Algebra For Physics Students

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Contents

1 Set theory

This section includes neccessary knowledge about the set theory. I will not talk anything about the ZFC axioms, or any axiom system of constructing sets. I assume that we already know what are sets and some basic operations, such as the union, the intersection, the complement, etc.

1.1 Sets

The first notion I want to mention is the **ordered pair**. For two elements a, b, the ordered pair (a, b) is defined as,

$$(a, b) := \{\{a, b\}, \{b\}\}. \tag{1.1}$$

We can see that $(a, b) \neq (b, a)$ if $a \neq b$, that's why it's called an ordered pair. We can easily generalize it to an ordered collection of n elements recursively as

$$(a_1, a_2, \ldots, a_n) := \{\{a_1, a_2, \ldots, a_n\}\} \cup (a_2, a_2, \ldots, a_n), \tag{1.2}$$

which is called a n-**tuple**. By this definition, we can see if we change any two nonequivalent elements, we will get another n-tuple. Thus this definition gives us an ordered object from the pure set theory. Now we define the **Cartesian product** of two sets X, Y as

$$X \times Y := \{(x, y) | x \in X, y \in Y\}. \tag{1.3}$$

And for n sets, we can also define

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) | x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n\}. \tag{1.4}$$

In particular, if the n sets are all the same, we write it as,

$$X^n \equiv X \times X \times \cdots \times X.$$

1.2 Relations

A **relation** in the set X is a subset of $X \times X$. For example, we can define the $<$ relation in the set $X = \{1, 2, 3\}$ as the subset

$$<_X = \{(1, 2), (1, 3), (2, 3)\}. \tag{1.5}$$

We usually denote $(x, y) \in \triangleright \subset X \times X$, for $\triangleright$ a relation in X, as $x \triangleright y$.

There is a special kind of relations called **equivalence relation**. We call a relation $\sim$ an equivalence relation in the set X if:

(1) $x \sim x, \forall x \in X$;

(2) $x \sim y \Rightarrow y \sim x$, $\forall x, y \in X$;

(3) $x \sim y$ and $y \sim z \Rightarrow x \sim z$, $\forall x, y, z \in X$.

If we define an equivalence relation in the set X, we call the subset

$$[x] := \{y \in X | y \sim x\}. \tag{1.6}$$

the equivalence class of x. We can prove if $x, y \in X$, $[x] \cap [y] = \emptyset$ or $[x] = [y]$. Then we can define the quotient set $X/\sim$ as the set of all equivalence classes of X.

1.3 Maps

A **map** from the set X to Y is a subset of $f \subset X \times Y$, such that

$$\forall x \in X, \exists! p \in f, \pi_X(p) = x. \tag{1.7}$$

Usually we denote the map f from X to Y, as $f : X \to Y$, and $(x, y) \in f$ as $f : x \mapsto y$ or $f(x) = y$. A map $f : X \to Y$ is said to be injective if:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \tag{1.8}$$

to be surjective if:

$$\forall y \in Y, \exists x \in X, \text{such that } f(x) = y. \tag{1.9}$$

A map is called bijective if it is both injective and surjective.

2 Vectors

2.1 Vector spaces

$\mathbb{F} = \mathbb{R}$ of $\mathbb{C}$.

2.2 Inner products

$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$

2.3 Linear maps

2.4 Tensors

A multilinear map is a map f from the Cartesian product of vector spaces $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ to $\mathcal{W}$, all vector spaces are over the same field $\mathbb{F}$, satisfying,

$$\begin{aligned} & f(\mathbf{v}_1, \mathbf{v}_2, \cdots, a\mathbf{v}_i + b\mathbf{v}'_i, \cdots, \mathbf{v}_n) \\ & = af(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_i, \cdots, \mathbf{v}_n) + bf(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}'_i, \cdots, \mathbf{v}_n) \end{aligned} \tag{2.1}$$

We have already know what are a vector space $\mathcal{V}$ and its dual space $\mathcal{V}^*$. the tensor of type (r, s) of vector space $\mathcal{V}$ as a multilinear mapping

$$T^r_s : \underbrace{\mathcal{V}^* \times \cdots \times \mathcal{V}^*}_{r \text{ times}} \times \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{s \text{ times}} \to \mathbb{F}. \tag{2.2}$$

The set containing all (r, s)-tensor is denoted as $\mathscr{T}^r_s$. The components of an (r, s)-tensor T^r_s are defined as,

$$T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} := T^r_s(\epsilon^{\mu_1}, \ldots, \epsilon^{\mu_r}, \mathbf{e}_{\nu_1}, \ldots, \mathbf{e}_{\nu_s}). \tag{2.3}$$

We can show

$$T^r_s = T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} \mathbf{e}_{\mu_1} \otimes \cdots \otimes \mathbf{e}_{\mu_r} \otimes \epsilon^{\nu_1} \otimes \cdots \otimes \epsilon^{\nu_s}. \tag{2.4}$$

Thus, sometimes we also consider the (r, s)-tensors as elements in the vector space

$$\underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{r \text{ times}} \otimes \underbrace{\mathcal{V}^* \otimes \cdots \otimes \mathcal{V}^*}_{s \text{ times}}. \tag{2.5}$$

2.5 Complexification

For a vector space $\mathcal{V}$ over $\mathbb{R}$, the complexification of it is $\mathcal{V} \otimes \mathbb{C}$, here $\mathbb{C}$ is 2-dimensional real vector space, The basis for this space is,

$$\mathbf{e}_j \otimes 1, \quad \mathbf{e}_j \otimes i \tag{2.6}$$

where $\{e_j\}_{j=1}^{\dim \mathcal{V}}$ is the set of basis of $\mathcal{V}$. We define,

$$a(\mathbf{v} \otimes b) = \mathbf{v} \otimes (ab), \quad a, b \in \mathbb{C}. \tag{2.7}$$

Now the basis of $\mathcal{V}^{\mathbb{C}} = \mathcal{V} \otimes \mathbb{C}$ becomes

$$\mathbf{e}_j \otimes 1 \tag{2.8}$$

since $\mathbf{e}_j \otimes 1$ and $\mathbf{e}_j \otimes i$ are linear dependent now and the scalar becomes $\mathbb{C}$. For a complex vector space $\mathcal{V}$, we can defind an anti-linear involusion $\sigma : \mathcal{V} \to \mathcal{V}$, satisfying

$$\sigma^2 = Id \tag{2.9}$$

which is the meaning of a involution. An anti-linear map f can also be fully determined by the behaviour of the basis. If we know $\mathbf{e}_j \mapsto f(\mathbf{e}_j)$, $j = 1, \ldots, n$, for any $\mathcal{V} \ni \mathbf{v} = a_i\mathbf{e}_1 + \cdots a_n\mathbf{e}_n$, we have

$$f(\mathbf{v}) = a_1^*\mathbf{e}_1 + \cdots + a_n^*\mathbf{e}_n. \tag{2.10}$$

The simplest example of antilinear involusion is the antilinear "identity" $Id(\mathbf{e}_j) = \mathbf{e}_j$. A non-trivial example in two-dimensional complex vector space is given by $\sigma(\mathbf{e}_1) = -\mathbf{e}_2, \sigma(\mathbf{e}_2) = -\mathbf{e}_1$. The subspace

$$\mathrm{Fix}(\sigma) := \{\mathcal{V} | \sigma(\mathbf{v}) = \mathbf{v}, \mathbf{v} \in \mathcal{V}\} \tag{2.11}$$

is

3 Operators

4 Algebras

An algebra A is a vector space over $\mathbb{F}$ with an additional binary operation, $\star : A \times A \to A$, satisfying

$$\begin{aligned}(x+y) \star z =& a \star z + y \star z \\ z \star (x+y) =& z \star a + z \star y \\ (ax) \star (by) =& (ab)(x \star y).\end{aligned} \tag{4.1}$$

for all $a, b \in \mathbb{F}$ and $x, y, z \in A$. We may add more axioms to get associative, unital, commutative algebra.

4.1 Exterior algebra

In subsection 2.4, we have defined tensors.

$$\omega = \frac{1}{p!}\sum_{i_1,\cdots,i_p}^{D} \omega_{i_1\cdots i_p} e^{i_1}\wedge\cdots\wedge e^{i_p} \tag{4.2}$$

For

$$\Omega = \omega\wedge\eta \tag{4.3}$$

$$\Omega_{i_1\cdots i_{p+q}} = \frac{(p+q)!}{p!q!}\omega_{i_1\cdots i_p}\eta_{i_{p+1}\cdots i_{p+q}} = \binom{p+q}{p}\omega_{i_1\cdots i_p}\eta_{i_{p+1}\cdots i_{p+q}} \tag{4.4}$$

The parameter $\binom{p+q}{p}$ can be understood as the Shuffle sum

$$\Omega_{i_1\cdots i_{p+q}} = \sum_{\sigma\in Sh(p+q)} (-1)^{|\sigma|}\omega_{i_{\sigma_1}\cdots i_{\sigma_p}}\eta_{i_{\sigma_{p+1}}\cdots i_{\sigma_{p+q}}} \tag{4.5}$$

4.2 Clifford algebra

4.3 Lie algebra

4.3.1 Complexification of the Lie algebra

For a complex Lie algebra $\mathfrak{g}$ with basis $\{T_k\}_{k=1}^d$ and structure constants f^{ijk} we can consider it as a real Lie algebra $\mathfrak{g}_\mathbb{R}$ with basis $\{T_k^{re}, T_k^{im}\}_{k=1}^d$, where T_k^{re}, T_k^{im} are not T_k, iT_k in general. But since they are in the same Lie algebra, we need the closure of Lie braket as,

$$[T_i^{re}, T_j^{re}] = f^{ijk}T_j^{re}, \quad [T_i^{re}, T_j^{im}] = f^{ijk}T_k^{im}, \quad [T_i^{im}, T_j^{im}] = -f^{ijk}T_k^{re}. \tag{4.6}$$

The right hand sides of these three equation are all well-defined.

4.3.2 Real forms of $\mathfrak{sl}(2;\mathbb{C})\oplus\mathfrak{sl}(2;\mathbb{C})$

The Cartan-Weyl basis of $\mathfrak{sl}(2,\mathbb{C})$ are

$$H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad E = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad F = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{4.7}$$

And for two $\mathfrak{sl}(2,\mathbb{C})$, we may use $\{H_L, E_L, F_L\}$ and $\{H_R, E_R, F_R\}$ respectively. Another often used basis are,

$$S_1 = \frac{1}{2}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}, \quad S_2 = \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad S_3 = \frac{1}{2}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} \tag{4.8}$$

which are just $S_k = \frac{\sigma_k}{2i}$.

$$\begin{aligned} S_1^{(L)} &= -\frac{i}{2}(E_L+F_L), \quad S_2^{(L)} = \frac{1}{2}(F_L-E_L), \quad S_3^{(L)} = -\frac{i}{2}H_L\\ S_1^{(R)} &= -\frac{i}{2}(E_R+F_R), \quad S_2^{(R)} = \frac{1}{2}(F_R-E_R), \quad S_3^{(R)} = -\frac{i}{2}H_R. \end{aligned} \tag{4.9}$$

If

$$\sigma(X,Y) = (-X^\dagger, -Y^\dagger), \tag{4.10}$$

For $\sigma(X,Y) = (X,Y)$, we have two such equations

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \begin{pmatrix} -\alpha^* & -\gamma^* \\ -\beta^* & \alpha^* \end{pmatrix} \tag{4.11}$$

The general solution is,

$$\frac{a}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} + \frac{b}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{c}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \equiv aJ_1 + bJ_2 + cJ_3 \tag{4.12}$$

for $a, b, c \in \mathbb{R}$. We can see,

$$[J_i, J_j] = \epsilon_{ijk} J_k. \tag{4.13}$$

Thus we have the direct sum of two independent $\mathfrak{su}(2)$,

$$\mathrm{Fix}(\sigma) = \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{4.14}$$

I write down the basis relation.

$$J_k^{(L)} = S_k^{(L)}, \quad J_k^{(R)} = S_k^{(R)} \tag{4.15}$$

If

$$\sigma(X,Y) = (\bar{X}, \bar{Y}) \tag{4.16}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & -\alpha^* \end{pmatrix} \tag{4.17}$$

The solution is just

$$a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.18}$$

which represents the 2-dimensional traceless matrix, the genertors of the group $SL(2,\mathbb{R})$. Thus we have the direct sum of two independent $\mathfrak{sl}(2,\mathbb{R})$,

$$\mathrm{Fix}(\sigma) = \mathfrak{sl}(2;\mathbb{R}) \oplus \mathfrak{sl}(2;\mathbb{R}) \tag{4.19}$$

If

$$\sigma(X,Y) = (\bar{Y}, \bar{X}) \tag{4.20}$$

we have

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} \oplus \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & -\alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_2^* & \beta_2^* \\ \gamma_2^* & -\alpha_2^* \end{pmatrix} \oplus \begin{pmatrix} \alpha_1^* & \beta_1^* \\ \gamma_1^* & -\alpha_1^* \end{pmatrix}, \tag{4.21}$$

the general solution is

$$\begin{gathered} a_0(H_L + H_R) - a_1(iH_L - iH_R) + b_0(E_L + E_R) - b_1(iE_L - iE_R) + \\ c_0(F_L + F_R) - c_1(iF_L - iF_R) \in \mathrm{Fix}(\sigma). \end{gathered} \tag{4.22}$$

We define

$$H^{re}, E^{re}, F^{re} := H_L + H_R, E_L + E_R, F_L + F_R \tag{4.23}$$

$$H^{im}, E^{im}, F^{im} := iH_R - iH_L, iE_R - iE_L, iF_R - iF_L. \tag{4.24}$$

We can show this new set of basis satisfying condition (4.6) of $\mathfrak{sl}(2, \mathbb{C})$, thus

$$\mathrm{Fix}(\sigma) = \mathfrak{sl}(2; \mathbb{C})_{\mathbb{R}} \tag{4.25}$$

which is a 6-dimensional real Lie algebra. We often use another basis of $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$,

$$\begin{aligned} J_1 &= -\frac{i}{2}(E^{re} + F^{re}), \quad J_2 = \frac{1}{2}(F^{re} - E^{re}), \quad J_3 = -\frac{i}{2}H^{re} \\ K_1 &= -\frac{i}{2}(E^{im} + F^{im}), \quad K_2 = \frac{1}{2}(F^{im} - E^{im}), \quad K_3 = -\frac{i}{2}H^{im}. \end{aligned} \tag{4.26}$$

Expressed in the conventions here, J_i is S_i^{re} and K_i is S_i^{im}. The commutation relation is given by,

$$\begin{aligned} [J_a, J_b] &= i\epsilon_{abc}J_c, \\ [J_a, K_b] &= i\epsilon_{abc}K_c, \\ [K_a, K_b] &= -i\epsilon_{abc}J_c, \end{aligned} \tag{4.27}$$

which is just the commutation relation of $\mathfrak{so}(1, 3)$. So $\mathfrak{so}(1, 3) \cong \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$.