1 General

$$A^{\mathrm{ALSM}}(\tau_n)=(\pi i)^{n-2}=\sum_{\rho\in S_{n-2}}S[\rho(23\ldots n-1)|\rho(23\ldots n-1)]_1 m[1,\rho(2,\ldots,n-1),n|\tau_n] \tag{1}$$

$$S[A,j|B,j,C]_i=(k_iB\cdot k_j)S[A|B,C]_i,\qquad S[\emptyset|\emptyset]\equiv 1 \tag{2}$$

$$k_iB\equiv k_i+k_{b_1}+\cdots+k_{b_{|B|}}$$

$$m(P,n|Q,n)=s_P\phi_{P|Q} \tag{3}$$

$$\phi_{P|Q}=\frac{1}{s_P}\sum_{XY=P}\sum_{AB=Q}\Big(\phi_{X|A}\phi_{Y|B}-(X\leftrightarrow Y)\Big),\qquad \phi_{i|j}=\delta_{ij} \tag{4}$$

$$s_P=k_P^2=(k_{P_1}+k_{P_2}+\cdots+k_{P_{|P|}})^2$$

An algorithm for Berends-Giele double-currents,

$$\phi_i=1,\qquad \phi_P=\frac{1}{s_P}\sum_{XY=P}\phi_X\phi_Y,\qquad X,Y\neq\emptyset. \tag{5}$$

$$\begin{aligned}
\phi_{12}=&\phi_{21}=\frac{1}{s_{12}}\\
\phi_{123}=&\phi_{321}=\frac{1}{s_{123}}\left(\frac{1}{s_{12}}+\frac{1}{s_{23}}\right)\\
\phi_{132}=&\phi_{231}=\frac{1}{s_{123}}\left(\frac{1}{s_{13}}+\frac{1}{s_{23}}\right)\\
\phi_{213}=&\phi_{312}=\frac{1}{s_{123}}\left(\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)
\end{aligned} \tag{6}$$

Shouten identity,

$$\langle ij\rangle\langle kl\rangle+\langle ik\rangle\langle lj\rangle+\langle il\rangle\langle jk\rangle=0 \tag{7}$$

$$[ij][kl]+[ik][lj]+[il][jk]=0 \tag{8}$$

$$\langle ij\rangle[kl]+\langle ik\rangle[lj]+\langle il\rangle[jk]=0 \tag{9}$$

KLT relation,

$$\mathcal{M}_m^{\mathrm{tree}}=-i\sum_{\sigma,\rho\in S_{m-3}(2,\ldots,m-2)}A_m^{\mathrm{tree}}(1,\sigma,m-1,m)S[\sigma|\rho]\tilde{A}_m^{\mathrm{tree}}(1,\rho,m,m-1) \tag{10}$$

This $S[\sigma|\rho]$ is just the former $S[|]_1$.

2 NLSM Theory

Four-point

$$A^{\mathrm{NLSM}}(\tau_4)=(\pi i)^2\Big\{S[23|23]_1 m[1,2,3,4|\tau_4]+s[32|32]_1 m[1,3,2,4|\tau_4]\Big\} \tag{11}$$

$$s[23|23]_1=(s_{13}+s_{23})s_{12}=-s_{12}^2 \tag{12}$$

$$s[32|32]_1=(s_{12}+s_{32})s_{13}=-s_{13}^2 \tag{13}$$

since $m(-|-)$ is cyclically symmetric in both, we need only consider,

$$\tau_4=1234,1324,2134,2314,3124,3214 \tag{14}$$

(i) $\tau_4=1234$ or 3214 (the result is the same for both cases since $\phi_{\sigma(123)|123}=\phi_{\sigma(123)|321}$ for any $\sigma\in S_3$),

$$\begin{aligned}
A^{\mathrm{NLSM}}(1234)=A^{\mathrm{NLSM}}(3214)=&(\pi i)^2\Big\{-s_{12}^2 s_{123}\phi_{123|123}-s_{12}^2 s_{123}\phi_{132|123}\Big\}\\
=&(\pi i)^2\Big\{-s_{12}^2\left(\frac{1}{s_{12}}+\frac{1}{s_{23}}\right)-s_{13}^2\frac{-1}{s_{23}}\Big\}\\
=&-(\pi i)^2 s_{13}
\end{aligned} \tag{15}$$

In the last line, $s_{ij} = k_i \cdot k_j$ is used.

(ii) $\tau_4 = 1324$ or 2314,

$$A^{\mathrm{NLSM}}(1324) = A^{\mathrm{NLSM}}(2314) = -(\pi i)^2 s_{12} \tag{16}$$

(iii) $\tau_4 = 2134$ or 3124,

$$A^{\mathrm{NLSM}}(2134) = A^{\mathrm{NLSM}}(3124) = -(\pi i)^2 s_{23} \tag{17}$$

Six-point [1608.02569,1304.3048]

$$A_6^{\mathrm{NLSM}}(1,2,3,4,5,6) = s_{12} - \frac{(s_{12}+s_{23})(s_{45}+s_{56})}{2s_{123}} + \mathrm{cyc}(1,2,3,4,5,6) \tag{18}$$

3 SDYM Theory

$$\begin{aligned} n^{sd}_{1|23|4} &= \langle \eta r\rangle^4 \left(\prod_{i=1}^{4} \frac{1}{\langle \eta i\rangle}\right) X_{1,2}X_{1+2,3} \\ &= (common) \times \Big(X_{1,2}X_{1,3} + X_{1,2}X_{2,3}\Big) \end{aligned} \tag{19}$$

$$\begin{aligned} n^{sd}_{1|32|4} &= (common) \times X_{1,3}X_{1+3,2} \\ &= (common) \times \Big(X_{1,2}X_{1,3} - X_{1,3}X_{2,3}\Big) \end{aligned} \tag{20}$$

$$\begin{aligned} A_4^{sd}(\underline{1},3,4,2) =& \frac{n^{sd}_{1|23|4}}{s_{12}} + \frac{n^{sd}_{1|32|4}}{s_{13}} \\ =& (common) \times \left(- \frac{X_{1,2}X_{1,3}s_{23}}{s_{12}s_{23}} + X_{23}\frac{X_{1,2}s_{13} - X_{1,3}s_{12}}{s_{12}s_{13}} \right) \\ =& \frac{(common)}{s_{12}s_{13}} \times \Big(\langle 1\eta\rangle [12] \langle 2\eta\rangle \langle \eta 1\rangle [13] \langle 3\eta\rangle \langle 23\rangle [32] + \langle \eta 2\rangle [23] \langle 3\eta\rangle \langle \eta 1\rangle [12][13] \Big(\langle 2\eta\rangle \langle 31\rangle - \langle 3\eta\rangle \langle 21\rangle \Big)\Big) \\ =& \frac{(common)}{s_{12}s_{13}} \times \langle 1\eta\rangle \langle 2\eta\rangle \langle 3\eta\rangle [12][13][23] \Big(\langle \eta 1\rangle \langle 32\rangle + \langle \eta 2\rangle \langle 13\rangle + \langle \eta 3\rangle \langle 21\rangle \Big) \\ =& \, 0 \end{aligned} \tag{21}$$

In the last line, the Shouten identity is used.

5-point SDYM tree amplitude,

$$\begin{aligned} A_5^{\mathrm{SDYM}}(1,2,3,4,5) =& \frac{X_{1,2}X_{1+2,3}X_{1+2+3,4}}{s_{12}s_{123}} + (\mathrm{cyc.}) \\ =& \frac{X_{1,2}X_{1+2,3}X_{4,5}}{s_{12}s_{45}} + (\mathrm{cyc.}) \\ =& \frac{1}{2}\left(\frac{X_{1,2}X_{1+2,3}X_{4,5}}{s_{12}s_{45}} - \frac{X_{1,2}X_{4+5,3}X_{4,5}}{s_{12}s_{45}} \right) + (\mathrm{cyc.}) \\ =& \frac{1}{2}\left(\frac{X_{1,2}X_{1+2,3}X_{4,5}}{s_{12}s_{45}} - \frac{X_{4,5}X_{2+3,1}X_{2,3}}{s_{45}s_{23}} \right) + (\mathrm{cyc.}) \end{aligned} \tag{22}$$

The minus term in the last line is from the fourth term in (cyc.) of the third line.

Lemma 1.

$$\frac{X_{i,j}X_{i+j,l}}{s_{ij}} - \frac{X_{j,l}X_{j+l,i}}{s_{jl}} = \frac{X_{i,j}X_{j,l}s_{ijl}}{s_{ij}s_{jl}} \tag{23}$$

Proof.

$$\begin{aligned}
&\frac{X_{i,j}X_{i+j,l}}{s_{ij}}-\frac{X_{j,l}X_{j+l,i}}{s_{jl}}\\
=&X_{i,j}X_{j,l}\frac{s_{ij}+s_{jl}}{s_{ij}s_{jl}}+X_{i,l}\frac{X_{i,j}s_{jl}+X_{j,l}s_{ij}}{s_{ij}s_{jl}}\\
=&X_{i,j}X_{j,l}\frac{s_{ij}+s_{jl}}{s_{ij}s_{jl}}+\langle\eta i\rangle[il]\langle l\eta\rangle\frac{\langle\eta i\rangle[ij]\langle j\eta\rangle\langle jl\rangle[lj]+\langle\eta j\rangle[jl]\langle l\eta\rangle\langle ij\rangle[ji]}{s_{ij}s_{jl}}\\
=&X_{i,j}X_{j,l}\frac{s_{ij}+s_{jl}}{s_{ij}s_{jl}}+\frac{\langle\eta i\rangle[il]\langle l\eta\rangle\langle j\eta\rangle\langle\eta j\rangle\langle li\rangle[jl][ij]}{s_{ij}s_{jl}}\\
=&X_{i,j}X_{j,l}\frac{s_{ij}+s_{jl}}{s_{ij}s_{jl}}+\frac{X_{i,j}X_{j,l}s_{il}}{s_{ij}s_{jl}}\\
=&\frac{X_{i,j}X_{j,l}s_{ijl}}{s_{ij}s_{jl}}.
\end{aligned}\tag{24}$$

By using Lemma 1, we can get,

$$A_5^{\text{SDYM}}(1,2,3,4,5)=\frac{1}{2}\frac{X_{1,2}X_{2,3}X_{4,5}}{s_{12}s_{23}}+(\text{cyc.})\tag{25}$$

Now I want to construct some cyclic symmetric term, which can be considered as a common factor.

$$\frac{X_{1,2}X_{2,3}X_{4,5}}{s_{12}s_{23}}=\frac{\langle\eta 1\rangle\langle\eta 2\rangle\langle\eta 3\rangle\langle\eta 4\rangle\langle\eta 5\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\times s_{45}\langle\eta 2\rangle\langle 34\rangle\langle 51\rangle\tag{26}$$

So we have,

$$A_5^{\text{SDYM}}(1,2,3,4,5)\propto s_{45}\langle\eta 2\rangle\langle 34\rangle\langle 51\rangle+(\text{cyc.})\tag{27}$$

I have proved it is indeed vanishing by momentum conservation, I may type later.

4 BI Theory

[0808.2598]

Helicity selection-rule, the amplitudes is non-vanishing when number of positive helicity particle is the same as the number of the negative helicity particle.

$$M_4^{\text{BI}}(1+,2+,3-,4-)\propto[12]^2\langle 34\rangle^2\tag{28}$$

5 Double-Copy Part

$\mathcal{M}^{\text{BI}}=\mathcal{A}^{\text{YM}}\otimes\mathcal{A}^{\text{NLSM}}$.

Four-point tree-level KLT relation,

$$M_4(1+,2+,3-,4-)=-iA_4^{\text{YM}}(1+,2+,3-,4-)S[2|2]A_4^{\text{NLSM}}(1,2,4,3)\tag{29}$$

$$LHS=[12]^2\langle 34\rangle^2\tag{30}$$

$$\begin{aligned}
RHS=&-\frac{\langle 34\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}s_{12}s_{23}\\
=&\frac{\langle 34\rangle^3[12][23]}{\langle 14\rangle}
\end{aligned}\tag{31}$$

we know

$$[23]\langle 34\rangle=[2|k_3|4\rangle=-[2|k_1+k_2+k_4|4\rangle=-[2|k_1|4\rangle=[12]\langle 14\rangle\tag{32}$$

Thus,

$$RHS=\langle 34\rangle^2[12]^2=LHS\tag{33}$$

Five-point tree level MHV,

$$0=M_5(++---)=\tag{34}$$

Six-point tree level,

$$0 = M_6(++----) = \tag{35}$$

$$\text{gravity} = \text{gauge} \otimes \text{gauge} \tag{36}$$

$$\text{BI} = \text{YM} \otimes \text{NLSM} \tag{37}$$

$$c_i \leftrightarrow n_i$$

$$f^{a_1 a_2 a_3} = -f^{a_2 a_1 a_3} \qquad F_{k_1 k_2 k_3} = \delta(k_1 + k_2 + k_3)\langle \eta | k_1 k_2 | \eta \rangle$$

$$F_{k_1 k_2 k_3} = -F_{k_2 k_1 k_3}$$