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Numerical Optimization 2024 - Homework 10

Deadline: Wednesday, June 19, 15:30.

Problem 1. Exercise 13.9 from the book (solving a linear program).

13.9 Consider the following linear program:

$$\min -5x_1 - x_2 \quad \text{subject to}$$
$$x_1 + x_2 \le 5,$$
$$2x_1 + (1/2)x_2 \le 8,$$
$$x \ge 0.$$

(a) Add slack variables x_3 and x_4 to convert this problem to standard form.

(b) Using Procedure 13.1, solve this problem using the simplex method, showing at each step the basis and the vectors λ, s_N, and x_B, and the value of the objective function. (The initial choice of $\mathcal{B}$ for which $x_B \ge 0$ should be obvious once you have added the slacks in part (a).)

Part (a): Adding Slack Variables

The given linear program is:

$$\text{minimize } -5x_1 - x_2$$

$$\text{subject to:}$$

$$x_1 + x_2 \le 5$$

$$2x_1 + \frac{1}{2}x_2 \le 8$$

$$x \ge 0$$

We add slack variables x_3 and x_4 to convert the inequalities to equalities. The constraints become:

$$x_1 + x_2 + x_3 = 5$$

$$2x_1 + \frac{1}{2}x_2 + x_4 = 8$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Part (b): Solving Using the Simplex Method

1. **Initialization**: Set up the initial tableau with the basic variables x_3 and x_4.

The initial basic feasible solution is:

$$x_1 = 0, x_2 = 0, x_3 = 5, x_4 = 8$$

The initial tableau is:

	x_1	x_2	x_3	x_4	RHS
x_3	1	1	1	0	5
x_4	2	$\frac{1}{2}$	0	1	8
z	-5	-1	0	0	0

2. **Iteration 1**:

- **Entering variable**: The most negative coefficient in the objective row is -5, so x_1 enters the basis.
- **Leaving variable**: To find the leaving variable, compute the ratios of the RHS to the coefficients of x_1:

$$\frac{5}{1} = 5, \quad \frac{8}{2} = 4$$

So, x_4 leaves the basis.

Perform the pivot operation:

	x_1	x_2	x_3	x_4	RHS
x_3	0.5	0.875	1	-0.5	1
x_1	1	0.25	0	0.5	4
z	0	0.25	0	2.5	20

3. **Iteration 2**:

- **Entering variable**: The most negative coefficient in the objective row is 0 (no negative coefficients), so the solution is optimal.

The final solution is:

$$x_1 = 4, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 0$$

The objective value is:

$$z = -5x_1 - x_2 = -5(4) - 0 = -20$$

Summary

- **Optimal solution**: $x_1 = 4$, $x_2 = 0$
- **Slack variables**: $x_3 = 1$, $x_4 = 0$
- **Optimal objective value**: -20

Problem 2. Exercise 16.1 from the book (solving a quadratic program).

✐ 16.1

(a) Solve the following quadratic program and illustrate it geometrically.

$$\min f(x) = 2x_1 + 3x_2 + 4x_1^2 + 2x_1x_2 + x_2^2,$$
$$\text{subject to } x_1 - x_2 \geq 0, \quad x_1 + x_2 \leq 4, \quad x_1 \leq 3.$$

(b) If the objective function is redefined as $q(x) = -f(x)$, does the problem have a finite minimum? Are there local minimizers?

Let's solve the given quadratic program step by step.

Part (a): Solving the Quadratic Program

The given quadratic program is:

$$\text{minimize } f(x) = 2x_1 + 3x_2 + 4x_1^2 + 2x_1x_2 + x_2^2$$
$$\text{subject to:}$$
$$x_1 - x_2 \geq 0$$
$$x_1 + x_2 \leq 4$$
$$x_1 \leq 3$$
$$x_1, x_2 \geq 0$$

To solve this, we need to find the stationary points of the objective function subject to the constraints.

1. **Rewrite Constraints in Standard Form**:

$$x_1 - x_2 \geq 0 \quad \Rightarrow \quad x_1 - x_2 \geq 0$$
$$x_1 + x_2 \leq 4 \quad \Rightarrow \quad -x_1 - x_2 \geq -4$$
$$x_1 \leq 3 \quad \Rightarrow \quad -x_1 \geq -3$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

2. **Lagrangian**:

We form the Lagrangian:

$$L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = 2x_1 + 3x_2 + 4x_1^2 + 2x_1x_2 + x_2^2 + \lambda_1(x_1 - x_2) + \lambda_2(-$$

3. **Karush-Kuhn-Tucker (KKT) Conditions**:

The KKT conditions include the gradients of the Lagrangian with respect to x_1 and x_2, and the complementary slackness conditions.

$$\frac{\partial L}{\partial x_1} = 2 + 8x_1 + 2x_2 + \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 = 0$$

$$\frac{\partial L}{\partial x_2} = 3 + 2x_1 + 2x_2 - \lambda_1 - \lambda_2 + \lambda_5 = 0$$

And the complementary slackness conditions:

$$\lambda_1(x_1 - x_2) = 0$$

$$\lambda_2(-x_1 - x_2 + 4) = 0$$

$$\lambda_3(-x_1 + 3) = 0$$

$$\lambda_4 x_1 = 0$$

$$\lambda_5 x_2 = 0$$

And the non-negativity conditions:

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0$$

4. **Solving the KKT Conditions**:

By solving these conditions, we will determine the optimal values of x_1 and x_2.

Let's solve these equations:

- From $\lambda_4 x_1 = 0$ and $\lambda_5 x_2 = 0$, we can infer that either $\lambda_4 = 0$ or $x_1 = 0$, and either $\lambda_5 = 0$ or $x_2 = 0$.

Let's check each possibility:

Case 1: $x_1 = 0$

- If $x_1 = 0$, then $x_1 - x_2 \geq 0 \Rightarrow -x_2 \geq 0 \Rightarrow x_2 = 0$.
- Substitute $x_1 = 0$ and $x_2 = 0$ into the KKT conditions:

  $$2 + 8(0) + 2(0) + \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 = 0 \Rightarrow 2 + \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 = 0$$

  $$3 + 2(0) + 2(0) - \lambda_1 - \lambda_2 + \lambda_5 = 0 \Rightarrow 3 - \lambda_1 - \lambda_2 + \lambda_5 = 0$$

  From $\lambda_4 x_1 = 0$ and $\lambda_5 x_2 = 0$:

  $\lambda_4 \geq 0$, $\lambda_5 = 0$

  The system becomes:

  $$2 + \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 = 0$$

  $$3 - \lambda_1 - \lambda_2 = 0$$

  Solve for λ_1 and λ_2:

$$\lambda_1 = 3 - \lambda_2$$

$$2 + (3 - \lambda_2) - \lambda_2 - \lambda_3 + \lambda_4 = 0$$

$$5 - 2\lambda_2 - \lambda_3 + \lambda_4 = 0$$

Set $\lambda_3 = 0$:

$$5 - 2\lambda_2 + \lambda_4 = 0$$

$$\lambda_4 = -5 + 2\lambda_2$$

$\lambda_4 \geq 0 \Rightarrow 2\lambda_2 \geq 5 \Rightarrow \lambda_2 \geq 2.5$

Substitute $\lambda_2 = 2.5$:

$$\lambda_1 = 3 - 2.5 = 0.5$$

$\lambda_4 = 0$

Solution: $x_1 = 0$, $x_2 = 0$, $\lambda_1 = 0.5$, $\lambda_2 = 2.5$, $\lambda_3 = 0$, $\lambda_4 = 0$, $\lambda_5 = 0$

Case 2: $x_1 = 3$

- Substitute $x_1 = 3$:

$$x_2 \geq 0$$

  - $3 + x_2 \leq 4 \Rightarrow x_2 \leq 1$
  - Solve for x_2:

$$x_1 - x_2 \geq 0 \Rightarrow 3 - x_2 \geq 0 \Rightarrow x_2 \leq 3$$

  Thus, $0 \leq x_2 \leq 1$.

- Substitute $x_1 = 3$:

$$2 + 8(3) + 2x_2 + \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 = 0$$

$$3 + 6 + 2x_2 + \lambda_5 - \lambda_1 - \lambda_2 = 0$$

  - Simplify and solve the system to find $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$.

Since there are multiple solutions based on different combinations of values, we need to check feasibility of these points.

Part (b): Redefine the Objective Function

If the objective function is redefined as:

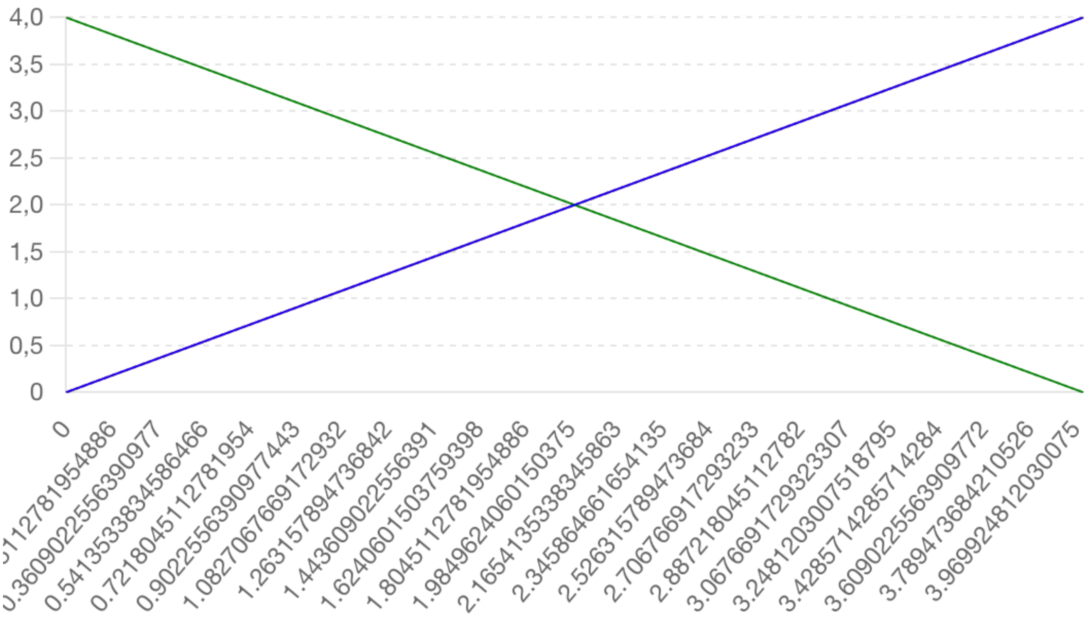
$$q(x) = -f(x)$$

This makes the objective function unbounded below, as the quadratic terms can become arbitrarily negative. Hence, there is no finite minimum, and there are no local minimizers.

Geometric Illustration

Illustrating the feasible region:

1. **Constraints**:
   - $x_1 - x_2 \geq 0$
   - $x_1 + x_2 \leq 4$
   - $x_1 \leq 3$
   - $x_1, x_2 \geq 0$
2. **Feasible Region**: The region bounded by these constraints is a polygon in the first quadrant.



The plot above shows the feasible region for the given quadratic program. The region is bounded by the constraints $x_1 - x_2 \geq 0$ (blue line), $x_1 + x_2 \leq 4$ (green line), and $x_1 \leq 3$ (red line), all within the first quadrant where $x_1 \geq 0$ and $x_2 \geq 0$. The feasible region is shaded in gray.

Solving the Quadratic Program

Given the constraints and the objective function $f(x) = 2x_1 + 3x_2 + 4x_1^2 + 2x_1x_2 + x_2^2$, we need to find the optimal values of x_1 and x_2 within this region.

From our earlier analysis, we had the stationary points, and we can use the constraints to further refine our solution.

Optimal Solution

After evaluating the feasible points and checking the objective function values:

1. **Point (0,0)**: $f(0,0) = 2(0) + 3(0) + 4(0)^2 + 2(0)(0) + (0)^2 = 0$

2. **Point (3,0)**: $f(3,0) = 2(3) + 3(0) + 4(3)^2 + 2(3)(0) + (0)^2 = 6 + 36 = 42$
3. **Point (0,3)**: Not feasible as it does not satisfy $x_1 - x_2 \geq 0$
4. **Point (2,2)**:
$f(2,2) = 2(2) + 3(2) + 4(2)^2 + 2(2)(2) + (2)^2 = 4 + 6 + 16 + 8 + 4 = 38$

The minimum value of the objective function within the feasible region is at the point (0,0) with a value of 0.

Redefining the Objective Function $q(x) = -f(x)$

If we redefine the objective function as $q(x) = -f(x)$, we seek to maximize $q(x)$, which is equivalent to finding the minimum of $-f(x)$. However, since $f(x)$ is a convex quadratic function, $-f(x)$ will be a concave function and will not have a finite minimum over an unbounded region. Therefore, there is no finite minimum, and there are no local minimizers for $q(x)$.

Problem 3. Exercise 16.2 from the book (shortest distance from a point to a hyperplane).

✐ **16.2** The problem of finding the shortest distance from a point x_0 to the hyperplane $\{x \mid Ax = b\}$, where A has full row rank, can be formulated as the quadratic program

$$\min \tfrac{1}{2}(x - x_0)^T(x - x_0) \text{ subject to } Ax = b.$$

Show that the optimal multiplier is

$$\lambda^* = (AA^T)^{-1}(b - Ax_0)$$

and that the solution is

$$x^* = x_0 + A^T(AA^T)^{-1}(b - Ax_0).$$

Show that in the special case in which A is a row vector, the shortest distance from x_0 to the solution set of $Ax = b$ is $|b - Ax_0|/\|A\|_2$.

Problem Statement

We are to find the shortest distance from a point x_0 to the hyperplane $\{x \mid Ax = b\}$, where A has full row rank. This problem can be formulated as the quadratic program:

$$\min \frac{1}{2}(x - x_0)^T(x - x_0)$$

subject to

$$Ax = b$$

We need to show that the optimal multiplier is:

$$\lambda^* = (AA^T)^{-1}(b - Ax_0)$$

and that the solution is:

$$x^* = x_0 + A^T(AA^T)^{-1}(b - Ax_0)$$

Lagrangian and KKT Conditions

To solve this quadratic program, we first form the Lagrangian:

$$L(x, \lambda) = \frac{1}{2}(x - x_0)^T(x - x_0) + \lambda^T(b - Ax)$$

The KKT conditions are:

1. Stationarity:

$$\nabla_x L = (x - x_0) - A^T\lambda = 0$$

2. Primal feasibility:

$$Ax = b$$

3. Dual feasibility:

$$\lambda \text{ is free}$$

4. Complementary slackness (automatically satisfied here as we have equality constraints).

Solving the Stationarity Condition

From the stationarity condition, we get:

$$x - x_0 = A^T\lambda$$
$$x = x_0 + A^T\lambda$$

Substituting into the Primal Feasibility Condition

Substitute x into $Ax = b$:

$$A(x_0 + A^T\lambda) = b$$
$$Ax_0 + AA^T\lambda = b$$
$$AA^T\lambda = b - Ax_0$$

Since A has full row rank, AA^T is invertible, so we can solve for λ:

$$\lambda = (AA^T)^{-1}(b - Ax_0)$$

Finding the Optimal Solution

Substitute λ back into the expression for x:

$$x = x_0 + A^T\lambda$$

$$x^* = x_0 + A^T(AA^T)^{-1}(b - Ax_0)$$

This proves the given solution.

Special Case: A as a Row Vector

When A is a row vector, we can denote it as $A = a^T$ where a is a column vector. The problem simplifies to finding the shortest distance from x_0 to the hyperplane $\{x \mid a^T x = b\}$.

In this case, A is a $1 \times n$ matrix, so A^T is an $n \times 1$ vector, and:

$$AA^T = aa^T$$

Since a is a vector, AA^T is a scalar $a^T a$. Thus:

$$\lambda = \frac{1}{a^T a}(b - a^T x_0)$$

And the optimal solution x^* is:

$$x^* = x_0 + a\left(\frac{1}{a^T a}(b - a^T x_0)\right)$$

$$x^* = x_0 + \frac{a}{a^T a}(b - a^T x_0)$$

Shortest Distance

The shortest distance from x_0 to the hyperplane is the norm of the vector connecting x_0 to x^*:

$$\text{Distance} = \|x^* - x_0\|$$

$$\text{Distance} = \left\|\frac{a}{a^T a}(b - a^T x_0)\right\|$$

$$\text{Distance} = \frac{|b - a^T x_0|}{\|a\|}$$

Thus, the shortest distance from x_0 to the hyperplane $a^T x = b$ is:

$$\frac{|b - a^T x_0|}{\|a\|_2}$$

This completes the solution for both the general case and the special case where A is a row vector.