1 RG Flow[1, 2]

Broadly speaking, physical theories are scale-dependent. As we move from theories at smaller scales to those at larger scales, a decouping process occurs, which means information from the smaller scales is effectively packaged and passed on as parameters in the theory at the larger scale. Here scale has the dimension of distance. Later it given in energy dimension.

In simple terms, renormalization is used to handle the decouping in quantum theories. Quantum theories are described by actions. Each scale admits many possible theories, and we use correlation function as the basis for determining whether two theories are connected. An action S_{eff} is called an effective action at the scale Λ_1 of S at the scale Λ if the correlation function derived form them are the same.

Now we can define a map in the "theory space" $R_{\Lambda_1\Lambda_2} : (S_1, \Lambda_1) \mapsto (S_2, \Lambda_2)$, where S_2 is the effective action at the scale Λ_2 of S_1 at the scale Λ_1. It is easy to see,

$$R_{\Lambda_1\Lambda_2} R_{\Lambda_2\Lambda_3} = R_{\Lambda_1\Lambda_3}, \tag{1}$$

and

$$R_{\Lambda\Lambda} = \text{identity map}, \tag{2}$$

which satisfies the definition of the semi-group. That's the reason for the name, renormalization "group". If we start from a theory at the scale Λ in the space of theories and following this map, we obtain a series of theories forming something like a flow, which is called the **RG flow**. In mathematical language,

$$t(S, \Lambda) = (R_{\Lambda,t\Lambda} S, t\Lambda), \qquad t \in \mathbb{R}^+. \tag{3}$$

The decouping process from the action S at the scale Λ to its effective Lagrangian $\mathcal{L}_{eff}$ at the scale Λ_1 can be described as,

$$e^{-S_{eff}(\phi)} = \int e^{-S(\phi+\eta)} \mathcal{D}\eta. \tag{4}$$

The integral is taken over all fields η whose Fourier transform is supported in $\Lambda_1 \leq |p| \leq \Lambda$.

2 β Fuction[3, 4]

2.1 Callan-Symanzik Equation

In general, equation (4) cannot guarantee the form of Lagrangian $\mathcal{L}$ and $\mathcal{L}_{eff}$ are identical. However, we can focus on theories where the Lagrangian structure remains the same, with only the parameters varying with the scales. In this case, studying how these parameters evolve with scale fully determines how the theory changes under scale changes. The behavior of there parameters is described by the **Callan-Symanzik equation**,

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+n\gamma(g)\right]G^{(n)}=0, \tag{5}$$

where $G^{(n)}$ is the n-point renormalized Green's function, μ is the scale, g is the coupling constant and,

$$\beta\equiv\mu\frac{\mathrm{d}g}{\mathrm{d}\mu}\quad\text{and}\quad\gamma=\frac{Z}{2\mu}\frac{\mathrm{d}Z}{\mathrm{d}\mu}, \tag{6}$$

where Z is the wave function renormalization factor for the field in this theory. For theories with more coupling constants or fields, more terms need to be added.

Next, I will compute the one-loop β-function for QCD.

2.2 QCD

The Lagrangian for Yang-Mills theory coupling with fermions is

$$\mathcal{L}=-\frac{1}{2}\operatorname{tr}\left(F^{\mu\nu}F_{\mu\nu}\right)-\frac{1}{2\xi}(\partial\cdot A^a)^2+\bar{c}^a(-\partial^\mu D_\mu^{ac})c^c+\bar{\psi}(i\not{D}-m)\psi, \tag{7}$$

where

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu, \tag{8}$$

and

$$D^{ac}_\mu=\delta^{ac}\partial_\mu+gf^{abc}A^b_\mu. \tag{9}$$

ξ is the gauge parameter, which will be chosen as 1 in the beta function computation part. After expanding,

$$\begin{aligned}\mathcal{L}=&-\frac{1}{4}(\partial_\mu A^a_{0\nu}-\partial_\nu A^a_{0\mu})^2-g_0f^{abc}(\partial_\mu A^a_{0\nu})A^{\mu b}_0A^{\nu c}_0-\frac{1}{2\xi}(\partial^\mu A^a_{0\mu})^2\\&-\frac{1}{4}g_0^2(f^{eab}A^{\mu a}_0A^{\nu b}_0)(f^{ecd}A^c_{0\mu}A^d_{0\nu})-\bar{c}^a_0\partial^2c^a_0-g_0\bar{c}^a_0f^{abc}\partial^\mu A^b_{0\mu}c^c_0\\&+\bar{\psi}_0(i\not{\partial}-m_0)\psi_0+g_0A^a_{0\mu}\bar{\psi}_0\gamma^\mu t^a\psi_0.\end{aligned} \tag{10}$$

The subscript 0 means they are bare quantities. Changing to renormalized quantities and doing the dimensional regularization ($d = 4 - \epsilon$),

$$\begin{aligned} A_0 &= Z_3^{1/2} A, \\ \psi_0 &= Z_2^{1/2} \psi, \\ c_0 &= Z_{2c}^{1/2} c, \\ g_0 &= Z_g g \mu^{-\epsilon/2}, \\ m_0 &= \frac{m_0}{m} m. \end{aligned} \tag{11}$$

$$\begin{aligned} \mathcal{L} = &- \frac{1}{4} Z_3 (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - g\mu^{-\epsilon/2} Z_g Z_3^{2/3} f^{abc} (\partial_\mu A_\nu^a) A^{\mu b} A^{\nu c} - \frac{1}{2\xi} Z_3 (\partial^\mu A_\mu^a)^2 \\ &- \frac{1}{4} g^2 \mu^{-\epsilon} Z_g^2 Z_3^2 (f^{eab} A^{\mu a} A^{\nu b})(f^{ecd} A_\mu^c A_\nu^d) - Z_{2c} \bar{c}^a \partial^2 c^a - g\mu^{-\epsilon/2} Z_g Z_{2c} (Z_3)^{1/2} \bar{c}^a f^{abc} \partial^\mu A_\mu^b c^c \\ &+ Z_2 \bar{\psi} (i\partial\!\!\!/ - \frac{m_0}{m} m) \psi + g\mu^{-\epsilon/2} Z_g Z_2 (Z_3)^{1/2} A_\mu^a \bar{\psi} \gamma^\mu t^a \psi. \end{aligned} \tag{12}$$

This Lagrangian can be splitted to the renormalized part and counter term. renormalized part is simple and I just write down the $\mathcal{L}_{ct}$,

$$\begin{aligned} \mathcal{L}_{ct} = &- \frac{1}{4} \delta_3 (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - g\mu^{-\epsilon/2} \delta_{A^3} f^{abc} (\partial_\mu A_\nu^a) A^{\mu b} A^{\nu c} - \frac{1}{2\xi} \delta_3 (\partial^\mu A_\mu^a)^2 \\ &- \frac{1}{4} g^2 \mu^{-\epsilon} \delta_{A^4} (f^{eab} A^{\mu a} A^{\nu b})(f^{ecd} A_\mu^c A_\nu^d) - \delta_{2c} \bar{c}^a \partial^2 c^a - g\mu^{-\epsilon/2} \delta_{1c} \bar{c}^a f^{abc} \partial^\mu A_\mu^b c^c \\ &+ \bar{\psi} (i\delta_2 \partial\!\!\!/ - \delta_m) \psi + g\mu^{-\epsilon/2} \delta_1 A_\mu^a \bar{\psi} \gamma^\mu t^a \psi. \end{aligned} \tag{13}$$

by comparing (12) and (13), we can get,

$$\begin{aligned} \delta_3 &= Z_3 - 1, \qquad \delta_{A^3} = Z_3^{2/3} Z_g - 1, \qquad \delta_{A^4} = Z_3^2 Z_1^2 - 1, \\ \delta_{2c} &= Z_{2c} - 1, \qquad \delta_{1c} = Z_{2c} (Z_3)^{1/2} Z_g - 1, \qquad \delta_2 = Z_2 - 1, \\ \delta_m &= Z_2 m_0 - m, \qquad \delta_1 = Z_2 (Z_3)^{1/2} Z_g - 1. \end{aligned} \tag{14}$$

All these couterterms can be computed by the standard perturbative method (Feynman diagrams). The last equation can be written as,

$$g_0 = g \frac{1 + \delta_1}{Z_2 (Z_3)^{1/2}} \mu^{\epsilon/2}. \tag{15}$$

Now we can get a simple formula for β function from the scale independence of bare quantities ($\frac{\mathrm{d}g_0}{\mathrm{d}\mu} = 0$),

$$\begin{aligned}\beta(g) =&\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g = g\left[\left(-\frac{\epsilon}{2}\right) - \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right)\right] + \cdots \\ =&g\left[\left(-\frac{\epsilon}{2}\right) - \mu\frac{\mathrm{d}g}{\mathrm{d}\mu}\frac{\partial}{\partial g}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right)\right] + \cdots \\ =&g\left[\left(-\frac{\epsilon}{2}\right) - \beta(g)\frac{\partial}{\partial g}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right)\right] + \cdots\end{aligned} \tag{16}$$

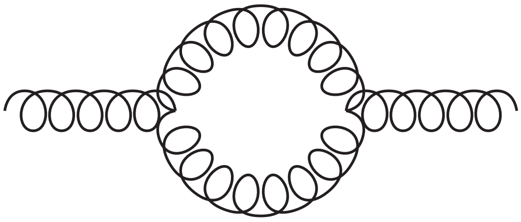
At one-loop, $\delta \propto g^2$. So if I only compute $\beta(g)$ up to g^3, the $\beta(g)$ in the right hand side can be taken as $-\frac{\epsilon}{2}g$, and the higher-order terms are thrown into the dots, i.e.

$$\beta(g) = -\frac{\epsilon}{2}g + \frac{\epsilon}{2}g^2\frac{\partial}{\partial g}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right) + \cdots \tag{17}$$

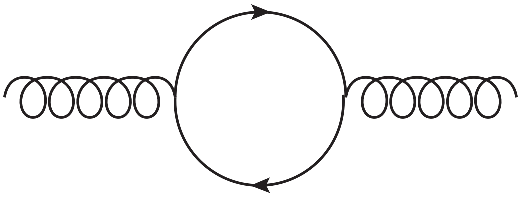
2.3 One-Loop β Function For QCD

Now I begin to compute the three couterterms δ_1, δ_2 and δ_3.

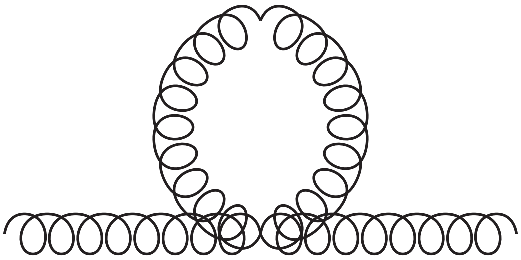
δ_3 is the counterterm for the gluon self-energy. At one-loop,



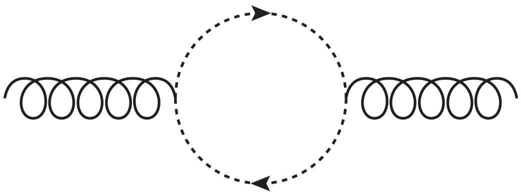
$= \mathcal{M}_3^{ab\mu\nu}$



$= \mathcal{M}_F^{ab\mu\nu}$



$= \mathcal{M}_4^{ab\mu\nu}$



$= \mathcal{M}_{gh}^{ab\mu\nu}$

and our counterterm δ_3. Since Feynman rules are not the focus here, I will omit them. Nevertheless, we can use them to get the expressions for these diagrams.

$$\begin{aligned}
& i\mathcal{M}_F^{ab\mu\nu} \\
&= -\operatorname{tr}\Big[T^a T^b\Big](ig)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2} \frac{i}{k^2 - m^2} \operatorname{tr}\Big[\gamma^\mu (\not{k} - \not{p} + m)\gamma^\nu (\not{k} + m)\Big] \\
&= -\frac{1}{2}\delta^{ab} g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{4[-p^\mu k^\nu - k^\mu p^\nu + 2k^\mu k^\nu + g^{\mu\nu}(-k^2 + p\cdot k + m^2)]}{[(p-k)^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon]}
\end{aligned} \tag{18}$$

We know Feynman parameters,

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}. \tag{19}$$

By using this,

$$\mathcal{M}_F^{ab\mu\nu} = 2i\delta^{ab}g^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{-p^\mu k^\nu - k^\mu p^\nu + 2k^\mu k^\nu + g^{\mu\nu}(-k^2 + p\cdot k + m^2)}{[k^2 + p^2x - 2k\cdot px - m^2 + i\epsilon]^2}. \tag{20}$$

We do not want to see the $k\cdot p$ term in the denominator, so we use $k'^\mu = k^\mu - xp^\mu$ to replace k. Since this is a full integration over momentum, this transformation will not affect the overall result of the integral.

$$\mathcal{M}_F^{ab\mu\nu} = 2i\delta^{ab}g^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{2k^\mu k^\nu - g^{\mu\nu}[k^2 - p^2x(1-x) - m^2] + 2x(x-1)p^\mu p^\nu}{[k^2 + p^2x(1-x) - m^2 + i\epsilon]^2}. \tag{21}$$

Here k means k'. I omit the pk'-form terms since they are the odd function of k and vanishing after integral. By using dimensional regularization and $k^\mu k^\nu = \frac{1}{d}g^{\mu\nu}k^2$ in d dimensions,

$$\mathcal{M}_F^{ab\mu\nu} = 2i\delta^{ab}g^2\mu^{4-d} \int \frac{\mathrm{d}^dk}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{-g^{\mu\nu}[(1-\frac{2}{d})k^2 - p^2x(1-x) - m^2] + 2x(x-1)p^\mu p^\nu}{[k^2 + p^2x(1-x) - m^2 + i\epsilon]^2}. \tag{22}$$

We have formula for this kind of integral,

$$\int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta + i\epsilon)^2} = -\frac{d}{2} \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Delta^{1-\frac{d}{2}}} \Gamma\Big(\frac{2-d}{2}\Big) \tag{23}$$

$$\int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta + i\epsilon)^2} = \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Delta^{2-\frac{d}{2}}} \Gamma\Big(\frac{4-d}{2}\Big) \tag{24}$$

Now we have,

$$
\begin{aligned}
\mathcal{M}_F^{ab\mu\nu} =& 2i\delta^{ab}g^2\mu^{4-d}\times \\
&\Bigg\{ - g^{\mu\nu}\Big(1-\frac{d}{2}\Big)\Gamma\Big(1-\frac{d}{2}\Big)\frac{i}{(4\pi)^{\frac{d}{2}}}\int_0^1 \mathrm{d}x\frac{1}{\Delta^{1-\frac{d}{2}}} + g^{\mu\nu}\Gamma\Big(2-\frac{d}{2}\Big)\frac{i}{(4\pi)^{\frac{d}{2}}}\int_0^1 \mathrm{d}x\frac{p^2x(1-x)}{\Delta^{2-\frac{d}{2}}} \\
&+ g^{\mu\nu}\Gamma\Big(2-\frac{d}{2}\Big)\frac{i}{(4\pi)^{\frac{d}{2}}}\int_0^1 \mathrm{d}x\frac{m^2}{\Delta^{2-\frac{d}{2}}} + p^\mu p^\nu\Gamma\Big(2-\frac{d}{2}\Big)\frac{i}{(4\pi)^{\frac{d}{2}}}\int_0^1 \mathrm{d}x\frac{x(x-1)}{\Delta^{2-\frac{d}{2}}}\Bigg\} \\
=& -2\delta^{ab}g^2\mu^{4-d}\Gamma\Big(2-\frac{d}{2}\Big)\frac{1}{(4\pi)^{\frac{d}{2}}}\times \\
&\Bigg\{g^{\mu\nu}\int_0^1 \mathrm{d}x\frac{-\Delta+p^2x(1-x)+m^2}{\Delta^{2-\frac{d}{2}}} + p^\mu p^\nu\int_0^1 \mathrm{d}x\frac{x(x-1)}{\Delta^{2-\frac{d}{2}}}\Bigg\} \\
=& -2\delta^{ab}g^2\mu^{4-d}\Gamma\Big(2-\frac{d}{2}\Big)\frac{1}{(4\pi)^{\frac{d}{2}}}(p^2g^{\mu\nu}-p^\mu p^\nu)\times\int_0^1 \mathrm{d}x\frac{2x(1-x)}{(m^2-p^2x(1-x))^{2-\frac{d}{2}}} \\
=&\frac{-\delta^{ab}g^2}{8\pi^2}\Gamma\Big(\frac{\epsilon}{2}\Big)(p^2g^{\mu\nu}-p^\mu p^\nu)\times\int_0^1 \mathrm{d}x\,2x(1-x)\Big(\frac{4\pi\mu^2}{m^2-p^2x(1-x)}\Big)^{\frac{\epsilon}{2}}
\end{aligned}
\tag{25}
$$

with $\Delta = m^2 - p^2x(1-x)$ and $\epsilon = 4-d$. In the limit $\epsilon \to 0$,

$$
\begin{aligned}
&\mathcal{M}_F^{ab\mu\nu} \\
=&\frac{-\delta^{ab}g^2}{8\pi^2}(p^2g^{\mu\nu}-p^\mu p^\nu)\Big(\frac{2}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\Big)\int_0^1 \mathrm{d}x\,2x(1-x)\Big[1+\frac{\epsilon}{2}\ln\Big(\frac{4\pi\mu^2}{m^2-p^2x(1-x)}\Big)+\mathcal{O}(\epsilon^2)\Big] \\
=&\frac{-\delta^{ab}g^2}{8\pi^2}(p^2g^{\mu\nu}-p^\mu p^\nu)\int_0^1 \mathrm{d}x\,2x(1-x)\Big[\frac{2}{\epsilon}+\ln\Big(\frac{4\pi e^{-\gamma_E}\mu^2}{m^2-p^2x(1-x)}\Big)+\mathcal{O}(\epsilon)\Big] \\
=&\frac{\delta^{ab}g^2}{(4\pi)^2}(p^2g^{\mu\nu}-p^\mu p^\nu)\Big[-\frac{4}{3}\frac{1}{\epsilon}+(finite)\Big],
\end{aligned}
\tag{26}
$$

where $x^{\epsilon/2} = 1+\frac{\epsilon}{2}\ln x+\cdots$ and $\Gamma(\epsilon/2) = \frac{2}{\epsilon}-\gamma_E+\cdots$ are used. If we have n_f species of fermions in the same representation, the total contribution will be,

$$
\frac{\delta^{ab}g^2}{(4\pi^2)^2}(p^2g^{\mu\nu}-p^\mu p^\nu)\Big[-\frac{4}{3}n_f\frac{1}{\epsilon}+(finite)\Big]. \tag{27}
$$

Now the contribution from the gluon bubble,

$$
i\mathcal{M}_3^{ab\mu\nu} = \frac{g^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-i}{k^2}\frac{-i}{(k-p)^2}f^{acd}f^{bdc}N^{\mu\nu}. \tag{28}
$$

The numerator is,

$$
\begin{aligned}
N^{\mu\nu} = & \left[g^{\mu\alpha}\left(p+k\right)^{\rho} + g^{\alpha\rho}\left(p-2k\right)^{\mu} + g^{\rho\mu}\left(k-2p\right)^{\alpha}\right] g_{\alpha\beta} g_{\rho\sigma} \\
& \times \left[g^{\nu\beta}(p+k)^{\sigma} - g^{\beta\sigma}(2k-p)^{\nu} - g^{\sigma\nu}(2p-k)^{\beta}\right].
\end{aligned} \tag{29}
$$

Also introducing Feynman parameters with $\Delta = x(x-1)p^2$ and using dimensional regularization, we can get,

$$
\begin{aligned}
\mathcal{M}_3^{ab\mu\nu} = & -\frac{g^2}{2}\frac{\mu^{4-d}}{(4\pi)^{d/2}}\delta^{ab}C_A\int_0^1 \mathrm{d}x\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \times \left\{g^{\mu\nu}3(d-1)\Gamma\left(1-\frac{d}{2}\right)\Delta\right. \\
& + p^{\mu}p^{\nu}\left[6\left(x^2-x+1\right) - d\left(1-2x\right)^2\right]\Gamma\left(2-\frac{d}{2}\right) \\
& \left. + g^{\mu\nu}p^2\left[\left(-2x^2+2x-5\right)\Gamma\left(2-\frac{d}{2}\right)\right]\right\}
\end{aligned} \tag{30}
$$

where C_A is defined by $f^{acd}f^{bcd} = C_A\delta^{ab}$.

Then the contribution from the four-point vertex, given by Feynman rule,

$$
\begin{aligned}
i\mathcal{M}_4^{ab\mu\nu} = & -\frac{ig^2}{2}\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-ig_{\rho\sigma}}{k^2+i\varepsilon} \\
& \times \left[f^{abe}f^{cde}\delta^{cd}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})\right. \\
& \left. + f^{ace}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\right] \\
= & -g^2\delta^{ab}g^{\mu\nu}C_A(d-1)\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2+i\varepsilon} = 0,
\end{aligned} \tag{31}
$$

Even though this integral is vanishing, I write it in the Feynman parameter for further computation,

$$
\begin{aligned}
\mathcal{M}_4^{ab\mu\nu} = & -g^2\delta^{ab}C_A\frac{\mu^{4-d}}{(4\pi)^{d/2}}g^{\mu\nu}\int_0^1\mathrm{d}x\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}(d-1) \\
& \times\left[-\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)\Delta + (1-x)^2p^2\Gamma\left(2-\frac{d}{2}\right)\right],
\end{aligned} \tag{32}
$$

where $\Delta = x(x-1)p^2$ as well.

The final contribution to the gluon self-energy is the ghost bubble,

$$
i\mathcal{M}_{gh}^{ab\mu\nu} = -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{(k-p)^2}\frac{i}{k^2}f^{cad}k^{\mu}f^{dbc}(k-p)^{\nu}, \tag{33}
$$

again,

$$\mathcal{M}_{\text{gh}}^{ab\mu\nu} = g^2 \frac{\mu^{4-d}}{(4\pi)^{d/2}} \delta^{ab} C_A \int_0^1 \mathrm{d}x \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \left\{ g^{\mu\nu} \left[\frac{1}{2}\Gamma\left(1-\frac{d}{2}\right)\Delta\right] \right.$$
$$\left. + p^\mu p^\nu \left[x(1-x)\Gamma\left(2-\frac{d}{2}\right)\right]\right\}. \tag{34}$$

Also using the $\epsilon = 4-d$ and in the $\epsilon \to 0$ limit,

$$\mathcal{M}_3^{ab\mu\nu} + \mathcal{M}_4^{ab\mu\nu} + \mathcal{M}_{gh}^{ab\mu\nu} = \frac{\delta^{ab} g^2}{(4\pi)^2}(p^2 g^{\mu\nu} - p^\mu p^\nu)\left[\frac{10}{3} C_A \frac{1}{\epsilon} + (finite)\right]. \tag{35}$$

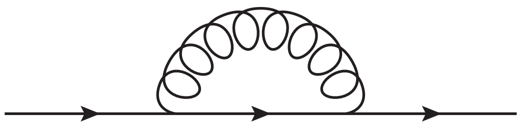
The full gluon self energy with counterterm is,

$$\mathcal{M}^{ab\mu\nu} = \delta^{ab}(p^2 g^{\mu\nu} - p^\mu p^\nu)\left[\frac{g^2}{16\pi^2}\left(\frac{10}{3}C_A - \frac{4}{3}n_f\right)\frac{1}{\epsilon} - \delta_3\right] + (finite) \tag{36}$$

Now we can get the value of counterterm δ_3,

$$\delta_3 = \frac{1}{\epsilon}\frac{g^2}{16\pi^2}\left(\frac{10}{3}C_A - \frac{4}{3}n_f\right). \tag{37}$$

The one-loop quark self-energy diagram is



$$= i\,\Sigma_2^{ij}(\gamma^\mu p_\mu)$$

The Feynman rule gives,

$$i\Sigma_2^{ij}(\not{p}) = (t^a)^{il}(t^a)^{lj}(ig)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k}+m)}{k^2 - m^2 + i\epsilon} \gamma_\mu \frac{-i}{(k-p)^2 + i\epsilon}. \tag{38}$$

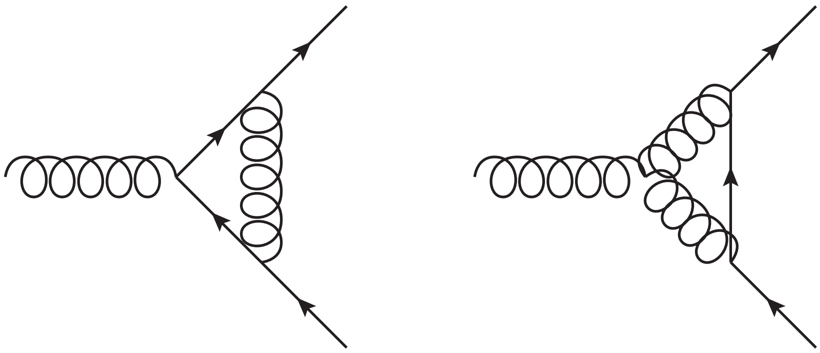
Adding counterterms δ_2, δ_m and computing this integral,

$$\Sigma_2^{ij}(\not{p}) = \delta^{ij}\left\{\frac{g^2}{16\pi^2}C_F\left(\frac{2\not{p} - 8m}{\varepsilon}\right) + \delta_2\not{p} - (\delta_m + \delta_2)m\right\} + (finite), \tag{39}$$

where C_F is defined by $(t^a)_{il}(t^a)_{lj} = C_F \delta_{ij}$, which is $C_F = \frac{N^2-1}{2N}$ for $SU(N)$-gauge theory. Now we can get the counterterm δ_2,

$$\delta_2 = \frac{1}{\epsilon}\frac{g^2}{16\pi^2}(-2C_F). \tag{40}$$

Finally, the counterterm δ_1 is for the gluon-fermion interaction, which has two diagrams in one-loop level,



After computation, the counterterm δ_1 is,

$$\delta_1 = \frac{1}{\epsilon}\frac{g^2}{16\pi^2}(-2C_F - 2C_A). \tag{41}$$

After getting all counterterms we need in equation (17), the beta-function at one-loop is,

$$\begin{aligned}\beta(g) &= -\frac{\epsilon}{2}g + \frac{1}{2}g^2\frac{\partial}{\partial g}\left[\left(-\frac{11}{3}C_A + \frac{2}{3}n_f\right)\frac{g^2}{16\pi^2}\right] \\ &= -\frac{\epsilon}{2}g - \frac{g^3}{16\pi}\left[\frac{11}{3}C_A - \frac{2}{3}n_f\right].\end{aligned} \tag{42}$$

The Casimir of the adjoint representation $C_A = N$ for $SU(N)$-gauge theory. In QCD case, $N = 3$ and the flavour number $n_f = 6$,

$$\beta_{\mathrm{QCD}}(g) = -\frac{\epsilon}{2}g - \frac{7g^3}{16\pi^2} < 0 \tag{43}$$

According to the definition of the beta function, when it is negative, it indicates that the coupling constant decreases at high energy. This is known as asymptotic freedom.

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