Topics in Quantum Field Theories

Qing-Jie Yuan

December 23, 2025

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Chapter 1

Spinors

1.1 Motivation

In the universe, there exist particles (fermions) that, under a spacetime transformation which leaves the spacetime invariant, may acquire a factor of -1. From the viewpoint of group representations, this seems impossible. But in physics, we actually work with the (unitary) projective representation, which is a map $\rho : G \to U(\mathcal{H})$, here G is a connected Lie group, satisfying,

$$\rho(g_1)\rho(g_2) = e^{i\phi(g_1,g_2)}\rho(g_1 g_2), \tag{1.1.1}$$

where $e^{i\phi(g_1,g_2)} \in U(1)$, and the associativity condition

$$\rho(g_1)\big(\rho(g_2)\rho(g_3)\big) = \big(\rho(g_1)\rho(g_2)\big)\rho(g_3). \tag{1.1.2}$$

Two projective representations $\tilde{\rho}, \rho$ are said to be equivalent[1] if

$$\tilde{\rho}(g) = e^{i\alpha(g)}\rho(g). \tag{1.1.3}$$

It is easy to see non-equivalent projective representations are classified by the second cohomology group $H^2(G, U(1))$. However, projective representations is hard to use. We prefer linear representations, which is well-established and familiar to physicists.

Here we use a result (I think it is only true for specific cases. At least it is true

[1] If you don't know the definitions of the equivalence relation, see my algebra notes.

for the cases we need.),

$$H^2(G, U(1)) \cong \mathrm{Hom}(\pi_1(G), U(1)), \tag{1.1.4}$$

which tells us the non-trivial projective representations are encodes in its fundamental group. The fundamental group determines the universal covering of G. Therefore, we can consider the universal covering group $\tilde{G}$ of G, which may be got by the central extension:

$$1 \to \pi_1(G) \to \tilde{G} \to G \to 1. \tag{1.1.5}$$

I cannot make sure this procedure is always feasible. But for the Lie groups we need, we can do this. Now what was originally represented through the fundamental group can now be handled through the center of the simply connected group. We can say: fermions are particles that can detect the topological structure of the spacetime isometry group.

A spinor is simply an element of the vector space that carries the spinor representation, and this representation is a representation of the universal covering group. In general, the spinor representation does not necessarily have non-identity matrix corresponding to the center. The essential point is that, once we have the simply connected Lie group, we can identify those representations whose action of the center is nontrivial. We have some results to know this: for compact semisimple Lie groups, given their Lie algebra and the Cartan-Weyl classification, the irreducible representations with nontrivial action on the center correspond to those whose highest weights lie outside the root lattice.

For a intuitive understanding, we can think of an element of a Lie group as a path in the group manifold starting from the identity. From the standpoint of the Lie group itself, a group element is independent of the specific path: only the endpoint matters. Different loops from and to the identity are indistinguishable in a linear representation. But fermions can. Why?

Anyone familiar with homotopy theory will immediately suspect that this must be related to the fundamental group of the Lie group, since the fundamental group precisely classifies different loops. This is indeed the case. Further, since the effect of such a transformation on a fermion is either $+1$ or -1. the fundamental group should be $\mathbb{Z}_2$ (one might even speculate that if the fundamental group were larger, the universe would contain more kinds of particles!).

For the spacetimes most commonly used in physics, n-dimensional Euclidean and Minkowski spaces, the corresponding isometry (sub)groups are $SO(n)$ and $SO^+(1, n-1)$ respectively. We know,

$$\pi_1(SO(n)) \cong \begin{cases} \mathbb{Z}_2 & , \quad n \geqslant 3, \\ \mathbb{Z} & , \quad n = 2, \\ \{e\} & , \quad n = 1. \end{cases} \tag{1.1.6}$$

The maximal compact subgroup of $SO^+(p, q)$ is $SO(p) \times SO(q)$, which gives the same homotopy groups:

$$\pi_1(SO^+(p,q)) = \pi_1(SO(p) \times SO(q)) = \pi_1(SO(p)) \times \pi_1(SO(q)) \tag{1.1.7}$$

Thus,

$$\pi_1(SO^+(1, n-1)) \cong \begin{cases} \mathbb{Z}_2 & , \quad n \geqslant 4, \\ \mathbb{Z} & , \quad n = 3 \\ \{e\} & , \quad n = 2. \end{cases} \tag{1.1.8}$$

Most cases that we care are indeed $\mathbb{Z}_2$. As a side note, it is worth noting the group $SO^+(1, 2)$ has a center $\mathbb{Z}$, which is why anyons can appear in $(2+1)$-dimensional systems.

In what follows, we will explicitly compute the cases of $SO(4)$ and $SO^+(1, 3)$. The logic for other dimensions is completely analogous: one finds representations of the universal covering of the corresponding isometry group. The more systematic way to handle $\mathbb{Z}_2$ extension is through the Clifford algebra. After that, we will introduce the spinor bundle and study how to construct spinors in curved spacetime, followed by a discussion of the Dirac operator.

1.2 Spinors in Four-dimensional Flat Spacetime

In physics, we are mostly considering the 4-dimensional flat spacetime, Euclidean (usually got by wick rotation) or Minkowski. And spinors in this kind of manifolds can be defined directly. We will talk about both in this section.

1.2.1 Euclidean Space

The metric for four-dimensional Euclidean spacetime is given by

$$g_{\mu\nu} = \delta_{\mu\nu} \equiv \begin{cases} 1, & \mu = \nu \\ 0, & \text{otherwise.} \end{cases} \tag{1.2.1}$$

The isometric subgroup we want to discuss here is $SO(4)$. Following the logic in section 1, we want to find the representations of its simply connected covering group $Spin(4)$. If you don't know the spin group, just consider it as an abstract notion for the simply connected covering group of special orthogonal group. We don't need the details of it in this section. And I will give the definition in the next section. We find their Lie algebra firstly. Let $J_{1,2,3}$ be the generators corresponding to the rotations in the $(x_2, x_3).(x_3, x_1)$ and (x_1, x_2) respectively, and $K_{1,2,3}$ the generators to the rotations in $(x_1, x_4), (x_2, x_4)$ and (x_3, x_4) respectively. The commutation relations are:

$$\begin{aligned} [J_a, J_b] =& i\epsilon_{abc} J_c, \\ [J_a, K_b] =& i\epsilon_{abc} K_c, \\ [K_a, K_b] =& i\epsilon_{abc} J_c, \end{aligned} \tag{1.2.2}$$

for $a, b, c = 1, 2, 3$. In mathematical context, the structure constants are usually real. The difference is, $T^a_{phy} = iT^a_{math}$, and therefore the expotential map is different,

$$\begin{aligned} \text{Physics: } & \exp\Big(-i\theta^a T^a_{phy}\Big) \\ \text{Mathematics: } & \exp(\theta^a T^a_{math}). \end{aligned} \tag{1.2.3}$$

We can define,

$$S_a^{(L)} = \frac{1}{2}(J_a + K_a) \text{ and } S_a^{(R)} = \frac{1}{2}(J_a - K_a). \tag{1.2.4}$$

Their commutation relations are given by:

$$\begin{aligned} [S_a^{(L)}, S_b^{(L)}] =& i\epsilon_{abc} S_c^{(L)}, \\ [S_a^{(R)}, S_b^{(R)}] =& i\epsilon_{abc} S_c^{(R)}, \\ [S_a^{(L)}, S_b^{(R)}] =& 0, \end{aligned} \tag{1.2.5}$$

which means

$$\mathfrak{spin}(4) \cong \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{1.2.6}$$

We know the irreducible representation of $\mathfrak{su}(2)$ is characterized by a half-integer $s = \frac{1}{2}, 1, \frac{3}{2}, ...,$

$$\rho^{(s)} : \mathfrak{su}(2) \to \mathfrak{gl}(2s+1, \mathbb{R}). \tag{1.2.7}$$

Here I give a short review to show how to get an irreducible representation of the direct sum of two Lie algebras. For Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, we can write the element in $\mathfrak{g}$ as $(X_1, X_2), X_1 \in \mathfrak{g}_1, X_2 \in \mathfrak{g}_2$, the ordered pairs in Cartensian product; or, as $X_1 + X_2$ since they are in different algebras (vector spaces), which will not lead ambiguity. Both $\mathfrak{g}_1, \mathfrak{g}_2$ equip representations,

$$\rho_1 : \mathfrak{g}_1 \to \mathfrak{gl}(V_1) \text{ and } \rho_2 : \mathfrak{g}_2 \to \mathfrak{gl}(V_2). \tag{1.2.8}$$

From these, we can define a representation of $\mathfrak{g}$ on $V_1 \otimes V_2$ by the action on the basis as,

$$\rho(X_1, X_2) \cdot (\mathbf{e}^{(1)} \otimes \mathbf{e}^{(2)}) := (\rho_1(X_1) \cdot \mathbf{e}^{(1)}) \otimes (\rho_2(X_2) \cdot \mathbf{e}^{(2)}) \tag{1.2.9}$$

Make sure you fully understand what I have written here. If you are not familiar with the definitions of the direct sum and tensor product of vector spaces, you can refer to my linear algebra notes. Thus we can define,

$$\rho^{(s_L, s_R)} : \mathfrak{so}(4) = \mathfrak{su}_L(2) \oplus \mathfrak{su}_R(2) \to \mathfrak{gl}(\mathbb{R}^{2s_L+1} \otimes \mathbb{R}^{2s_R+1}). \tag{1.2.10}$$

The representation of a Lie algebra is one-to-one correspondence with its simply connected global form, thus we can get the representations of

$$Spin(4) \cong SU(2) \times SU(2). \tag{1.2.11}$$

1.2.2 Minkowski Space

The metric for Minkowski spacetime is given by

$$g_{\mu\nu} = \eta_{\mu\nu} \equiv \begin{cases} 1 & , \quad \mu = \nu = 0 \\ -1 & , \quad \mu = \nu = 1, 2, \ldots, n-1 \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{1.2.12}$$

The isometry group component in the four-dimensional space we want to discuss is $SO^+(1,3)$. Then, physicists always write down the commutation relations of its Lie algebra $\mathfrak{so}^+(1,3)$,

$$\begin{aligned} [J_a, J_b] =& i\epsilon_{abc} J_c, \\ [J_a, K_b] =& i\epsilon_{abc} K_c, \\ [K_a, K_b] =& -i\epsilon_{abc} J_c, \end{aligned} \tag{1.2.13}$$

then define a new set of basis as

$$S_a^{(L)} = \frac{1}{2}(J_a + iK_a), \quad S_a^{(R)} = \frac{1}{2}(J_a - iK_a), \tag{1.2.14}$$

and say that they are two $\mathfrak{su}(2)$s. Then all are the same. But it is not a strict treatment, since both $\mathfrak{so}(1,3)$ and $\mathfrak{su}(2)$ are real Lie algebra, how can we write down the complex linear combinations. To be strict, we need complexify $\mathfrak{so}(1,3)$ firstly, then we can say,

$$\mathfrak{spin}(1,3)_{\mathbb{C}} \cong \mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}), \tag{1.2.15}$$

since the complexification of $\mathfrak{su}(2)$ is $\mathfrak{sl}(2,\mathbb{C})$. Now we can choose the new set of basis (1.2.14). Finding the representations of this Lie algebra becomes standard because $\mathfrak{sl}(2,\mathbb{C})$ is just the A_1 in Cartan-Weyl classification. You may realize: the complexification of $\mathfrak{so}_4$ is also isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. To know how to get different real spaces from one complex space, you may refer to my linear algebra notes.

Here I will perform the explicit calculations. For $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ case, the antilinear involusion is $\sigma(X,Y) = (\hat{Y}, \hat{X})$, which mixes two components. We firstly talk about $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, in which two components can be handled separently. For a single $\mathfrak{sl}(2,\mathbb{C})$, its Cartan-Weyl basis $\{H,E,F\}$ in representation $\rho^{(s)}$ is given by

$$\begin{aligned}(\mathbf{e}_{m'},H^{(s)}\mathbf{e}_m) =& m\delta_{m'm}\\ (\mathbf{e}_{m'},E^{(s)}\mathbf{e}_m) =& \sqrt{(s+m+1)(s-m)}\delta_{m',m+1}\\ (\mathbf{e}_{m'},F^{(s)}\mathbf{e}_m) =& \sqrt{(s+m)(s-m+1)}\delta_{m',m-1}.\end{aligned} \tag{1.2.16}$$

Here we only consider $s=\frac{1}{2}$. In this representation,

$$H^{(\frac{1}{2})}=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad E^{(\frac{1}{2})}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad F^{(\frac{1}{2})}=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{1.2.17}$$

Later we will omit the indices $(\frac{1}{2})$. So for its real form got by $\sigma(X,Y)=(-X^\dagger,-Y^\dagger)$, how can get the representation of $\mathrm{Fix}(\sigma)$. For $\mathbb{C}^2$, we have an antilinear map $\tau:\mathbb{C}^2\to\mathbb{C}^2$. They should satisfy

$$\tau(\rho(X)\cdot\mathbf{v})=\rho(\sigma(X))\cdot\tau(\mathbf{v}). \tag{1.2.18}$$

We can solve this condition. The components are expressed as,

$$\begin{aligned}\mathbf{v} =& \alpha_1\mathbf{e}_+ + \alpha_2\mathbf{e}_-\\ X =& x_1H+x_2E+x_3F\\ \tau(\mathbf{e}_+) =& \tau_{11}\mathbf{e}_+ + \tau_{21}\mathbf{e}_-\\ \tau(\mathbf{e}_-) =& \tau_{12}\mathbf{e}_+ + \tau_{22}\mathbf{e}_-\end{aligned} \tag{1.2.19}$$

By solving (1.2.18), we get,

$$\tau(\alpha_1\mathbf{e}_+ + \alpha_2\mathbf{e}_-)=\alpha_2^*\mathbf{e}_+ - \alpha_1^*\mathbf{e}_-. \tag{1.2.20}$$

The general solution is $\alpha\mathbf{e}_+ - \alpha^*\mathbf{e}_-$, which has two real degrees of freedom. Now we have a representation $\rho:\mathrm{Fix}(\sigma)\to\mathfrak{gl}(\mathrm{Fix}(\tau))$, expressed in matrix form,

$$\begin{pmatrix}ic & -ia-b\\ -ia+b & -ic\end{pmatrix}\begin{pmatrix}\alpha\\ -\alpha^*\end{pmatrix}=\begin{pmatrix}ic\alpha-ia\alpha^*-b\alpha^*\\ -ia\alpha+b\alpha+ic\alpha^*\end{pmatrix}\in\mathrm{Fix}(\tau) \tag{1.2.21}$$

For $\sigma(X) = \bar{X}$, I only write the result

$$\tau(\alpha_1 \mathbf{e}_+ + \alpha_2 \mathbf{e}_-) = \alpha_1^* \mathbf{e}_+ \alpha_2^* \mathbf{e}_-. \tag{1.2.22}$$

After understanding the logic, we can talk our case $\sigma(X, Y) = (\bar{Y}, \bar{X})$.
$(\frac{1}{2}, 0)$ rep.

$$(x_1 H_L + x_2 E_L + x_3 F_L) \otimes (y_1 \mathbb{I} + y_2 \mathbb{I} + y_3 \mathbb{I}) \cdot (\alpha_1 \mathbf{e}_+^L \otimes \mathbf{e}^R + \alpha_2 \mathbf{e}_-^L \otimes \mathbf{e}^R). \tag{1.2.23}$$

$(0, \frac{1}{2})$ rep.

$$(x_1 \mathbb{I} + x_2 \mathbb{I} + x_3 \mathbb{I}) \otimes (y_1 H_R + y_2 E_R + y_3 F_R) \cdot (\beta_1 \mathbf{e}^L \otimes \mathbf{e}_+^R + \beta_2 \mathbf{e}^L \otimes \mathbf{e}_-^R). \tag{1.2.24}$$

We are not able to take real forms from them, and they are called the left handed and right handed spinors respectively.

1.3 Clifford Algebras

1.4 Spinor Bundles

1.5 The Dirac Operator

Chapter 2

The Construction of Gauge Theoriy

2.1 Mathematical Preliminary

2.2 Classical Gauge Theory

2.A a

2.B b

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Chapter 3

Examples of Gauge Theories

3.1 Yang-Mills Theory

3.2 Chern-Simons Theory

3.2.1 Abelian Chern-Simons Theory

I will consider the simplest case to give the logic and then generalize the discussion to other cases.

The Abelian Chern-Simons action on the smooth 3-manifold Σ^3 is given by

$$S_{\text{CS}}[A, \Sigma^3] = \frac{k}{4\pi} \int_{\Sigma^3} A \wedge dA, \tag{3.2.1}$$

where A is a $U(1)$ gauge field and k is a numerical parameter. The equation of motion

$$F \equiv dA = 0, \tag{3.2.2}$$

saying that the gauge connections are flat. Under the gauge transformation $A \to A + d\lambda$, the action transforms as

$$S_{\text{CS}} \to S_{\text{CS}} + \frac{k}{4\pi} \int_{\Sigma^3} d(\lambda \wedge dA) \tag{3.2.3}$$

We couple the gauge fields with a source J, the partition function is

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} \equiv \frac{1}{\mathcal{Z}[0]} \int \mathcal{D}A \exp\left[i \int_{\Sigma^3} \frac{k}{4\pi} A \wedge dA + A \wedge \star J\right] \tag{3.2.4}$$

For

$$\star J = q_1 \delta(\gamma_1) + q_2 \delta(\gamma_2), \tag{3.2.5}$$

the partition function gives

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \langle W_{q_1}(\gamma_1) W_{q_2}(\gamma_2) \rangle \,, \tag{3.2.6}$$

where W_q denotes the Wilson operator:

$$W_q(\gamma) \equiv \exp\left(iq \oint_\gamma A \right) \tag{3.2.7}$$

We can solve the classical equation of motion $\delta S_{\mathrm{CS}}(A, J) = 0$,

$$A^{\mathrm{cl}} = -\frac{2\pi}{k} \left(q_1 \beta_{\delta(\gamma_1)} + q_2 \beta_{\delta(\gamma_2)} \right) \tag{3.2.8}$$

where β_ω for the differential form ω satisfies $d\beta_\omega = \omega$, called the primitive of ω. By using the result from the appendix 3.A,

$$\begin{aligned} \langle W(\gamma_1) W(\gamma_2) \rangle &= \exp\left[\frac{i}{2} \int_{\Sigma^3} A^{cl} \wedge \star J \right] \\ &= \exp\left[-\frac{\pi i}{k} \int_{\Sigma^3} \left(q_1 \beta_{\delta(\gamma_1)} + q_2 \beta_{\delta(\gamma_2)} \right) \wedge \left(q_1 \delta(\gamma_1) + q_2 \delta(\gamma_2) \right) \right] \\ &= \exp\left[-\frac{2\pi i q_1 q_2}{k} \mathrm{Link}(\gamma_1, \gamma_2) \right]. \end{aligned} \tag{3.2.9}$$

To get the result, we need remove the self-interaction terms, which will be discussed later.

The Abelian CS theories may contain more than one $U(1)$ gauge connections. The action is given by

$$S_{\mathrm{CS}}[A] = \frac{K_{IJ}}{4\pi} \int A^I \wedge dA^J. \tag{3.2.10}$$

We can also couple it to a current,

$$* J_I = q_{1,I} \delta(\gamma_1) + q_{2,I} \delta(\gamma_2). \tag{3.2.11}$$

Now the partition function gives

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \langle W_{q_1}(\gamma_1) W_{q_2}(\gamma_2) \rangle \,, \tag{3.2.12}$$

where $W_q(\gamma)$ is the Wilson operator

$$W_q(\gamma) \equiv \exp\left(i q_I \int_\gamma A^I \right). \tag{3.2.13}$$

When $q_I = \delta_{II_1}$, for later convenience, we denote the Wilson operator as

$$W^{(I_1)}(\gamma) = \exp\left(q \int_\gamma A^{I_1} \right) \tag{3.2.14}$$

We can solve the equation of motion

$$\begin{aligned} &\frac{K_{IJ}}{2\pi} dA^{J,\mathrm{cl}} + \Big(q_{1,I}\delta(\gamma_1) + q_{2,I}\delta(\gamma_2) \Big) = 0 \\ \Rightarrow & dA^{J,\mathrm{cl}} = -2\pi (K^{-1})^{JI} \Big(q_{1,I}\delta(\gamma_1) + q_{2,I}\delta(\gamma_2) \Big) \\ \Rightarrow & A^{J,\mathrm{cl}} = -2\pi \Big((K^{-1})^{JI} q_{1,I} \beta_{\delta(\gamma_1)} + (K^{-1})^{JI} q_{2,I} \beta_\delta(\gamma_2) \Big). \end{aligned} \tag{3.2.15}$$

This correlators of linked Wilson operators be computed by a similar way:

$$\begin{aligned} &\langle W_{q_1}(\gamma_1) W_{q_2}(\gamma_2) \rangle = \exp\left[\frac{i}{2} \int_{\Sigma^3} A^{I,cl} \wedge \star J_I \right] \\ =& \exp\left[-\pi i \int_{\Sigma^3} \Big((K^{-1})^{IJ} q_{1,J} \beta_{\delta(\gamma_1)} + (K^{-1})^{IJ} q_{2,J} \beta_\delta(\gamma_2) \Big) \wedge \Big(q_{1,I}\delta(\gamma_1) + q_{2,I}\delta(\gamma_2) \Big) \right] \\ =& \exp\left[-2\pi i (K^{-1})^{IJ} q_{1,I} q_{2,J} \mathrm{Link}(\gamma_1, \gamma_2) \right]. \end{aligned} \tag{3.2.16}$$

In both (3.2.9) and (3.2.16), we ingore the terms related to

$$\int_{\Sigma_3} \beta_{\delta(\gamma)} \wedge \delta(\gamma), \tag{3.2.17}$$

which is divergent. This divergence can be cured by framing the line γ. I may discuss it later.

The result is called the framing number $\mathrm{Fram}(\gamma)$ or self-linking number. It also appears in the correlator of a single Wilson operator

$$\langle W_q(\gamma)\rangle = \exp\left[-2\pi i \frac{q^2}{2k}\mathrm{Fram}(\gamma)\right]. \tag{3.2.18}$$

We call $\frac{q^2}{2k}$ the spin of the line operator, denoted as $h(W_q)$. Thus for (3.2.9), we have an extra phase,

$$\exp\Big[-2\pi i\Big[h(W_{q_1})\mathrm{Fram}(\gamma_1) + h(W_{q_2})\mathrm{Fram}(\gamma_2)\Big]\Big]. \tag{3.2.19}$$

For the K-matrix Abelian CS theory, the spin of the line operator W_q is given by

$$h(W_q) = \frac{q_I (K^{-1})^{IJ} q_J}{2}. \tag{3.2.20}$$

The extra phase for (3.2.16) is

$$\exp\Big[-2\pi i\Big[h(W_{q_1})\mathrm{Fram}(\gamma_1) + h(W_{q_2})\mathrm{Fram}(\gamma_2)\Big]\Big] \tag{3.2.21}$$

3.2.2 Application 1: FQHE

We couple the CS to a backgroud field A, the Lagrangian

$$S[a] = -\frac{K_{IJ}}{4\pi}\int a^I \wedge da^J + \frac{1}{2\pi} t_I A \wedge da^I. \tag{3.2.22}$$

EOM:

$$da^I = (K^{-1})^{IJ} t_J dA = (K^{-1})^{IJ} t_J F \tag{3.2.23}$$

The current

$$J = \frac{t_I}{2\pi} da^I = \frac{t_I (K^{-1})^{IJ} t_J}{2\pi} F \tag{3.2.24}$$

The

$$J^x = J_{02} = \frac{t_I (K^{-1})^{IJ} t_J}{2\pi} F_{02} = \frac{t_I (K^{-1})^{IJ} t_J}{2\pi} E^y \tag{3.2.25}$$

We have

$$\sigma_{xy} = \frac{t_I (K^{-1})^{IJ} t_J}{2\pi} = \frac{t_I (K^{-1})^{IJ} t_J e^2}{2\pi\hbar} \tag{3.2.26}$$

3.2.3 Non-Abelian Chern-Simons Theory

3.3 Dijkgraaf-Witten Theory

3.A Path Integrals for Quadratic From Action

If the action is a quadratic form of the field ϕ,

$$S[\phi, J] = \frac{1}{2}\langle\phi, K\phi\rangle + \langle J, \phi\rangle, \tag{3.A.1}$$

the partition function can be computed simply. We first write down the classical solution

$$\frac{\delta S}{\delta\phi} = K\phi + J = 0 \Rightarrow \phi^{\mathrm{cl}} = -K^{-1}J. \tag{3.A.2}$$

Then we define a new field as $\eta = \phi - \phi^{\mathrm{cl}}$, and substitute it into the action

$$\begin{aligned} S[\phi, J] =& \frac{1}{2}\langle\eta + \phi^{\mathrm{cl}}, K\eta + \phi^{\mathrm{cl}}\rangle + \langle J, \eta + \phi^{\mathrm{cl}}\rangle \\ =& \frac{1}{2}\langle\eta, K\eta\rangle + \frac{1}{2}\langle\eta, K\phi^{\mathrm{cl}}\rangle + \frac{1}{2}\langle\phi^{\mathrm{cl}}, K\eta\rangle + \frac{1}{2}\langle\phi^{\mathrm{cl}}, K\phi^{\mathrm{cl}}\rangle + \langle J, \eta\rangle + \langle J, \phi^{\mathrm{cl}}\rangle \\ =& \frac{1}{2}\langle\eta, K\eta\rangle + \frac{1}{2}\langle\phi^{\mathrm{cl}}, K\phi^{\mathrm{cl}}\rangle + \langle J, \phi^{\mathrm{cl}}\rangle \\ =& S[\eta, 0] - \frac{1}{2}\langle J, K^{-1}J\rangle. \end{aligned} \tag{3.A.3}$$

Sometimes the term $\frac{1}{2}\langle\phi^{\mathrm{cl}}, K\phi^{\mathrm{cl}}\rangle + \langle J, \phi^{\mathrm{cl}}\rangle = -\frac{1}{2}\langle J, K^{-1}J\rangle$ is written as $S[\phi^{\mathrm{cl}}, J]$ to indicate it is the classical contribution. The partition function now is

$$\begin{aligned} \mathcal{Z}[J] = \int \mathcal{D}\phi e^{iS[\phi,J]} = e^{iS[\phi^{\mathrm{cl}},J]}\int \mathcal{D}\phi e^{iS[\eta,0]} \\ = e^{iS[\phi^{\mathrm{cl}},J]}\int \mathcal{D}\eta e^{iS[\eta,0]} = \exp\left[-\frac{i}{2}\langle J, K^{-1}J\rangle\right]\mathcal{Z}[0], \end{aligned} \tag{3.A.4}$$

where we assume the path integral measure $\mathcal{D}\eta = \mathcal{D}\phi$. In physics, the operator K is usually a differential one, so K^{-1} is the corresponding Green's function. Sometimes we can construct the classical solution directly. For this case, we have

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \exp\left[\frac{i}{2}\left\langle J, \phi^{\text{cl}}\right\rangle\right] \tag{3.A.5}$$

This result can be generalized for more fields, expressed as $\mathbf{\Phi} = (\phi_1, ..., \phi_N)$. The quadratic form action is give by

$$S[\mathbf{\Phi}] = \frac{1}{2}\left\langle \phi_I, \hat{K}^{IJ}\phi_J\right\rangle + \left\langle \phi_I, J^I\right\rangle, \tag{3.A.6}$$

where $\hat{K}^{IJ}$ can be decomposed to two parts

$$\hat{K}^{IJ} = K^{IJ} \times \hat{D}. \tag{3.A.7}$$

K^{IJ} is a symmetric $N \times N$ matrix and $\hat{D}$ is defined on the Hilbert space. We expand the inner products

$$S[\phi, J] = \int dx \frac{1}{2}\phi_I(x)K^{IJ}\hat{D}_x\phi_J(x) + \phi_I(x)J^I(x) \tag{3.A.8}$$

The equation of motion is

$$K^{IJ}\hat{D}\phi_J + J_I = 0. \tag{3.A.9}$$

The solution is given by

$$\phi_I^{\text{cl}}(x) = -(K^{-1})_I{}^J \int dy G(x,y) J_J(y), \tag{3.A.10}$$

where $G(x, y)$ is the Green's function for $\hat{D}$, which means

$$\hat{D}G(x,y) = \delta(x-y). \tag{3.A.11}$$

The effective action

$$\begin{aligned} S[\mathbf{\Phi}^{\text{cl}}] =& \frac{1}{2}\left\langle \phi_I^{\text{cl}}, \hat{K}^{IJ}\phi_J^{\text{cl}}\right\rangle + \left\langle \phi_I^{\text{cl}}, J^I\right\rangle \\ =& -\frac{1}{2}\int dx dy J_I(x)(K^{-1})^{IJ} G(x,y) J_J(y). \end{aligned} \tag{3.A.12}$$

Chapter 4

Transformations in Field Theory

Let M be a d-dimensional spacetime manifold. A transformation is labelled by an elemnet in the diffeomorphism group of M, i.e.

$$diff(M) \ni S : M \to M. \tag{4.0.1}$$

If we have a coordinate system, $f : M \to \mathbb{R}^d$, we can use S to construct another coordinate system g as,

$$g := f \circ S. \tag{4.0.2}$$

And we can also use S to define a diffeomorphism in $\mathbb{R}^d$ as,

$$\hat{S} := f \circ S \circ f^{-1}. \tag{4.0.3}$$

Here we firstly consider a classical transformation. A classical (r, s)-type tensor field T is a section of the tensor bundle, which can be understood as,

$$T : M \to \mathcal{T}^r_s(M), \tag{4.0.4}$$

satisfying,

$$T(\mathcal{P}) \in \mathcal{T}^r_s(\mathcal{P}), \;\; \mathcal{P} \in M. \tag{4.0.5}$$

And we can use the pushforward and pullback of the coordinate to construct the map,

$$\tilde{f} : \mathcal{T}^r_s(M) \to \mathbb{R}^{d^{(r+s)}}. \tag{4.0.6}$$

And we can also define a transformation $\tilde{S}$ in $\mathbb{R}^{d^{(r+s)}}$, satisfying $\tilde{g} = \tilde{S} \circ \tilde{f}$. The commutative diagram is,

$$\begin{tikzcd}
\mathbb{R}^d \arrow[d, "\hat{S}"'] & M \arrow[l, "f"'] \arrow[r, "T"] \arrow[dl, "g"] \arrow[d, "S"] & \mathcal{T}^r_s(M) \arrow[r, "\tilde{f}"] \arrow[dr, "\tilde{g}"'] & \mathbb{R}^{d^{(r+s)}} \arrow[d, "\tilde{S}"] \\
\mathbb{R}^d & M \arrow[l, "f"] & & \mathbb{R}^{d^{(r+s)}}
\end{tikzcd}$$

We define a map from $\mathbb{R}^d$ to $\mathbb{R}^{d^{(r+s)}}$ as,

$$\phi := \tilde{f} \circ T \circ f^{-1}, \tag{4.0.7}$$

written in components,

$$\phi^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}(x_1, \ldots, x_n) \in \mathbb{R}. \tag{4.0.8}$$

the passive transformation labelled by S is the coordinate transformation $\hat{S}$, and the map becomes,

$$\phi_{pa} = \tilde{g} \circ T \circ g^{-1}, \tag{4.0.9}$$

written in components,

$$\phi_{pa}^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}(x_1, \ldots, x_n) = (\tilde{g} \circ T \circ g^{-1})^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}(x_1, \ldots, x_n). \tag{4.0.10}$$

The actual field $T : M \to \mathcal{T}^r_s$ is invariant under this perspective of transformation. And the active transformation means we change the field to $T' : M \to \mathcal{T}^r_s$ satisfying some condition, which will be given latter. In this case, we do not have different coordinates, and we can use S to define a transformation S_T in the tensor bundles $\mathcal{T}^r_s$, we can prove it satisfies,

$$\tilde{f} \circ S_T = \tilde{S} \circ \tilde{f}. \tag{4.0.11}$$

I do this step since $\tilde{S}$ is related to the coordinate while S_T I put all items and the condition for T' in one commutative diagram,

$$\begin{tikzcd}
\mathbb{R}^d \arrow[d, "\hat{S}"'] & M \arrow[l, "f"'] \arrow[r, "T"] \arrow[dl, "g"] \arrow[d, "S"] & \mathcal{T}^r_s(M) \arrow[r, "\tilde{f}"] \arrow[d, "S_T"'] \arrow[dr, "\tilde{g}"] & \mathbb{R}^{d(r+s)} \arrow[d, "\tilde{S}"] \\
\mathbb{R}^d & M \arrow[l, "f"] \arrow[r, "T'"'] & \mathcal{T}^r_s(M) \arrow[r, "\tilde{f}"'] & \mathbb{R}^{d(r+s)}
\end{tikzcd}$$

which says

$$T' \circ S = S_T \circ T. \tag{4.0.12}$$

We define another map from $\mathbb{R}^d$ to $\mathbb{R}^{d^{(r+s)}}$ as,

$$\phi_{ac} = \tilde{f} \circ T' \circ f. \tag{4.0.13}$$

Thus the condition (4.0.12) can be understood as,

$$\phi_{ac} = \phi_{pa}. \tag{4.0.14}$$

I review the definition chain here,

$$\{S, f\} \Rightarrow \{g\} \Rightarrow \{\hat{S}\} \tag{4.0.15}$$

$$\{f, g\} \Rightarrow \{\tilde{f}, \tilde{g}\} \Rightarrow \{\tilde{S}\} \Rightarrow \{S_T\} \tag{4.0.16}$$

$$\{T, S_T, \tilde{S}\} \Rightarrow T' \tag{4.0.17}$$

The quantum transformation is more complicated and I will give the detials now.

$$\begin{tikzcd}
\mathbb{R}^d \arrow[d, "\hat{S}"'] & M \arrow[l, "f"'] \arrow[r, "T"] \arrow[dl, "g"] \arrow[d, "S"] & \mathcal{T}^r_s(M) \times GL(\mathcal{H}) \arrow[r, "\mathrm{id}\times\psi"] & \mathcal{T}^r_s(M) \times \mathbb{C} \arrow[r, "\tilde{f}\times\mathrm{id}"] \arrow[dr, "\tilde{g}\times\mathrm{id}"] & \mathbb{R}^{d(r+s)} \times \mathbb{C} \arrow[d, "\tilde{S}\times\mathrm{id}"] \\
\mathbb{R}^d & M \arrow[l, "f"] \arrow[r, "T'"] \arrow[dr, "T"'] & \mathcal{T}^r_s(M) \times GL(\mathcal{H}) \arrow[r, "\mathrm{id}\times\psi"] & \mathcal{T}^r_s(M) \times \mathbb{C} \arrow[r, "\tilde{f}\times\mathrm{id}"'] & \mathbb{R}^{d(r+s)} \times \mathbb{C} \\
 & & \mathcal{T}^r_s(M) \times GL(\mathcal{H}) \arrow[u, "\mathrm{id}\times U"] \arrow[ur, "\mathrm{id}\times\psi'"'] & &
\end{tikzcd}$$

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Chapter 5

The Renoramlization Group

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Chapter 6

Spontaneously broken symmetries

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Chapter 7

Anomalies