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# Simple chaotic flows with a line equilibrium

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ABSTRACT

Using a systematic computer search, nine simple chaotic flows with quadratic nonlinearities were found that have the unusual feature of having a line equilibrium. Such systems belong to a newly introduced category of chaotic systems with hidden attractors that are important and potentially problematic in engineering applications.



## 1. Introduction

It is widely recognized that mathematically simple systems of nonlinear differential equations can exhibit chaos. With the advent of fast computers, it is now possible to explore the entire parameter space of these systems with the goal of finding parameters that result in some desired characteristics of the system.

Recent research has involved categorizing periodic and chaotic attractors as either self-excited or hidden [1–10]. A self-excited attractor has a basin of attraction that is associated with an unstable equilibrium, whereas a hidden attractor has a basin of attraction that does not intersect with small neighborhoods of any equilibrium points. The classical attractors of Lorenz, Rössler, Chua, Chen, Sprott systems (cases B–S) and other widely-known attractors are those excited from unstable equilibria. From a computational point of view this allows one to use a numerical method in which a trajectory started from a point on the unstable manifold in the neighborhood of an unstable equilibrium, reaches an attractor and identifies it [7]. Hidden attractors cannot be found by this method and are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or an airplane wing.

The chaotic attractors in dynamical systems without any equilibrium points or with only stable equilibria are hidden attractors. That is the reason such systems are rarely found, and only a few such examples have been reported in the literature [11–20].

In this paper, we introduce a new category of chaotic systems with hidden attractors: systems with a line equilibrium. Although in such systems the basin of attraction may intersect the line equilibrium in some sections, there are usually uncountably many points on the line that lie outside the basin of attraction of the chaotic attractor, and thus it is impossible to identify the chaotic attractor for sure by choosing an arbitrary initial condition in the vicinity of the unstable equilibria. In other words, from a computational point of view these attractors are hidden, and knowledge about equilibria does not help in their localization. On the other hand, to the best of our knowledge, although there are dynamical systems with a line equilibrium in the literature [21–24], only one chaotic example has been reported [25], and it is artificial because it is a four-dimensional system that can be reduced to a three-dimensional system in which the line equilibrium vanishes. The goal of this paper is to describe a new category of hidden attractor and expand the list of known mathematically simple hidden chaotic attractors. Thus we perform a systematic computer search for chaos in three-dimensional autonomous systems with quadratic nonlinearities which have been designed so that there will be a line equilibrium, and we ensure that the line equilibrium cannot be made to vanish by reduction to a system of lower dimension.

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**Table 1**
Six simple chaotic flows with line equilibrium.

| Case | Equations | $(a,b)$ | Equilibrium | Eigenvalues | LEs | $D_{KY}$ | $(x_0, y_0, z_0)$ |
|---|---|---|---|---|---|---|---|
| $LE_1$ | $\dot{x} = y$ | $a = 15$ | 0 | $\frac{z\pm\sqrt{z^2-4}}{2}$ | 0.0717 | 2.1371 | 0 |
| | $\dot{y} = -x + yz$ | $b = 1$ | 0 | 0 | 0 | | 0.5 |
| | $\dot{z} = -x - axy - bxz$ | | $z$ | | −0.5232 | | 0.5 |
| $LE_2$ | $\dot{x} = y$ | $a = 17$ | 0 | $\frac{z\pm\sqrt{z^2-4}}{2}$ | 0.0564 | 2.1927 | 0 |
| | $\dot{y} = -x + yz$ | $b = 1$ | 0 | 0 | 0 | | 0.4 |
| | $\dot{z} = -y - axy - bxz$ | | $z$ | | −0.2927 | | 0 |
| $LE_3$ | $\dot{x} = y$ | $a = 18$ | 0 | $\frac{z\pm\sqrt{z^2-4}}{2}$ | 0.0556 | 2.1714 | 0 |
| | $\dot{y} = -x + yz$ | $b = 1$ | 0 | 0 | 0 | | −0.4 |
| | $\dot{z} = x^2 - axy - bxz$ | | $z$ | | −0.3245 | | 0.5 |
| $LE_4$ | $\dot{x} = y$ | $a = 4$ | 0 | $\frac{z\pm\sqrt{z^2-4}}{2}$ | 0.0539 | 2.1712 | 0.2 |
| | $\dot{y} = -x + yz$ | $b = 0.6$ | 0 | 0 | 0 | | 0.7 |
| | $\dot{z} = -axy - bxy - yz$ | | $z$ | | −0.3147 | | 0 |
| $LE_5$ | $\dot{x} = y$ | $a = 1.5$ | 0 | $\frac{z\pm\sqrt{z^2-4a}}{2}$ | 0.1386 | 2.1007 | 0.7 |
| | $\dot{y} = -ax + yz$ | $b = 5$ | 0 | 0 | 0 | | 1 |
| | $\dot{z} = -x^2 - y^2 - bxz$ | | $z$ | | -1.3764 | | 0 |
| $LE_6$ | $\dot{x} = y$ | $a = 0.04$ | 0 | $\frac{z\pm\sqrt{z^2-4}}{2}$ | 0.0543 | | 12 |
| | | $b = 0.1$ | 0 | 0 | 0 | | 2 |
| | $\dot{y} = -x + yz$ | | $z$ | | −0.6314 | 2.0860 | 0 |
| | $\dot{z} = -ay^2 - xy - bxz$ | | | | | | |
| $LE_7$ | $\dot{x} = z$ | $a = 1.85$ | 0 | $\frac{-0.3y\pm\sqrt{0.09y^2-4y}}{2}$ | 0.1144 | 2.0140 | 5.1 |
| | $\dot{y} = x + yz$ | | $y$ | 0 | 0 | | 7 |
| | $\dot{z} = -ax^2 - xy - byz$ | $b = 0.3$ | 0 | | −1.0270 | | 0 |
| $LE_8$ | $\dot{x} = z$ | $a = 3$ | 0 | $\pm\sqrt{y}$ | 0.0521 | 2.0647 | 0 |
| | $\dot{y} = -x - yz$ | $b = 1$ | $y$ | 0 | 0 | | −0.3 |
| | $\dot{z} = ax^2 - xy - bxz$ | | 0 | | −0.8053 | | −1 |
| $LE_9$ | $\dot{x} = z$ | $a = 1.62$ | $x$ | $\frac{-0.62\pm\sqrt{6.8644-4x^2}}{2}$ | 0.0642 | 2.0939 | 0 |
| | $\dot{y} = -ay + xz$ | | 0 | 0 | 0 | | 1 |
| | $\dot{z} = z - bz^2 + xy$ | $b = 0.2$ | 0 | | −0.6842 | | 0.8 |

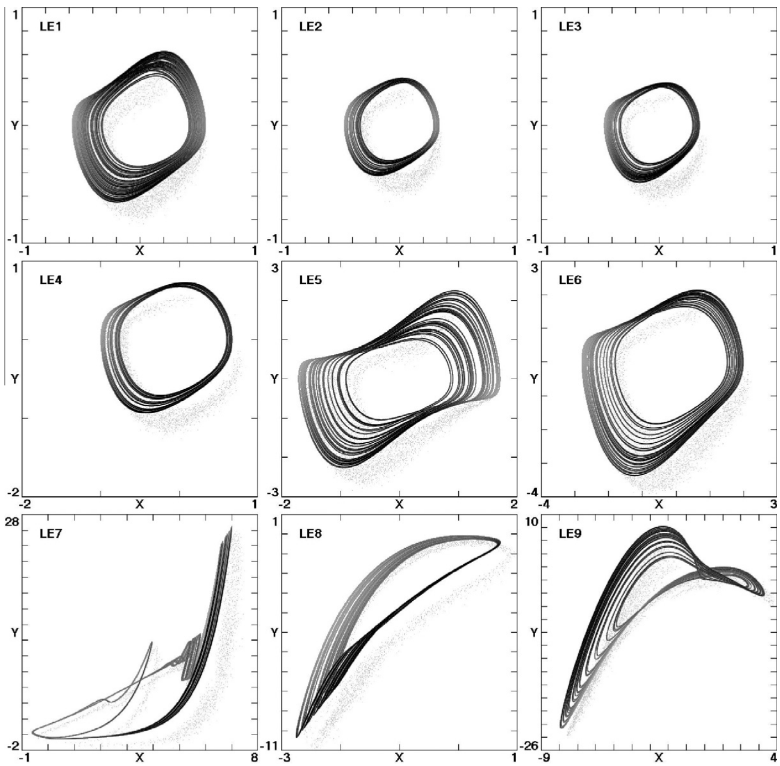


**Fig. 1.** State space plots of the cases in Table 1 projected onto the $xy$-plane.

## 2. Simple chaotic flows with a line equilibrium

In the search for chaotic flows with a line equilibrium, we were inspired by the structure of the conservative Sprott case A system [26],

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x + yz\\ \dot{z} &= 1 - y^2\end{aligned} \tag{1}$$

This system is the oldest and best-known example of a chaotic system with no equilibria, but it does not have a strange attractor since it is conservative. It is an important system since it is a special case of the Nose–Hoover oscillator [27] which describes many natural phenomena [28], and thus it suggests that such systems may have practical as well as theoretical importance.

We consider a general parametric form of Eq. (1) with quadratic nonlinearities of the form

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= a_1x + a_2yz\\ \dot{z} &= a_3x + a_4y + a_5x^2 + a_6y^2 + a_7xy + a_8xz + a_9yz\end{aligned} \tag{2}$$

As can be seen, this system has a line equilibrium in $(0, 0, z)$ with no other equilibria (in other words the $z$-axis is the line equilibrium of this system).

An exhaustive computer search was done considering millions of combinations of the coefficients $a_1$ through $a_9$ and initial conditions, seeking dissipative cases for which the largest Lyapunov exponent is greater than 0.001. For each case that was found, the space of coefficients was searched for values that are deemed "elegant" [29], by which we mean that as many coefficients as possible are set to zero with the others set to ±1 if possible or otherwise to a small integer or decimal fraction with the fewest possible digits. Cases $LE_1$–$LE_6$ in Table 1 are six simple cases found in this way with only six terms. With some similar procedure, three other similar cases $LE_7$–$LE_9$ are included in the table.

In addition to the cases in the table, dozens of additional cases were found, but they were either equivalent to one of the cases listed by some linear transformation of variables, or they were extensions of these cases with more than six terms.

All these cases are dissipative with attractors projected onto the $xy$-plane as shown in Fig. 1. The equilibria, eigen-

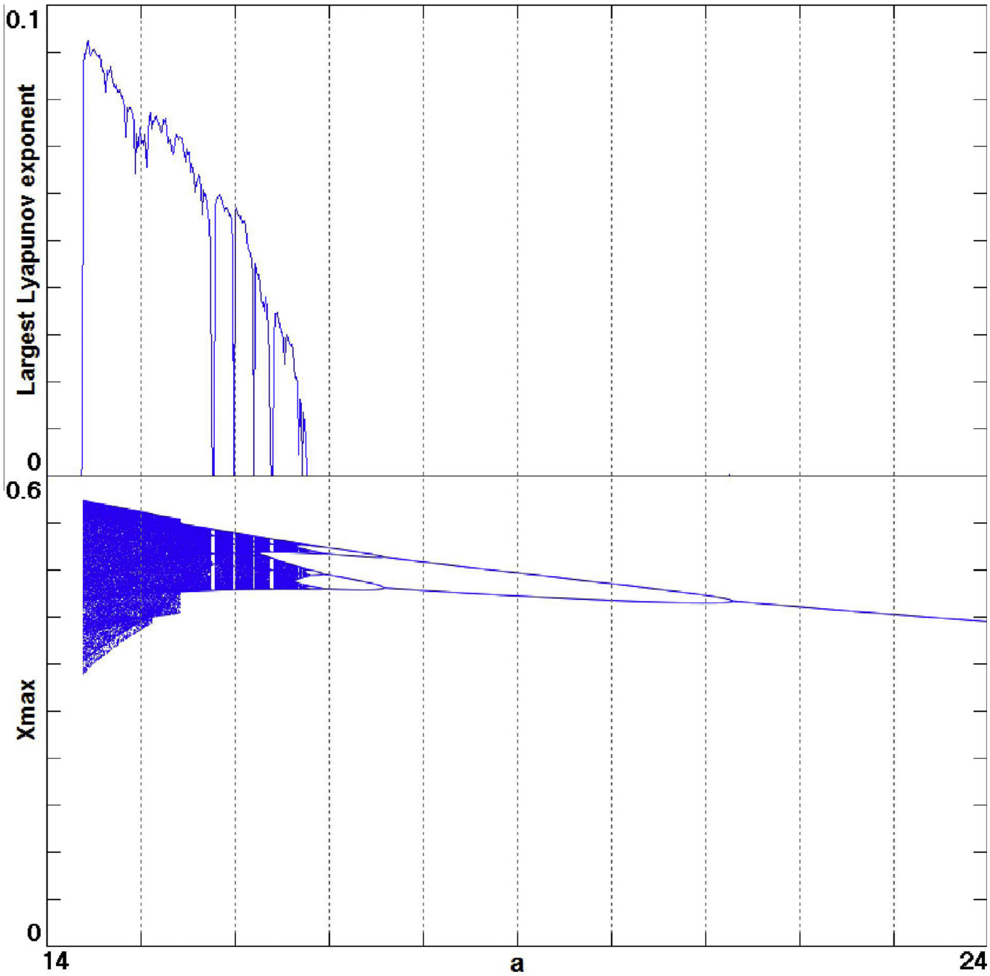


**Fig. 2.** The largest Lyapunov exponent and bifurcation diagram of case $LE_1$ showing a period-doubling route to chaos

values, Lyapunov exponent spectra, and Kaplan–Yorke dimensions are shown in Table 1 along with initial conditions that are close to the attractor. As is usual for strange attractors from three-dimensional autonomous systems, the attractor dimension is only slightly greater than 2.0, the largest of which is $LE_2$ with $D_{KY} = 2.1927$, although no effort was made to tune the parameters for maximum chaos. All the cases appear to approach chaos through a succession of period-doubling limit cycles, a typical example of which ($LE_1$) is shown in Fig. 2 with decreasing $a$ for $b = 1$. As $a$ decreases further, the strange attractor is destroyed in a boundary crisis.

Fig. 3 shows a cross section in the $xz$-plane at $y = 0$ of the basin of attraction for the two attractors for the typical case $LE_1$. Note that the cross section of the strange attractor nearly touches its basin boundary as is typical of low-dimensional chaotic flows.

For $LE_1$ we can obtain the eigenvalues from $\lambda(\lambda^2 - z\lambda + 1) = 0$. One of the eigenvalues is zero. The other two depend on $z$. Using the Routh–Hurwitz stability criterion for the other two, we can say for $z > 0$ the eigenvalues have a positive real part, and thus the positive $z$-axis is unstable. As shown in Fig. 3, the basin of attraction of the chaotic attractor intersects the line equilibrium in some portions. However there are other parts of the $z$-axis (for $z > 0$) which lie in the basin of the stable equilibrium or that attract to infinity. So the strange attractor is hidden in the sense that there are uncountable unstable points on the line equilibrium of which only a tiny portion intersects the basin of the chaotic attractor. In other words, for computational purposes, the attractor is hidden and knowledge about the line equilibrium does not help in its localization.

The reason the stable equilibrium for $z < -2$ does not appear in the basin plot is because it is a node rather than a focus in the $xy$-plane for $z < -2$. Hence the trajectory never crosses the $y = 0$ plane for $z < -2$. Orbits that start to the left of the equilibrium ($x < -2$) are pulled in the $+z$ direction for $z > -2$ and in the $-z$ direction for $z < -2$. However, they do not go to infinity unless they start far from the equilibrium. Rather, they asymptotically approach it from the $-x$ side and converge to a point on the line that depends on the initial condition. Thus the entire negative $z$-axis is an attractor, but it is nonlinearly contracting along its length for $-2 < z < 0$ and nonlinearly expanding for $z < -2$.

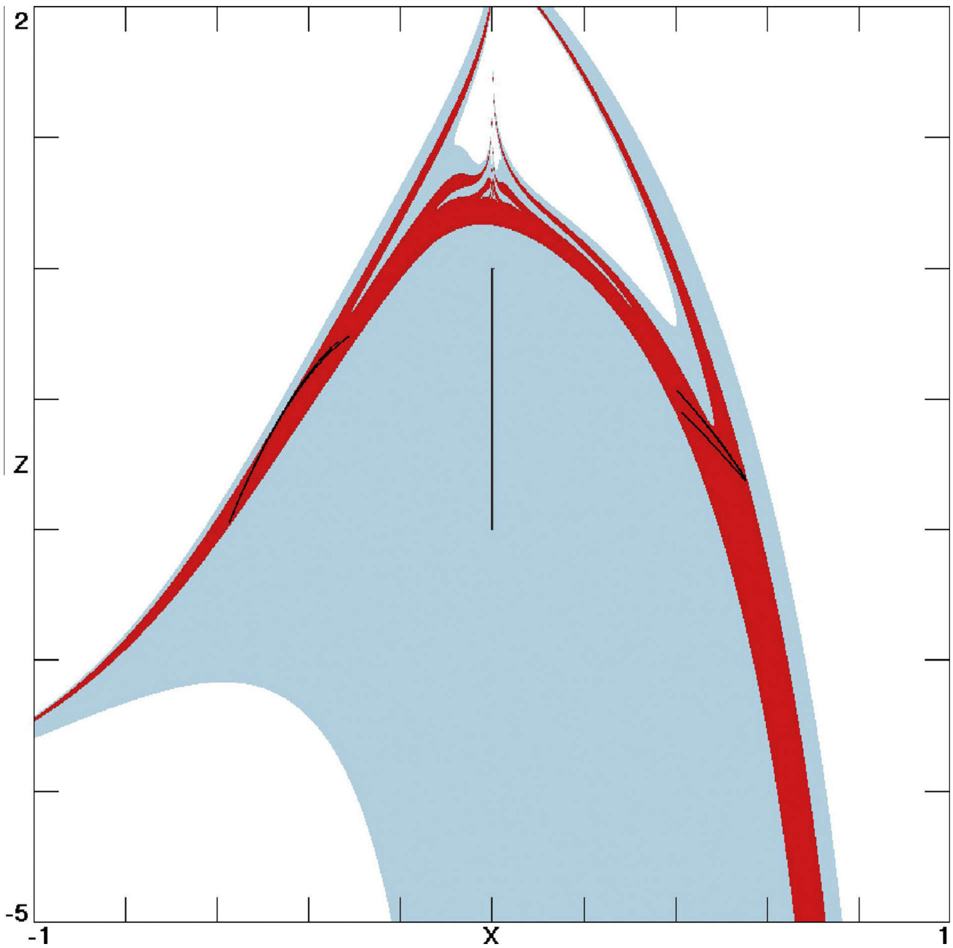


**Fig. 3.** Cross section of the basins of attraction of the two attractors in the $xz$-plane at $y = 0$ for case $LE_1$. Initial conditions in the white region lead to unbounded orbits, those in the red region lead to the strange attractor, and those in the light blue region lead to the line equilibrium.(For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

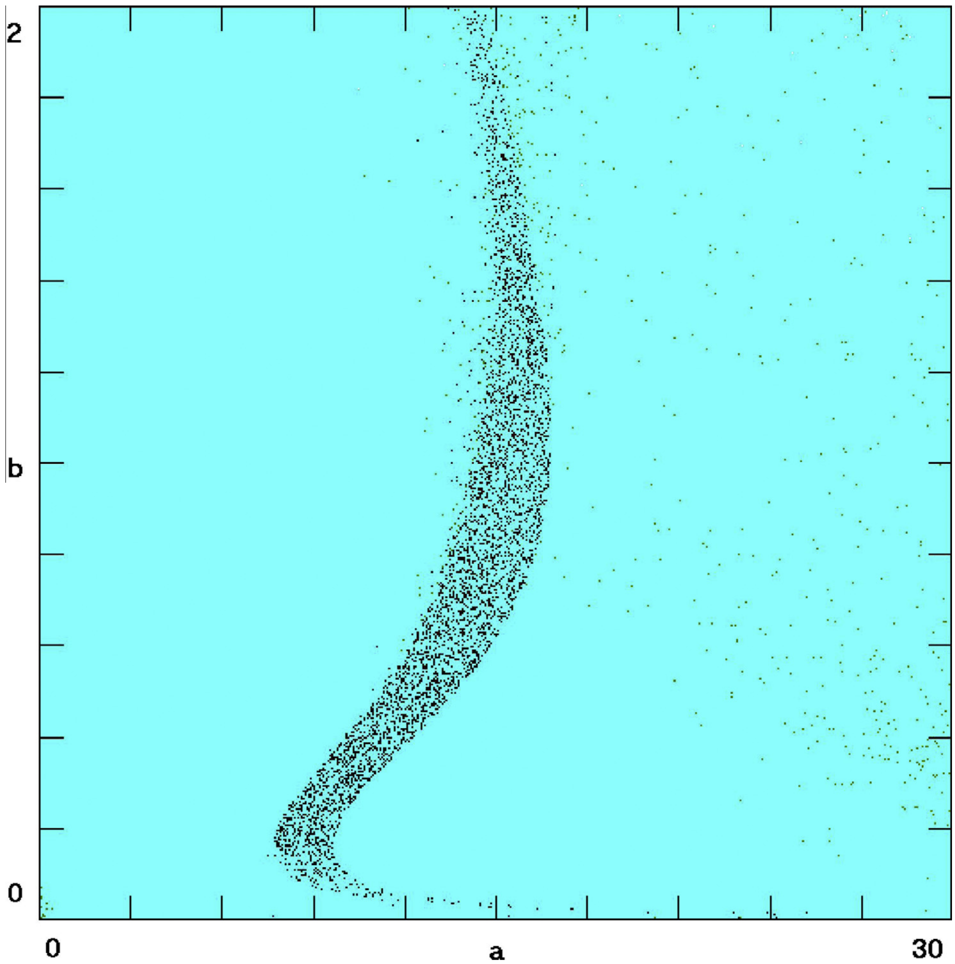


**Fig. 4.** Regions of different dynamic behavior in parameter space for case $LE_1$. Light blue represents a static equilibrium, and the black dots correspond to regions of chaos. Each pixel uses a different random condition thereby indicating the coexistence of static and chaotic attractors. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

To show that the behavior described about is not dependent on the particular choice of parameters, Fig. 4 shows the regions of different dynamical behavior in the *ab*-parameter space. In this plot each pixel represents a different initial condition chosen from a Gaussian distribution with zero mean and unit variance. Thus the region in which the strange attractor (black dots) coexists with the stable line equilibrium (light blue background) extends throughout much of the parameter space. The other eight cases in Table 1 shows similar behavior.

Also of interest is the fact that in cases $LE_1$–$LE_8$, the strange attractor surrounds the line equilibrium, while in case $LE_9$, the line equilibrium lies just outside the strange attractor. In none of the cases does the line equilibrium intersect the attractor, and thus we would not expect homoclinic orbits.

## 3. Conclusion

In conclusion, it is apparent that simple chaotic systems with a line equilibrium that seemed to be rare, may in fact be rather common. These systems belong to the newly introduced class of chaotic systems with hidden attractors. In fact they are a new category of them which have not been previously described.

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