Homework Assignment 2

Due September 12, 8:30am

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1 Problems

1. What are the matrices that implement the elemental matrix inversion operations on *columns* instead of rows (i.e. scale columns, transpose columns, add one column to another)? Do these matrices multiply the target matrix on the left or on the right?

2. Let $\mathcal{X} = \{\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \ldots, \underline{\mathbf{x}}_N\}$ be an arbitrary set of equal-length vectors. Prove that the set of all possible linear combinations of the vectors in $\mathcal{X}$ is a vector space.

3. Decide whether or not the following vectors are linearly independent by looking for non-zero scalar multipliers of each vector such that they sum to zero. Justify your answer.

$$\underline{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \underline{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \underline{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \underline{\mathbf{v}}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

4. Describe geometrically the subspace of $\mathbb{R}^3$ spanned by
   (a) $(0,0,0), \quad (0,1,0), \quad (0,2,0)$
   (b) $(0,0,1), \quad (0,1,1), \quad (0,2,1)$
   (c) All six of the vectors. Which 2 of them form an orthonormal basis for the subspace?

(d) The set of all vectors with only positive components

5. consider the following 4 vectors

$$\underline{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \underline{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \underline{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad \underline{\mathbf{v}}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \tag{2}$$

(a) Are these four vectors all mutually orthogonal? Justify your answer

(b) Can these vectors be scaled such that they form an orthonormal basis? If so, by what factor is each vector scaled?

(c) Let $\underline{\mathbf{u}}_i = \alpha_i \underline{\mathbf{v}}_i$ where α_i are the scale factors you chose above. Let

$$A = \begin{bmatrix} \underline{\mathbf{u}}_1^T \\ \underline{\mathbf{u}}_2^T \\ \cdots \\ \underline{\mathbf{u}}_N^T \end{bmatrix} \tag{3}$$

What is A^T? (write it out)

(d) What is AA^T?

(e) Let $\underline{\mathbf{u}}_i$ be a set of orthonormal vectors. Let A be the matrix with the $\underline{\mathbf{u}}_i^T$ as its rows. We demonstrated above that $AA^T = I$ and hence that $A^{-1} = A^T$. We call A a transform matrix, and its inverse is A^T. Note that $A\underline{\mathbf{x}}$ computes the projection of $\underline{\mathbf{x}}$ onto each basis vector (make sure you understand this!). The transform above is the Hadamard transform. It is useful because it is similar to a DFT, i.e., the rows are rather "sinusoidal" in appearance. It is nice because it can be computed with only additions and subtractions. Sketch a graph of the 4 basis functions of the Hadamard transform.