Reading Notes of the Mirror Symmetry

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Contents

1 Ch9. QFT in $d = 0$

The fields X are maps from the spacetime manifold, a point for $d = 0$, to reals $X : M \to \mathbb{R}$. The partition function,

$$Z \equiv \int dX e^{-S[X]}. \tag{1.1}$$

The correlation function,

$$\langle f(X) \rangle \equiv \frac{1}{Z} \int dX f(X) e^{-S[X]}. \tag{1.2}$$

For Lagrangian,

$$S(X_i, M) = \frac{1}{2} X^i M_{ij} X^j, \tag{1.3}$$

The partition function,

$$Z(M) = \int \prod_i dX^i e^{-\frac{1}{2} X^i M_{ij} X^j} = \frac{(2\pi)^{N/2}}{\sqrt{\det(M)}}. \tag{1.4}$$

1.1 Fermions

The Grassmann variables ψ^a,

$$X^i \psi^a = \psi^a X^i,\ \psi^a \psi^b = -\psi^b \psi^a. \tag{1.5}$$

The rules of integration over Grassmann variables are defined by,

$$\int d\psi = 0,\ \int \psi d\psi = 1. \tag{1.6}$$

We have,

$$\int \psi^1 \cdots \psi^n d\psi^1 \cdots d\psi^n = 1 \tag{1.7}$$

The partition functions for theories containing both bosonic fields and fermionic fileds are,

$$Z = \int \prod_i dX^i \prod_a d\psi^a e^{-S(X,\psi)}. \tag{1.8}$$

For the action,

$$S(\psi) = \frac{1}{2} \psi^i M_{ij} \psi^j, \tag{1.9}$$

the partition function is,

$$Z = \int \prod_k d\psi^k e^{-\frac{1}{2} \psi^i M_{ij} \psi^j} = \mathrm{Pf}(M) \tag{1.10}$$

The most general action of one bosonic variable and two fermionic variables,

$$S(X,\psi^1,\psi^2) = S_0(X) - \psi^1\psi^2 S_1(X). \tag{1.11}$$

The partition function (the result)

$$Z = \int dX e^{-S_0} S_1(X) \tag{1.12}$$

For a special choice of S_0 and S_1,

$$S_0(X) = \frac{1}{2}(\partial h)^2 \text{ and } S_1(X) = \partial^2 h. \tag{1.13}$$

where $\partial h \equiv \frac{\partial h}{\partial X}$ for a real function h of X. I write the action explicitly for further convenience,

$$S(X,\psi^1,\psi^2) = \frac{1}{2}(\partial h)^2 - \psi^1\psi^2\partial^2 h. \tag{1.14}$$

The above theory has supersymmetry, which means the action is invariant under the transformations parameterized by ϵ^i, $i=1,2$ (though expressed as ϵ, not infinitesimal),

$$\begin{aligned}\delta X =&\epsilon^1\psi^1 + \epsilon^2\psi^2,\\ \delta\psi_1 =&\epsilon^2\partial h,\\ \delta\psi_2 =&-\epsilon^1\partial h.\end{aligned} \tag{1.15}$$

Here ϵ^i and ψ_i are Grassmann odd variables. We have,

$$\delta(\partial h) = \frac{\partial(\partial h)}{\partial X}\delta X = \partial^2 h\delta X = \partial^2 h(\epsilon^1\psi^1 + \epsilon^2\psi^2) \tag{1.16}$$

and

$$\delta X\psi_1\psi_2 = 0. \tag{1.17}$$

We can prove,

$$\begin{aligned}\delta S =&\partial h\delta(\partial h) - \delta\psi_1\psi_2\partial^2 h - \psi_1\delta\psi_2\partial^2 h\\ =&\partial h\partial^2 h(\epsilon^1\psi_1 + \epsilon^2\psi_2) - \epsilon^2\partial h\psi_2\partial^2 h + \psi_1\epsilon^1\partial h\partial^2 h\\ =&0\end{aligned} \tag{1.18}$$

1.2 Localization

We choose supersymmetry transformation to set one of the fermions in action to be zero. For action (1.11), we can choose $\epsilon^1 = \epsilon^2 = -\psi_1/\partial h$, if $\partial h \neq 0$.

The action is invariant, which means,

$$S(X,\psi_1,\psi_2) = S\Big(X - \frac{\psi_1\psi_2}{\partial h(X)}, 0, \psi_1+\psi_2\Big). \tag{1.19}$$

Define new variables,

$$\begin{aligned}\widehat{X} \equiv& X - \frac{\psi_1\psi_2}{\partial h(X)}\\ \widehat{\psi}_1 \equiv& \alpha(X)\psi_1\\ \widehat{\psi}_2 \equiv& \psi_1+\psi_2,\end{aligned} \tag{1.20}$$

where $\alpha(X)$ is an arbitrary function of X. And action in new variables is $S(\hat{X},0,\hat{\psi}_2)$. The measure,

$$dXd\psi_1 d\psi_2 = \Big(\alpha(\widehat{X}) - \frac{\partial^2 h(\widehat{X})}{(\partial h(\widehat{X}))^2}\widehat{\psi}_1\widehat{\psi}_2\Big)d\widehat{X}d\widehat{\psi}_1\widehat{\psi}_2. \tag{1.21}$$

We can see,

$$
\begin{aligned}
Z =& \int e^{-S(X,\psi_1,\psi_2)} dX d\psi_1 d\psi_2 \\
=& \int d\widehat{\psi}_1 (\text{no } \hat{\psi}_1 \text{ term}) - \int e^{-S(\hat{X},0,\hat{\psi}_2)} \frac{\partial^2 h(\widehat{X})}{(\partial h(\widehat{X}))^2} \widehat{\psi}_1 \widehat{\psi}_2 d\hat{X} d\widehat{\psi}_1 \widehat{\psi}_2 \\
=& 0 - \int e^{-\frac{1}{2}(\partial h(\hat{X}))^2} \frac{\partial^2 h(\widehat{X})}{(\partial h(\widehat{X}))^2} dX \\
=&
\end{aligned}
\tag{1.22}
$$

We can also consider the case when $\partial h = 0$ for some X_c. This result means the partition function only get contributions from the critical points of h. Near the critical point,

$$
h(X) = h(X_c) + \frac{\alpha_c}{2}(X - X_c)^2 + \cdots . \tag{1.23}
$$

The partition function localizes at the critical points, so we can consider the infinitesimal neiborhood of such points and keep only the leading terms.

$$
\begin{aligned}
Z =& \sum_{X_c} \int \frac{dX d\psi^1 d\psi^2}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha_c^2 (X-X_c)^2 + \alpha_c \psi^1 \psi^2} \\
=& \sum_{X_c} \int \frac{dX}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha_c^2 (X-X_c)^2} \\
=& \sum_{X_c} \frac{\alpha_c}{|\alpha_c|} = \sum_{X_c} \frac{h''(X_c)}{|h''(X_c)|}.
\end{aligned}
\tag{1.24}
$$

If h is a polynomial of X of odd order, $Z = 0$. And for even order, $Z = \pm 1$, the sign depending on whether the leading term in h is positive or negative.