Assignment 1

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Question 1

First let us define the payoff matrix:

$$U = \begin{pmatrix}
1,1 & 1,2 & 1,3 & \dots 1,49 & 1,50 & 1,51 & \dots & 1,98 & 1,99 & 1,99 \\
2,1 & 2,2 & 2,3 & \dots 2,49 & 2,50 & 2,51 & \dots & 2,98 & 2,98 & 2,98 \\
3,1 & 1,2 & 1,3 & \dots 3,49 & 3,50 & 3,51 & \dots & 3,97 & 3,97 & 3,97 \\
\vdots & \dots & \dots & \ddots & \vdots & \dots & \dots & \dots & \dots & \\
49,1 & 49,2 & 49,3 & \dots 49,49 & 49,50 & 49,51 & \dots & 49,51 & 49,51 & 49,51 \\
50,1 & 50,2 & 50,3 & \dots 50,49 & 50,50 & 50,50 & \dots & 50,50 & 50,50 & 50,50 \\
51,1 & 51,2 & 51,3 & \dots 51,49 & 50,50 & 50,50 & \dots & 51,49 & 51,49 & 51,49 \\
\vdots & \dots & \dots & \dots & \vdots & \dots & \ddots & \dots & \dots & \\
98,1 & 98,2 & 97,3 & \dots 51,49 & 50,50 & 49,51 & \dots & 50,50 & 98,2 & 98,2 \\
99,1 & 98,2 & 97,3 & \dots 51,49 & 50,50 & 49,51 & \dots & 2,98 & 50,50 & 99,1 \\
99,1 & 98,2 & 97,3 & \dots 51,49 & 50,50 & 49,51 & \dots & 2,98 & 1,99 & 50,50
\end{pmatrix}$$

Let us start with Player 1, It is obvious that row 2 dominates row 1, hence we can cross out row 1. It is also obvious that row 3 dominates row 2, hence we can cross out that one as well. We can continue with this till row 50 which dominates row 49. How ever row 51 does not dominated row 50. We can do the same for Player 2 with the columns till we reach a sub matrix of 51 by 51.

At this point we don't have any dominating strategies so we will look for the best action.
If player 1 chooses 100 player 2 best action is 99. If player 1 chooses 99 player 2 best action is 98 and so meaning if player 1 choose n player 2 best action would be $n-1$, this holds for $51 \leq n \leq 100$. the same for the player 2, for any action n player 1 best action is $n-1$ for $51 \leq n \leq 100$. From this we find that the only NE that are relevant in the 50 by 50 game is player one chooses 51 and player 2 chooses 51. Now let have a look at what happens if player 1 chooses 50, player 2 best action will be any action (they yield 50) and the other way around. This gives us that another NE is if player 1 chooses 50 and player 2 chooses 50.

To conclude, the NE's are $(50,50)$, $(51,51)$.

Question 2

Lets consider a direct auction where all the buyers including the seller (auctioneer) bid. The winning rule is that the highest bid wins, and the payment rule is that the highest bid pays the second highest bid (same as in the second price auction).
The difference between this auction and the second price auction is that if the highest bid belongs to the seller, non of the buyers will get the goods.
Lets prove that being truthful is a weakly dominant strategy: Suppose the for buyer i value is v_i, and he considers biding $b_i > v_i$ or $b_i < v_i$. Lets denote the highest bid $\hat{b}$ of the other players including the auctioneer/seller.

There are 6 possible outcomes:
$b_i > v_i > \hat{b}$: The buyer will get the goods, the profit is $v_i - \hat{b}$, and could have got the same result bidding $b_i = v_i$.
$v_i > b_i > \hat{b}$:The buyer will get the goods, the profit is $v_i - \hat{b}$, and could have got the same result bidding $b_i = v_i$.
$b_i > \hat{b} > v_i$: The buyer gets the goods but the the utility is negative, could have done better if he had bid $b_i = v_i$.
$v_i > \hat{b} > b_i$: The buyer did not get the goods and could have done better if he had bid v_i (would get a utility of $v_i - \hat{b}$ instead of 0)
$\hat{b} > v_i > b_i$: The buyer did not get the goods, would have done the same if had bid v_i.
$\hat{b} > b_i > v_i$: The buyer did not get the goods, would have done the same if had bid v_i.

To conclude the player does better if he bids his actual value v_i then to bid higher or lower. Till now we referred to the auctioneer as a player, and this is true unless his value is the highest, when this occurs (the seller has the highest bid) he gets the goods and no other player, so this does not affect the weakly dominating strategy of bidding truthfully. This type of auction satisfies all the demands and disputes the claim.

Question 3

First let us define the utility function

$$u(v_i, b_i) = \begin{cases} v_i - b_i & \text{if } b_i > max\{b_j\}; \\ \frac{v_i - b_i}{k} & \text{if k ties;} \\ 0 & \text{else} \end{cases}$$

We assume any v_i is drawn from f and has a CDF of F and is *iid*.
We also assume that the b_i is a function of v_i and that all the player use the same policy to maximize their gain. Meaning $b_i(v) = b_j(v)$ (for the same v).

The player wishes to maximize his gain hence he will play $b^*(v)$ that maximizes the following:

$$Pr(win|b^*(v))(v - b^*(v)) = Pr(b^*(v) > \max\{b_j(v_j)\})(v - b^*(v))$$

Because of the *iid* assumption we can say

$$\Pi\{Pr(b^*(v) > b_j(v_j)\}(v - b^*(v)) = \{Pr(b^*(v) > b(v))\}^{n-1}(v - b^*(v))$$

We will assume that $b(v)$ is an a non-decreasing function in v and we get

$$Pr(win|b^*(v))(v - b^*(v)) = \{F(b^{-1}(b^*(v)))\}^{n-1}(v - b^*(v))$$

Lets derive and compare to zero (assuming $F' = f$):

$$\frac{(n-1)f(b^{-1}(b^*(v)))\{F(b^{-1}(b^*(v)))\}^{n-2}}{b'(b^{-1}(b^*(v))}(v - b^*(v)) - \{F(b^{-1}(b^*(v)))\}^{n-1} = 0$$

We are looking for the equilibrium and hence $b^*(v) = b(v)$ so we can say the following:

$$\frac{F^{n-1}(v)}{b'(v)}(v - b(v)) - F^{n-1}(v)$$

Integration by parts will give us:

$$b^*(v) = v - \frac{\int_0^v F^{n-1}(x)dx}{F^{n-1}(v)}$$

Now we need to use the CDF that is given

$$F^{n-1}(x) = \begin{cases} 0 & \text{if } v < a; \\ (\frac{x-a}{b-a})^{n-1} & \text{if } a \le x \le b; \\ 1 & \text{else} \end{cases}$$

And after a bit of calculus we get:

For any v that satisfies $a \le v \le b$ we get

$$\int_0^v F^{n-1}(x)dx = \int_a^v F^{n-1}(x)dx$$

$$\frac{(b-a)}{n}(\frac{v-a}{b-a})^n - \frac{(b-a)}{n}(\frac{a-a}{b-a})^n = \frac{(b-a)}{n}(\frac{v-a}{b-a})^n =$$

$$= \frac{1}{n}\frac{(v-a)^n}{(b-a)^{n-1}}$$

$$\frac{\int_0^v F^{n-1}(x)dx}{F^{n-1}(v)} = \frac{(v-a)^n}{n(b-a)^{n-1}} \cdot (\frac{b-a}{v-a})^{n-1} = \frac{v-a}{n}$$

And we finally get:

$$b^*(v) = v - \frac{v-a}{n} = \frac{(n-1)v+a}{n}$$

Lets make a quick sanity check with $a = 0$ like we have seen in class:

$$b^*(v) = \frac{(n-1)}{n}v$$

yey!

Question 4

In this question the parameters are $n = 3$, $a = 5$, $b = 10$, $v = 8$. we get

$$b^*(v) = \frac{(n-1)v+a}{n} = \frac{(3-1)8+5}{3} = \frac{21}{3} = 7$$

Now let us calculate the expected utility:

$$\mathbb{E}_{v_2\sim U[5,10],v_3\sim U[5,10]}\left[v_1 - b_1\right]$$

$$= (v_1 - b_1)\int_5^{10}\int_5^{10}\mathbb{P}(b_1 > \frac{2v_2+5}{3}, b_1 > \frac{2v_2+5}{3})dv_2dv_3$$

Now lets calculate the the expected utility for the different bids. Lets start witht the equilibrium $b_1 = 7$:

$$(8-7)\int_5^{10}\int_5^{10}\mathbb{P}(7 > \frac{2v_2+5}{3})\mathbb{P}(7 > \frac{2v_2+5}{3})dv_2dv_3$$

$$= \int_5^{10}\mathbb{P}(v_2 < 8)\int_5^{10}\mathbb{P}(v_3 < 8) = \frac{3}{5}\frac{3}{5} = \frac{9}{25}$$

Now with $b_1 = 6$

$$(8-6)\int_5^{10}\int_5^{10}\mathbb{P}(6 > \frac{2v_2+5}{3})\mathbb{P}(6 > \frac{2v_2+5}{3})dv_2dv_3$$

$$= 2\int_5^{10}\mathbb{P}(v_2 < 6.5)\int_5^{10}\mathbb{P}(v_3 < 6.5) = 2\frac{3}{10}\frac{3}{10} = 2\frac{9}{100} = \frac{9}{50}$$

And now with $b_1 = 8$

$$(8-8)\int_5^{10}\int_5^{10}\mathbb{P}(8 > \frac{2v_2+5}{3})\mathbb{P}(8 > \frac{2v_2+5}{3})dv_2dv_3 = 0$$

It is clear that $b_1 = 7$ is the best bid.