_x R_{90} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1 \cdot 0) + (0 \cdot 1) & (1 \cdot -1) + (0 \cdot 0) \\ (0 \cdot 0) + (-1 \cdot 1) & (0 \cdot -1) + (-1 \cdot 0) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = T'$$



verify:

$$x' = 0x - 1y$$
$$y' = -1x + 0y$$
$$T' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$M_x R_{90} S = T' S = S'$$

3. Let $M = \begin{bmatrix} 1/2 & 1/4 & 1/2 \\ 0 & 1/4 & 1/2 \\ m_{31} & 1/2 & m_{33} \end{bmatrix}$ be a stochasic matrix.

(a) Interpret this as a set of movement rules between three locations by identifying what fractions of those at each location at $t = k$ will move to the other locations at $t = k + 1$.

- 1/2 of those at location a move to location c.
- 1/4 of those at location b move to location a, and 1/2 move to location c.
- 1/2 of those at location c move to location a, and 1/2 move to location b.

(b) Find m_{31} and m_{33}.

Since each columns must sum to one, it follows that

$$\begin{aligned} m_{11} + m_{21} + m_{31} &= 1 \\ 1/2 + 0 + m_{31} &= 1 \\ m_{31} &= \underline{1/2} \end{aligned} \qquad \begin{aligned} m_{31} + m_{32} + m_{33} &= 1 \\ 1/2 + 1/2 + m_{33} &= 1 \\ m_{33} &= \underline{0} \end{aligned}$$

(c) Find M^2, where m_{ij} corresponds to the fraction of those in location i at $t = k$ who will be in location j at $t = k + 2$.

	a	b	c
$t = k$	a_k	b_k	c_k
$t = k+1$	$\frac{a_k}{2} + \frac{b_k}{4} + \frac{c_k}{2}$	$\frac{b_k}{4} + \frac{c_k}{2}$	$\frac{a_k}{2} + \frac{b_k}{2}$
$t = k+2$	$\frac{\frac{a_k}{2}+\frac{b_k}{4}+\frac{c_k}{2}}{2} + \frac{\frac{b_k}{4}+\frac{c_k}{2}}{4} + \frac{\frac{a_k}{2}+\frac{b_k}{2}}{2}$	$\frac{\frac{b_k}{4}+\frac{c_k}{2}}{4} + \frac{\frac{a_k}{2}+\frac{b_k}{2}}{2}$	$\frac{\frac{a_k}{2}+\frac{b_k}{4}+\frac{c_k}{2}}{2} + \frac{\frac{b_k}{4}+\frac{c_k}{2}}{2}$

$$\begin{aligned}
a_{k+2} &= \frac{a_{k+1}}{2} + \frac{b_{k+1}}{4} + \frac{c_{k+1}}{2} = \frac{\frac{a_k}{2} + \frac{b_k}{4} + \frac{c_k}{2}}{2} + \frac{\frac{b_k}{4} + \frac{c_k}{2}}{4} + \frac{\frac{a_k}{2} + \frac{b_k}{2}}{2} \\
&= \frac{2a_k + b_k + 2c_k}{8} + \frac{b_k + 2c_k}{16} + \frac{a_k + b_k}{4} \\
&= \frac{4a_k + 2b_k + 4c_k + b_k + 2c_k + 4a_k + 4b_k}{16} \\
&= \frac{8a_k + 7b_k + 6c_k}{16} = \frac{a_k}{2} + \frac{7b_k}{16} + \frac{3c_k}{8} \\
b_{k+2} &= \frac{b_{k+1}}{4} + \frac{c_{k+1}}{2} = \frac{\frac{b_k}{4} + \frac{c_k}{2}}{4} + \frac{\frac{a_k}{2} + \frac{b_k}{2}}{2} = \frac{b_k + 2c_k}{16} + \frac{a_k + b_k}{4} \\
&= \frac{4a_k + 5b_k + 2c_k}{16} = \frac{a_k}{4} + \frac{5b_k}{16} + \frac{c_k}{8} \\
c_{k+2} &= \frac{a_{k+1}}{2} + \frac{b_{k+1}}{4} + \frac{c_{k+1}}{2} = \frac{\frac{a_k}{2} + \frac{b_k}{4} + \frac{c_k}{2}}{2} + \frac{\frac{b_k}{4} + \frac{c_k}{2}}{2} \\
&= \frac{2a_k + b_k + 2c_k}{8} + \frac{b_k + 2c_k}{8} = \frac{a_k}{4} + \frac{b_k}{4} + \frac{c_k}{2}
\end{aligned}$$

therefore

$$M^2 = \begin{bmatrix} 1/2 & 7/16 & 3/8 \\ 1/4 & 5/16 & 1/8 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

(d) Explain how you found the number of people in location c at $t = k + 2$.

First, we find $c_{k+1} = Mc_k =$ the number of people at location c at time $k+1$ by applying the matrix M to c_k. Then apply the matrix M to c_{k+1} to find c_{k+2}.

(e) The preceeding problem suggests that if A, B are 2×2 matrices, the product AB is the matrix whose entries correspond to the dot product of the rows of A with the columns of B. Verify that this works on 3×3 matrices.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.

$$\begin{aligned}
\text{Thus } AB &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} \\
&= \begin{bmatrix} \vec{\boldsymbol{a_1}} \cdot \vec{\boldsymbol{b_1}} & \vec{\boldsymbol{a_1}} \cdot \vec{\boldsymbol{b_2}} & \vec{\boldsymbol{a_1}} \cdot \vec{\boldsymbol{b_3}} \\ \vec{\boldsymbol{a_2}} \cdot \vec{\boldsymbol{b_1}} & \vec{\boldsymbol{a_2}} \cdot \vec{\boldsymbol{b_2}} & \vec{\boldsymbol{a_2}} \cdot \vec{\boldsymbol{b_3}} \\ \vec{\boldsymbol{a_3}} \cdot \vec{\boldsymbol{b_1}} & \vec{\boldsymbol{a_3}} \cdot \vec{\boldsymbol{b_2}} & \vec{\boldsymbol{a_3}} \cdot \vec{\boldsymbol{b_3}} \end{bmatrix}
\end{aligned}$$

Thus $AB_{ij} = \vec{\boldsymbol{a_i}} \cdot \vec{\boldsymbol{b_j}}$ where $\{\vec{\boldsymbol{a_i}}\}_{i \in I}$ are the rows of A, and $\{\vec{\boldsymbol{b_j}}\}_{j \in J}$ are the columns of B where $I, J = \{1, 2, 3\}$.

4. The identity transformation is simply $(x, y) \to (x, y)$. We'll continue to use M_x and R_{90} from above.

   (a) Write down the coefficient matrix corresponding to the identity transformation.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   (b) Given a matrix B, the inverse matrix B^{-1} satisfies $B^{-1}B = I$. First, describe the geometric transformation corresponding to the matrix M_x^{-1}, then find M_x^{-1}.

Since $M_x\langle x, y\rangle = \langle x, -y\rangle$, We should define M_x^{-1} such that $M_x^{-1}\langle x, -y\rangle = \langle x, y\rangle$. In the case of a mirroring transformation like M_x, we can see that $M_xM_x = I$ since

$$\begin{aligned} M_xM_x\langle x, y\rangle &= M_x\langle x, -y\rangle \\ &= \langle x, y\rangle \end{aligned}$$

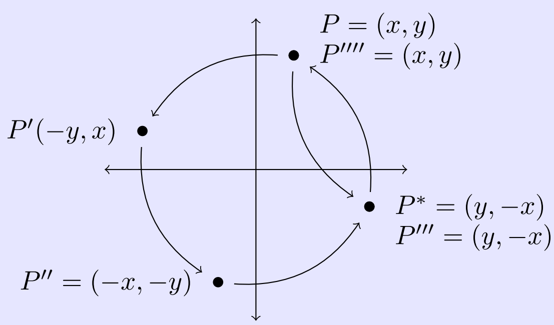
therefore

$$M_x^{-1} = M_x$$

   (c) Explain, without computing or otherwise referring to the values of the matrices involved, why $R_{90}^3 = R_{90}^{-1}$.

Since $R_{90}^4P = P$ it follows that $R_{90}^4 = I$ therefore

$$\begin{aligned} R_{90}^{-1}R_{90}^4 &= R_{90}^{-1}I \\ R_{90}^3 &= R_{90}^{-1} \end{aligned}$$



$$\begin{aligned} R_{90}^3P &= P''' \\ x''' &= 0x + 1y \\ y''' &= -1x + 0y \\ R_{90}^{-1}P &= P^* \\ x^* &= 0x + 1y \\ y^* &= -1x + 0y \\ R_{90}^{-1}P &= R_{90}^3P \end{aligned}$$

   (d) Find R_{90}^{-1}.

By the answer above, $R_{90}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

   (e) Suppose A, B are matrices corresponding to some geometric transformations.
   What is $(AB)^{-1}$? Explain.

For some matrices A, B such that $T = AB$, the inverse $T^{-1} = (AB)^{-1}$ is the matrix such that $T^{-1}T = (AB)^{-1}AB = I$.

$$\begin{aligned} (AB)^{-1}AB &= I \\ (AB)^{-1}ABB^{-1} &= IB^{-1} = B^{-1} \\ (AB)^{-1}AI &= (AB)^{-1}A = B^{-1} \\ (AB)^{-1}AA^{-1} &= B^{-1}A^{-1} \\ (AB)^{-1}I &= (AB)^{-1} = \underline{B^{-1}A^{-1}} \end{aligned}$$

5. Answer the following questions.

(a) Reduce the following matrix to row echelon form:

$$\left[\begin{array}{ccc|c} 3 & 1 & 5 & 1 \\ 2 & -1 & 3 & -2 \\ 1 & 4 & 0 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 5 & 1 \\ 2 & -1 & 3 & -2 \\ 1 & 4 & 0 & 3 \end{array}\right] \overset{\substack{R_1 - R_2 \to R_1 \\ R_2 - 2R_3 \to R_2 \\ R_3 - R_1 \to R_3}}{=} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -9 & 3 & -8 \\ 0 & 2 & -2 & 0 \end{array}\right] \overset{9R_3 + 2R_2 \to R_3}{=} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -9 & 3 & -8 \\ 0 & 0 & -12 & -16 \end{array}\right] \overset{\text{normalize}}{=} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & -1/3 & 8/9 \\ 0 & 0 & 1 & 4/3 \end{array}\right]$$

(b) Let $\vec{\boldsymbol{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\boldsymbol{v}} = \langle v_1, v_2, v_3 \rangle$. Prove or disprove: $|\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}| \leq |\vec{\boldsymbol{u}}| + |\vec{\boldsymbol{v}}|$.

Note that $a \leq b$ if and only if $a^2 \leq b^2$ for all $a, b \in [0, \infty)$. Also note that $|\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}| \in [0, \infty)$, and that $|\vec{\boldsymbol{u}}|, |\vec{\boldsymbol{v}}| \in [0, \infty)$.
Let $a = |\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}|$, $b = |\vec{\boldsymbol{u}}| + |\vec{\boldsymbol{v}}|$, and $\theta \in [0, \pi]$ be the angle between $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$. Thus

$$\begin{aligned}
a^2 &= \left(\sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2 + (u_3 + v_3)^2}\right)^2 \\
&= (u_1^2 + 2u_1v_1 + v_1^2) + (u_2^2 + 2u_2v_2 + v_2^2) + (u_3^2 + 2u_3v_3 + v_3^2) \\
&= (u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) + 2(u_1v_1 + u_2v_2 + u_3v_3) \\
&= |\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2 + 2(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}) = |\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2 + 2\left(|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos\theta\right) \\
b^2 &= \left(\sqrt{u_1^2 + u_2^2 + u_3^2} + \sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 \\
&= \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 + 2\sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)} + \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 \\
&= |\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2 + 2|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|
\end{aligned}$$

Since $-1 \leq \cos\theta \leq 1$ it follows that $2|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos\theta \leq 2|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|$, thus

$$|\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2 + 2|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos\theta \leq |\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2 + 2|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \text{ thus } a^2 \leq b^2$$

and subsequently

$$a \leq b \text{ thus } |\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}| \leq |\vec{\boldsymbol{u}}| + |\vec{\boldsymbol{v}}| \text{ for all vectors } \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^3. \quad \square$$

(c) Let $\vec{\boldsymbol{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\boldsymbol{v}} = \langle v_1, v_2, v_3 \rangle$. Prove or disprove: $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{u}}$.

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = u_1v_1 + u_2v_2 + u_3v_3$$

since $ab = ba$ for all $a, b \in \mathbb{R}$ it follows that

$$u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3$$

and subsequently

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{u}} \quad \square$$

(d) Let $\vec{\boldsymbol{p}} = \langle 3, -1, 1\rangle$. Find all vectors that are perpendicular to $\vec{\boldsymbol{p}}$.

Let $\vec{\boldsymbol{u}} = \langle u_1, u_2, u_3\rangle$ such that $\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{u}} = 0$. Thus

$$\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{u}} = 3u_1 + (-1)u_2 + 1u_3 = 0$$

To parameterize, let

$$\begin{aligned} u_2 &= 3s \\ u_3 &= 3t \end{aligned}$$

It follows through substitution that

$$\begin{aligned} 3u_1 + (-1)(3s) + (3t) &= 0 \\ 3u_1 &= 3s - 3t \\ u_1 &= s - t \end{aligned}$$

Thus we can write

$$U = \{\vec{\boldsymbol{u}} \in \mathbb{R}^3 : \vec{\boldsymbol{u}} = s\langle 1, 3, 0\rangle + t\langle -1, 0, 3\rangle \text{ for all } s, t \in \mathbb{R}\}$$

Therefore the set U is a plane containing all vectors perpendicular to $\vec{\boldsymbol{p}}$.