Time stepping in land surface scheme

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Abstract

This document discusses the core set of equations of the land surface scheme, their discretization in time, and the solution technique for the discretized system of equations on each time step.

Contents

1. Mass and energy balance equations

The vegetated land exchanges mass and energy between vegetation, ground (which can be snow-covered), canopy air, and atmosphere. We assume that turbulent mass and heat exchange occurs between canopy air and vegetation, and between canopy air and ground surface; only canopy air exchanges sensible heat and mass with the atmosphere.

1a. *Canopy air*

To write the canopy air balance equations, let's assume that the only processes that perceptibly affect energy and mass balance of the canopy air are turbulent exchanges with surrounding environment. In other words, let's assume that the thermal radiation emitted and absorbed by the canopy air, the short-wave radiation absorption in the canopy air, phase changes of water inside the canopy air (i.e. fog formation), and heat and moisture exchange between the falling precipitation and canopy air can all be neglected.

In this assumption, the canopy air receives the flux of sensible heat from the canopy H_v and the sensible heat flux from ground surface H_g, and passes the sensible heat flux H_a up to the atmosphere. Similarly, the canopy air receives turbulent water vapor fluxes from the vegetation E_v and from ground E_g, and passes flux E_a up to the atmosphere; the water vapor also carries with it sensible heat proportional to its temperature and heat capacity. The mass and energy balances of canopy air can therefore be written simply as following[1]:

$$m_c \frac{dq_c}{dt} = E_v + E_g - E_a \tag{1.1}$$ eq:cana-mb

$$m_c \frac{d}{dt}\big((1-q_c)c_p T_c + q_c c_v T_c\big) = H_v + H_g - H_a + c_v(T_v E_v + T_g E_g - T_c E_a) \tag{1.2}$$ eq:cana-eb

where m_c is the mass of the canopy air, c_p and c_v are specific heat capacities of dry air and water vapor at constant pressure, T_c is the canopy air temperature, and q_c is the canopy air specific humidity.

Naturally, the full water vapor flux from the canopy to the canopy air is equal to the sum of transpiration, evaporation of intercepted water and sublimation of intercepted snow:

$$E_v \equiv E_t + E_l + E_s \tag{1.3}$$

[1] Equation (1.2) is inconsistent with the equation of canopy energy balance in P. C. D. Milly's notes (page 8, section 2.1.2) where the terms associated with heat transported by water vapor are absent.

Note that for simplicity we assumed that the mass of the canopy air m_c does not change with time, and therefore can be moved outside of the derivatives in the left-hand side of (1.1) and (1.2).

1b. *Vegatation canopy*

The processes controlling mass and energy balances of the vegetation canopy are somewhat more complicated, because the overall mass and heat capacity of the canopy can change due to interception of rainfall P_l and snowfall P_s, evaporation of intercepted water E_l and sublimation of intercepted snow E_s. Water and snow can also drip from the canopy to the ground with the rates D_l and D_s respectively. Besides, the intercepted snow can melt with the rate M_i (or the intercepted liquid water can freeze with the rate $-M_i$). The resuling mass balance equations for the intercepted liquid water w_l and snow w_s are:

$$\frac{dw_l}{dt} = f_{il}P_l - D_l - E_l + M_i \tag{1.4}$$

eq:vegn-mb-l

$$\frac{dw_s}{dt} = f_{is}P_s - D_s - E_s - M_i \tag{1.5}$$

eq:vegn-mb-f

where f_{il} and f_{is} are fractions of the total rainfall and snowfall that canopy intercepts.

Strictly speaking, the mass of the dry canopy also changes, for example due to vegetation growth or seasonal changes. Besides, the chemical processes involved in the vegetation growth (for example, carbon uptake and respiration) can also contribute to the energy balance of the vegeation, and probably to the energy balance of the ground. However, let's assume (at least for now) that those processes are not very important for the mass and energy balance and can be safely ignored here.

With the equations of mass balance (1.4) and (1.5) in mind, and assuming that the intercepted water, snow, and canopy always have identical temperature T_v, the canopy energy balance equation can be written as:

$$\begin{aligned}\frac{d}{dt}(C_vT_v + w_lc_lT_v + w_sc_sT_v) = R_{Sv} + R_{Lv} - H_v \\ - E_t(L_{e0} + c_vT_v - c_lT_u) \\ - E_l(L_{e0} + c_vT_v) \\ - E_s(L_{s0} + c_vT_v) \\ + c_l(T_{pl}f_{il}P_l - T_vD_l) + c_s(T_{ps}f_{is}P_s - T_vD_s) - L_fM_i\end{aligned} \tag{1.6}$$

eq:vegn-eb

where C_v is the total heat capacity of dry vegetation, R_{Sv} and R_{Lv} are the net short-wave and long-wave radiative balances of the canopy, c_l and c_s are the specific heats of liquid and solid (frozen) water, E_t is the transpiration through the canopy (which we naturally assume to be liquid water), L_{e0}, L_{s0} are specific heats of evaporation and sublimation extrapolated to $T = 0$, L_f is the specific heat of fusion, T_{pl} and T_{ps} are the temperatures of liquid and frozen precipitation.

For discussion of vaporization-related terms in the equation (1.6) see section A on page 24.

1c. *Ground surface*

The energy balance of the ground surface can be written as:

$$R_{Sg} + R_{Lg} - H_g - L_g E_g - G - L_f M_g = 0 \tag{1.7}$$

eq:grnd-eb

where R_{Sg} and R_{Lg} ar net short-wave and long-wave radiative balances of the ground surface, L_g is latent heat of evaporation or transpiration, G is the ground heat flux from the surface, and M_g is melting rate of the ground surface. Note that since the water vapor flux E_g can come from a mixture of evaporation and sublimation, the effective heat of vaporization L_g is also a mixture of L_e and L_s.

Equations (1.1) – (1.7), together with functional dependencies of the fluxes on variables and parameters of the land model represent the core equation system of the land surface scheme, at least from the point of view of the fast time-scale interaction with the atmosphere.

Typically the heat capacities in the equations (1.1) – (1.7) are small compared to contributions of the terms in the right-hand sides of the equations per time step. Therefore a straightforward explicit approach to the numerical solution of the system would lead to unacceptable instabilities, or oscillatory behavior. Following the example of the atmospheric model set in Held (2001), and previous versions of the land surface scheme, a fully implicit scheme is used to solve the system.

An important distinctive feature of the system (1.1) – (1.7) is, however, the fact that the equation (1.7) is significantly nonlinear. In particular, the ground melt term M_g depends non-linearly on the temeparture, on the sign of the sum of the rest of the energy balance components, and on the amount of frozen water actually available for melting (or liquid water available for freezing). Therefore, strictly speaking the numerical solution of equations (1.1) – (1.7) can't be implemented as a straightforward implicit time stepping scheme. To work around the problem, we linearize and solve the reduced system of equations (1.1) – (1.6), obtaining as a result the tendencies of canopy and canopy air prognostic variables expressed in terms of the surface prognostic variable T_g. That allows us to solve a single non-linear equation of surface energy balance for the tendency of ground temperature, and then use this tendency to get the final values of the canopy variables at the end of the time step.

2. Explicitly treated terms

Since solving of linearized equations implicitly is a rather cumbersome process, one would like to simplify the system as much as possible before going ahead to numerical solution. One way to do this is to exclude some terms that can't cause numerical instability if treated explicitly.

The separation of the terms is fairly arbitrary, but in general the right-hand side terms that can be left out for explicit treatment should be the ones that are either sufficiently small, or structured so that the can't cause numerical instabilities.

In our case, we choose to exclude the intercepted water phase change M_i and ground phase change rate M_g. See more on the explicit treatment of canopy water in section 3g on page 11.

3. Linearized and discretized equations

sect:linearization

3a. *Prognostic variables and dependence of fluxes on them*

The prognostic variables of the above system are three temperatures T_c, T_v, and T_g, specific humidity in the canopy air q_c, and masses of intrecepted liquid and frozen water w_l, and w_s. Each of the fluxes between components and atmosphere depends on a subset of prognostic variables:

$$H_v = H_v(T_v, T_c) \qquad H_g = H_g(T_g, T_c) \qquad H_a = H_a(T_c) \tag{3.1a}$$

$$E_t = E_t(T_v, q_c) \qquad E_l = E_l(T_v, q_c, w_l, w_s) \qquad E_s = E_s(T_v, q_c, w_l, w_s) \tag{3.1b}$$

$$E_g = E_g(T_g, q_c, \psi_g) \qquad E_a = E_a(q_c) \tag{3.1c}$$

$$R_{Lv} = R_{Lv}(T_v, T_g) \qquad R_{Lg} = R_{Lg}(T_g, T_v) \tag{3.1d}$$

$$G = G(T_g) \tag{3.1e}$$

To solve the energy balance equations implicitly we must linearize the fluxes in terms of prognostic variables around the current state. That is, suppose that some flux is described by some, possibly nonlinear expression

$$F = F(x_1, x_2, \cdots, x_n) \tag{3.2}$$

eq:flux-exact

and suppose that the current state of the prognostic variables is $(x_1^0, x_2^0, \cdots, x_n^0)$. Then we replace the exact expression for the flux (3.2) with it's Taylor decomposition to the first order:

$$F(x_1, x_2, \cdots, x_n) \approx F(x_1^0, x_2^0, \cdots, x_n^0) + \frac{\partial F}{\partial x_1}\Delta x_1 + \frac{\partial F}{\partial x_2}\Delta x_2 + \cdots \frac{\partial F}{\partial x_n}\Delta x_n \tag{3.3}$$

where $\Delta x_i = x_i - x_i^0$ and all partial derivatives are calculated at the current state of prognostic variables $(x_1^0, x_2^0, \cdots, x_n^0)$.

sect:discr-lin

3b. *Discretization and linearization*

The equation of canopy air mass balance (1.1), after substituting linearized terms, yields:

$$\begin{aligned} m_c \frac{\Delta q_c}{\Delta t} = E_{v0} &+ \left(\frac{\partial E_v}{\partial T_v}\right)\Delta T_v + \left(\frac{\partial E_v}{\partial q_c}\right)\Delta q_c \\ &+ \left(\frac{\partial E_v}{\partial w_l}\right)\Delta w_l + \left(\frac{\partial E_v}{\partial w_s}\right)\Delta w_s \\ &+ E_{g0} + \left(\frac{\partial E_g}{\partial T_g}\right)\Delta T_g + \left(\frac{\partial E_g}{\partial \psi_g}\right)\Delta \psi_g + \left(\frac{\partial E_g}{\partial q_c}\right)\Delta q_c \\ &- E_{a0} - \left(\frac{\partial E_a}{\partial q_c}\right)\Delta q_c \end{aligned} \tag{3.4}$$

eq:cana-mb-lin

where $\Delta x \equiv x^{t+1} - x^t$ is the change of variable x during the time step, and Δt is the time step.

Equation (1.2) is more complicated:

$$
\begin{aligned}
& m_c\Big[\big(c_p + q_c(c_v - c_p)\big)\frac{\Delta T_c}{\Delta t} + (c_v - c_p)T_c\frac{\Delta q_c}{\Delta t} + (c_v - c_p)\frac{\Delta q_c \Delta T_c}{\Delta t}\Big] \\
&\quad = H_{v0} + \left(\frac{\partial H_v}{\partial T_v}\right)\Delta T_v + \left(\frac{\partial H_v}{\partial T_c}\right)\Delta T_c \\
&\quad + H_{g0} + \left(\frac{\partial H_g}{\partial T_g}\right)\Delta T_g + \left(\frac{\partial H_g}{\partial T_c}\right)\Delta T_c \\
&\quad - H_{a0} - \left(\frac{\partial H_a}{\partial T_c}\right)\Delta T_c \\
&\quad + c_v T_v\Big[E_{v0} + \frac{\partial E_v}{\partial q_c}\Delta q_c + \frac{\partial E_v}{\partial T_v}\Delta T_v + \frac{\partial E_v}{\partial w_l}\Delta w_l + \frac{\partial E_v}{\partial w_s}\Delta w_s\Big] \\
&\quad + c_v T_g\Big[E_{g0} + \frac{\partial E_g}{\partial q_c}\Delta q_c + \frac{\partial E_g}{\partial \psi_g}\Delta \psi_g + \frac{\partial E_g}{\partial T_g}\Delta T_g\Big] \\
&\quad - c_v T_c\Big[E_{a0} + \frac{\partial E_a}{\partial q_c}\Delta q_c\Big]
\end{aligned} \tag{3.5}
$$

eq:cana-eb-lin

Note that for the sake of simplicity, we omit the linearization of the terms in front of the water vapor fluxes in equation (1.2) and (1.6). For the sake of completeness, however, the full equations are presented in the section B on page 25.

Similarly, from equations (1.4), (1.5), and (1.6) with "explicit" terms excluded we have:

$$
\frac{\Delta w_l}{\Delta t} = f_{il}P_l - D_l - E_{l0} - \frac{\partial E_l}{\partial T_v}\Delta T_v - \frac{\partial E_l}{\partial q_c}\Delta q_c - \frac{\partial E_l}{\partial w_l}\Delta w_l - \frac{\partial E_s}{\partial w_s}\Delta w_s \tag{3.6}
$$

eq:vegn-mb-l-lin

$$
\frac{\Delta w_s}{\Delta t} = f_{is}P_s - D_s - E_{s0} - \frac{\partial E_s}{\partial T_v}\Delta T_v - \frac{\partial E_s}{\partial q_c}\Delta q_c - \frac{\partial E_s}{\partial w_l}\Delta w_l - \frac{\partial E_s}{\partial w_s}\Delta w_s \tag{3.7}
$$

eq:vegn-mb-f-lin

$$
\begin{aligned}
& (C_v + c_l w_l + c_s w_s)\frac{\Delta T_v}{\Delta t} + c_l T_v\frac{\Delta w_l}{\Delta t} + c_s T_v\frac{\Delta w_s}{\Delta t} + \frac{\Delta T_v(c_l\Delta w_l + c_s\Delta w_s)}{\Delta t} \\
&\quad = R_{Sv} + R_{Lv0} + \frac{\partial R_{L_v}}{\partial T_v}\Delta T_v + \frac{\partial R_{Lv}}{\partial T_g}\Delta T_g \\
&\quad - H_{v0} - \frac{\partial H_v}{\partial T_v}\Delta T_v - \frac{\partial H_v}{\partial T_c}\Delta T_c \\
&\quad - L_t'\Big[E_{t0} + \frac{\partial E_t}{\partial q_c}\Delta q_c + \frac{\partial E_t}{\partial T_v}\Delta T_v + \frac{\partial E_t}{\partial w_l}\Delta w_l + \frac{\partial E_t}{\partial w_s}\Delta w_s\Big] \\
&\quad - L_e'\Big[E_{l0} + \frac{\partial E_l}{\partial q_c}\Delta q_c + \frac{\partial E_l}{\partial T_v}\Delta T_v + \frac{\partial E_l}{\partial w_l}\Delta w_l + \frac{\partial E_l}{\partial w_s}\Delta w_s\Big] \\
&\quad - L_s'\Big[E_{s0} + \frac{\partial E_s}{\partial q_c}\Delta q_c + \frac{\partial E_s}{\partial T_v}\Delta T_v + \frac{\partial E_s}{\partial w_l}\Delta w_l + \frac{\partial E_s}{\partial w_s}\Delta w_s\Big] \\
&\quad + c_l\big[f_{il}P_l T_{pl} - (T_v + \Delta T_v)D_l\big] + c_s\big[f_{is}P_s T_{ps} - (T_v + \Delta T_v)D_s\big]
\end{aligned} \tag{3.8}
$$

eq:vegn-eb-lin

where

$$L'_t = L_{e0} + c_v T_v - c_l T_u \tag{3.9a}$$

$$L'_e = L_{e0} + c_v T_v \tag{3.9b}$$

$$L'_s = L_{s0} + c_v T_v \tag{3.9c}$$

Finally, from the energy balance of the ground (1.7), we have:

$$\begin{aligned} R_{Sg} &+ R_{Lg0} + \left(\frac{\partial R_{Lg}}{\partial T_g}\right)\Delta T_g + \left(\frac{\partial R_{Lg}}{\partial T_v}\right)\Delta T_v \\ &- H_{g0} - \left(\frac{\partial H_g}{\partial T_g}\right)\Delta T_g - \left(\frac{\partial H_g}{\partial T_c}\right)\Delta T_c \\ &- L_g\left[E_{g0} + \left(\frac{\partial E_g}{\partial T_g}\right)\Delta T_g + \left(\frac{\partial E_g}{\partial \psi_g}\right)\Delta \psi_g + \left(\frac{\partial E_g}{\partial q_c}\right)\Delta q_c\right] \\ &- G_0 - \left(\frac{\partial G}{\partial T_g}\right)\Delta T_g \end{aligned} \tag{3.10}$$

eq:grnd-eb-lin

Note that in the right-hand side of the equations (3.5) and (3.8) there are terms that make them nonlinear with respect to variable tendencies. They can't be left out altogether, since they are essential for the energy conservation, but we can't tolerate them in the linearized system either, since they make the solution awkward. We propose, however, to split them out for now just like the "explicit" terms, and deal with them on a later stage, after linear implicit time step is done (see Section 3e on page 10).

The system of equations (3.4) – (3.10) with the nonlinear terms excluded is the system we are going to solve each time step for the tendencies Δq_c, ΔT_c, ΔT_v, Δw_l, Δw_s, and ΔT_g.

3c. *Canonical form of equations*

It seems useful to convert the system (3.4) – (3.10) to the 'canonical' form – that is, assemble all the unknowns in the left-hand side and free terms in the right-hand side.

Equation of canopy air mass balance (1.1):

$$\begin{aligned} \Delta q_c\left[\frac{m_c}{\Delta t} - \frac{\partial E_v}{\partial q_c} - \frac{\partial E_g}{\partial q_c} + \frac{\partial E_a}{\partial q_c}\right] - \Delta T_v\frac{\partial E_v}{\partial T_v} - \Delta w_l\frac{\partial E_v}{\partial w_l} - \Delta w_s\frac{\partial E_v}{\partial w_s} \\ - \Delta T_g\frac{\partial E_g}{\partial T_g} - \Delta\psi_g\frac{\partial E_g}{\partial \psi_g} = E_{v0} + E_{g0} - E_{a0} \end{aligned} \tag{3.11}$$

eq:cana-mb-canon

Equation of canopy air energy balance (1.2):

$$
\begin{aligned}
&\Delta q_c\left[\frac{m_c(c_v-c_p)T_c}{\Delta t}-c_vT_v\frac{\partial E_v}{\partial q_c}-c_vT_g\frac{\partial E_g}{\partial q_c}+c_vT_c\frac{\partial E_a}{\partial q_c}\right]\\
&+\Delta T_c\left[\frac{m_c\big(c_p+q_c(c_v-c_p)\big)}{\Delta t}-\frac{\partial H_v}{\partial T_c}-\frac{\partial H_g}{\partial T_c}+\frac{\partial H_a}{\partial T_c}\right]\\
&-\Delta T_v\left[\frac{\partial H_v}{\partial T_v}+c_vT_v\frac{\partial E_v}{\partial T_v}\right]\\
&-\Delta w_l\, c_vT_v\frac{\partial E_v}{\partial w_l}\\
&-\Delta w_s\, c_vT_v\frac{\partial E_v}{\partial w_s}\\
&-\Delta T_g\left[\frac{\partial H_g}{\partial T_g}+c_vT_g\frac{\partial E_g}{\partial T_g}\right]\\
&-\Delta\psi_g c_vT_g\frac{\partial E_g}{\partial \psi_g}\\
&\qquad\qquad = H_{v0}+H_{g0}-H_{a0}+c_v(T_vE_{v0}+T_gE_{g0}-T_cE_{a0})
\end{aligned}
\tag{3.12}
$$

cana-eb-canon

Equation of vegetation energy balance (1.6):

$$
\begin{aligned}
&\Delta q_c\left[L_t'\frac{\partial E_t}{\partial q_c}+L_e'\frac{\partial E_l}{\partial q_c}+L_s'\frac{\partial E_s}{\partial q_c}\right]+\Delta T_c\frac{\partial H_v}{\partial T_c}\\
&+\Delta T_v\left[\frac{C_v+c_lw_l+c_sw_s}{\Delta t}-\frac{\partial R_{Lv}}{\partial T_v}+\frac{\partial H_v}{\partial T_v}\right.\\
&\qquad\qquad\left.+L_t'\frac{\partial E_t}{\partial T_v}+L_e'\frac{\partial E_l}{\partial T_v}+L_s'\frac{\partial E_s}{\partial T_v}+c_lD_l+c_sD_s\right]\\
&+\Delta w_l\left[\frac{c_lT_v}{\Delta t}+L_t'\frac{\partial E_t}{\partial w_l}+L_e'\frac{\partial E_l}{\partial w_l}+L_s'\frac{\partial E_s}{\partial w_l}\right]\\
&+\Delta w_s\left[\frac{c_sT_v}{\Delta t}+L_t'\frac{\partial E_t}{\partial w_s}+L_e'\frac{\partial E_l}{\partial w_s}+L_s'\frac{\partial E_s}{\partial w_s}\right]\\
&-\Delta T_g\frac{\partial R_{Lv}}{\partial T_g}\\
&\qquad\qquad = R_{Sv}+R_{Lv0}-H_{v0}-L_t'E_{t0}-L_e'E_{l0}-L_s'E_{s0}\\
&\qquad\qquad\quad +c_lf_{il}P_lT_{pl}-c_lD_lT_v+c_sf_{is}P_sT_{ps}-c_sD_sT_v
\end{aligned}
\tag{3.13}
$$

eq:vegn-eb-canon

Equation of intercepted water balance (1.4):

$$
\Delta q_c\frac{\partial E_l}{\partial q_c}+\Delta T_v\frac{\partial E_l}{\partial T_v}+\Delta w_l\left[\frac{1}{\Delta t}+\frac{\partial E_l}{\partial w_l}\right]+\Delta w_s\frac{\partial E_l}{\partial w_s}=f_{il}P_l-E_{l0}-D_l \tag{3.14}
$$

eq:vegn-mb-l-canon

Equation of intercepted snow balance (1.5):

$$
\Delta q_c\frac{\partial E_s}{\partial q_c}+\Delta T_v\frac{\partial E_s}{\partial T_v}+\Delta w_l\frac{\partial E_s}{\partial w_l}+\Delta w_s\left[\frac{1}{\Delta t}+\frac{\partial E_s}{\partial w_s}\right]=f_{is}P_s-E_{s0}-D_s \tag{3.15}
$$

eq:vegn-mb-f-canon

Equation of ground surface energy balance (1.5):

$$\begin{aligned} L_g \Delta q_c \frac{\partial E_g}{\partial q_c} + \Delta T_c \frac{\partial H_g}{\partial T_c} - \Delta T_v \frac{\partial R_{lg}}{\partial T_v} + \Delta \psi_g L_g \frac{\partial E_g}{\partial \psi_g} \\ + \Delta T_g \left[-\frac{\partial R_{Lg}}{\partial T_g} + \frac{\partial H_g}{\partial T_g} + L_g \frac{\partial E_g}{\partial T_g} + \frac{\partial G}{\partial T_g} \right] \\ = R_{Sg} + R_{Lg0} - H_{g0} - L_g E_{g0} - G_0 \end{aligned} \tag{3.16}$$

eq:grnd-eb-canon

The equations (3.11) – (3.16) can be solved analytically, with a lengthy chain of substitutions. Alternatively, the resulting system of equations (3.11) – (3.16) can be solved numerically on each time step using one of well-known and efficient techniques. Besides being less error-prone and more easily testable, this latter method has an advantage of being more general: when any new prognostic variables and/or interdependencies are introduced in the system, then it easier to modify the system and let the numerical solver deal with the extended equations, rather than re-do the tedious analytic solution by hand[2].

sect:constraints

3d. *Material and physical constraints*

The equations (3.11) – (3.16) are written in assumption of linearity, or near-linearity of the system. Typically, the land surface system can be assumed to be approximately linear, but there are some important cases when the linearized equations would give a less then satisfactory solution. One example of such a situation is a time step where there is not enough material to satisfy flux demand – for example $E_g \Delta t > w_g$, where w_g is the amount of water available for evaporation at the beginning of time step. This probably happens most frequently on the ground surface, where the amount of surface water is typically small, and downregulation of evaporability may not be strong enough to sufficiently restrict the water vapor flux. Another example of problematic situation would occur if predicted temperature would go higher that freezing while there is snow on the ground (or go below freezing point while there is some freezable liquid water).

Strictly speaking, to resolve such situations in a general way, we need to solve non-linear system of equations with constraints on each time step, or may be use something similar to linear optimization techniques.

However, in some limited cases when we allow the non-linear limitations only at the ground surface the linearized system can be generalized and solved without expensive computations.

Let's consider a reduced system (3.11) – (3.13); while we can't directly solve it obtain the final values of the increments of prognostic variables, we can solve it partially, expressing each increment in terms of ΔT_g, so that for each variable X

$$\Delta X = b_X^{(0)} + b_X^{(1)} \Delta T_g + b_X^{(2)} \Delta \psi_g \tag{3.17}$$

eq:vars-through-tg

[2] Yet another alternative would be to create a "solver generator" – a program that, given a matrix of the canonical system written out analytically, would produce a piece of code that solves it using, say, elimination technique but entirely skipping the terms that are a priori equal to zero. Since the matrix is approximately half-empty, this solver just by construction should be more efficient than the numerical one.

In the long run, that latter approach is probably the best possible option, since it provides essentially the same flexibility for future changes as numerical solution, but doesn't incur associated performance penalties. The only difficulty is coding the "solver generator", but it can be done.

Then, substituting these expressions for Δq_c, ΔT_c, and ΔT_v in (3.16), we get:

$$
\begin{aligned}
L_g \frac{\partial E_g}{\partial q_c}\left(b_{qc}^{(0)} + b_{qc}^{(1)}\Delta T_g + b_{qc}^{(2)}\Delta\psi_g\right) + \frac{\partial H_g}{\partial T_c}\left(b_{Tc}^{(0)} + b_{Tc}^{(1)}\Delta T_g + b_{Tc}^{(2)}\Delta\psi_g\right) \\
- \frac{\partial R_{lg}}{\partial T_v}\left(b_{Tv}^{(0)} + b_{Tv}^{(1)}\Delta Tg + b_{Tv}^{(2)}\Delta\psi_g\right) \\
+ \Delta T_g\left[-\frac{\partial R_{Lg}}{\partial T_g} + \frac{\partial H_g}{\partial T_g} + L_g\frac{\partial E_g}{\partial T_g} + \frac{\partial G}{\partial T_g}\right] + \Delta\psi_g L_g \frac{\partial E_g}{\partial \psi_g} \\
= R_{Sg} + R_{Lg0} - H_{g0} - L_g E_{g0} - G_0
\end{aligned}
\tag{3.18}
$$

Gathering similar terms, we get:

$$B_g + \frac{DB_g}{DT_g}\Delta T_g + \frac{DB_g}{D\psi_g}\Delta\psi_g = 0 \tag{3.19}$$

where B_g is effective energy balance of the surface:

$$B_g = R_{Sg} - G_0 + R_{Lg0} + b_{Tv}^{(0)}\frac{\partial R_{lg}}{\partial T_v} - H_{g0} - b_{Tc}^{(0)}\frac{\partial H_g}{\partial T_c} - L_g\left[E_{g0} + b_{qc}^{(0)}\frac{\partial E_g}{\partial q_c}\right] \tag{3.20}$$

DB_g/DT_g is full derivative of this balance with respect to surface temperature:

$$\frac{DB_g}{DT_g} = \frac{\partial R_{Lg}}{\partial T_g} + b_{Tv}^{(1)}\frac{\partial R_{lg}}{\partial T_v} - \frac{\partial H_g}{\partial T_g} - b_{Tc}^{(1)}\frac{\partial H_g}{\partial T_c} - L_g\left[\frac{\partial E_g}{\partial T_g} + b_{qc}^{(1)}\frac{\partial E_g}{\partial q_c}\right] - \frac{\partial G}{\partial T_g} \tag{3.21}$$

and $DB_g/D\psi_g$ is full derivative of this balance with respect to surface water potential:

$$\frac{DB_g}{D\psi_g} = b_{Tv}^{(2)}\frac{\partial R_{lg}}{\partial T_v} - b_{Tc}^{(2)}\frac{\partial H_g}{\partial T_c} - L_g\left[\frac{\partial E_g}{\partial \psi_g} + b_{qc}^{(2)}\frac{\partial E_g}{\partial q_c}\right] \tag{3.22}$$

B_g has a physical meaning of the surface energy balance after the time step if the temperature of the ground surface and the soil water potential do not change.

Given this equation, and linearized expressions for surface fluxes expressed in terms of ΔT_g, we can easily obtain the final value of ΔT_g that satisfies the material and physical constraints at the surface. Substituting the result in the expressions (3.17) we get the final values of ΔX.

sect:residual

3e. *Energy residuals*

Now let's recall that we dropped two nonlinear second-order terms from the left-hand sides of the equations (3.5) and (3.8), namely $m_c(c_v - c_p)\frac{\Delta q_c \Delta T_c}{\Delta t}$ and $\frac{\Delta T_v(c_l \Delta w_l + c_s \Delta w_s)}{\Delta t}$.

There are two problems with them: first, one would hope that these terms are not big compared to the other terms, because otherwise the solution of linearized system would not make much sense. Second, even if they are not big we must take them into account somehow, because failure to do so would lead to violation of energy conservation law.

The first problem can be examined in real situations by just saving the second-order term estimates calculated after the linear time step and comparing them to the other energy balance components. A short two-day run started in July showed that the values of residuals are fairly small, typically on the order of few thousandths $\mathrm{W/m^2}$, although the maximum value of canopy energy balance residual reached about $0.1\,\mathrm{W/m^2}$ in an isolated point near Tibet. There is a pronounced diurnal cycle with spikes when the changes in temperatures and masses are the fastest.

There are many possible ways to address the second problem. For example, it is possible to add the estimated energy imbalance immediately to some other component of the system – preferably some mostly linear and highly thermally inertial component, like ground. Alternatively, it is possible to store the imbalance and add it to the right-hand side of the same equations on the next time step. It's not immediately clear which way is better, since both have their disadvantages.

3f. *Degenerate cases*

In certain cases some equations of the system (3.11) – (3.16), due to specifics of locations and specified parameter values, turn into trivial identities. In particular, if there is no vegetation, then all terms in both sides of the equation (3.13), except terms in front of Δw_l and Δw_s are equal to zero. At the same time, the equations (3.14) and (3.15) turn into simply $\Delta w_l = 0$ and $\Delta w_s = 0$. That means that the matrix of (3.11) – (3.16) in this case is singular (because equations (3.14), (3.15), and (3.13), are linearly dependent), and therefore can't be solved.

While it is possible to exclude (3.13) and then solve the reduced system for this special case, it would be nice not to have to change the rank of the equation system on the fly. Obviously, this can be easily achieved if one specifies an arbitrary dummy value for coefficient in front of ΔT_v, ensuring that the entire equation (3.13) effectively turns into $\Delta T_v = 0$ in this special case. The simplest way to do that is to set vegetation heat capacity C_v to some value, for example 1, when there is no vegetation present, or there is no leaves.

Note that in contrast to the canopy energy balance, the equations of canopy mass balance (3.14) and (3.15) never turn singular, since the term $1/\Delta t$ never vanishes. Of course, one has to make sure that for consistency the fluxes E_l and E_s and their derivatives are equal to zero in the case when the canopy is absent.

3g. *Explicit treatment of canopy water phase change*

sect:melt

Suppose there is some snow on the canopy, but the temperature of the canopy after the implicit time step is above freezing point. In this case, to get physically consistent solution, we should melt some water and reduce temperature accordingly, to conserve energy. The full energy of the canopy and canopy water before phase change can be expressed as:

$$U^0 = \left[C_v + c_l w_l + c_s w_s\right](T_v^0 - T_f) - L_f w_s \tag{3.23}$$

eq:before-melt

where T_f is freezing point of water, and T_v^0 is the canopy temperature after the implicit time step. Then the amount of water that melts is:

$$\Delta w = \min(w_s, \frac{C_v + c_l w_l + c_s w_s}{L_f}(T_v^0 - T_f)) \tag{3.24}$$

and the temperature after the melt will satisfy the energy equation:

$$U^0 = U^1 = \left[C_v + c_l(w_l + \Delta w) + c_s(w_s + \Delta w)\right](T_v^1 - T_f) - L_f(w_s - \Delta w) \quad (3.25)$$ eq:after-melt

After some simple conversions of equations (3.23) and (3.25) we finally get:

$$T_v^1 = T_f + \frac{(T_v^0 - T_f)\left[C_v + c_l w_l + c_s w_s\right] - L_f \Delta w}{C_v + c_l w_l + c_s w_s + \Delta w(c_l - c_s)} \quad (3.26)$$ eq:melt-t-change

Equation (3.26) definitely conserves energy, but unfortunately in some situations results in underestimate of the melt, and, as a result, in extended periods of time when canopy temperature stays at T_f even though the turbulent exchange fluxes toward the canopy should melt the snow pretty quickly.

To illustrate the problem, let's consider a simplified equation:

$$C\frac{dT}{dt} = B(T, \dots) \quad (3.27)$$

where B represents the full energy balance of the canopy. The liearized discretized version will look like that:

$$C\frac{\Delta T}{\Delta t} = B_0 + k\Delta T \quad (3.28)$$

where for brevity we introduced notation $k = \frac{\partial B}{\partial T}$. Suppose, for simplicity, that the temperature at the beginning of the time step is equal to the water freezing point T_f, and B_0 happens to be positive (i.e. the explicit estimate of the net heat flux is directed toward the canopy). Then the implicit temperature change during the time step will be

$$\Delta T = \frac{B_0}{C/\Delta t - k} \quad (3.29)$$

and the amount of heat available for phase change is equal to

$$M = B_0 \Delta t \frac{C}{C - k\Delta t} \quad (3.30)$$

On the other hand, if we assume that the snow melts during the entire time step (as it does in real world), so that the temperature stays at the freezing point, the amount of heat spent on melting would be

$$M' = B_0 \Delta t \quad (3.31)$$

It is now easy to notice that M is always smaller than M', since the exchange coefficient k is always negative – the net energy balance of the canopy goes down as the temperature goes up. Clearly the rate of snow melt is always underestimated by the explicit formulas. To get an idea about the magnitude of the effect, let's make a simple calculation for a typical case. Let's assume that the amount of snow on canopy is $1\,\mathrm{kg/m^2}$, giving $C = 2090\,\mathrm{J/(m^2\,K)}$; typical value of k is, say, $-5\,\mathrm{W/(m^2\,K)}$, and $\Delta t = 3600\,\mathrm{s}$. That gives

$$\frac{M}{M'} = \frac{2090}{2090 + 5 \times 3600} \approx 0.104$$

That is, the explicit formula underestimates the melt rate by about an order of magnitude. Consequently the life time of snow or ice on canopy is going to be overestimated by approximately the same factor, and during all that time the simulated temperature of canopy will be reset to T_f, with detrimental consequences for the canopy temperature evolution, in particular diurnal cycle. The same back-of-the-envelope calculation can be repeated for freezing of canopy liquid water, with similar conclusions.

In the light of the above calculations, it looks like it's better to avoid using the explicit treatment of the snow melt and water freezing in the canopy. On the other hand the full implicit treatment of canopy water phase change would require solving a non-linear system of the equations, which is much more computationally expensive than solving linearized system. Arguably, dropping the phase change calculations from the canopy altogether seems at the moment as the least of two evils. In this approach, the snow/water would drop to the ground when the canopy temperature reaches freezing point T_f and change phase there. Note that there is no similar problem on the ground since the melt on ground surface is treated as a nonlinear term in any case.

4. Fluxes and their derivatives

4a. *Turbulent exchange fluxes between canopy air and atmosphere*

In principle, the exchange between atmosphere and the land obeys the standard bulk formulas:

$$H_a = \rho\, c_p C_D |\mathbf{v}| (T_c - T_a) \tag{4.1}$$

$$E_a = \rho\, C_D |\mathbf{v}| (q_c - q_a) \tag{4.2}$$

where C_D is a stability-dependant drag coefficient, and $\mathbf{v}$ is the wind speed at the top of the constant flux layer.

However, the values of fluxes and derivatives that the land gets from the atmosphere are not exactly the same that one would expect from the above formulas. Rather, input values of $\partial H_a/\partial T_c$ and $\partial E_a/\partial q_c$ are "effective" fluxes – that is, the fluxes obtained as a result of the downward pass of the implicit solution of vertical diffusion equation in the atmosphere.

4b. *Roughness properties of the land surface*

In `TURB_LM3V` treatment, the displacement height is calculated as [citation needed]

$$d_0 = 1.1 H \log\left(1 + \sqrt[4]{0.07(LAI + SAI)}\right) \tag{4.3}$$

and the roughness length for momentum as

$$z_{0m} = \begin{cases} 0.3(H - d_0) & LAI + SAI > 2.85 \\ z_{0g} + 0.3H\sqrt{0.07(LAI + SAI)} & \text{otherwise} \end{cases} \tag{4.4}$$

where H is the maximum of the vegetation height and hardcoded constant $0.1\,\mathrm{m}$, z_{0g} is the momentum roughness length of the underlying ground surface. The roughness length for the scalars (temperature and tracers, including specific humidity) $z_{0s} = z_{0m}\exp(-2)$.

Note that the roughness length has a discontinuity at $LAI + SAI = 2.85$.

4c. *Turbulence inside canopy*

To calculate the aerodynamic resistances between canopy and canopy air r_v, and between ground and canopy air r_g the model follows Bonan (1996) in assuming that the wind speed profile within canopy is exponential:

$$u(z) = u(H)\exp\Big(-a\frac{H-z}{H}\Big) \tag{4.5}$$

and the vertical profile of eddy diffusivity $K_h(z)$ (m^2/s) is also exponential:

$$K_h(z) = K_h(H)\exp\Big(-a\frac{H-z}{H}\Big) \tag{4.6}$$

where H is the height of the top of the canopy, and a is an empirical parameter, typically 3.

The formulation of aerodynamic conductance through leaf boundary layer can be traced through Bonan (1996) to Choudhury and Monteith (1988), to Jones (1983):

$$g_b(z) = \alpha\sqrt{\frac{u(z)}{d_l}} \tag{4.7}$$ eq:leaf-bl-conductance

where $\alpha = 0.01\,\mathrm{m/s}^{1/2}$, and d_l is the characteristic plant surface dimension in the direction of the wind flow. The expression (4.7) gives the value of aerodynamic conductance per unit leaf area. For an isothermal canopy we can easily calculate the total conductance as:

$$g_v = \int_0^H g_b(z)L(z)dz \tag{4.8}$$

where $L(z)$ is the vertical distribution of leaf area density. Assuming that $L(z)$ is uniform, we get:

$$\frac{1}{r_v} = g_v = LAI\frac{2\alpha}{a}\sqrt{\frac{u(H)}{d_l}}(1-e^{-a/2}) \tag{4.9}$$ eq:con_v_h

To estimate the wind speed at the top of the canopy, the following expression is used:

$$u(H) = \frac{u^*}{\kappa}\log\Big(\frac{H-d_0}{z_{0m}}\Big) \tag{4.10}$$ eq:wind-on-top

where u^* is friction velociy, κ is von Karman constant, d_0 is displacement height, and z_{0m} is land surface roughness length for momentum. It needs to be emphasized that u^* does not represent the friction velocity undercanopy; instead it is used to roughly estimate the magnitude of wind incident on the canopy.

Note that the model uses the leaf area index in the equation (4.9), not the sum of the leaf area and stem area incices as some other formulations do.

Under TURB_LM3V treatment, $a = a_{max}$ is just a constant value, prescribed globally, and H is the maximum of the vegetation height and a hard-coded constant 0.1 m.

If TURB_LM3W option is selected, then $a = a_{max} * cover$, and H is just a vegetation height

To calculate the resistance between two layers within the canopy, z_0 and z_1, we assume that the vertical heat flux in the canopy is constant, and recall the definition of the aerodynamic resistance:

$$F = -K_h(z)\frac{dT}{dz} = \frac{T(z_0) - T(z_1)}{r_g} \tag{4.11}$$

Rearranging the terms in the above equation, we get:

$$r_g \frac{dT}{dz} = \frac{T(z_1) - T(z_0)}{K_h(z)} \tag{4.12}$$

Integrating both sides from z_0 to z_1:

$$r_g = \int_{z_0}^{z_1} \frac{1}{K_h(z)} dz = \frac{H}{K_h(H)a}\left(e^{a(1-\frac{z_0}{H})} - e^{a(1-\frac{z_1}{H})}\right) \tag{4.13}$$

The wind on top of the canopy is given by expression (4.10), and the eddy diffusivity on top of the canopy is:

$$K_h(H) = \kappa u^*(H - d) \tag{4.14}$$

In TURB_LM3V treatment, z_0 is equal to the ground surface roughness length for scalars z_{g0s}, $z_1 = z_{0m} + d$, where z_{0m} is the land surface roughness length for momentum, d is displacement height, and the there are final adjustments that limit the extreme cases:

$$r_{g,LM3V} = \begin{cases} \min(r_g, 1250\,\mathrm{s/m}) & \text{if } d > 0.06\,\mathrm{m} \text{ and } LAI + SAI > 0.25 \\ 0.01\,\mathrm{s/m} & \text{otherwise} \end{cases} \tag{4.15}$$

In TURB_LM3W treatment, z_0 is the same as above, but $z_1 = z_{0s} + d$, z_{0s} being the land surface roughness length or heat, and no further adjustments are made.

Note that H and a are also defined differently for TURB_LM3V and TURB_LM3W, as described in section 4d.

ect:canopy-ca-exchange

4d. *Turbulent exchange fluxes between canopy and canopy air*

The fluxes of sensible heat and moisture between canopy and canopy air can be described by the following equations:

$$H_v = \frac{\rho c_p}{r_v}(T_v - T_c) \tag{4.16}$$

eq:vegn-sens

$$E_t = \rho \frac{f_t}{r_c + r_v} \big(q^*(T_v) - q_c \big) \tag{4.17}$$

where f_t is the effective fraction of the canopy area that transpires, r_v is the aerodynamic resistance between canopy and canopy air, r_c is the stomatal conductance of the canopy, and $q^*(T_v)$ is the saturated specific humidity at the canopy temperature T_v. Note than resistances r_v and r_c are bulk canopy quantities, not the values per unit leaf area.

To estimate the fraction of canopy area that can transpire, sometimes it is assumed that the stomata (openings on leaf surface through which the transpiration and the uptake of carbon dioxide occurs) are located on the lower side of the leaves, so that the intercepted water or snow never block them, and therefore f_t is always equal to 1.

In present work, however, we follow the formulation of Bonan (1996):

$$f_t = 1 - f_l - f_s \tag{4.18}$$

where f_l and f_s are the fractions of the canopy area covered by liquid water or snow, respectively.

It is arguable whether we should include a factor of two in the expression for sensible heat flux (4.16) since the leaves can lose sensible heat from both sides, while they transpire or evaporate from one side only.

Evaporation and sublimation from the surface of intercepted water or snow can be expressed as:

$$E_l = \rho \frac{f_l}{r_v} \big(q^*(T_v) - q_c \big) \tag{4.19}$$

$$E_s = \rho \frac{f_s}{r_v} \big(q^*(T_v) - q_c \big) \tag{4.20}$$

Note that in general case all three fractions f_l, f_s, and f_t are functions of both canopy water mass w_l and canopy snow mass w_s, that is, $f_{l,s,t} = f_{l,s,t}(w_s, w_l)$.

Again following Bonan (1996), we assume that in the absence of snow, the wet fraction of the canopy can be calculated as:

$$f_l^0 = \left(\frac{w_l}{W_{l,\max}} \right)^{p_l} \tag{4.21}$$

eq:frl-0

where $p_l = \frac{2}{3}$ and $W_{l,\max}$ is maximum water-holding capacity of the canopy. In addition, we assume that the snow-covered fraction of the canopy can be described by a similar power dependence:

$$f_s^0 = \left(\frac{w_s}{W_{s,\max}} \right)^{p_s} \tag{4.22}$$

eq:frs-0

where $p_s = \frac{2}{3}$, just like for the water, and $W_{s,\max}$ is the maximum snow-holding capacity of the canopy.

While equations (4.21) and (4.22) are sufficient to describe the situation when either snow or liquid water cover the canopy, sometimes both phases are present on the leaves simultaneously. In this case, to calculate the dry portion of the canopy and the area of the canopy water and snow exposed to the interaction with the canopy air, we must describe somehow the overlap between phases of the water on the canopy. For simplicity, we assume that (1) water and snow patches are distributed over the canopy independently of each other, and (2) snow is always on top of water where they overlap.

These assumptions are fairly arbitrary, and it's not clear how well they hold in real world. However, given that the range of conditions when both snow and water can be present on the canopy is not very wide, and the evaporation should not be very important in those cases because of relatively low temperature, it is probably OK to use this simple formulation.

With these assumptions in mind, it is easy to write expressions for exposed parts of snow:

$$f_s = f_s^0 = \left(\frac{w_s}{W_{s,\max}}\right)^{p_s} \tag{4.23}$$ `eq:frs`

liquid water:

$$f_l = f_l^0(1 - f_s^0) = \left(\frac{w_l}{W_{l,\max}}\right)^{p_l} \left[1 - \left(\frac{w_s}{W_{s,\max}}\right)^{p_s}\right] \tag{4.24}$$ `eq:frl`

and dry canopy:

$$f_t = (1 - f_l^0)(1 - f_s^0) = \left[1 - \left(\frac{w_l}{W_{l,\max}}\right)^{p_l}\right] \left[1 - \left(\frac{w_s}{W_{s,\max}}\right)^{p_s}\right] \tag{4.25}$$ `eq:frt`

If the water vapor flux is directed toward the surface, it shouldn't depend on the amount of water already present on the canopy – in this case, f_l or f_s should be assumed to be equal to 1. That makes expression for water flux significantly nonlinear for the dry (or, in general, not-completely-wet) canopy, but hopefully this should not result in pathological instabilities [3].

The saturated water vapor pressures over liquid and frozen water surfaces are different; however, the atmospheric model does not make this distinction (using some weighted combination of the two around water freezing point), and the land model uses exactly the same saturated humidity function as the atmosphere does. It may be not entirely justifiable, though, since the atmosphere seems to be basing the weighting function on the average distribution of liquid versus frozen phase in the clouds for a given temperature, which is probably not applicable to the typical distribution of phases on the land surface.

4e. *Turbulent exchange fluxes between ground surface and canopy air*

The expressions for sensible heat and water vapor fluxes between ground and canopy air can be written as follows:

$$H_g = \rho c_p \frac{1}{r_g}(T_g - T_c) \tag{4.26}$$

[3] It has been observed that switching from condensation to evaporation regime in the morning does produce spike in canopy evaporation; the spike can be high enough to momentarily produce negative canopy water content. Typically it is not a big deal since absolute value of negative canopy water is small, and the mass conservation is not violated. It is still an open problem, however.

$$E_g = \rho \frac{1}{r_g}(q_g - q_c) \tag{4.27}$$

where r_g is aerodynamic resistance between ground and canopy air, and q_g is near-ground specific humidity.

Surface specific humidity q_g is calculated based on the presence of snow (it is 100% if snow is present) and the amount of water available in the upper layer of the soil. The dependence of q_g on the soil water should be rather sharp, to avoid unrealistic downward water vapor fluxes for wet but not saturated soil, and therefore it also creates a potential numerical instability problem due to nonlinearity.

4f. *Ground heat flux*

The ground heat flux G is a heat conductance in the substrate near its surface, that is, the transport of heat from the surface underlying the canopy air. This underlying substrate can be anything that is in contact with the canopy air: soil, or glacier surface, or lake, or any of the above covered by the snow layer. Strictly speaking, this transport should include both diffusion and advection (with water), but we consider only diffusion here.

Let's adopt the convention that z increases downward, and the heat flux is also positive downward. The vertical diffusion equation for heat in the substrate is:

$$c\frac{\partial T}{\partial t} = -\frac{\partial}{\partial z}\lambda\frac{\partial T}{\partial z} \tag{4.28}$$

eq:vert-diff

and the ground heat flux that we want to find is the boundary condition at the top:

$$G_t = -\left(\lambda\frac{\partial T}{\partial z}\right)_{z=0} \tag{4.29}$$

We can also write similar boundary condition for the bottom of the media of thickness H:

$$G_b = -\left(\lambda\frac{\partial T}{\partial z}\right)_{z=H} \tag{4.30}$$

To solve the time stepping equations in the canopy air, we must know the value of G_t, and its derivative with respect to the surface temperature.

Note, however, that just like the fluxes to the atmosphere it is an "effective" flux – that is, the flux obtained as a result of forward elimination in the process of implicit solution of vertical diffusion equation.

Suppose that the substrate is vertically discretized in N layers, with layer 1 at the top and layer N at the bottom. The thickness of layer i is Δz_i, its total specific heat is c_i, and its heat conductance is λ_i. Properties c_i and λ_i depend on the state of the substrate: for example, on the amount of water in the soil, and on its phase state.

Discretizing the equation (4.28) in time and space gives:

$$c_i\frac{T_i^{t+1} - T_i^t}{\Delta t} = F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}} \tag{4.31}$$

eq:vert-diff-discr

We assume that the flux at the bottom $F_{N+1/2}$ and its derivative with respect to T_N are known:

$$F_{N+\frac{1}{2}} = G_b(T_N^0) + \frac{\partial G_b}{\partial T_N}(T_N - T_N^0) \tag{4.32}$$

Eventually, we want to find the diffusive heat flux at the top of the snow $F_{1/2} = G_t$ and its derivative with respect to the snow skin temperature T_1: this is going to be our "ground heat flux" for the coupling with the rest of the system.

The heat fluxes between layers can be written this way:

$$F_{i+\frac{1}{2}} = \Lambda_{i+\frac{1}{2}}(T_i - T_{i+1}), \quad i = 1, \ldots, N-1 \tag{4.33}$$

where

$$\Lambda_{i+\frac{1}{2}} = 2\left[\frac{\Delta z_i}{\lambda_i} + \frac{\Delta z_{i+1}}{\lambda_{i+1}}\right]^{-1} \tag{4.34}$$

eq:vert-diff-system

Discretizing the equation (4.31) in time, we get:

$$c_i \frac{\Delta T_i}{\Delta t} = E_i + \frac{\partial G_t}{\partial T_1}\Delta T_i \qquad - \Lambda_{i+\frac{1}{2}}(\Delta T_i - \Delta T_{i+1}), \quad i = 1 \tag{4.35a}$$

eq:layer-1

$$c_i \frac{\Delta T_i}{\Delta t} = E_i + \Lambda_{i-\frac{1}{2}}(\Delta T_{i-1} - \Delta T_i) - \Lambda_{i+\frac{1}{2}}(\Delta T_i - \Delta T_{i+1}), \quad i = 2, \ldots, N-1 \tag{4.35b}$$

$$c_i \frac{\Delta T_i}{\Delta t} = E_i + \Lambda_{i-\frac{1}{2}}(\Delta T_{i-1} - \Delta T_i) - \frac{\partial G_b}{\partial T_N}\Delta T_i, \qquad i = N \tag{4.35c}$$

where E_i are explicit estimates of the net heat balance of the layers:

$$E_i = G_t^0 \qquad - \Lambda_{i+\frac{1}{2}}(T_i^0 - T_{i+1}^0), \quad i = 1 \tag{4.36a}$$

$$E_i = \Lambda_{i-\frac{1}{2}}(T_{i-1}^0 - T_i^0) - \Lambda_{i+\frac{1}{2}}(T_i^0 - T_{i+1}^0), \quad i = 2, \ldots, N-1 \tag{4.36b}$$

$$E_i = \Lambda_{i-\frac{1}{2}}(T_{i-1}^0 - T_i^0) - G_b^0, \qquad i = N \tag{4.36c}$$

Equations (4.35) can be re-written in a simple canonical form [4]:

$$-A_i \Delta T_{i-1} + B_i \Delta T_i - C_i \Delta T_{i+1} = E_i \tag{4.37}$$

eq:vert-diff-canonical

where

$$A_i = \begin{cases} 0, & i = 1 \\ \Lambda_{i-\frac{1}{2}}, & i = 2, \ldots, N \end{cases} \tag{4.38}$$

$$B_i = \frac{c_i}{\Delta t} + \begin{cases} -\frac{\partial G_t}{\partial T_1} + \Lambda_{i+\frac{1}{2}}, & i = 1 \\ \Lambda_{i-\frac{1}{2}} + \Lambda_{i+\frac{1}{2}}, & i = 2, \ldots, N-1 \\ \Lambda_{i-\frac{1}{2}} + \frac{\partial G_b}{\partial T_N}, & i = N \end{cases} \tag{4.39}$$

$$C_i = \begin{cases} \Lambda_{i+\frac{1}{2}}, & i = 1, \ldots, N-1 \\ 0, & i = N \end{cases} \tag{4.40}$$

[4] Note that equation (4.37) does not follow the code of the model to the letter. In the code, the signs of A_i and C_i are inverted, and all coefficients A_i, B_i, and C_i are multiplied by $\frac{\Delta T}{c_i}$.

To solve the system (4.37) we can express the temperature tendency of the lower layer in terms of the tendency in the layer above it:

$$\Delta T_{i+i} = e_i \Delta T_i + f_i \tag{4.41}$$ eq:T-substitution

Since $C_N = 0$, we get the following expression for layer $N-1$:

$$e_{N-1} = \frac{A_N}{B_N}, \quad f_{N-1} = \frac{E_N}{B_N} \tag{4.42}$$

$$e_{i-1} = \frac{A_i}{B_i - C_i e_i}, \quad f_{i-1} = \frac{E_i + C_i f_i}{B_i - C_i e_i} \tag{4.43}$$

Substituting (4.41) in (4.35a), we obtain for the upper layer of the snow:

$$G_t^0 = \Lambda_{1+\frac{1}{2}}(T_1^0 + f_1 - T_2^0) \tag{4.44}$$

$$\frac{\partial G_t}{\partial T_1} = \Lambda_{1+\frac{1}{2}}(1 - e_1) + \frac{c_1}{\Delta t} \tag{4.45}$$ eq:dGdT_snow

i. Simple asymptotic case Let's assume that $\Delta z_i = \Delta z$ is the same for every layer, and also that all $\lambda_i = \lambda$ are the same, and all $c_i = c$ are the same. In this simple case

$$\Lambda_{i+\frac{1}{2}} = \Lambda = \frac{\lambda}{\Delta z} \tag{4.46}$$

and

$$A_i = C_i = \Lambda, \quad B_i = \frac{c}{\Delta t} + 2\Lambda \tag{4.47}$$

so that

$$e_{i-1} = \frac{\Lambda}{\frac{c}{\Delta t} + 2\Lambda - \Lambda e_i} = \frac{1}{2 + \frac{c}{\Delta t \Lambda} - e_i} \tag{4.48}$$

Suppose also that the number of layers is infinite (so that snow depth is infinite); in this case

$$e_i = \frac{1}{2 + \frac{c}{\Delta t \Lambda} - e_i} \tag{4.49}$$

and therefore

$$e_i^2 - \left(2 + \frac{c}{\Delta t \Lambda}\right) e_i + 1 = 0 \tag{4.50}$$

The solutions of this quadratic equation are:

$$\begin{aligned} e_i &= \frac{1}{2}\left(2 + \frac{c}{\Delta t \Lambda}\right) \pm \frac{1}{2}\sqrt{\left(\frac{c}{\Delta t \Lambda}\right)^2 + 4\frac{c}{\Delta t \Lambda}} \\ &= 1 + \frac{1}{2\Lambda}\left(\frac{c}{\Delta t} \pm \sqrt{\left(\frac{c}{\Delta t}\right)^2 + 4\frac{c\Lambda}{\Delta t}}\right) \end{aligned} \tag{4.51}$$ eq:e-for-infinite-N

The solution with the "$+$" sign is not stable, so we will only consider the case with "$-$". Equation (4.45) for the derivative of surface flux with respect to surface temperature becomes:

$$\begin{aligned}\frac{\partial G_t}{\partial T_1} &= -\frac{c}{2\Delta t} + \frac{1}{2}\sqrt{\left(\frac{c}{\Delta t}\right)^2 + 4\frac{c\Lambda}{\Delta t}} + \frac{c}{\Delta t} \\ &= \frac{c}{2\Delta t} + \frac{1}{2}\sqrt{\left(\frac{c}{\Delta t}\right)^2 + 4\frac{c\lambda}{\Delta z \Delta t}} \qquad (4.52)\end{aligned}$$

eq:dgdt-inf-N

It is easy to see that if the total heat capacity of a single layer c is proportional to its thickness Δz: $c = c_s \Delta z$, then

$$\frac{\partial G_t}{\partial T_1} \longrightarrow \sqrt{\frac{c_s \lambda}{\Delta t}}, \quad \text{as } \Delta z \longrightarrow 0 \tag{4.53}$$

Note that in the model, snow has a minimum finite "fictitious heat capacity", so that $c_s \Delta z \longrightarrow const$ as $\Delta z \longrightarrow 0$. The result is sometimes weird behavior of thin snow layers. In rare cases, it results in emergence of huge amounts of negative liquid water in the snow in just one time step, compensated by huge amounts of positive solid water.

4g. *Long-wave radiation fluxes and derivatives*

Suppose we know that the downward long-wave radiation at the top of the canopy is $I_a^{\downarrow}$, and we need to calculate the fluxes and derivatives of the long-wave radiative balance that appear in the linearized system of equations (3.11) – (3.16). To do that, we assume we know the radiative properties of the vegetation and land. For the land, the relevant radiative properties are long-wave reflectance α_g and emissivity ε_g. The vegetation is semi-transparent for the light, so in addition to its' reflectivity α_v and emissivity ε_v we need to know its transmissivity τ_v. From the energy conservation and Kirchhoff laws, $\alpha_g + \varepsilon_g = 1$ and $\alpha_v + \varepsilon_v + \tau_v = 1$. Given temperatures of the vegetation T_v and ground T_g, we can easily calculate the fluxes emitted by those components:

$$B_v^{\uparrow} = B_v^{\downarrow} = \varepsilon_v \sigma T_v^4 \tag{4.54}$$

$$B_g^{\uparrow} = \varepsilon_g \sigma T_g^4 \tag{4.55}$$

However, to calculate the net long-wave radiative balances of the vegetation and the ground surface we must take into account the re-reflection of the radiation between the two. Suppose the full downward flux at the lower boundary flux of the vegetation canopy is $I_v^{\downarrow}$, and the full upward flux at the ground surface is $I_g^{\uparrow}$. We can write the following relationship between the fluxes and the optical properties of the components of the system:

$$I_v^{\downarrow} = \tau_v I_a^{\downarrow} + B_v^{\downarrow} + \alpha_v I_g^{\uparrow} \tag{4.56}$$

eq:ivdn-1

$$I_g^{\uparrow} = B_g^{\uparrow} + \alpha_g I_v^{\downarrow} \tag{4.57}$$

eq:igup-1

Substituting (4.57) into (4.56) and solving the resulting linear equation, we get:

$$I_v^{\downarrow} = \frac{\tau_v I_a^{\downarrow} + B_v^{\downarrow} + \alpha_v B_g^{\uparrow}}{1 - \alpha_g \alpha_v} \tag{4.58}$$

eq:ivdn-2

Substituting (4.58) back into the (4.57), we get:

$$I_g^{\uparrow} = \frac{\alpha_g \tau_v I_a^{\downarrow} + \alpha_g B_v^{\downarrow} + B_g^{\uparrow}}{1 - \alpha_g \alpha_v} \tag{4.59}$$

eq:igup-2

Consequently, the net long-wave radiation at the surface is:

$$\begin{aligned} R_{lg} &= I_v^{\downarrow} - I_g^{\uparrow} = \\ &= (\tau_v I_a^{\downarrow} + B_v^{\downarrow}) \frac{1 - \alpha_g}{1 - \alpha_g \alpha_v} - B_g^{\uparrow} \frac{1 - \alpha_v}{1 - \alpha_g \alpha_v} \end{aligned} \tag{4.60}$$

eq:net-grnd-lw

In limiting case of black-body ground surface ($\varepsilon_g = 1$, and therefore $\alpha_g = 0$) expression (4.60) reduces to $\tau_v I_a^{\downarrow} + B_v^{\downarrow} - (1 - \alpha_v) B_g^{\uparrow}$, just as expected.

The expression for net radiative balance of canopy is more complicated, simply because there are more fluxes involved:

$$\begin{aligned} R_{lv} &= (1 - \alpha_v) I_a^{\downarrow} - B_v^{\uparrow} + (1 - \tau_v) I_g^{\uparrow} - I_v^{\downarrow} \\ &= (1 - \alpha_v) I_a^{\downarrow} - B_v^{\uparrow} \\ &\qquad + (1 - \tau_v) \frac{\alpha_g \tau_v I_a^{\downarrow} + \alpha_g B_v^{\downarrow} + B_g^{\uparrow}}{1 - \alpha_g \alpha_v} - \frac{\tau_v I_a^{\downarrow} + B_v^{\downarrow} + \alpha_v B_g^{\uparrow}}{1 - \alpha_g \alpha_v} \\ &= A_a I_a^{\downarrow} + A_v B_v + A_g B_g^{\uparrow} \end{aligned} \tag{4.61}$$

eq:net-vegn-lw-1

Gathering similar terms, we get:

$$\begin{aligned} A_a &= 1 - \alpha_v - \tau_v + \left[\tau_v + \frac{(1 - \tau_v) \alpha_g \tau_v}{1 - \alpha_g \alpha_v} - \frac{\tau_v}{1 - \alpha_g \alpha_v} \right] \\ &= (1 - \alpha_v - \tau_v) + \tau_v \frac{-\alpha_g \alpha_v + (1 - \tau_v) \alpha_g}{1 - \alpha_g \alpha_v} \\ &= (1 - \alpha_v - \tau_v) + \tau_v \alpha_g \frac{1 - \tau_v - \alpha_v}{1 - \alpha_g \alpha_v} \\ &= (1 - \alpha_v - \tau_v) \left[1 + \frac{\tau_v \alpha_g}{1 - \alpha_g \alpha_v} \right] = \varepsilon_v \left[1 + \frac{\tau_v \alpha_g}{1 - \alpha_g \alpha_v} \right] \end{aligned} \tag{4.62}$$

eq:Aa

For the black background conditions ($\alpha_g = 0$), this coefficient reduces to expected $1 - \alpha_v - \tau_v \equiv \varepsilon_v$; for transparent vegetation ($\alpha_v = 0$, $\tau_v = 1$), it is simply equal to zero.

$$\begin{aligned} A_v &= -1 + \frac{(1 - \tau_v) \alpha_g}{1 - \alpha_g \alpha_v} - \frac{1}{1 - \alpha_g \alpha_v} \\ &= -2 + \left[1 + \frac{(1 - \tau_v) \alpha_g}{1 - \alpha_g \alpha_v} - \frac{1}{1 - \alpha_g \alpha_v} \right] \\ &= -2 + \frac{\alpha_g (1 - \alpha_v - \tau_v)}{1 - \alpha_g \alpha_v} = -2 + \frac{\alpha_g \varepsilon_v}{1 - \alpha_g \alpha_v} \end{aligned} \tag{4.63}$$

eq:Av

$$A_g = \frac{1 - \alpha_v - \tau_v}{1 - \alpha_g \alpha_v} = \frac{\varepsilon_v}{1 - \alpha_g \alpha_v} \tag{4.64}$$

eq:Ag

Substituting (4.62), (4.63), and (4.64) into (4.61), we obtain the final expression for the net long-wave radiative balance of the canopy:

$$R_{lv} = \varepsilon_v \left[1 + \frac{\tau_v \alpha_g}{1 - \alpha_g \alpha_v}\right] I_a^{\downarrow} + \left[\frac{\alpha_g \varepsilon_v}{1 - \alpha_g \alpha_v} - 2\right] \varepsilon_v \sigma T_v^4 + \frac{\varepsilon_v}{1 - \alpha_g \alpha_v} \varepsilon_g \sigma T_g^4 \tag{4.65}$$

eq:net-vegn-lw

It is now simple to obtain the expression for derivatives of the net long-wave radiation:

$$\frac{\partial R_{lg}}{\partial T_g} = -4\varepsilon_g \sigma \frac{1 - \alpha_v}{1 - \alpha_g \alpha_v} T_g^3 \qquad \frac{\partial R_{lg}}{\partial T_v} = 4\varepsilon_v \sigma \frac{1 - \alpha_g}{1 - \alpha_g \alpha_v} T_v^3 \tag{4.66}$$

$$\frac{\partial R_{lv}}{\partial T_g} = 4\varepsilon_g \sigma \frac{\varepsilon_v}{1 - \alpha_g \alpha_v} T_g^3 \qquad \frac{\partial R_{lv}}{\partial T_v} = 4\varepsilon_v \sigma \left[\frac{\alpha_g \varepsilon_v}{1 - \alpha_g \alpha_v} - 2\right] T_v^3 \tag{4.67}$$

The expression for the outgoing long-wave radiation at the canopy top is a sum of three components: the scattered (re-emitted?) part of the downward long-wave radiation, the emission of the canopy itself, and the part of the upward radiation flux below the canopy that passes through the canopy:

$$I_a^{\uparrow} = \alpha_v I_a^{\downarrow} + B_v + \tau_v I_g^{\uparrow} \tag{4.68}$$

Substituting the expression for $I_g^{\uparrow}$ (4.59), and assuming that the amounts of energy emitted by canopy upward and downward are the same (that is, $B_v^{\uparrow} = B_v^{\downarrow} = B_v$), we get:

$$\begin{aligned} I_a^{\uparrow} &= \alpha_v I_a^{\downarrow} + B_v + \tau_v \frac{\tau_v I_a^{\downarrow} + B_v + \alpha_v B_g^{\uparrow}}{1 - \alpha_g \alpha_v} \\ &= \left[\alpha_v + \frac{\alpha_g \tau_v^2}{1 - \alpha_g \alpha_v}\right] I_a^{\downarrow} + \left[1 + \frac{\alpha_g \tau_v}{1 - \alpha_g \alpha_v}\right] B_v + \frac{\tau_v}{1 - \alpha_g \alpha_v} B_g^{\uparrow} \end{aligned} \tag{4.69}$$

Finally, recalling the expressions for the ground surface and vegetation emissions in terms of respective temperatures:

$$I_a^{\uparrow} = \left[\alpha_v + \frac{\alpha_g \tau_v^2}{1 - \alpha_g \alpha_v}\right] I_a^{\downarrow} + \left[1 + \frac{\alpha_g \tau_v}{1 - \alpha_g \alpha_v}\right] \varepsilon_v \sigma T_v^4 + \frac{\tau_v}{1 - \alpha_g \alpha_v} \varepsilon_g \sigma T_g^4 \tag{4.70}$$

eq:iaup

eq:iaup-derivs

And for derivatives:

$$\frac{\partial I_a^{\uparrow}}{\partial T_v} = 4\varepsilon_v \sigma \left[1 + \frac{\alpha_g \tau_v}{1 - \alpha_g \alpha_v}\right] T_v^3 \tag{4.71a}$$

$$\frac{\partial I_a^{\uparrow}}{\partial T_g} = 4\varepsilon_g \sigma \frac{\tau_v}{1 - \alpha_g \alpha_v} T_g^3 \tag{4.71b}$$

Expressions (4.70) and (4.71) can be easily verified if one recalls that according to the energy conservation law, $R_{lv} + R_{lg} = I_a^{\downarrow} - I_a^{\uparrow}$, and therefore $I_a^{\uparrow} = I_a^{\downarrow} - R_{lv} - R_{lg}$.

4h. *Water and snow drip*

We assume that water and snow drip rates from the canopy are proportional to the amount of substances on the canopy, except the cases of extremely intense precipitation, when the amount intercepted by the canopy exceeds the maximum allowed values. The expression for liquid water:

$$D_l = \begin{cases} \frac{w_l}{\tau_l} & w_l + f_{il} P_l \Delta t < W_{l,\max} \\ \frac{w_l - W_{l,\max}}{\Delta t} + f_{il} P_l & \text{otherwise} \end{cases} \tag{4.72}$$

and snow:

$$D_s = \begin{cases} \frac{w_s}{\tau_s} & w_s + f_{is} P_s \Delta t < W_{s,\max} \\ \frac{w_s - W_{s,\max}}{\Delta t} + f_{is} P_s & \text{otherwise} \end{cases} \tag{4.73}$$

Maximum water-holding capacity $W_{l,\max}$ and snow-holding capacity $W_{s,\max}$ of the canopy are directly proportional to the LAI with coefficients that can be specified independently.

A. Equation of energy balance for wet canopy

sect:vegn-eb

To explain the vaporization-related terms in the equation (1.6), let's consider a leaf with amount of water m on top of it; for simplicity let's assume that the heat capacity of the leaf itself is zero.

The amount of heat δQ that comes to the canopy during time dt is spent on evaporation of the water, and on the heating of the remaining water; one can imagine the process as splitting out the amount of water dm, vaporizing it at the current temperature T, and using the remaining heat $\delta Q - L(T)dm$ to heat the water remaining on the leaf:

$$c_l(m - dm)dT = \delta Q - L(T)dm \tag{A.1}$$

Recalling the latent heat of vapoization dependence on temperature $L(T) = L(T_0) + (c_v - c_l)(T - T_0)$ (where T_0 is some reference temperature), and omitting vanishingly small term $dm\, dT$ we can write:

$$c_l m dT + c_l(T - T_0)dm = \delta Q - [L(T_0) + (T - T_0)c_v]dm \tag{A.2}$$

or, equivalently:

$$c_l d(m(T - T_0)) = \delta Q - [L(T_0) + (T - T_0)c_v]dm \tag{A.3}$$

Dividing the above equation by dt, and recalling that $\frac{dm}{dt} \equiv E$ we finally get:

$$c_l \frac{d(T - T_0)m}{dt} = R - [L(T_0) + (T - T_0)c_v]E \tag{A.4}$$

The value of arbitrary constant T_0 was chosen to be 0 in equation (1.6).

The contribution of the transpiration to the energy balance can be written in essentially the same way, except that the water that participates in transpiration comes to the leave at the uptake temperature T_u with the rate equal to transpiration E_t, and therefore some energy is spent on heating it up to the temperture of leaf T before varporization.

B. Fully linearized equations

sect:full-lin

Recall that in the section 3b we neglected to linearize the terms in front of the water vapor fluxes. This section presents the full equations, for the sake of completeness. These equations are currently not used in the model code.

Ba. *Linearized equations*

Canopy air energy balance (1.2)

$$\begin{aligned}
& m_c\big(c_p + q_c(c_v - c_p)\big)\frac{\Delta T_c}{\Delta t} + m_c(c_v - c_p)T_c\frac{\Delta q_c}{\Delta t} + m_c(c_v - c_p)\frac{\Delta q_c \Delta T_c}{\Delta t} \\
&\quad = H_{v0} + \left(\frac{\partial H_v}{\partial T_v}\right)\Delta T_v + \left(\frac{\partial H_v}{\partial T_c}\right)\Delta T_c \\
&\quad + H_{g0} + \left(\frac{\partial H_g}{\partial T_g}\right)\Delta T_g + \left(\frac{\partial H_g}{\partial T_c}\right)\Delta T_c \\
&\quad - H_{a0} - \left(\frac{\partial H_a}{\partial T_c}\right)\Delta T_c \\
&\quad + c_v(T_v + \Delta T_v)\left[E_{v0} + \frac{\partial E_v}{\partial q_c}\Delta q_c + \frac{\partial E_v}{\partial T_v}\Delta T_v + \frac{\partial E_v}{\partial w_l}\Delta w_l + \frac{\partial E_v}{\partial w_s}\Delta w_s\right] \\
&\quad + c_v(T_g + \Delta T_g)\left[E_{g0} + \frac{\partial E_g}{\partial q_c}\Delta q_c + \frac{\partial E_g}{\partial T_g}\Delta T_g\right] \\
&\quad - c_v(T_c + \Delta T_c)\left[E_{a0} + \frac{\partial E_a}{\partial q_c}\Delta q_c\right]
\end{aligned} \tag{B.1}$$

eq:cana-eb-lin-full

Canopy energy balance (1.6)

$$\begin{aligned}
& (C_v + c_l w_l + c_s w_s)\frac{\Delta T_v}{\Delta t} + c_l T_v\frac{\Delta w_l}{\Delta t} + c_s T_v\frac{\Delta w_s}{\Delta t} + \frac{\Delta T_v(c_l\Delta w_l + c_s\Delta w_s)}{\Delta t} \\
&\quad = R_{Sv} + R_{Lv0} + \frac{\partial R_{L_v}}{\partial T_v}\Delta T_v + \frac{\partial R_{Lv}}{\partial T_g}\Delta T_g \\
&\quad - H_{v0} - \frac{\partial H_v}{\partial T_v}\Delta T_v - \frac{\partial H_v}{\partial T_c}\Delta T_c \\
&\quad - (L_t' + c_v\Delta T_v)\left[E_{t0} + \frac{\partial E_t}{\partial q_c}\Delta q_c + \frac{\partial E_t}{\partial T_v}\Delta T_v + \frac{\partial E_t}{\partial w_l}\Delta w_l + \frac{\partial E_t}{\partial w_s}\Delta w_s\right] \\
&\quad - (L_e' + c_v\Delta T_v)\left[E_{l0} + \frac{\partial E_l}{\partial q_c}\Delta q_c + \frac{\partial E_l}{\partial T_v}\Delta T_v + \frac{\partial E_l}{\partial w_l}\Delta w_l + \frac{\partial E_l}{\partial w_s}\Delta w_s\right] \\
&\quad - (L_s' + c_v\Delta T_v)\left[E_{s0} + \frac{\partial E_s}{\partial q_c}\Delta q_c + \frac{\partial E_s}{\partial T_v}\Delta T_v + \frac{\partial E_s}{\partial w_l}\Delta w_l + \frac{\partial E_s}{\partial w_s}\Delta w_s\right] \\
&\quad + c_l\big[f_{il}P_l T_{pl} - (T_v + \Delta T_v)D_l\big] + c_s\big[f_{is}P_s T_{ps} - (T_v + \Delta T_v)D_s\big]
\end{aligned} \tag{B.2}$$

eq:vegn-eb-lin-full

Bb. *Linearized equations in canonical form*

Canopy air energy balance (B.1)

$$\begin{aligned}
&\Delta q_c\left[\frac{m_c(c_v-c_p)T_c}{\Delta t}-c_vT_v\frac{\partial E_v}{\partial q_c}-c_vT_g\frac{\partial E_g}{\partial q_c}+c_vT_c\frac{\partial E_a}{\partial q_c}\right]\\
&+\Delta T_c\left[\frac{m_c\big(c_p+q_c(c_v-c_p)\big)}{\Delta t}-\frac{\partial H_v}{\partial T_c}-\frac{\partial H_g}{\partial T_c}+\frac{\partial H_a}{\partial T_c}+c_vE_{a0}\right]\\
&-\Delta T_v\left[\frac{\partial H_v}{\partial T_v}+c_vE_{v0}+c_vT_v\frac{\partial E_v}{\partial T_v}\right]\\
&-\Delta w_l\, c_vT_v\frac{\partial E_v}{\partial w_l}\\
&-\Delta w_s\, c_vT_v\frac{\partial E_v}{\partial w_s}\\
&-\Delta T_g\left[\frac{\partial H_g}{\partial T_g}+c_vE_{g0}+c_vT_g\frac{\partial E_g}{\partial T_g}\right]\\
&\qquad\qquad=H_{v0}+H_{g0}-H_{a0}+c_v(T_vE_{v0}+T_gE_{g0}-T_cE_{a0})
\end{aligned}\tag{B.3}$$

Canopy energy balance (B.2)

$$\begin{aligned}
&\Delta q_c\left[L_t'\frac{\partial E_t}{\partial q_c}+L_e'\frac{\partial E_l}{\partial q_c}+L_s'\frac{\partial E_s}{\partial q_c}\right]+\Delta T_c\frac{\partial H_v}{\partial T_c}\\
&+\Delta T_v\left[\frac{C_v+c_lw_l+c_sw_s}{\Delta t}-\frac{\partial R_{Lv}}{\partial T_v}+\frac{\partial H_v}{\partial T_v}\right.\\
&\qquad\qquad+L_t'\frac{\partial E_t}{\partial T_v}+L_e'\frac{\partial E_l}{\partial T_v}+L_s'\frac{\partial E_s}{\partial T_v}+c_lD_l+c_sD_s\\
&\qquad\qquad\left.+c_v(E_{t0}+E_{l0}+E_{s0})\right]\\
&+\Delta w_l\left[\frac{c_lT_v}{\Delta t}+L_t'\frac{\partial E_t}{\partial w_l}+L_e'\frac{\partial E_l}{\partial w_l}+L_s'\frac{\partial E_s}{\partial w_l}\right]\\
&+\Delta w_s\left[\frac{c_sT_v}{\Delta t}+L_t'\frac{\partial E_t}{\partial w_s}+L_e'\frac{\partial E_l}{\partial w_s}+L_s'\frac{\partial E_s}{\partial w_s}\right]\\
&-\Delta T_g\frac{\partial R_{Lv}}{\partial T_g}\\
&\qquad\qquad=R_{Sv}+R_{Lv0}-H_{v0}-L_t'E_{t0}-L_e'E_{l0}-L_s'E_{s0}\\
&\qquad\qquad\quad+c_lf_{il}P_lT_{pl}-c_lD_lT_v+c_sf_{is}P_sT_{ps}-c_sD_sT_v
\end{aligned}\tag{B.4}$$

Bc. *Additional nonlinear terms*

In the fully linearized equations (B.1) and (B.1), in addition to the terms mentioned in section 3e, there are also several non-linear terms in the right-hand side.

In the equation (B.1):

$$\begin{aligned}
&c_v \Delta T_v \left[\frac{\partial E_v}{\partial q_c} \Delta q_c + \frac{\partial E_v}{\partial T_v} \Delta T_v + \frac{\partial E_v}{\partial w_l} \Delta w_l + \frac{\partial E_v}{\partial w_s} \Delta w_s \right] \\
&+ c_v \Delta T_g \left[\frac{\partial E_g}{\partial q_c} \Delta q_c + \frac{\partial E_g}{\partial T_g} \Delta T_g \right] \\
&- c_v \Delta T_c \left[\frac{\partial E_a}{\partial q_c} \Delta q_c \right]
\end{aligned} \tag{B.5}$$

and (B.2):

$$- c_v \Delta T_v \left[\frac{\partial E_v}{\partial q_c} \Delta q_c + \frac{\partial E_v}{\partial T_v} \Delta T_v + \frac{\partial E_v}{\partial w_l} \Delta w_l + \frac{\partial E_v}{\partial w_s} \Delta w_s \right] \tag{B.6}$$

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