Topics in Quantum Field Theories

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Chapter 3

Examples of Gauge Theories

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3.2.1 Abelian Chern-Simons Theory

I will consider the simplest case to give the logic and then generalize the discussion to other cases.

The Abelian Chern-Simons action on the smooth 3-manifold Σ^3 is given by

$$S_{\text{CS}}[A, \Sigma^3] = \frac{k}{4\pi} \int_{\Sigma^3} A \wedge dA, \tag{3.2.1}$$

where A is a $U(1)$ gauge field and k is a numerical parameter. The equation of motion

$$F \equiv dA = 0, \tag{3.2.2}$$

saying that the gauge connections are flat. Under the gauge transformation $A \to A + d\lambda$, the action transforms as

$$S_{\text{CS}} \to S_{\text{CS}} + \frac{k}{4\pi} \int_{\Sigma^3} d(\lambda \wedge dA) \tag{3.2.3}$$

We couple the gauge fields with a source J, the partition function is

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} \equiv \frac{1}{\mathcal{Z}[0]} \int \mathcal{D}A \exp\left[i \int_{\Sigma^3} \frac{k}{4\pi} A \wedge dA + A \wedge \star J\right] \tag{3.2.4}$$

For

$$\star J = q_1\delta(\gamma_1) + q_2\delta(\gamma_2), \tag{3.2.5}$$

the partition function gives

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \langle W_{q_1}(\gamma_1)W_{q_2}(\gamma_2)\rangle\,, \tag{3.2.6}$$

where W_q denotes the Wilson operator:

$$W_q(\gamma) \equiv \exp\left(iq\oint_\gamma A\right) \tag{3.2.7}$$

We can solve the classical equation of motion $\delta S_{\mathrm{CS}}(A, J) = 0$,

$$A^{\mathrm{cl}} = -\frac{2\pi}{k}\left(q_1\beta_{\delta(\gamma_1)} + q_2\beta_{\delta(\gamma_2)}\right) \tag{3.2.8}$$

where β_ω for the differential form ω satisfies $d\beta_\omega = \omega$, called the primitive of ω. By using the result from the appendix 3.A,

$$\begin{aligned}\langle W(\gamma_1)W(\gamma_2)\rangle &= \exp\left[\frac{i}{2}\int_{\Sigma^3} A^{cl}\wedge\star J\right]\\ &= \exp\left[-\frac{\pi i}{k}\int_{\Sigma^3}\left(q_1\beta_{\delta(\gamma_1)} + q_2\beta_{\delta(\gamma_2)}\right)\wedge\left(q_1\delta(\gamma_1) + q_2\delta(\gamma_2)\right)\right]\\ &= \exp\left[-\frac{2\pi i q_1 q_2}{k}\mathrm{Link}(\gamma_1,\gamma_2)\right].\end{aligned} \tag{3.2.9}$$

To get the result, we need remove the self-interaction terms, which will be discussed later.

The Abelian CS theories may contain more than one $U(1)$ gauge connections. The action is given by

$$S_{\mathrm{CS}}[A] = \frac{K_{IJ}}{4\pi}\int A^I\wedge dA^J. \tag{3.2.10}$$

We can also couple it to a current,

$$*J_I = q_{1,I}\delta(\gamma_1) + q_{2,I}\delta(\gamma_2). \tag{3.2.11}$$

Now the partition function gives

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \langle W_{q_1}(\gamma_1) W_{q_2}(\gamma_2) \rangle \,, \tag{3.2.12}$$

where $W_q(\gamma)$ is the Wilson operator

$$W_q(\gamma) \equiv \exp\Big(i q_I \int_\gamma A^I \Big). \tag{3.2.13}$$

When $q_I = \delta_{I I_1}$, for later convenience, we denote the Wilson operator as

$$W^{(I_1)}(\gamma) = \exp\Big(q \int_\gamma A^{I_1} \Big) \tag{3.2.14}$$

We can solve the equation of motion

$$\begin{aligned} &\frac{K_{IJ}}{2\pi} dA^{J,\mathrm{cl}} + \Big(q_{1,I}\delta(\gamma_1) + q_{2,I}\delta(\gamma_2) \Big) = 0 \\ \Rightarrow & dA^{J,\mathrm{cl}} = -2\pi (K^{-1})^{JI} \Big(q_{1,I}\delta(\gamma_1) + q_{2,I}\delta(\gamma_2) \Big) \\ \Rightarrow & A^{J,\mathrm{cl}} = -2\pi \Big((K^{-1})^{JI} q_{1,I} \beta_{\delta(\gamma_1)} + (K^{-1})^{JI} q_{2,I} \beta_\delta(\gamma_2) \Big). \end{aligned} \tag{3.2.15}$$

This correlators of linked Wilson operators be computed by a similar way:

$$\begin{aligned} &\langle W_{q_1}(\gamma_1) W_{q_2}(\gamma_2) \rangle = \exp\Big[\frac{i}{2} \int_{\Sigma^3} A^{I,cl} \wedge \star J_I \Big] \\ =& \exp\Big[-\pi i \int_{\Sigma^3} \Big((K^{-1})^{IJ} q_{1,J} \beta_{\delta(\gamma_1)} + (K^{-1})^{IJ} q_{2,J} \beta_\delta(\gamma_2) \Big) \wedge \Big(q_{1,I}\delta(\gamma_1) + q_{2,I}\delta(\gamma_2) \Big) \Big] \\ =& \exp\Big[-2\pi i (K^{-1})^{IJ} q_{1,I} q_{2,J} \mathrm{Link}(\gamma_1, \gamma_2) \Big]. \end{aligned} \tag{3.2.16}$$

In both (3.2.9) and (3.2.16), we ingore the terms related to

$$\int_{\Sigma_3} \beta_{\delta(\gamma)} \wedge \delta(\gamma), \tag{3.2.17}$$

which is divergent. This divergence can be cured by framing the line γ. I may discuss it later.

The result is called the framing number $\mathrm{Fram}(\gamma)$ or self-linking number. It also appears in the correlator of a single Wilson operator

$$\langle W_q(\gamma)\rangle = \exp\left[-2\pi i\frac{q^2}{2k}\mathrm{Fram}(\gamma)\right]. \tag{3.2.18}$$

We call $\frac{q^2}{2k}$ the spin of the line operator, denoted as $h(W_q)$. Thus for (3.2.9), we have an extra phase,

$$\exp\Big[-2\pi i\Big[h(W_{q_1})\mathrm{Fram}(\gamma_1)+h(W_{q_2})\mathrm{Fram}(\gamma_2)\Big]\Big]. \tag{3.2.19}$$

For the K-matrix Abelian CS theory, the spin of the line operator W_q is given by

$$h(W_q) = \frac{q_I(K^{-1})^{IJ}q_J}{2}. \tag{3.2.20}$$

The extra phase for (3.2.16) is

$$\exp\Big[-2\pi i\Big[h(W_{q_1})\mathrm{Fram}(\gamma_1)+h(W_{q_2})\mathrm{Fram}(\gamma_2)\Big]\Big] \tag{3.2.21}$$

3.2.2 Application 1: FQHE

We couple the CS to a backgroud field A, the Lagrangian

$$S[a] = -\frac{K_{IJ}}{4\pi}\int a^I\wedge da^J + \frac{1}{2\pi}t_I A\wedge da^I. \tag{3.2.22}$$

EOM:

$$da^I = (K^{-1})^{IJ}t_J dA = (K^{-1})^{IJ}t_J F \tag{3.2.23}$$

The current

$$J = \frac{t_I}{2\pi}da^I = \frac{t_I(K^{-1})^{IJ}t_J}{2\pi}F \tag{3.2.24}$$

The

$$J^x = J_{02} = \frac{t_I(K^{-1})^{IJ}t_J}{2\pi}F_{02} = \frac{t_I(K^{-1})^{IJ}t_J}{2\pi}E^y \tag{3.2.25}$$

We have

$$\sigma_{xy} = \frac{t_I(K^{-1})^{IJ}t_J}{2\pi} = \frac{t_I(K^{-1})^{IJ}t_Je^2}{2\pi\hbar} \tag{3.2.26}$$

3.2.3 Non-Abelian Chern-Simons Theory

3.3 Dijkgraaf-Witten Theory

3.A Path Integrals for Quadratic From Action

If the action is a quadratic form of the field ϕ,

$$S[\phi, J] = \frac{1}{2}\langle \phi, K\phi\rangle + \langle J, \phi\rangle, \tag{3.A.1}$$

the partition function can be computed simply. We first write down the classical solution

$$\frac{\delta S}{\delta \phi} = K\phi + J = 0 \Rightarrow \phi^{\mathrm{cl}} = -K^{-1}J. \tag{3.A.2}$$

Then we define a new field as $\eta = \phi - \phi^{\mathrm{cl}}$, and substitute it into the action

$$\begin{aligned} S[\phi, J] =& \frac{1}{2}\langle \eta + \phi^{\mathrm{cl}}, K\eta + \phi^{\mathrm{cl}}\rangle + \langle J, \eta + \phi^{\mathrm{cl}}\rangle \\ =& \frac{1}{2}\langle \eta, K\eta\rangle + \frac{1}{2}\langle \eta, K\phi^{\mathrm{cl}}\rangle + \frac{1}{2}\langle \phi^{\mathrm{cl}}, K\eta\rangle + \frac{1}{2}\langle \phi^{\mathrm{cl}}, K\phi^{\mathrm{cl}}\rangle + \langle J, \eta\rangle + \langle J, \phi^{\mathrm{cl}}\rangle \\ =& \frac{1}{2}\langle \eta, K\eta\rangle + \frac{1}{2}\langle \phi^{\mathrm{cl}}, K\phi^{\mathrm{cl}}\rangle + \langle J, \phi^{\mathrm{cl}}\rangle \\ =& S[\eta, 0] - \frac{1}{2}\langle J, K^{-1}J\rangle. \end{aligned} \tag{3.A.3}$$

Sometimes the term $\frac{1}{2}\langle \phi^{\mathrm{cl}}, K\phi^{\mathrm{cl}}\rangle + \langle J, \phi^{\mathrm{cl}}\rangle = -\frac{1}{2}\langle J, K^{-1}J\rangle$ is written as $S[\phi^{\mathrm{cl}}, J]$ to indicate it is the classical contribution. The partition function now is

$$\begin{aligned} \mathcal{Z}[J] = \int \mathcal{D}\phi e^{iS[\phi, J]} &= e^{iS[\phi^{\mathrm{cl}}, J]}\int \mathcal{D}\phi e^{iS[\eta, 0]} \\ &= e^{iS[\phi^{\mathrm{cl}}, J]}\int \mathcal{D}\eta e^{iS[\eta, 0]} = \exp\left[-\frac{i}{2}\langle J, K^{-1}J\rangle\right]\mathcal{Z}[0], \end{aligned} \tag{3.A.4}$$

where we assume the path integral measure $\mathcal{D}\eta = \mathcal{D}\phi$. In physics, the operator K is usually a differential one, so K^{-1} is the corresponding Green's function. Sometimes we can construct the classical solution directly. For this case, we have

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \exp\left[\frac{i}{2}\left\langle J, \phi^{\mathrm{cl}}\right\rangle\right] \tag{3.A.5}$$

This result can be generalized for more fields, expressed as $\mathbf{\Phi} = (\phi_1, ..., \phi_N)$. The quadratic form action is give by

$$S[\mathbf{\Phi}] = \frac{1}{2}\left\langle \phi_I, \hat{K}^{IJ}\phi_J\right\rangle + \left\langle \phi_I, J^I\right\rangle, \tag{3.A.6}$$

where $\hat{K}^{IJ}$ can be decomposed to two parts

$$\hat{K}^{IJ} = K^{IJ} \times \hat{D}. \tag{3.A.7}$$

K^{IJ} is a symmetric $N \times N$ matrix and $\hat{D}$ is defined on the Hilbert space. We expand the inner products

$$S[\phi, J] = \int dx \frac{1}{2}\phi_I(x)K^{IJ}\hat{D}_x\phi_J(x) + \phi_I(x)J^I(x) \tag{3.A.8}$$

The equation of motion is

$$K^{IJ}\hat{D}\phi_J + J_I = 0. \tag{3.A.9}$$

The solution is given by

$$\phi_I^{\mathrm{cl}}(x) = -(K^{-1})_I{}^J \int dy G(x,y)J_J(y), \tag{3.A.10}$$

where $G(x, y)$ is the Green's function for $\hat{D}$, which means

$$\hat{D}G(x,y) = \delta(x-y). \tag{3.A.11}$$

The effective action

$$\begin{aligned} S[\mathbf{\Phi}^{\mathrm{cl}}] =& \frac{1}{2}\left\langle \phi_I^{\mathrm{cl}}, \hat{K}^{IJ}\phi_J^{\mathrm{cl}}\right\rangle + \left\langle \phi_I^{\mathrm{cl}}, J^I\right\rangle \\ =& -\frac{1}{2}\int dxdy J_I(x)(K^{-1})^{IJ}G(x,y)J_J(y). \end{aligned} \tag{3.A.12}$$