Water uptake by roots

Technical Note

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1. Linear treatment

Normalized vertical distribution of roots (for historical reasons, called `uptake_frac_max` in the code) is calculated as:

$$f_i^r = \frac{\exp(-z_{i-1}/\zeta) - \exp(-z_i/\zeta)}{1 - \exp(-z_N/\zeta)} \tag{1}$$

where z_i is the depth of the interface between layer i and $i+1$ (so that $z_0 = 0$ and z_N is the total depth of soil) and ζ is e-folding depth of roots.

It's easy to see that

$$\sum_{i=1}^{N} f_i^r = 1 \tag{2}$$

An analog of the bucket-style β factor can be calculated for each layer:

$$\beta_i = f_i^r \max\left(0, \min\left(1, \frac{\theta_i - \theta_{wilt}}{0.75(\theta_{fc} - \theta_{wilt})}\right)\right) \tag{3}$$

and normalized

$$f_i^u = \frac{\beta_i}{\sum_{i=1}^{N} \beta_i} \tag{4}$$

so that

$$\sum_{i=1}^{N} f_i^u = 1 \tag{5}$$

f_i^u is called `soil%uptake_frac` in the code, it's calculated in `soil_step_1` and then used in `soil_step_2`.

Given the total transpiration E_t, the uptake from each layer U_i (kg/(m^2 s)) is calculated as

$$U_i = E_t f_i^u \tag{6}$$

Water supply

$$U_{max} = \sum_{i=1}^{N} D_{fr} C_r \rho B_r f_i^r \max(0, \theta_i - \theta_{wilt}) \tag{7}$$

where D_{fr} is "diameter of fine roots," C_r is a scaling factor, ρ is the density of water, and B_r is toatal biomass of fine roots, $\mathrm{kg_C/m^2}$.

2. Treatment based on Darcy law

Use standard 2D radial flow, Darcy formulation. Define soil-controlled flux as that with root potential at plant permanent wilting point. Use quasi-steady state approximation.

Let u be water uptake per unit length of fine root [$\mathrm{kg/(m\,s)}$]; R – characteristic radial half-distance to the next root [m], r_r – root radius [m], and r – "microscopic" distance from root axis [m].

For steady flow toward the root,

$$u = 2\pi r K \frac{d\psi}{dr} \tag{8}$$

where $K = K(\psi)$ is unsaturated hydraulic conductivity [$\mathrm{kg/(m^2\,s)}$],

$$K(\psi) = \begin{cases} K_s \left(\frac{\psi}{\psi_*}\right)^{-(2+3/b)} & \psi \le \psi_* \\ K_s & \psi > \psi_* \end{cases} \tag{9}$$

where ψ is soil water matric head [m], and ψ_* is air entry water potential. Note that the flow is assumed to be steady-state, u doesn't depend on r.

Integrating from root-soil interface to "bulk" soil (with matric head ψ_s at the distance R from the root axis):

$$\int_{r_r}^{R} \frac{u\,dr}{2\pi r} = \int_{\psi_r}^{\psi_s} K(\psi)d\psi \tag{10}$$

or, equivalently:

$$u = \frac{2\pi}{\ln(R/r_r)} \int_{\psi_r}^{\psi_s} K(\psi)d\psi \tag{11}$$ $\boxed{\texttt{u1}}$

This relationship is assumed to hold at a macroscopic point, i.e., a model layer in our case.

The characteristic half-distance between roots R can be expressed in terms of specific root length λ, [$\mathrm{m/kg_C}$] (SRL, length of fine roots per unit mass of carbon) and the volumetric density of root biomass b_r, [$\mathrm{kg_C/m^3}$]. The total length of roots per unit volume is λb_r; therefore the area of soil cross-section surrounding the root is $A = \pi R^2 = 1/\lambda b_r$, giving

$$R = \frac{1}{\sqrt{\pi\lambda b_r}} \tag{12}$$

To evaluate the integral in (11) let's assume (for now) that $\psi_r < \psi_s$ and $\psi_r < \psi_*$; then:

$$\int_{\psi_r}^{\psi_s} K(\psi)\, d\psi = \int_{\psi_r}^{\min(\psi_s,\psi_*)} K_s \left(\frac{\psi}{\psi_*}\right)^{-(2+3/b)} d\psi + \int_{\min(\psi_s,\psi_*)}^{\psi_s} K_s\, d\psi$$

$$= \frac{K_s\psi_*}{n}\left[\left(\frac{\min(\psi_s,\psi_*)}{\psi_*}\right)^n - \left(\frac{\psi_r}{\psi_*}\right)^n\right] + K_s(\psi_s - \min(\psi_s,\psi_*)) \tag{13}$$ $\boxed{\texttt{u:integral}}$

where we introduced notation $n = -(1+3/b)$. In general case, if our assumptions about value of ψ_r are invalid, we can always write

$$\int_{\psi_r}^{\psi_s} K(\psi)\, d\psi = \int_{\psi_m}^{\psi_s} K(\psi)\, d\psi - \int_{\psi_m}^{\psi_r} K(\psi)\, d\psi \tag{14}$$ $\boxed{\texttt{u4}}$

where ψ_m is some arbitrary low value of water potential, such that $\psi_m \le \min(\psi_r, \psi_s, \psi_*)$. Applying (13) to both terms in the right-hand side of (14), and noting that $\psi - \min(\psi, \psi_*) = \max(0, \psi - \psi_*)$, we finally get the general expression for the steady-state flow toward the root:

$$u = \frac{2\pi K_s}{\ln(R/r_r)}\left\{\frac{\psi_*}{n}\left[\left(\frac{\min(\psi_s,\psi_*)}{\psi_*}\right)^n - \left(\frac{\min(\psi_r,\psi_*)}{\psi_*}\right)^n\right] + \max(0, \psi_s - \psi_*) - \max(0, \psi_r - \psi_*)\right\} \tag{15}$$ $\boxed{\texttt{u:soil}}$

On the other hand, the water flux through the root skin per unit length of root is

$$u = 2\pi r_r K_r(\psi_r - \psi_x) \tag{16}$$ $\boxed{\texttt{u:root}}$

where K_r is permeability of root membrane per unit membrane area [kg/(m^3 s)], and ψ_x is the water potential inside the root (xylem water potential) [m].

Combining (15) and (16), we get:

$$r_r K_r(\psi_r - \psi_x) = \frac{2\pi K_s}{\ln(R/r_r)}\left\{\frac{\psi_*}{n}\left[\left(\frac{\min(\psi_s,\psi_*)}{\psi_*}\right)^n - \left(\frac{\min(\psi_r,\psi_*)}{\psi_*}\right)^n\right] + \max(0, \psi_s - \psi_*) - \max(0, \psi_r - \psi_*)\right\} \tag{17}$$ $\boxed{\texttt{u:final}}$

Given xylem water potential ψ_x and soil water potential ψ_s, we can solve[1] equation (17) to get the water potential at the root-soil interface ψ_r, and, consequently, the water flux per unit root length $u = u(\psi_r, \psi_s)$.

[1] The model solves (17) and (19) numerically, using a robust variant of Newton-Raphson solver (Press et al. 1997). It should be noted that the solution is facilitated by the fact that derivative of equation (15) can be calculated simply as:

$$\frac{\partial u}{\partial \psi_r} = -\frac{2\pi}{\ln(R/r_r)} K(\psi_r)$$

Species	SRL (m/kg$_C$)	Root radius (10^{-3} m)
C3 Grass	236.0	0.11
C4 Grass	236.0	0.11
Temperate Deciduous Trees	24.4	0.29
Tropical Trees	24.4	0.29
Evergreen Trees	24.4	0.29

Table 1: Root uptake parameters, Jackson et al. (1997). SRL values were converted to carbon units using carbon-to-biomass factor of 2.

To calculate the total water uptake, we should note that the xylem potential increases with depth so that $\psi_x = \psi_{x0} + z$, where ψ_{x0} is the xylem potential at the surface. The total uptake will be then the sum of layer values, properly weighted:

$$U(\psi_{x0}) = \sum_{1}^{N} u(\psi_{x0} + z_i, \psi_i) L_i S_i \tag{18}$$

u:total

where z_i is the depth of the layer, ψ_i is the soil water potential in the layer, and L_i is the total length of roots in the layer. The additional factor S_i is used to turn off uptake when certain conditions are met: when there is ice in the layer; when the uptake is negative (optional, when one-way-uptake is requested); and when the soil is saturated (optional, when uptake-from-sat is not requested).

The maximum soil water supply to the vegetation U_{max} is calculated as the value of the uptake for the xylem water potential at the surface equal to the permanent wilting point ψ_{wilt}: $U_{max} = U(\psi_{wilt})$.

To obey the mass conservation law, the transpiration by plants E_t must be exactly equal to the total uptake U, so after the transpiration is calculated (given the limiting value of U_{max} and other physical and physiological factors) we solve the equation

$$E_t = U(\psi_{x0}) \tag{19}$$

utotal

to find the value of ψ_{x0} that satisfies the mass conservation condition.

3. Linearized equation for uptake

While the equation (19) gives the exact (within the assumptions) solution for the uptake by roots, its solution is awkward: it is nonlinear, can't be solved analytically, and therefore must be solved iteratively. Moreover, the equation (17) is transcendental, and therefore also must be solved numerically to get uptake u for given ψ_i and $\psi_x = \psi_{x0} + z_i$. Together, these two facts result in very slow performance of the straightforward implementation[2].

[2] Not so straightforward implementation might tabulate the solution of (17) for range of inputs and parameters; however, the parameter space is many-dimensional, making such tabulation not very easy.

Let's linearize (15) around some known value $\psi_r^{(0)}$:

$$u = u^{(0)} + u^{(1)}(\psi_r - \psi_r^{(0)}) \tag{20}$$

u:soil-lin

where

$$u^{(0)} = u(\psi_r^{(0)}), \quad u^{(1)} = \frac{\partial u}{\partial \psi_r} \tag{21}$$

and the partial derivative is calculated for the current values of prognostic variables and initial guess of root/soil interface water potential $\psi_r^{(0)}$. Combining the equation (20) with (16), and introducing notation $\mathbb{K}_r = 2\pi r_r K_r$, we get

$$u^{(0)} + u^{(1)}(\psi_r - \psi_r^{(0)}) = \mathbb{K}_r(\psi_r - \psi_x) \tag{22}$$

or, rearranging terms:

$$\psi_r \left(u^{(1)} - \mathbb{K}_r \right) + u^{(0)} - \psi_r^{(0)} u^{(1)} + \mathbb{K}_r \psi_x = 0 \tag{23}$$

and, therefore:

$$\psi_r = \frac{u^{(0)} - \psi_r^{(0)} u^{(1)} + \mathbb{K}_r \psi_x}{\mathbb{K}_r - u^{(1)}} \tag{24}$$

Note that $u^{(1)} < 0$, and $\mathbb{K}_r$ is positive, so that the denominator is never zero. Finally, we get the following expression for the water flow toward the root per unit root length:

$$\begin{aligned} u =& \mathbb{K}_r \left[\frac{u^{(0)} - \psi_r^{(0)} u^{(1)} + \mathbb{K}_r \psi_x}{\mathbb{K}_r - u^{(1)}} - \psi_x \right] \\ =& \frac{\mathbb{K}_r}{\mathbb{K}_r - u^{(1)}} \left[u^{(0)} - u^{(1)}(\psi_r^{(0)} - \psi_x) \right] \end{aligned} \tag{25}$$

u:full-lin

We can apply the equation (25) at every soil layer, and, from equations (18), (19) get:

$$E_t = \sum_1^N \frac{\mathbb{K}_r L_i S_i}{\mathbb{K}_r - u_i^{(1)}} \left(u_i^{(0)} - u_i^{(1)}(\psi_{r,i}^{(0)} - \psi_{x0} - z_i) \right) \tag{26}$$

lin:general

The above equation is still nonlinear, since in general case triggers S_i depend on the difference between $\psi_{x,i} = \psi_{x0} + z_i$ and $\psi_{s,i}$; therefore it must be solved numerically. In special case where $S_i = 1$, the solution can be obtained analytically:

$$\psi_{x0} = \left[E_t - \sum_1^N \frac{\mathbb{K}_r L_i}{\mathbb{K}_r - u_i^{(1)}} \left(u_i^{(0)} - u_i^{(1)}(\psi_{r,i}^{(0)} - z_i) \right) \right] \left[\sum_1^N \frac{\mathbb{K}_r u_i^{(1)} L_i}{\mathbb{K}_r - u_i^{(1)}} \right]^{-1} \tag{27}$$

lin:twoway

The model solves equation (26) numerically for one-way linearized option, and uses more efficient expression (27) for two-way linearized Darcy uptake. For simplicity of implementation, the model assumes $\psi_{r,i}^{(0)} = \psi_{s,i}$.

References

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