Integrals in AGT conjecture

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1 Review of AFLT integral [1012.2929]

The Selberg integral,

$$I^{(N)}(\mathcal{O};u,v,\beta) := \frac{1}{N!}\int_{[0,1]^N} \mathcal{O}(x_1,\dots,x_N)\prod_{i=1}^{N} x_i^u(1-x_i)^v \prod_{i<j}|x_i-x_j|^{2\beta}\mathrm{d}x_1\cdots\mathrm{d}x_N. \tag{1.1}$$

The Selberg average,

$$\langle\mathcal{O}\rangle^{(N)} := \frac{I_N(\mathcal{O};u,v,\beta)}{I_N(1;u,v,\beta)}. \tag{1.2}$$

The AFLT integral,

$$I^{(N)}_{Y,W}(u,v,\beta) := I^{(N)}\Big(\mathbf{P}^{(1/\beta)}_Y[p_k]\mathbf{P}^{(1/\beta)}_W[p_k+(v-\beta+1)/\beta];u,v,\beta\Big). \tag{1.3}$$

For $y=(y_1,\dots,y_{N-1})$, $x=(x_1,\dots,x_N)$, satisfy the interlacing property,

$$x_1<y_1<x_2<y_2<\cdots<x_{N-1}<y_{N-1}<x_N \tag{1.4}$$

denoted by $y\prec x$.

$$\begin{aligned}
&\prod_{1\leqslant i<j\leqslant N}(x_j-x_i)^{2\beta-1}\mathbf{P}^{(1/\beta)}_Y(x_1,\dots,x_N)\\
=&\Lambda_Y(\beta)\times\int_{y\prec x}\mathbf{P}^{(1/\beta)}_Y(y_1,\dots,y_{N-1})\prod_{1\leqslant i<j\leqslant N-1}(y_j-y_i)\prod_{i=1}^{N-1}\prod_{j=1}^{N}|y_i-x_j|^{\beta-1}\mathrm{d}y_1\dots\mathrm{d}y_{N-1}.
\end{aligned} \tag{1.5}$$

For $x=(x_1,\dots,x_N)$, $y=(0,y_1,\dots,y_{N-1},1)$, satisfy the interlacing property,

$$0<x_1<y_1<x_2<y_2<\cdots<x_{N-1}<y_{N-1}<x_N<1 \tag{1.6}$$

denoted by $x\prec y$,

$$\begin{aligned}
&\int_{x\prec y}\mathbf{P}^{(1/\beta)}_Y[p_k(x)+(v+1-\beta)/\beta]\prod_{1\leqslant i<j\leqslant N}(x_j-x_i)\prod_{i=1}^{N}x_i^u(1-x_i)^v\prod_{i=1}^{N-1}\prod_{j=1}^{N}|y_i-x_j|^{\beta-1}\mathrm{d}x_1\cdots\mathrm{d}x_N\\
=&\Xi_Y(u,v,\beta)\prod_{i=1}^{N-1}y_i^{u+\beta}(1-y_i)^{v+\beta}\prod_{1\leqslant i<j\leqslant N-1}(y_j-y_i)^{2\beta-1}\mathbf{P}^{(1/\beta)}_Y[p_k(y)+(v+1)/\beta].
\end{aligned} \tag{1.7}$$

By using (B.2) then (1.5) then (1.7), we can get a recursive relation,

$$\begin{aligned}
I_{Y,W}^{(N)}(u,v,\beta) =&\Lambda_{Y^1}(\beta)\Xi_W(u+Y_N,v,\beta)I_{Y^1,W}^{(N-1)}(u+\beta,v+\beta,\beta)\\
=&\frac{[N\beta]_{Y^1}}{[(N-1)\beta]_{Y^1}}\frac{\Gamma(N\beta)\Gamma(u+Y_N+1)\Gamma(v+1)}{\Gamma(\beta)\Gamma(u+v+(N-1)\beta+Y_N+2)}\frac{(u+v+(N-2)\beta+Y_N+2,\beta)_W}{(u+v+(N-1)\beta+Y_N+2,\beta)_W}\\
&\times I_{Y^1,W}^{(N-1)}(u+\beta,v+\beta,\beta)\\
=&\frac{\mathbf{P}_Y^{(1/\beta)}[N]}{\mathbf{P}_{Y^1}^{(1/\beta)}[N-1]}\frac{\Gamma(N\beta)\Gamma(u+Y_N+1)\Gamma(v+1)}{\Gamma(\beta)\Gamma(u+v+(N-1)\beta+Y_N+2)}\frac{(u+v+(N-2)\beta+Y_N+2,\beta)_W}{(u+v+(N-1)\beta+Y_N+2,\beta)_W}\\
&\times I_{Y^1,W}^{(N-1)}(u+\beta,v+\beta,\beta).
\end{aligned} \tag{1.8}$$

In the first step, $N\to N-1,\ u\to u+\beta,\ v\to v+\beta,\ Y\to Y^1$. After N steps,

$$\begin{aligned}
I_{Y,W}^{(N)}(u,v,\beta) =&\mathbf{P}_Y^{(1/\beta)}[N]\mathbf{P}_W^{(1/\beta)}[N+(v-\beta+1)/\beta]\prod_{i=1}^{N}\frac{\Gamma(i\beta)\Gamma(u+Y_i+(N-i)\beta+1)\Gamma(v+(i-1)\beta+1)}{\Gamma(\beta)\Gamma(u+v+)}\\
&\times
\end{aligned} \tag{1.9}$$

A Conventions and notations

For a partition $Y=(Y_1,\dots,Y_N)$,

$$\begin{aligned}
Y^1 :=&(Y_1-Y_N,\dots,Y_{N-1}-Y_N,0)\\
Y^n :=&(Y^{n-1})^1,\ \text{ for } n\leqslant N.
\end{aligned} \tag{A.1}$$

Pochhammer symbol

$$(x)_N := \frac{\Gamma(x+N)}{\Gamma(x)}, \tag{A.2}$$

which can be generalized to be indexed by a partition,

$$(x;\beta)_Y := \prod_{i\geqslant 1}(x+(1-i)\beta)_{Y_i} = \prod_{(i,j)\in Y}(x+(1-i)\beta+(j-1)) \tag{A.3}$$

$$\Lambda_Y(\beta) := \frac{\Gamma(N\beta)}{\Gamma^N(\beta)}\frac{(N\beta,\beta)_Y}{((N-1)\beta,\beta)_Y}. \tag{A.4}$$

$$\Xi_Y(u,v,\beta) := \frac{\Gamma(u+1)\Gamma(v+1)\Gamma^{N-1}(\beta)}{\Gamma(u+v+(N-1)\beta+2)}\frac{(u+v+(N-2)\beta+2,\beta)_Y}{(u+v+(N-1)\beta+2,\beta)_Y} \tag{A.5}$$

B Properties of Jack polynomials

$$\mathbf{P}_Y^{1/\beta}(0,\dots,0,x_1,\dots,x_N) = \mathbf{P}_Y^{1/\beta}(x_1,\dots,x_N) \tag{B.1}$$

For a partition $Y=(Y_1,\dots,Y_N)$, the Jack polynomial,

$$\mathbf{P}_Y^{1/\beta}[p_k(x)] = (x_1\cdots x_N)^{Y_N}\mathbf{P}_{Y^1}^{1/\beta}[p_k(x)] \tag{B.2}$$

$$\frac{\mathbf{P}_Y^{(1/\beta)}[N]}{\mathbf{P}_{Y^1}^{(1/\beta)}[N-1]} = \frac{(N\beta,\beta)_{Y^1}}{((N-1)\beta,\beta)_{Y^1}} \tag{B.3}$$