Dueling bandits

immediate

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We define the set $T_p = \{2^{p-1}, \ldots, 2^p - 1\} = \{s \in \mathbb{N} : \lfloor \log_2 s \rfloor = p - 1\}$.

Algorithm 1: Improved Doubler

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initialization x_1 fixed in X, L = {x_1}, f̂_0 = 0;
t ← 1;
p ← 1;
while true do
    for j = 1 to 2^{p−1} do
        choose x_t uniformly from L;
        y_t ← advance(S);
        play (x_t; y_t), observe choice b_t;
        feedback(S; b_t + f̂_{p−1});
        t ← t + 1;
    L the multi-set of arms played as y_t in the last for-loop;
    f̂_p ← f̂_p + ∑_{s∈T_p} b_s/2^{p−1} − 1/2;
    p ← p + 1;
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Observe that if $t \in T_p$

$$\mathbb{E}\left[b_t \middle| \left\{y_s, s \in T_{p-1}\right\}, y_t\right] = \sum_{s \in T_{p-1}} \frac{\mu(y_t) - \mu(y_s) + 1}{2^{p-1}} = \frac{\mu(y_t) + 1}{2} - \sum_{s \in T_{p-1}} \frac{\mu(y_s)}{2^{p-1}},$$

and that

$$\mathbb{E}\left[\sum_{s \in T_{p-1}} b_s/2^{p-2} - 1/2 \middle| \bigcup_{r=p-2}^{p-1} \{y_s, s \in T_r\}\right] = \sum_{s \in T_{p-1}} \frac{\mu(y_s)}{2^{p-1}} - \sum_{s \in T_{p-2}} \frac{\mu(y_s)}{2^{p-2}}.$$

Let us denote $f_t = b_t + \hat{f}_{p-1} = b_t + \sum_{r=1}^{\lfloor \log_2 t \rfloor} \sum_{s \in T_r} b_s/2^{r-1} - \lfloor \log_2 t \rfloor/2$ the feedback that we introduce in S. Using the recurrence defining $\hat{f}_p$ we obtain

$$\mathbb{E}\left[f_t \middle| x_1, \bigcup_{r=1}^{p-1}\{y_s, s\in T_r\}, y_t\right] \quad = \quad \frac{\mu(y_t)-\mu(x_1)+1}{2} \; .$$

Since the above right term is $\sigma(x_1, y_t)$- measurable we conclude that

$$\mathbb{E}\left[f_t | x_1, y_t\right] \quad = \quad \frac{\mu(y_t)-\mu(x_1)+1}{2} \; .$$

Let $y_{t_1} = \ldots = y_{t_k}$ and let $f = \sum_{j=1}^{k} f_{t_j}/k$.
Observe that $f = \sum_{s=1}^{t_k} a_s b_s/t_k - \sum_{j=1}^{k} \log_2 t_j/2k$ with $a_s \in [0, A_s]$. We will later specify this bound. Since the $a_s b_s$ are independent and $\mathbb{P}[a_s b_s \in [0, A_s]] = 1$ we can apply the Hoeffding's inequality:

$$\mathbb{P}\left[f - \mathbb{E}[f] \geq \varepsilon \middle| x_1, y_{t_1} = \ldots = y_{t_k}\right] \leq \exp\left(-\frac{2t_k^2\varepsilon^2}{\sum_{s=1}^{t_k} A_s^2}\right)$$

Assume the convention $t_0 = 1$ and set $S_j = \{s \in \mathbb{N} : 2^{\lfloor \log_2 t_{j-1}\rfloor} \leq s < 2^{\lfloor \log_2 t_j \rfloor}\}$ for $j = 1, \ldots, k$ and $S_{k+1} = \{s \in \mathbb{N} : 2^{\lfloor \log_2 t_k \rfloor} \leq s \leq t_k\}$. For each $1 \leq j \leq k+1$, $s \in S_j$, $A_s = t_k((k-j+1)/2^{\lfloor \log_2 s\rfloor}+1)/k$ if $s = t_i$ for some $1 \leq i < j$ and $A_s = t_k(k-j+1)/(2^{\lfloor \log_2 s \rfloor}k)$ otherwise. We say that $t_j \in S_{t(j)}$. The function t is obviously non decreasing, $k+1 = t(k) \geq t(j) \geq j+1$.

$$\begin{aligned}
\frac{k^2}{t_k^2}\sum_{s=1}^{t_k} A_s^2 = \sum_{j=1}^{k+1}\sum_{s\in S_j}\frac{(k-j+1)^2}{2^{2\lfloor \log_2 s\rfloor}} + \sum_{j=1}^{k}\frac{k-t(j)+1}{2^{\lfloor \log_2 t_j\rfloor -1}} + k = \\
\sum_{j=1}^{k}(k-j+1)^2\left(\frac{1}{2^{\lfloor \log_2 t_{j-1}\rfloor-1}} - \frac{1}{2^{\lfloor \log_2 t_j\rfloor -1}}\right) + \sum_{j=1}^{k}\frac{k-t(j)+1}{2^{\lfloor \log_2 t_j\rfloor-1}} + k = \\
2k^2 - \sum_{j=1}^{k}\frac{2(k-j)+1}{2^{\lfloor \log_2 t_j\rfloor -1}} + \sum_{j=1}^{k}\frac{k-t(j)+1}{2^{\lfloor \log_2 t_j\rfloor-1}} + k \; .
\end{aligned}$$

This implies that

$$\begin{aligned}
3k^2 \geq 2k^2 - \sum_{j=1}^{k}\frac{k-j+1}{2^{\lfloor \log_2 t_j\rfloor-1}} + k \geq \frac{k^2}{t_k^2}\sum_{s=1}^{t_k} A_s^2 \geq \\
2k^2 - \sum_{j=1}^{k}\frac{2(k-j)+1}{2^{\lfloor \log_2 t_j\rfloor-1}} + k \geq 2k^2\left(1 - \frac{1}{2^{\lfloor \log_2 t_1\rfloor}}\right) + k \; .
\end{aligned}$$

To obtain the convergence of Improved Doubler using UCB at the same rate as UCB we need that $\sum_{s=1}^{t_k} A_s^2 = O(t_k^2/k)$, but we just showed that it is not possible.