Linear Algebra

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2.1 Harmonic Oscillator

The Hamiltonian for one-dimensional harmonic oscillator is

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2. \tag{2.1}$$

We define

$$a = \frac{1}{\sqrt{2\hbar}}\Big(\sqrt{m\omega}q + i\frac{p}{\sqrt{m\omega}}\Big) \text{ and } a^\dagger = \frac{1}{\sqrt{2\hbar}}\Big(\sqrt{m\omega}q - i\frac{p}{\sqrt{m\omega}}\Big). \tag{2.2}$$

We can get

$$H = \hbar\omega\Big(a^\dagger a + \frac{1}{2}\Big) \tag{2.3}$$

and

$$[a, a^\dagger] = 1. \tag{2.4}$$

There is a long discussion to get the spectrum of this Hamiltonian...

The result is

$$E_n = \hbar\omega\Big(n + \frac{1}{2}\Big); \quad n = 0, 1, 2, \dots\ . \tag{2.5}$$

We define

$$x = \sqrt{\frac{m\omega}{\hbar}} q. \tag{2.6}$$

Then

$$dx = \text{ and } \frac{d}{dx} = \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dq} = i\sqrt{\frac{1}{m\hbar\omega}} p. \tag{2.7}$$

The creation and annihilation operators in the x-representation are

$$a = \frac{1}{\sqrt{2}}\Big(x + \frac{d}{dx}\Big); \quad a^\dagger = \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big). \tag{2.8}$$

The lowest eigenstate is got by the equation

$$\Big(x + \frac{d}{dx}\Big)\varphi_o(x) = 0 \tag{2.9}$$

the general solution is given by

$$\varphi_0(x) = c_0 \exp\left(-\frac{x^2}{2}\right), \tag{2.10}$$

we can choose c_0 from the normalization condition

$$1 \overset{!}{=} \int_{-\infty}^{+\infty} dq |\varphi_0(q)|^2 = \sqrt{\frac{\hbar}{m\omega}} |c_0|^2 \int_{-\infty}^{+\infty} dx e^{-x^2} = \sqrt{\frac{\pi\hbar}{m\omega}} |c_0|^2 \tag{2.11}$$

as $c_0 = (\frac{m\omega}{\hbar\pi})^{1/4}$, and higher eigenwavefunctions can be got from

$$\begin{aligned} \varphi_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{n!}} \langle x|(a^\dagger)^n|0\rangle &= \frac{1}{\sqrt{2^n n!}}\Big(x - \frac{d}{dx}\Big)^n \varphi_0(x) \\ &= \frac{1}{\sqrt{2^n n!}}\Big(\frac{m\omega}{\hbar\pi}\Big)^{1/4}\Big(x - \frac{d}{dx}\Big)^n \exp\left(-\frac{x^2}{2}\right). \end{aligned} \tag{2.12}$$

The equation can be converted to the definition of the Hermite polynomials $\mathrm{H}_n(x)$. The relation is given by

$$\varphi_n(x) = \frac{1}{\sqrt{2^n n!}}\Big(\frac{m\omega}{\hbar\pi}\Big)^{1/4}\exp\left(-\frac{x^2}{2}\right)\mathrm{H}_n(x) \tag{2.13}$$

From (2.12), we get

$$\mathrm{H}_n(x) = \exp\left(\frac{x^2}{2}\right)\Big(x - \frac{d}{dx}\Big)^n \exp\left(-\frac{x^2}{2}\right), \tag{2.14}$$

which is equivalent to the definition of the Hermite polynomials:

$$\mathrm{H}_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}. \tag{2.15}$$

The equivalence can be seen by

$$\begin{aligned}
(-1)^n \exp\left(x^2\right)\frac{d^n}{dx^n}\exp\left(-x^2\right) =&(-1)^n e^{x^2}\frac{d}{dx}e^{-x^2}e^{x^2}\frac{d}{dx}e^{-x^2}\cdots e^{x^2}\frac{d}{dx}e^{-x^2}\\
=&(-1)^n\left[e^{x^2}\frac{d}{dx}e^{-x^2}\right]^n\\
=&(-1)^n\left[e^{\frac{x^2}{2}}e^{\frac{x^2}{2}}\frac{d}{dx}e^{-\frac{x^2}{2}}e^{-\frac{x^2}{2}}\right]^n\\
=&(-1)^n\left[e^{\frac{x^2}{2}}\left(\frac{d}{dx}-x\right)e^{-\frac{x^2}{2}}\right]^n\\
=&e^{\frac{x^2}{2}}\left(x-\frac{d}{dx}\right)e^{-\frac{x^2}{2}}e^{\frac{x^2}{2}}\left(x-\frac{d}{dx}\right)e^{-\frac{x^2}{2}}\cdots e^{\frac{x^2}{2}}\left(x-\frac{d}{dx}\right)e^{-\frac{x^2}{2}}\\
=&e^{\frac{x^2}{2}}\left(x-\frac{d}{dx}\right)^n e^{-\frac{x^2}{2}}.
\end{aligned} \tag{2.16}$$

The only non-trvial step is the operator relation

$$\frac{1}{f(x)}\frac{d}{dx}f(x) = \frac{d}{dx} + \frac{f'(x)}{f(x)}. \tag{2.17}$$

Now we give the general solution of the one-dimensional harmonic oscillator, which may have a non-trvial displacement d from the original point,

$$\varphi_n(q) = \frac{1}{\sqrt{2^n n!}}\Big(\frac{m\omega}{\hbar\pi}\Big)^{1/4}\exp\left(-\frac{(q+d)^2}{2l_\omega^2}\right)\mathrm{H}_n\Big(\frac{q+d}{l_\omega}\Big), \tag{2.18}$$

where $l_\omega \equiv \sqrt{\frac{\hbar}{m\omega}}$.

2.2 Landau Level

The Lagrangian for a particle of charge $-e$ and mass m moving in a magnetic vector potential is

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - e\dot{\mathbf{x}} \cdot \mathbf{A}. \tag{2.19}$$

The canonical momentum is

$$\mathbf{p} = m\dot{\mathbf{x}} - e\mathbf{A}. \tag{2.20}$$

The Hamiltonian is

$$H = \dot{\mathbf{x}} \cdot \mathbf{p} - L = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2. \tag{2.21}$$

Define

$$\pi \equiv \mathbf{p} + e\mathbf{A} = m\dot{\mathbf{x}}. \tag{2.22}$$

Possoin brackets are

$$\{x_i, p_j\} = \delta_{ij} \text{ and } \{x_i, x_j\} = \{p_i, p_j\} = 0. \tag{2.23}$$

We can get

$$\{\pi_i, \pi_j\} = \sum_{k=1}^{3} \frac{\partial \pi_i}{\partial x_k}\frac{\partial \pi_j}{\partial p_k} - \frac{\partial \pi_i}{\partial p_k}\frac{\partial \pi_j}{\partial x_k} = -e\epsilon_{ijk}B_k. \tag{2.24}$$

The quantization is given by

$$[x_i, p_j] = i\hbar\delta_{ij} \text{ and } [x_i, x_j] = [p_i, p_j] = 0. \tag{2.25}$$

The particle is constrainted in the x-y plane and the magnetic field is set to $\nabla \times \mathbf{A} = B\hat{\mathbf{z}}$, the Hamiltonian is

$$H = \frac{1}{2m}(\pi_x^2 + \pi_y^2), \tag{2.26}$$

and we have[1]

$$[\pi_x, \pi_y] = -ie\hbar B. \tag{2.27}$$

If we define

$$a = \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \text{ and } a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y), \tag{2.28}$$

we have

$$[a, a^\dagger] = 1 \tag{2.29}$$

Now the Hamiltonian is

$$H = \hbar\omega_B \left(a^\dagger a + \frac{1}{2}\right), \tag{2.30}$$

[1] We can see if we set $p = -\pi_y$ and $q = \pi_x/eB$, the Hamiltonian and commutator is just the same as that of the harmonic oscillator.

where $\omega_B \equiv eB/m$. This Hamiltonian has the same algebraic structure as the that of harmonic. Thus we have the spectrum

$$E_n = \hbar\omega_B \left(n + \frac{1}{2}\right), \tag{2.31}$$

which is called the Landau level. Then we compute the eigenstates in two gauges respectively.

2.2.1 Landau Gauge

The Landau gauge is the choice

$$\mathbf{A} = xB\hat{\mathbf{y}}. \tag{2.32}$$

Now the Hamiltonian is

$$H = \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_B^2\Big(x + \frac{p_y}{eB}\Big)^2. \tag{2.33}$$

There exists translation invariance in y direction, meaning $[p_y, H] = 0$, which tells us p_y eigenvalues are good quantum number to characterize states. We can use the ansatz

$$\psi_k(x, y) = e^{iky} f_k(x). \tag{2.34}$$

Apply the Hamiltonian to this state, we get

$$H\psi_k(x, y) = \Big[\frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_B^2\Big(x + \frac{\hbar k}{eB}\Big)^2\Big]\psi_k(x, y) \equiv e^{iky} H_k f_k(x), \tag{2.35}$$

where H_k can be interpreted as a harmonic oscillator with a displacement. By using the result (2.18), we get

$$\psi_k(x, y) \sim e^{iky}\mathrm{H}_n\Big(\frac{x + kl_B^2}{l_b}\Big)e^{-(x+kl_B^2)^2/2l_B^2} \tag{2.36}$$

where $l_B = \sqrt{\frac{\hbar}{m\omega_B}} = \sqrt{\frac{\hbar}{eB}}$.

2.2.2 Symmetric Gauge

The symmetric gauge is

$$\mathbf{A} = -\frac{yB}{2}\hat{\mathbf{x}} + \frac{xB}{2}\hat{\mathbf{y}}. \tag{2.37}$$

We have rotational symmetry about the origin in this choice of gauge, which means the angular momentum is a good quantum number.

3 Angular Momentum

4 Central Potential

5 Approximation Methods

6 Scattering Theory