1. In the following, let $\vec{v_1} = \langle 3, -2, -1\rangle$, $\vec{v_2} = \langle 1, 1, -1\rangle$, and $\vec{v_3} = \langle -3, 2, 6\rangle$ be vectors in $\mathbb{R}^3$. Find the following, if possible. If not possible, explain why.

   (a) $3\vec{v_1} - 2\vec{v_2} + 4\vec{v_3}$

$$\begin{aligned} 3\vec{v_1} - 2\vec{v_2} + 4\vec{v_3} &= \langle 3\times 3, 3\times -2, 3\times -1\rangle + \langle -2\times 1, -2\times 1, -2\times -1\rangle + \langle 4\times -3, 4\times 2, 4\times 6\rangle \\ &= \langle 9-2-12, -6-2+8, -3-2+24\rangle \\ &= \underline{\langle -5, 0, 19\rangle} \end{aligned}$$

   (b) $\vec{v_1} \cdot \vec{v_2}$

$$\begin{aligned} \vec{v_1} \cdot \vec{v_2} &= (3\times 1) + (-2\times 1) + (-1\times -1) \\ &= 3 - 2 + 1 \\ &= \underline{2} \end{aligned}$$

   (c) $2\vec{v_1} \cdot \vec{v_2} + 2\vec{v_3}$

This is not possible as written, as evaluating the dot product $2\vec{v_1} \cdot \vec{v_2}$ results in a scalar, and the subsequent addition of the scalar $2\vec{v_1} \cdot \vec{v_2}$ and the vector $2\vec{v_3}$ is not defined. However, if it were written $2\vec{v_1} \cdot (\vec{v_2} + 2\vec{v_3})$ then it would evaluate to a scalar.

   (d) $\vec{v_1} \cdot \vec{v_2} \cdot \vec{v_3}$

This is not possible as written, as evaluating the dot product $\vec{v_1} \cdot \vec{v_2}$ results in a scalar, and the subsequent dot product between the scalar $\vec{v_1} \cdot \vec{v_2}$ and the vector $\vec{v_3}$ is undefined.

   (e) $\vec{v_1} \cdot (3\vec{v_2} - 2\vec{v_3})$

$$\begin{aligned} \vec{v_1} \cdot (3\vec{v_2} - 2\vec{v_3}) &= \langle 3, -2, -1\rangle \cdot (\langle 3\times 1, 3\times 1, 3\times -1\rangle + \langle -2\times -3, -2\times 2, -2\times 6\rangle) \\ &= \langle 3, -2, -1\rangle \cdot \langle 3+6, 3-4, -3-12\rangle \\ &= \langle 3, -2, -1\rangle \cdot \langle 9, -1, -15\rangle \\ &= (3\times 9) + (-2\times -1) + (-1\times -15) \\ &= 27 + 2 + 15 \\ &= \underline{44} \end{aligned}$$

   (f) $|\vec{v_1}|$

$$\begin{aligned} |\vec{v_1}| &= |\langle 3, -2, -1\rangle| \\ &= \sqrt{3^2 + (-2)^2 + (-1)^2} \\ &= \sqrt{9+4+1} \\ &= \underline{\sqrt{14}} \end{aligned}$$

   (g) $|5\vec{v_1}|$

$$\begin{aligned} |5\vec{v_1}| &= |\langle 5\times 3, 5\times -2, 5\times -1\rangle| \\ &= |\langle 15, -10, -5\rangle| \\ &= \sqrt{15^2 + (-10)^2 + (-5)^2} \\ &= \sqrt{225 + 100 + 25} \\ &= \sqrt{350} \\ &= \underline{5\sqrt{14}} \end{aligned}$$

2. In the following, assume $\vec{\boldsymbol{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\boldsymbol{v}} = \langle v_1, v_2, v_3 \rangle$, and θ is the angle between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

(a) Prove: $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos\theta$

Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ be non-zero vectors in $\mathbb{R}^3$ such that $\theta \in [0, \pi)$ is the angle between them. The law of cosines states that

$$|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2 = |\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2 - 2|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos\theta$$

is true for all non-zero vectors. Thus

$$\begin{aligned}
|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2 &= |\langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle|^2 \\
&= (\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2})^2 \\
&= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2) \\
&= (\sqrt{u_1^2 + u_2^2 + u_3^2})^2 + (\sqrt{v_1^2 + v_2^2 + v_3^2})^2 - 2(u_1v_1 + u_2v_2 + u_3v_3) \\
&= |\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2 - 2(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})
\end{aligned}$$

and we can rewrite the law of cosines as

$$|\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2 - 2(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}) = |\vec{\boldsymbol{u}}|^2 + |\vec{\boldsymbol{v}}|^2 - 2|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos\theta$$

simplifying this, we get

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos\theta$$

Therefore, we have shown that if $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}} \in \mathbb{R}^3$ are non-zero vectors and $\theta \in [0, \pi)$ is the angle between them, then $|\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}| = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos\theta$ $\square$

(b) Let $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ be two sides of a triangle. Find the area of the triangle.

The area of a triangle is defined as $A = \frac{bh}{2}$ where $b, h \in [0, \infty)$ are the base length and height of the triangle, respectively. Let $b = adjacent = |\vec{\boldsymbol{u}}|$ and $hypotenuse = |\vec{\boldsymbol{v}}|$. Using the definition of $\sin\theta = \frac{opposite}{hypotenuse}$, and considering the $h = height = opposite$, we find that $\sin\theta = \frac{h}{|\vec{\boldsymbol{v}}|}$ and subsequently $h = |\vec{\boldsymbol{v}}| \sin\theta$.

Since $\sin^2\theta + \cos^2\theta = 1$ and $\cos\theta = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|}$ it follows that

$$\begin{aligned}
\sin\theta &= \sqrt{1 - \left(\frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|}\right)^2} \text{ for } \theta \in [0, \pi) \\
&= \frac{1}{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|} \sqrt{|\vec{\boldsymbol{u}}|^2|\vec{\boldsymbol{v}}|^2 - (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})^2} \\
&= \frac{1}{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|} \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)(u_1v_1 + u_2v_2 + u_3v_3)} \\
&= \frac{1}{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|} \sqrt{(u_1^2v_2^2 - 2u_1v_2u_2v_1 + u_2^2v_1^2) + (u_1^2v_3^2 - 2u_1v_3u_3v_1 + u_3^2v_1^2) + (u_2^2v_3^2 - 2u_2v_3u_3v_2 + u_3^2v_2^2)} \\
&= \frac{1}{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|} \sqrt{(u_1v_2 - u_2v_1)^2 + (u_1v_3 - u_3v_1)^2 + (u_2v_3 - u_3v_2)^2} \\
&= \frac{|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|}{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|}
\end{aligned}$$

and subsequently

$$A = \frac{bh}{2} = \frac{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \sin\theta}{2} = \frac{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}||\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|}{2|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|} = \frac{|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|}{2}$$

3. Let $P = (1, 4, -3)$, $Q = (5, 1, -3)$, and $R = (-1, 1, 2)$ be points in $\mathbb{R}^3$.

(a) Explain how you would write the parametric equation of the line through X, where $\vec{\boldsymbol{v}}$ gives the direction of a line.

I would take a starting point $X = x_0$ and add $\vec{\boldsymbol{v}}$ scaled by the parameter s for all $s \in \mathbb{R}$.

(b) Find the parametric equation of the lines $\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{QR}$

Let $\vec{\boldsymbol{u}} = \overrightarrow{PQ} = \langle 4, -3, 0 \rangle$ and $\vec{\boldsymbol{v}} = \overrightarrow{PR} = \langle -2, -3, 5 \rangle$ and $\vec{\boldsymbol{w}} = \overrightarrow{QR} = \langle -6, 0, 5 \rangle$

$$\begin{aligned} PQ(s) &= p_0 + s\vec{\boldsymbol{u}} = (1, 4, -3) + s\langle 4, -3, 0 \rangle \\ PR(s) &= p_0 + s\vec{\boldsymbol{v}} = (1, 4, -3) + s\langle -2, -3, 5 \rangle \\ QR(s) &= q_0 + s\vec{\boldsymbol{w}} = (5, 1, -3) + s\langle -6, 0, 5 \rangle \end{aligned}$$

for all $s \in \mathbb{R}$

(c) Explain why any linear combination of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ must be in the same plane as P, Q, R.

A linear combination of vectors is the sum of those vectors each multiplied by a corresponding scalar value. As scaling a vector does not change its direction, any linear combination of two vectors will form a new vector that is co-planar with both original vectors. As long as the two vectors are not co-linear and neither is the zero vector, all possible linear combinations form a plane. If the two are co-linear then they are also co-planar.

(d) Use the proceeding idea to write the parametric equation of the plane PQR.

Let $p_0 = P = (1, 4, -3)$ be a point on the plane PQR, and let $\vec{\boldsymbol{u}} = \overrightarrow{PQ} = \langle 4, -3, 0 \rangle$ and $\vec{\boldsymbol{v}} = \overrightarrow{PR} = \langle -2, -3, 5 \rangle$ be vectors on the plane PQR.

$$PQR(s, t) = \{p : p = p_0 + s\vec{\boldsymbol{u}} + t\vec{\boldsymbol{v}} \text{ for all } s, t \in \mathbb{R}\} = (1, 4, -3) + s\langle 4, -3, 0 \rangle + t\langle -2, -3, 5 \rangle$$

(e) Find the area of the triangle ΔPQR.

Let $\vec{\boldsymbol{u}} = \overrightarrow{PQ} = \langle 4, -3, 0 \rangle$ and $\vec{\boldsymbol{v}} = \overrightarrow{PR} = \langle -2, -3, 5 \rangle$ be vectors in $\mathbb{R}^3$ forming two sides of ΔPQR.

$$\begin{aligned} area(\Delta PQR) &= \frac{|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|}{2} \\ &= \frac{1}{2}\sqrt{[(4)(-3) - (-3)(-2)]^2 + [(4)(5) - (0)(-2)]^2 + [(-3)(5) - (0)(-3)]^2} \\ &= \frac{1}{2}\sqrt{(-18)^2 + (20)^2 + (-15)^2} = \frac{\sqrt{949}}{2} \approx \underline{15.403} \end{aligned}$$

(f) Find the distance between the point R and the line $\overrightarrow{PQ}$.

Let $\vec{\boldsymbol{u}} = \overrightarrow{PQ} = \langle 4, -3, 0 \rangle$ and $\vec{\boldsymbol{v}} = \overrightarrow{PR} = \langle -2, -3, 5 \rangle$ be vectors in $\mathbb{R}^3$, and let $\theta \in [0, \pi)$ be the angle between them. The distance between R and $\overrightarrow{PQ}$ is equivalent to the triangle height that we calculated in question 2(b) above, that is:

$$\begin{aligned} dist(R, \overrightarrow{PQ}) &= |\vec{\boldsymbol{v}}| \sin\theta = \frac{|\vec{\boldsymbol{v}}||\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|}{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|} = \frac{|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|}{|\vec{\boldsymbol{u}}|} \\ &= \frac{\sqrt{[(4)(-3) - (-3)(-2)]^2 + [(4)(5) - (0)(-2)]^2 + [(-3)(5) - (0)(-3)]^2}}{\sqrt{4^2 + (-3)^2 + 0^2}} \\ &= \frac{\sqrt{949}}{5} \approx \underline{6.161} \end{aligned}$$

4. Suppose M is the coefficient matrix for a system of equations augmented by the constants of the equation. For each of the following, identify the corresponding operation on a system of equations; then state whether the operation is allowable or forbidden. Assume $c \neq 0$.

   (a) Multiplying every term of a row by c.

   Multiplying every term in a row by a scalar would not result in a change of direction of the underlying vector of that row, and thus would have no effect on the system as a whole. Thus this is allowed.

   (b) Switching two columns.

   Switching two columns results in a switching of coefficients between variables, resulting in a possibly different value for the system. Thus this operation is forbidden.

   (c) Switching two rows.

   Switching two rows results in a null operation on the system, as each row represents a discrete equation in the system, and there is no defined ordering of the equations in the system. Therefore this operation is allowed.

   (d) Adding c to each term in a row.

   Adding a constant to all terms in a row would result in a different value for the equation, for example

   $$\begin{aligned} 3x &= 15 \\ x &= 5 \end{aligned}$$

   is not equivalent to

   $$\begin{aligned} 5x &= 17 \\ x &\approx 3.4 \end{aligned}$$

   Therefore this operation is forbidden.

   (e) Multiplying every term in a row by c, then adding the corresponding terms to another row.

   If we were to conceptualize the equations of a system of linear of equations as being co-planar, any linear combination of equations in the system would form a new equation that is co-planar with the existing equations, and thus is readily acceptable as a new equation in the system. Therefore this operation is allowed.

   (f) Suppose that, after applying a sequence of allowable operations to M, you end up with a row consisting of all zeroes except the last entry, which is non-zero. What does this say about the original system of equations?

   If by "last entry" you mean the augmented column, then you have a system of equations with no viable solutions, as there is no way that a combination of variables all scaled by zero can equal a non-zero value.

5. Find a parameterization of the following systems of equations.

(a) $$3x + 2y = 5$$

Let $x = 1 + 2s$,
Then:

$$3(1+2s) + 2y = 5 \qquad \rightarrow 2y = 5 - (3+6s) \qquad \rightarrow y = \frac{2-6s}{2} \qquad \rightarrow y = 1 - 3s$$

Thus $(x, y) = (1, 1) + s\langle 2, -3\rangle$ for all $s \in \mathbb{R}$

(b)
$$\begin{aligned} 3w - 2x - 4y + 2z &= 0 \\ 3x + 2y - 5z &= 0 \\ 2y - 1z &= 0 \end{aligned}$$

Let $z = 18s$.
Then:

$$\begin{aligned} 2y - 18s = 0 &\qquad \rightarrow 2y = 18s &\qquad \rightarrow y = 9s \\ 3x + 2(9s) - 5(18s) = 0 &\qquad \rightarrow 3x = 72s &\qquad \rightarrow x = 24s \\ 3w - 2(24s) - 4(9s) + 2(18s) = 0 &\qquad \rightarrow 3w = 48s &\qquad \rightarrow w = 16s \end{aligned}$$

Thus $(w, x, y, z) = (0, 0, 0, 0) + s\langle 16, 24, 9, 18\rangle$ for all $s \in \mathbb{R}$

(c)
$$\begin{aligned} 2w + 3x - 5y - 2z &= 0 \\ x + 2y - 1z &= 0 \end{aligned}$$

Let $z = 4s$,
and $x = 4t$.
Then:

$$\begin{aligned} 4t + 2y - 4s = 0 &\qquad \rightarrow 2y = 4s - 4t &\qquad \rightarrow y = 2s - 2t \\ 2w + 3(4t) - 5(2s - 2t) - 2(4s) = 0 &\qquad \rightarrow 2w = 18s - 22t &\qquad \rightarrow w = 9s - 11t \end{aligned}$$

Thus $(w, x, y, z) = (0, 0, 0, 0) + s\langle 9, 0, 2, 4\rangle + t\langle -11, 4, -2, 0\rangle$ for all $s, t \in \mathbb{R}$.