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A Category Theory

This appendix outlines the essential knowledge of category theory required to understand the main discussion. Readers can find more details in [1, 2].

A.1 Basic Concepts of Category Theory

Definition A.1. A *category* $\mathcal{C}$ consists of:

- a collection C_0 of *objects*;
- a collection C_1 of *morphisms*;
- there is an operation s, which assigns to each morphism f an object $s(f)$, called its *source* or *domain*;

- there is an operation t, which assigns to each morphism f an object $t(f)$, called its *target* or *codomain*;
- there is an operation 1, which assigns to each object c an morphisms 1_c, called the *identity morphism* on c;
- there is a composition operation $\circ$, which assigns a pair of morphisms f and g, $t(f) = s(g)$, a morphisms $g \circ f$, called their *composite*;
- such that the following properties are satisfied:
  - $s(1_c) = c = t(1_c)$;
  - $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$;
  - associativity of composition, $(h \circ g) \circ f = h \circ (g \circ f)$;
  - composition satisfies the unit law, fo $a \xrightarrow{f} b$, $f \circ 1_a = f = 1_b \circ f$.

we may write $c \in \mathcal{C}$ to indicate c is an object of $\mathcal{C}$. And people often write $\mathrm{Hom}(a, b)$, $\mathrm{Hom}_{\mathcal{C}}(a, b)$, $\mathcal{C}(a, b)$ or $\mathcal{C}(a \to b)$ for the collection of morphisms $f : a \to b$, which is called the hom-set of objects a and b.

Definition A.2. An *isomorphism* between two objects a, b in category $\mathcal{C}$ is a morphism $f : a \to b$, and there exsits a morphism $g : b \to a$, such that $f \circ g = 1_b$ and $g \circ f = 1_a$. a, b are *isomorphic* if there exsits an isomorphism between them. The *isomorphism class* of object c is the set of objects isomorphic to c.

Definition A.3. For categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* T is a morphism with source $\mathcal{C}$ and target $\mathcal{D}$, consists of,

- an *object function* T_0, which assigns to each $c \in \mathcal{C}$ an object $T_0 c \in \mathcal{D}$;
- a *morphism function* T_1, which assigns to each morphism $f : c \to c'$ of $\mathcal{C}$ an morphisms $T_1 f : T_0 c \to T_0 c'$ of $\mathcal{D}$;
- such that,

$$T_1(1_c) = 1_{T_0 c}, \qquad T_1(g \circ f) = T_1 g \circ T_1 f. \tag{1}$$

I will write both T_0 and T_1 as T in this article.

Definition A.4. A natural transformation

Definition A.5. For two given categories $\mathcal{B}$ and $\mathcal{C}$, a new category $\mathcal{B} \times \mathcal{C}$, called the *product* of them, is defined as follows. An object of $\mathcal{B} \times \mathcal{C}$ is a pair $\langle b, c \rangle$ of $b \in \mathcal{B}$ and $c \in \mathcal{C}$. A morphism $\langle b, c \rangle \to \langle b', c' \rangle$ in $\mathcal{B} \times \mathcal{C}$ is a pair $\langle f, g \rangle$ of $f : b \to b'$ and $g : c \to c'$. The composition operation in $\mathcal{B} \times \mathcal{C}$ is defined as,

$$\langle f', g' \rangle \circ \langle f, g \rangle = \langle f' \circ f, g' \circ g \rangle \tag{2}$$

A.2 Unitary Fusion Category

Main reference for this subsection is [3].

Definition A.6. An *additive category* $\mathcal{C}$ is a category satisfying,

- Every hom-set is equipped with a structure of an abelian group such that the the composition operation is biadditive;

- There exsits a zero object $0 \in \mathcal{C}$ such that $\mathcal{C}(0,0) = 0$;
- For any objects $c_1, c_2 \in \mathcal{C}$, there exsits an object $c \in \mathcal{C}$ and morphisms $p_1 : c \to c_1$, $p_2 : c \to c_2$, $i_1 : c_1 \to c$, $i_2 : c_2 \to c$ such that $p_1 \circ i_1 = 1_{c_1}$, $p_2 \circ i_2 = 1_{c_2}$ and $i_1 \circ p_1 + i_2 \circ p_2 = 1_c$. The object c is unique up to a unique isomorphism, is denoted by $c_1 \oplus c_2$, and called the *direct sum* of c_1 and c_2. $\oplus$ can be viewed as a functor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. And we can easily generalize this definition to the case for more than two objects.

Definition A.7. If $\mathbb{F}$ is a field, the $\mathbb{F}$-linear category $\mathcal{C}$ is an additive category whose hom-sets are all linear spaces over $\mathbb{F}$, and whose composition operation is $\mathbb{F}$-linear. In physics, the field $\mathbb{F}$ is always chosen to be the complex number $\mathbb{C}$. Unless stated otherwise, all linear spaces in this article are over $\mathbb{C}$.

Definition A.8. An *idempotent* in category $\mathcal{C}$ is a morphisms e such that $e \circ e = e$. A *splitting* for an idempotent $e : c \to c$ is an triple (a, r, s) where $a \in \mathcal{C}, r \in \mathcal{C}(c, a)$, and $s \in \mathcal{C}(a, c)$ such that $s \circ r = e$ and $r \circ s = 1_a$. A lineae category is called *idempotent complete* if every idempotent admits a splitting.

Definition A.9. A linear category $\mathcal{C}$ is called *semisimple* if,

- it admits all finite direct sum;
- it is idempotents complete;
- there exsits objects c_i labeled by an index set I such that,
  - for any $i, j \in I$, we have
    $$\mathcal{C}(c_i, c_j) \cong \delta_{ij}\mathbb{C}, \tag{3}$$
    such objects are called *simple*;
  - for any pair of objects $a, b \in \mathcal{C}$,
    $$\bigoplus_{i \in I} \mathcal{C}(a, c_i) \otimes \mathcal{C}(c_i, b) \to \mathcal{C}(a, b) \tag{4}$$
    is an isomorphism. Here $\oplus$ and $\otimes$ are direct sum and tensor product for linear spaces respectively.

$\mathcal{C}$ is called *finitely semisimple* is in addition $\mathcal{C}$ has finitely many isomorphism classes of simple objects.

I state the following proposition without proof, as it will be used in the text.

Proposition A.10. Every object of semisimple linear category is a direct sum of simple objects c_i .

Theorem A.11. dagger structure

Definition A.12. $*$-algebra

Theorem A.13. unitary algebra condition

Definition A.14. unitary algebra

References

[1] S.M. Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer, 1998.

[2] E. Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2017.

[3] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor Categories*. Mathematical Surveys and Monographs. American Mathematical Society, 2015.