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Magnets at the corners of polygons

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We consider the statics and dynamics of N identical magnetic dipoles, with inertia, pinned at the corners of a regular polygon with N sides. We show that at equilibrium all dipoles are aligned tangent to the circumscribed circle. We find the normal modes and frequencies for oscillations about equilibrium, and discuss the dispersion relation. In the highest frequency normal mode the dipoles oscillate in phase with equal amplitudes, so no magnetic dipole radiation is emitted. In the limit $N\to\infty$, the equilibrium energy and the frequency of the symmetric mode are both related to $\zeta(3)$, where ζ is the Riemann zeta function.

I. INTRODUCTION

When magnets are on a line they spontaneously order with the north pole of one magnet facing the south pole of the next, i.e., with their magnetic moments parallel to each other. But suppose the magnets are arranged in a plane, with each magnet free to rotate about a fixed axis through its center. What is the equilibrium configuration then?

In practice one can approximately answer this question by mechanical construction with pivots and magnets. To study this problem we did that for an equilateral triangle, a square, and a regular pentagon and hexagon. But to solve the general problem analytically, or even numerically, is difficult because one must find the minimum of the system's magnetic potential energy, which is a function of many variables. Even for only three magnets at the corners of an arbitrary triangle the analytical problem is difficult.

In this paper we discuss a special case of this problem, namely, identical magnetic dipoles at the corners of a regular polygon with N sides. We find the equilibrium energy, the normal modes and frequencies for oscillations about equilibrium, and discuss the dispersion curve and other interesting properties of these oscillations.

The more basic problem of two pinned dipoles oscillating in each other's fields, including the effects of radiation, radiation reaction, and retardation, has been treated in a recent paper.[1] A related problem, of the equilibrium structure of colloidal suspensions of magnetic particle aggregates, has been considered by Jund *et al.*[2] In our terms these are effectively unpinned magnets with vanishing inertia and hard-core interactions. They found, using molecular dynamics techniques, that for $N>3$ the equilibrium configuration is a regular polygon.

II. EQUILIBRIUM AND NORMAL MODES OF OSCILLATION

A. The potential energy of interaction

The basis of our calculation of magnetic potential energy for this configuration is the interaction energy between two dipoles. The magnetic field of a dipole $\mathbf{m}$ is $\mathbf{B}=(\mu_0/4\pi r^3)\times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}]$, and the potential energy of a dipole in the field $\mathbf{B}$ is $U=-\mathbf{m}\cdot\mathbf{B}$.[3] Thus, if we consider two dipoles, say, $\mathbf{m}_n$ and $\mathbf{m}_{n'}$ at the respective positions $\mathbf{r}_n$ and $\mathbf{r}_{n'}$ from the origin, then their mutual potential energy may be written

$$U_{nn'}=-\frac{\mu_0 m_n m_{n'}}{4\pi r_{nn'}^3}[2\cos\phi_n\cos\phi_{n'}-\sin\phi_n\sin\phi_{n'}]. \tag{1}$$

In Eq. (1), ϕ_n and $\phi_{n'}$ are the respective angles that $\mathbf{m}_n$ and $\mathbf{m}_{n'}$ make with the separation vector $\mathbf{r}_{nn'}=\mathbf{r}_{n'}-\mathbf{r}_n$, which points from $\mathbf{r}_n$ to $\mathbf{r}_{n'}$. Equation (1) assumes that the interaction between the dipoles is instantaneous. This is justified because a previous calculation shows that retardation effects are negligible for physically realistic systems.[1]

Figure 1 shows schematically the system we will study. At each corner of a regular polygon, which has N sides each of length d, there is centered a magnetic dipole $\mathbf{m}_n$, where $n=0,1,2,\ldots,N-1$. The circle that circumscribes the polygon is also shown. Its radius is R, and $d=2R\sin(\pi/N)$. The dipole $\mathbf{m}_0$ is on the x axis. Correspondingly, the dipole $\mathbf{m}_n$ is located at angle $\theta_n=2\pi n/N$ from the x axis. All the magnetic dipole moments have the same magnitude, m. Each dipole is free to oscillate in the x,y plane about an axis through its center and perpendicular to the plane. At equilibrium the dipoles align themselves tangent to the circumscribed circle and all point in the same sense, taken in Fig. 1

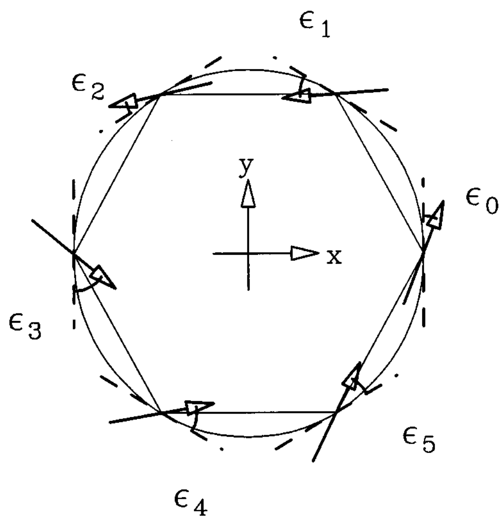

Fig. 1. At each vertex of a regular polygon with N sides, each of length d, a point magnetic dipole of magnitude m is free to rotate in the plane of the polygon. Dashed lines at each vertex show the equilibrium positions of the dipoles; these are tangent to the circumscribed circle, whose radius is R. The angle ϵ_n is the angular displacement of the nth dipole from its equilibrium position, so the moment of the nth dipole is $\mathbf{m}_n = -\hat{\mathbf{i}}m \sin(2\pi n/N+\epsilon_n)+\hat{\mathbf{j}}m \cos(2\pi n/N+\epsilon_n)$.

as counterclockwise. The equilibrium positions are shown by dashed lines at the corners of the polygon. The angular deviation of $\mathbf{m}_n$ from its equilibrium position is ϵ_n. In this problem, ϵ_n are the small oscillation angles.

From Fig. 1 we see that the distance between $\mathbf{m}_n$ and $\mathbf{m}_{n'}$ is

$$r_{nn'} = 2R \sin \frac{\pi}{N} |n-n'| = d\rho(n-n'), \tag{2}$$

where we define

$$\rho(\nu) = \frac{\sin \frac{\pi}{N} |\nu|}{\sin \frac{\pi}{N}}. \tag{2a}$$

The angles ϕ_n and $\phi_{n'}$ in Eq. (1), expressed in terms of ϵ_n and $\epsilon_{n'}$ are[4]

$$\phi_n = \epsilon_n + \frac{\pi}{N}(n-n'), \quad \phi_{n'} = \epsilon_{n'} + \frac{\pi}{N}(n'-n). \tag{3}$$

The total magnetic energy of the system of N magnetic dipoles is then

$$U = \frac{1}{2} \sum_{n,n'=0, n\neq n'}^{N-1} U_{nn'} = \sum_{n=0}^{N-1} \sum_{n'>n}^{N-1} U_{nn'}. \tag{4}$$

For the interesting case of small oscillations,[5] we expand U in powers of the ϵ_n's as

$$U = U^{(0)} + U^{(1)} + U^{(2)} + \cdots, \tag{5}$$

where $U^{(0)}$ is the value of U when $\epsilon_n=0$ for all n, $U^{(1)}$ contains the terms linear in ϵ_n, and $U^{(2)}$ contains the quadratic terms of the forms ϵ_n^2 and $\epsilon_n\epsilon_{n'}$.

1. The equilibrium potential energy

At static equilibrium $\epsilon_n=0$ for all n so that $U^{(0)}$, the equilibrium energy of our system, may be written

$$U^{(0)} = \frac{1}{2} N \sum_{n'=1}^{N-1} (U_{0n'})_{\epsilon=0}. \tag{6}$$

In writing Eq. (6) we have used the fact that, at equilibrium, the interaction energy of each of the N dipoles with all of the others is the same as the interaction energy of the zeroth dipole with all of the others. We now evaluate $U_{0n'}$ from Eq. (1) so that Eq. (6) becomes

$$U^{(0)} = -\frac{\mu_0 m^2 N}{8\pi} \sum_{n'=1}^{N-1} \frac{1}{r_{0n'}^3} \left[2\cos^2 \frac{\pi}{N} n' + \sin^2 \frac{\pi}{N} n' \right]$$

$$= -\frac{\mu_0 m^2 N}{8\pi d^3} Z_N, \tag{7}$$

where

$$Z_N = \sum_{n'=1}^{N-1} \left[\frac{1+\cos^2 \frac{\pi}{N} n'}{\rho^3(n')} \right]. \tag{8}$$

From Eq. (7) we obtain the equilibrium potential energy per dipole, $U^{(0)}/N = -(\mu_0 m^2/8\pi d^3)Z_N$. It is interesting to consider the limit of large N, *keeping the distance d between neighboring dipoles constant as N varies.* [The radius of the circumscribed circle increases with N, because $R = d/2 \sin(\pi/N)$.] In the limit of large N, the quantity Z_N approaches an interesting limit: $\lim_{N\to\infty} Z_N = 4\zeta(3) = 4.808\,24\ldots$, where $\zeta(3) = \Sigma_{\nu=1}^{\infty}(1/\nu^3)$ is the Riemann zeta function evaluated at 3.[6,7] As N increases, $U^{(0)}/N$ converges fairly rapidly to this limit; thus, for example, $Z_3=2.50$, $Z_4 = 3.36$, $Z_5=3.84$, and $Z_{10}=4.55$.

2. Potential energy terms that are linear in the angles of oscillation

For any configuration the torque on the nth dipole is $\partial U/\partial\epsilon_n$ or $\partial U/\partial\phi_n$, where the derivatives are evaluated for the actual angle to which the dipole is turned. We will now show that for the particular configuration in which all the ϵ_n's are zero, i.e., when the dipoles are tangent to the circumscribed circle and point in the same sense, $(\partial U/\partial\epsilon_n)_{\epsilon=0}=0$, so that there is no torque on any dipole. Therefore, this is an equilibrium configuration. In fact, this configuration minimizes the potential energy. It also then follows that $U^{(1)}=0$. We notice that in the equilibrium configuration we have $\Sigma_{n=0}^{N-1}\mathbf{m}_n=0$, so that the magnetic moment of the whole system is zero.

Because of the symmetry of the problem it will be sufficient to consider only the zeroth dipole (i.e., $n=0$). Using Eq. (4), we have

$$\left(\frac{\partial U}{\partial\epsilon_0}\right)_{\epsilon=0} = \sum_{n'=1}^{N-1} \left(\frac{\partial U_{0n'}}{\partial\epsilon_0}\right)_{\epsilon=0}$$

$$= \sum_{n'=1}^{[N/2]} \left[\left(\frac{\partial U_{0n'}}{\partial\epsilon_0}\right)_{\epsilon=0} + \left(\frac{\partial U_{0,N-n'}}{\partial\epsilon_0}\right)_{\epsilon=0} \right], \tag{9}$$

where $[N/2]$ is the largest integer which is smaller than $N/2$. For the case of even N, the $n'=N/2$ term appears in the sum after the first equals sign but has been omitted from the sum after the second equals sign because $(\partial U_{0,N/2}/\partial\epsilon_0)_{\epsilon=0}=0$.

We now show that the terms in the square brackets in Eq. (9) cancel out in pairs. We evaluate first, using Eq. (1),

$$\left(\frac{\partial U_{0n'}}{\partial\epsilon_0}\right)=\frac{\mu_0 m^2}{4\pi r_{0n'}^3}\,[2\sin\phi_0\cos\phi_{n'}+\cos\phi_0\sin\phi_{n'}],\tag{10}$$

and, using Eq. (3),

$$\left(\frac{\partial U_{0n'}}{\partial\epsilon_0}\right)_{\epsilon=0}=-\frac{\mu_0 m^2}{4\pi r_{0n'}^3}\sin\frac{\pi}{N}n'\cos\frac{\pi}{N}n'.\tag{11}$$

Similarly, we have

$$\left(\frac{\partial U_{0,N-n'}}{\partial\epsilon_0}\right)_{\epsilon=0}=\frac{\mu_0 m^2}{4\pi r_{0,N-n'}^3}\sin\frac{\pi}{N}n'\cos\frac{\pi}{N}n'.\tag{12}$$

Because of the symmetry, $r_{0n'}=r_{0,N-n'}$. Therefore, the square bracket in Eq. (9) vanishes.

3. Potential energy terms that are quadratic in the angles of oscillation

The term $U^{(2)}$ may be written

$$U^{(2)}=\frac{1}{2}\sum_{n,n'=0}^{N-1}\epsilon_n M_{nn'}\epsilon_{n'},\tag{13}$$

where

$$M_{nn'}=\left(\frac{\partial^2 U}{\partial\epsilon_n\partial\epsilon_{n'}}\right)_{\epsilon=0}=\left(\frac{\partial^2 U}{\partial\phi_n\partial\phi_{n'}}\right)_{\epsilon=0}\tag{14}$$

is a symmetric matrix. Because of the cyclic symmetry of the problem the elements of $M_{nn'}$ depend only on the difference $n-n'$. Therefore,

$$M_{nn'}=V_{|n-n'|}\tag{15}$$

is a Toeplitz matrix. After some algebra we find

$$V_\nu=\frac{\mu_0 m^2}{4\pi d^3}\left[\frac{1+\sin^2\frac{\pi}{N}\nu}{\rho^3(\nu)}\right]\quad\text{for }\nu\neq 0,$$

and

$$V_0=\frac{\mu_0 m^2}{4\pi d^3}Z_N.\tag{16}$$

B. Normal modes of oscillation

The normal modes and frequencies of oscillation for this problem are determined by considering the eigenvalue problem[5]

$$\sum_{n'=0}^{N-1}M_{nn'}a_{n'}^{(j)}=\lambda^{(j)}a_n^{(j)},\tag{17}$$

in which $\lambda^{(j)}$ is the jth eigenvalue of $M_{nn'}$ and $a_n^{(j)}$ is the nth component of the corresponding eigenvector. The kinetic energy for our system is $\mathrm{KE}_{\mathrm{rotational}}=(1/2)\Sigma_{n=0}^{N-1}I\dot{\epsilon}_n^2$, where I is the moment of inertia of each dipole, so the equations of motion are

$$I\ddot{\epsilon}_n=-\frac{\partial U^{(2)}}{\partial\epsilon_n}=-\sum_{n'=0}^{N-1}M_{nn'}\epsilon_{n'}.\tag{18}$$

Now, for motion in the jth mode, ϵ_n undergoes harmonic oscillation at angular frequency $\omega^{(j)}$ and is an eigenvector of $M_{nn'}$ so, combining Eqs. (17) and (18), we find that the actual normal frequencies of our problem are related to the $\lambda^{(j)}$'s by

$$\omega^{(j)}=\sqrt{\frac{\lambda^{(j)}}{I}}.\tag{19}$$

It is convenient to introduce a characteristic frequency Ω defined by

$$\Omega=\sqrt{\frac{\mu_0 m^2}{4\pi d^3 I}},\tag{20}$$

and express frequencies in terms of Ω.

Eigenvectors can be determined from symmetry. An interesting aspect of our problem is that we can construct the eigenvectors $a_n^{(j)}$ from a symmetry of the system.[8] Since the Lagrangian of the system of N dipoles is invariant under any *cyclic permutation* of the angles ϵ_n, the matrix $M_{nn'}$ commutes with the matrix operator $R_{nn'}$ that transforms $(\epsilon_0,\epsilon_1,\ldots,\epsilon_{N-1})$ into $(\epsilon_1,\epsilon_2,\ldots,\epsilon_{N-1},\epsilon_0)$; namely,

$$R_{nn'}=\delta(n',(n+1)\bmod N).\tag{21}$$

The eigenvectors of $R_{nn'}$, which are simultaneously eigenvectors of $M_{nn'}$ because these matrices commute, are

$$a_n^{(j)}=\exp\left(\frac{2\pi i}{N}jn\right),\tag{22}$$

where j is an integer. It is straightforward to verify that $\Sigma_{n'}R_{nn'}a_{n'}^{(j)}=a_n^{(j)}\exp(2\pi ij/N)$. It is also straightforward to verify that Eq. (22) satisfies the eigenvector equation, Eq. (17). After substituting Eqs. (15) and (22) into Eq. (17), and dividing the result by $\exp(2\pi inj/N)$, we find that the eigenvalue $\lambda^{(j)}$ is

$$\sum_{n'=0}^{N-1}V_{|n-n'|}\exp\left(\frac{2\pi i}{N}j(n-n')\right)=\lambda^{(j)}.\tag{23}$$

The sum on the left-hand side of Eq. (23) is indeed independent of n, by the cyclic symmetry $V_{N-\nu}=V_\nu$ for $0\leqslant\nu\leqslant N$, which verifies that Eq. (22) does satisfy the eigenvector equation. Also, $\lambda^{(j)}$ is real, as is necessary for small oscillations. Finally, substituting for $V_{|n-n'|}$ from Eq. (16) we determine an explicit formula for the eigenvalue

$$\lambda^{(j)}=I\Omega^2\left(Z_N+\sum_{\nu=1}^{N-1}\left[\frac{1+\sin^2\frac{\pi\nu}{N}}{\rho^3(\nu)}\right]\cos\frac{2\pi}{N}j\nu\right).\tag{24}$$

The eigenvector in Eq. (22) is complex. Since the matrix $M_{nn'}$ is real, the real and imaginary parts of $a_n^{(j)}$, which are

$$\cos\frac{2\pi nj}{N},\quad\sin\frac{2\pi nj}{N},$$

are degenerate eigenvectors of $M_{nn'}$, with eigenvalue $\lambda^{(j)}$. We obtain a complete set of N linearly independent eigenmodes by taking the cosine modes with

$$j=0,1,2,3,\ldots,N/2, \quad \text{for } N \text{ even},$$

$$j=0,1,2,3,\ldots(N-1)/2, \quad \text{for } N \text{ odd},$$

and the sine modes with

$$j=1,2,3,\ldots,\frac{N}{2}-1, \quad \text{for } N \text{ even},$$

$$j=1,2,3,\ldots,(N-1)/2, \quad \text{for } N \text{ odd}.$$

These real eigenvectors of $M_{nn'}$ are the eigenmodes of the N dipoles. For example, for oscillations in the jth cosine mode the dipole angles are

$$\epsilon_n(t)=\epsilon\left(\cos\frac{2\pi nj}{N}\right)e^{i\omega^{(j)}t}, \tag{25}$$

where ϵ is a small constant. We also remark that any linear combination of cosine and sine modes with the same j is also a normal mode, because these modes are degenerate.

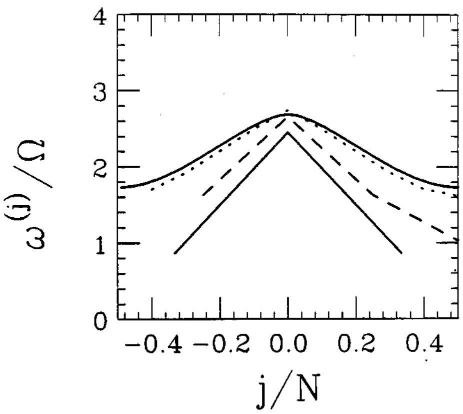

Fig. 2. The dispersion curve, in dimensionless form, for these systems. The ordinate is proportional to $\omega^{(j)}$, which is the frequency of the jth normal mode of oscillation as given in Eq. (19). The abscissa, j/N, in which N is the number of sides of the polygon, is proportional to $k^{(j)}=2\pi j/Nd$, which is the wave number of the jth mode. Results are shown for $N=3$ (solid line), $N=4$ (long dashes), $N=10$ (short dashes), and $N=100$ (solid line).

C. Equilateral triangle and square arrays

We will return to the general case in Sec. III, but it is interesting to consider first the two simplest regular polygons. For these cases, as for all other N-sided regular polygons, the equations of motion can be gotten by expanding the potential energy, and the normal mode frequencies and eigenvectors follow from Eqs. (22) and (24).

The case $N=3$, dipoles at the corners of an equilateral triangle, serves as a prototype for odd N. In terms of the oscillation angles ϵ_0, ϵ_1, and ϵ_2, the linearized equation of motion for ϵ_0 is $\ddot{\epsilon}_0=-\alpha^2(10\epsilon_0+7\epsilon_1+7\epsilon_2)$, where $\alpha=\Omega/2$. From this, the equations of motion for ϵ_1 and ϵ_2 can be obtained by the cyclic permutation $\epsilon_0\rightarrow\epsilon_1\rightarrow\epsilon_2\rightarrow\epsilon_0$. For an equilateral triangle the side is $d=R\sqrt{3}$.

For $N=3$, the highest frequency normal mode has $\omega^{(0)}=\Omega\sqrt{6}$, and the corresponding eigenvector is $(a_0^0\ a_1^0\ a_2^0)=(1\ 1\ 1)$, where a_0^0, a_1^0, and a_2^0, are the amplitudes of ϵ_0, ϵ_1, and ϵ_2, for harmonic oscillation at frequency $\omega^{(0)}$. In this mode, therefore, the dipoles oscillate with equal amplitudes and in phase with each other. The other two modes of oscillation are degenerate with $\omega^{(1)}=\Omega\sqrt{3}/2$, and the eigenvectors may be taken to be $(1\ -1\ 0)$ and $(1\ 0\ -1)$, which correspond to standing waves with wavelength $3d$ or $3d/2$; that is, one dipole is at rest and the other two oscillate 180 deg out of phase. [As written, these degenerate modes are not orthogonal but are rather linear combinations of the eigenvectors $(1\ -1/2\ -1/2)$ and $(0\ \sqrt{3}/2\ -\sqrt{3}/2)$, which are orthogonal. These latter are the real and imaginary parts of $a_n^{(1)}$.]

The case $N=4$, dipoles at the corners of a square, serves as a prototype for even N. In terms of the oscillation angles ϵ_0, ϵ_1, ϵ_2, and ϵ_3, the linearized equation of motion for ϵ_0 is $\ddot{\epsilon}_0=-\beta^2[(1+6\sqrt{2})\epsilon_0+3\sqrt{2}\epsilon_1+2\epsilon_2+3\sqrt{2}\epsilon_3]$, where $\beta=\Omega/2^{3/4}$. From this, the equations of motion for ϵ_1, ϵ_2, and ϵ_3, can be obtained by the cyclic permutation $\epsilon_0\rightarrow\epsilon_1\rightarrow\epsilon_2\rightarrow\epsilon_3\rightarrow\epsilon_0$. For a square the side is $d=R\sqrt{2}$.

For $N=4$, the highest frequency normal mode has $\omega^{(0)}=(\Omega/2)(24+3\sqrt{2})^{1/2}$, and the corresponding eigenvector is $(a_0^0\ a_1^0\ a_2^0\ a_3^0)=(1\ 1\ 1\ 1)$. In this case, as for $N=3$, all the dipoles oscillate with equal amplitude in phase with each other. As we shall see, this behavior for the mode with the highest frequency holds for any N.

The mode of oscillation with the lowest frequency for this case is $\omega^{(2)}=(\Omega/2)(3\sqrt{2})^{1/2}$. It has the eigenvector $(1\ -1\ 1\ -1)$, corresponding to a standing wave with wavelength $2d$. The two remaining modes of oscillation are degenerate with frequency, $\omega^{(1)}=(\Omega/2)(12-\sqrt{2})^{1/2}$. The eigenvectors for these modes may be taken to be $(1\ 0\ -1\ 0)$ and $(0\ 1\ 0\ -1)$, corresponding to standing waves with wavelength $4d$ or $4d/3$.

III. CONCLUSIONS

Finally, we discuss the general dispersion curve and its implications. The physics underlying it is not difficult to understand. Equation (24), which gives the relation between the eigenvalue of the jth mode and the other parameters of the problem, is essentially the dispersion relation, usually written $\omega=\omega(k)$. Here, the wave vector $k^{(j)}$ for the jth mode is $k^{(j)}=(2\pi j/Nd)$, as can be seen from Eq. (22): We write $a_n^{(j)}=\exp(ik^{(j)}x_n)$, where $x_n=nd$.

In Fig. 2 we have plotted the results of Eq. (24) for the cases $N=3$, 4, 10, and 100. We see that the $\omega^{(j)}$'s, and the $\lambda^{(j)}$'s, depend somewhat on N, but converge to finite values as $N\rightarrow\infty$. (Again, we keep d fixed as N increases.) It is interesting that the equilibrium energy and the frequencies of the zeroth and the $(N/2)$th modes all depend on $\zeta(3)$, where ζ is, again, the Riemann zeta function. This constant, $\zeta(3)$, is called Apéry's constant, as Apéry proved that it is irrational.[9] Thus, in the limit $N\rightarrow\infty$, $U^{(0)}/N\rightarrow-2I\Omega^2\zeta(3)$, and, as may be seen by taking the limit in Eq. (24) and substituting the result into Eq. (19), $\omega^{(0)}\rightarrow\Omega\sqrt{6\zeta(3)}$. We also note that, again in the limit $N\rightarrow\infty$, $\omega^{(N/2)}\rightarrow\Omega\sqrt{5\zeta(3)/2}$.

From Fig. 2 and Eq. (24) we see that for oscillating dipoles at the corners of any regular polygon, i.e., for any N, the mode with the highest frequency is the nondegenerate mode with $j=0$. For this mode the eigenvector $\phi_n^{(0)}$ is unity for all n, as we see by setting $j=0$ in Eq. (22). Thus, in this

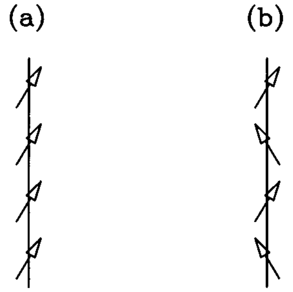

Fig. 3. Modes of oscillation of a line of dipoles: (a) shows the symmetric mode, corresponding to $j=0$, in which the dipoles' orientation angles are all equal, i.e., $\epsilon_n=\epsilon$; (b) shows the mode with the maximum wave vector, corresponding to $j=N/2$, for which the orientation angles are $\epsilon_n=\epsilon\cos(\pi n)$. Mode (a) has the highest frequency of oscillation, and mode (b) has the lowest.

mode, all the magnetic dipoles oscillate with equal amplitudes and in phase with each other. We also see from Fig. 2 that for systems with even N there is one other nondegenerate mode, besides the $j=0$ mode; this nondegenerate mode has $j=N/2$. In this mode, adjacent dipoles oscillate with equal amplitudes but one-half period out of phase. The other $N-2$ normal modes are doubly degenerate. For odd N, only the $j=0$ mode is nondegenerate; the other $N-1$ modes are doubly degenerate.

A. The dispersion curve has only an optical branch

Another interesting feature that can be seen from Fig. 2 is that the group velocity and the phase velocity have opposite signs (except at $k=0$). For example, for $k<0$ the phase velocity, ω/k, is negative and the group velocity, $d\omega/dk$, is positive. For $k>0$ both signs are changed. At $k=0$ the phase velocity is infinite and the group velocity is zero. Thus, in the language of lattice dynamics, only an optical branch is present and the acoustic branch is absent.

It is remarkable that in our system there is only an optical branch, because usually there is an acoustical branch present in a system of coupled oscillators. The physical reason for this may be understood from Fig. 3, which shows a few dipoles for the $N\rightarrow\infty$ limit of our system. The configuration of Fig. 3(a), with $k=0$ (i.e., $j=0$), is the configuration for which the restoring torque on each dipole is maximum; therefore that mode of oscillation has the highest frequency. For the configuration of Fig. 3(b), with $k=\pi/d$ (i.e., $j=N/2$), the restoring torques are minimum and therefore that mode has the lowest frequency. We may understand these torques by considering the interactions of just two neighboring dipoles in Fig. 3(a) and 3(b).[1,10] The torque the dipoles exert on each other is $\mathbf{N}=\mathbf{m}\times\mathbf{B}$ so that the magnitude of the, restoring, torque is proportional to the component of $\mathbf{B}$, which is perpendicular to $\mathbf{m}$. It is not difficult to show, by using $\mathbf{B}$ for a dipole, that for the case of parallel dipoles, as in Fig. 3(a), this component of $\mathbf{B}$ is three times larger than what it is for the configuration in Fig. 3(b).

What characterizes an acoustic mode is that, in the limit $k\rightarrow 0$, one also has $\omega\rightarrow 0$. In systems in which the interactions are isotropic, there must be an acoustical branch, because no energy and hence no restoring torque or force is associated with oscillations in this limit. But the interaction energy between two dipoles is $U_{21}=-\mathbf{m}_2\cdot\mathbf{B}_{21}=(\mu_0/4\pi r^3)\times[\mathbf{m}_1\cdot\mathbf{m}_2-3(\mathbf{m}_1\cdot\hat{\mathbf{r}}_{12})(\mathbf{m}_2\cdot\hat{\mathbf{r}}_{12})]$, which contains an anisotropic term, namely, $-3(\mathbf{m}_1\cdot\hat{\mathbf{r}}_{12})(\mathbf{m}_2\cdot\hat{\mathbf{r}}_{12})$, in addition to the isotropic term $\mathbf{m}_1\cdot\mathbf{m}_2$. In our system this anisotropic contribution causes the unusual behavior of the dispersion relation.

B. Dipole and quadrupole radiation

Finally, we will examine electromagnetic radiation from our system. This radiation has interesting and surprising properties.

If we take the polygon in Fig. 1 to be in the x,y plane, then the nth dipole makes an angle of $(2\pi n/N+\pi/2+\epsilon_n)$ with the x axis. The total magnetic dipole moment of the system is the sum of the individual dipoles, i.e., $\mathbf{M}_{\text{total}}=\sum_{n=0}^{N-1}\mathbf{m}_n(\epsilon_n)$ which, for small oscillations, can be expanded in powers of the ϵ_n's. We therefore write $\mathbf{M}_{\text{total}}=\mathbf{M}_0+\mathbf{M}_1(\epsilon_n)+O(\epsilon_n^2)$, where

$$\mathbf{M}_0=m\sum_{n=0}^{N-1}\left(-\hat{\mathbf{i}}\sin\frac{2\pi n}{N}+\hat{\mathbf{j}}\cos\frac{2\pi n}{N}\right), \tag{26a}$$

and

$$\mathbf{M}_1(\epsilon_n)=-m\sum_{n=0}^{N-1}\epsilon_n(t)\left(\hat{\mathbf{i}}\cos\frac{2\pi n}{N}+\hat{\mathbf{j}}\sin\frac{2\pi n}{N}\right). \tag{26b}$$

The magnetic moment of the system at equilibrium is zero, i.e., $\mathbf{M}_0=0$. This follows algebraically from Eq. (26a). One can also see it geometrically because when $\epsilon_n=0$ for all n, then the dipoles if placed head-to-tail just form a closed regular polygon. Furthermore, when the dipoles oscillate in the $j=0$ mode, in which, as Eq. (25) shows, all the dipoles oscillate with equal amplitudes at frequency $\omega^{(0)}$, the total dipole moment of the system remains zero, i.e., $\mathbf{M}_{\text{total}}=0$. This follows algebraically from Eq. (26b) but one can also see, geometrically, that because the system has zero magnetic moment at equilibrium, it will still have zero magnetic moment when all the dipoles oscillate in phase with equal amplitudes. Moreover, the small oscillations are harmonic, so we can write $\sum_{n=0}^{N-1}\ddot{\mathbf{m}}_n(t)=-(\omega^{(0)})^2\sum_{n=0}^{N-1}\mathbf{m}_n(t)=0$. Now the power emitted as magnetic dipole radiation by a system of N dipoles is $P_{\text{mag.dipole}}(t)=(1/6\pi\epsilon_0c^5)\times[\sum_{n=0}^{N-1}\ddot{\mathbf{m}}_n(t)]^2$. For our system this is zero, as we have seen, so when oscillating in the highest frequency mode, $j=0$, the system emits no magnetic dipole radiation.

One can analyze the magnetic dipole radiation properties for all the other modes by substituting $\epsilon_n(t)$ from Eq. (25), in the form of cosine and sine modes, into Eq. (26b). The surprising result is that for any N, *magnetic dipole radiation is emitted only for oscillations in the, doubly degenerate,* $j=1$ *mode*. For the sine mode with $j=1$, one has $\mathbf{M}_1(\epsilon_n)=-m\epsilon(N/2)\hat{\mathbf{j}}\sin\omega^{(1)}t$. For the corresponding cosine mode with $j=1$, one has $\mathbf{M}_1(\epsilon_n)=-m\epsilon(N/2)\hat{\mathbf{i}}\cos\omega^{(1)}t$. For all other modes of oscillation no magnetic dipole radiation is emitted. Then the radiation which is emitted is quadrupole, for which the power is proportional to $1/c^7$, or higher-order multipole radiation. Thus, if the system is set oscillating in any mode other than $j=1$, it will continue to oscillate for a very long time, assuming there is no damping of the motion except by radiation.

It is natural to ask next: For which modes does this system emit quadrupole radiation? We have solved this problem by using an expansion to quadrupolar terms of the magnetic scalar potential. The details will not be given here. We have shown that magnetic quadrupole radiation is emitted only in the, nondegenerate, $j=0$ mode and in the $j=2$ mode, which is nondegenerate only for the case of a square but is otherwise doubly degenerate. For all other modes of oscillation there is no magnetic quadrupole radiation.

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[4] Note that ϕ_n depends on both n and n', while ϵ_n depends only on n; likewise, $\phi_{n'}$ depends on both n and n', while $\epsilon_{n'}$ depends only on n'.

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[6] The sum in Eq. (8) is dominated by terms with $n' \ll N$ or $|n'-n| \ll N$. For these terms we may approximate $\cos(\pi n'/N) \approx \pm 1$ and $\rho(n') \approx n'$. Then $Z_N \approx 4\zeta(3)$.

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