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GROUP THEORY AND THE VIBRATING POLYGON

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The problem of determining the natural frequency spectrum, including the degeneracy structure, is considered for the in-plane vibrations of a system of n identical particles situated at the vertices of a regular nth order polygon and coupled together in pairs by identical light springs lying along the sides of the polygon. Group theoretical arguments are used firstly to determine the degeneracy structure of the spectrum and secondly to simplify the eigenvalue problem sufficiently to permit a complete and simple analytical solution to be obtained. The investigation illustrates the power of group theory as a tool in studying vibration problems.

1. INTRODUCTION

Group theory is well established as being a powerful tool in several branches of theoretical physics [1]. In particular, it is well known as a tool for classifying molecular vibrations [2]. Its usefulness does not, however, seem to be generally appreciated by workers in the more general field of vibrations. The close analogy between the operator formulation of quantum mechanics and classical elastic vibration theory would seem to suggest that, since group theory is widely used with great effect in quantum mechanics, its use in vibration studies should be of considerable benefit.

Most of the well known applications of group theory yield information of a purely qualitative nature. (For an example of such an application in vibration theory see reference [3].) This information however, in addition to being of interest in itself, can often be used to simplify the problem in hand. This simplification can sometimes be dramatic and may, for example, reduce a problem which is in practice insoluble analytically to one which can readily be solved. Such a problem is the one which is considered in detail in the present paper.

The problem of the in-plane vibrations of a regular polygon, which is considered in some detail in the present work, is of interest in itself and has not, so far as the author is aware, been solved elsewhere. The primary purpose is not, however, to present the solution of this problem but rather to use it as a vehicle to demonstrate the power of the group theoretical approach to vibration problems in systems with some symmetries. In particular, it is regarded as demonstrating that group theory can, at least indirectly, yield quantitative information. It is assumed that the reader is familiar with the elementary ideas of group theory and of the eigenvalue approach to the problem of small oscillations, but a brief summary of some of the more important ideas and results is given, especially where this is of help in defining notation. The reader who experiences difficulty is directed to reference [1] for the group theory and reference [4] for the theory of small oscillations. He is particularly advised to familiarize himself with the concepts of group, class, representation, irreducibility and character.

2. GENERAL FORMULATION

Consider the problem of determining the natural vibration spectrum for in-plane motions of a system consisting, at equilibrium, of n identical particles of mass m situated at the vertices of a regular nth order polygon and joined together in pairs by identical light springs with spring constant k lying along the sides of the polygon and with natural lengths equal to the lengths of the polygon sides. A portion of the system for the general case is shown in Figure 1. The particles are labelled $1, 2, \ldots, n$, starting from an arbitrary particle and going around the polygon anticlockwise.

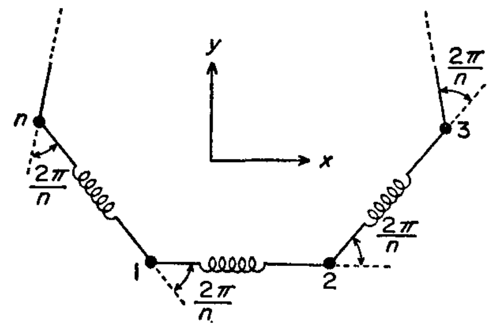

Figure 1. The vibrating regular polygon.

If the system undergoes an arbitrary small displacement from its equilibrium configuration but is kept in the plane of the original polygon then the new situation can be described by means of $2n$ generalized displacements η_i, $i = 1, 2, .., 2n$. These are the components of a $2n$ dimensional state vector $\boldsymbol{\eta}$ which describes the configuration of the whole system. In the present work, for convenience, the two generalized displacements for any given particle j are taken as the rectangular Cartesian coordinates (x_j, y_j) of the displaced particle referred to an origin at the equilibrium position of the particle and with the x-axis parallel to the polygon side joining the equilibrium positions of particles 1 and 2 as shown in Figure 1. The identifications $\eta_{2j-1} = x_j$ and $\eta_{2j} = y_j$, where $j = 1, 2, .., n$, are made. The usual convention is adopted of displaying S as a column matrix,

$$\boldsymbol{\eta} = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ \vdots \\ y_n \end{pmatrix}.$$

The kinetic and potential energies of the system are given by

$$T = \tfrac{1}{2}m \sum_{p=1}^{n} (\dot{x}_p{}^2 + \dot{y}_p{}^2),$$

$$V = \tfrac{1}{2}k \sum_{p=1}^{n} \left\{(x_{p+1} - x_p)\cos\left[(p-1)\frac{2\pi}{n}\right] + (y_{p+1} - y_p)\sin\left[(p-1)\frac{2\pi}{n}\right]\right\}^2, \tag{1}$$

respectively, where $V = 0$ at the equilibrium configuration. Thus one can write, deviating slightly from the usual approach for the sake of convenience,

$$T = \tfrac{1}{2}m \sum_{i=1}^{2n} \dot{\eta}_i{}^2 = \tfrac{1}{2}m\tilde{\dot{\boldsymbol{\eta}}}\dot{\boldsymbol{\eta}} \tag{2}$$

and

$$V = \tfrac{1}{2}k \sum_{i,j=1}^{2n} V_{ij}\eta_i\eta_j = \tfrac{1}{2}k\tilde{\boldsymbol{\eta}}\mathbf{V}\boldsymbol{\eta}. \tag{3}$$

$\mathbf{V}$ is the symmetric $2n \times 2n$ matrix whose elements are the V_{ij}, which are all zero except for the following, with $p = 1, 2, .., n$:

$$V_{2p-1,2p-1} = \cos^2\left[(p-1)\frac{2\pi}{n}\right] + \cos^2\left[(p-2)\frac{2\pi}{n}\right],$$

$$V_{2p,2p} = \sin^2\left[(p-1)\frac{2\pi}{n}\right] + \sin^2\left[(p-2)\frac{2\pi}{n}\right],$$

$$V_{2p-1,2p} = V_{2p,2p-1} = \sin\left[(p-1)\frac{2\pi}{n}\right]\cos\left[(p-1)\frac{2\pi}{n}\right]$$

$$+ \sin\left[(p-2)\frac{2\pi}{n}\right]\cos\left[(p-2)\frac{2\pi}{n}\right],$$

$$V_{2p-1,2p+1} = V_{2p+1,2p-1} = -\cos^2\left[(p-1)\frac{2\pi}{n}\right],$$

$$V_{2p,2p+2} = V_{2p+2,2p} = -\sin^2\left[(p-1)\frac{2\pi}{n}\right],$$

$$V_{2p-1,2p+2} = V_{2p+2,2p-1} = V_{2p,2p+1} = V_{2p+1,2p}$$

$$= -\sin\left[(p-1)\frac{2\pi}{n}\right]\cos\left[(p-1)\frac{2\pi}{n}\right]. \tag{4}$$

Since the system is conservative equations (2) and (3) may be used to form the Lagrangian for the system $L = T - V$, and Lagrange's equations then yield

$$m\ddot{\eta}_j + k\sum_{i=1}^{2n} V_{ji}\eta_i = 0, \qquad j = 1, 2, .., 2n. \tag{5}$$

Now for a normal mode $\eta_j = a_j e^{-i\omega t}, j = 1, 2, .., 2n$, where the a_j are amplitudes and ω is the angular frequency of the mode. Substituting this form into equations (5) gives

$$\sum_{i=1}^{2n} V_{ji}a_i = \lambda a_j$$

where

$$\lambda = \frac{m\omega^2}{k}: \tag{6}$$

i.e.,

$$\mathbf{Va} = \lambda\mathbf{a} \tag{7}$$

where

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n} \end{pmatrix}.$$

Thus the normal modes correspond to eigenvectors of $\mathbf{V}$ with eigenvalues which determine the frequencies. It is well known that in this kind of eigenvalue problem the $\mathbf{a}$ can always be chosen to form a complete orthonormal set in the sense that $\tilde{\mathbf{a}}_i\mathbf{a}_j = \delta_{i,j}$ even if degeneracy is present. The matrix $\mathbf{H}$, the columns of which are the orthonormal $\mathbf{a}$, then diagonalizes $\mathbf{V}$, the elements of the new matrix being the eigenvalues. Thus

$$\tilde{\mathbf{H}}\mathbf{V}\mathbf{H} = \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \lambda_{2n} \end{pmatrix}.$$

$\mathbf{H}$ also supplies the basis transformation connecting the original choice of generalized displacements to the normal coordinates. Thus

$$\phi_i = \tilde{\mathbf{a}}_i\boldsymbol{\eta} \qquad \text{or} \qquad \boldsymbol{\phi} = \tilde{\mathbf{H}}\boldsymbol{\eta},$$

where

$$2T = \sum_i \dot{\phi}_i^2,$$

$$2V = \sum_i \lambda_i \phi_i^2. \tag{8}$$

3. SYMMETRY CONSIDERATIONS

To obtain analytical expressions for the eigenvalues λ in the general case by solving equation (7) for the form of $\mathbf{V}$ given in equation (4) appears at first to be a prohibitively complicated exercise. There is, however, a good deal of information available which has not yet been utilised in the form of symmetries of the system. In the present section a brief review is given of the general situation with regard to the effect of symmetries in vibration problems. The actual use of the results is deferred to a subsequent section.

One must be careful, in principle at least, to be clear exactly what is meant by a symmetry operation in the present kind of problem. A symmetry operation on the polygon means some operation S which brings the polygon back into self coincidence—a simple example is a rotation of the polygon in its plane through an angle of $2\pi/n$. If the deformed system is subjected to a symmetry operation S of the undeformed polygon then a new configuration is obtained with the same potential energy. This new configuration may alternatively be obtained by using a new set of displacements of the original system. An imagined operation on and re-identification of the displacements alone is what is meant by a symmetry operation in a vibration problem. There is an isomorphism between the symmetry operations on the displacements and on the undisturbed system. The new set of displacements η'_i are related to the old set η_i by a linear transformation

$$\boldsymbol{\eta}' = \mathbf{S}\boldsymbol{\eta}, \tag{9}$$

where $\mathbf{S}$ is a $2n \times 2n$ real orthogonal matrix which depends upon the precise nature of S. It is straightforward to show that the S form a group and the $\mathbf{S}$ a representation of that group [1]. This is the "symmetry group" of the polygon.

Since any symmetry operation leaves V unchanged equation (3) gives

$$V = \tfrac{1}{2}k\tilde{\boldsymbol{\eta}}\mathbf{V}\boldsymbol{\eta} = \tfrac{1}{2}k\tilde{\boldsymbol{\eta}}'\mathbf{V}'\boldsymbol{\eta}'.$$

Hence, using equation (9) yields

$$\tilde{\boldsymbol{\eta}}\mathbf{V}\boldsymbol{\eta} = \tilde{\boldsymbol{\eta}}\tilde{\mathbf{S}}\mathbf{V}'\mathbf{S}\boldsymbol{\eta}.$$

Therefore, since $\mathbf{S}$ is orthogonal,

$$\mathbf{SV} = \mathbf{V'S}.$$

Now $\mathbf{V}$ is a characteristic of the system so a symmetry operation, which as previously emphasized involves only a change in the displacements in the present context, must leave its analytic form unchanged: i.e., $\mathbf{V} = \mathbf{V'}$. Thus it is seen that a symmetry operation S may be identified by the commuting of $\mathbf{S}$ with $\mathbf{V}$: i.e.,

$$\mathbf{SV} = \mathbf{VS}. \tag{10}$$

Consider now the effect of the symmetry operation S on the two sides of the eigenvalue equation (7). One has

$$\mathbf{S}(\mathbf{Va}) = \mathbf{S}(\lambda\mathbf{a}).$$

Therefore, because of equation (10),

$$\mathbf{V}(\mathbf{Sa}) = \lambda(\mathbf{Sa}).$$

Thus if $\mathbf{a}$ is an eigenvector of eigenvalue λ then so is $\mathbf{Sa}$ for all S in the symmetry group. It therefore follows that $\mathbf{Sa}$ can only be a linear combination of those linearly independent eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h$, say, with the same eigenvalue λ. Since this is true for each member of the degenerate set these eigenvectors span a vector space which transforms according to an h-dimensional representation D of the symmetry group. Clearly D must be irreducible [1] in most cases for if it were not then the degenerate eigenvectors could be split into two (or more) sets which would transform separately (span different invariant subspaces). It would then not be necessary for the two (or more) sets to have the same eigenvalue. Sets of eigenvectors which are not mixed by symmetry operations but which do happen to have the same eigenvalue, usually due to dynamical peculiarities, are said to be "accidentally" degenerate. This is in fact a rare occurrence. Thus, in summary, the following rule may be stated: "each degenerate set of eigenvectors of the matrix $\mathbf{V}$ spans a vector space which transforms according to an irreducible representation of the group of symmetry operations which leave $\mathbf{V}$ invariant, provided there are no accidental degeneracies." It follows that the eigenvectors can be classified according to symmetry properties.

4. THE SYMMETRY GROUP D_n OF THE REGULAR POLYGON

The symmetry group of the regular polygon of arbitrary order has been considered by various authors [5]. In the present work only those symmetries involving proper and improper rotations in the plane of the polygon are considered. These form the so-called dihedral group D_n for the polygon. The polygon does, of course, have other symmetries which this group does not contain. It is convenient to treat the cases of even and odd values of n separately, and the character tables [1] for these are shown in Tables 1 and 2 respectively. The operation R is the smallest rotation of the polygon in its plane in the positive sense which brings it back into self coincidence. This is a rotation of $2\pi/n$ anti-clockwise. Clearly $R^n = E$ where E is the identity operator. Inversion of the polygon is denoted by P so that $P^2 = E$. In the tables the notations $R^k \equiv R_k$ and $PR^k = R^{-k}P \equiv P_k$ are used. The order of the group D_n is $2n$ in all cases.

5. THE REPRESENTATION D

The various elements S of the group D_n induce linear transformations in the state vector $\boldsymbol{\eta}$. The matrices describing these form a $2n$-dimensional representation D of the group. If

TABLE 1

Character table for the group D_n with n even

Class	Characters					Comments
	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	$\chi^{(4)}$	$\ldots\chi^{(4+p)}\ldots$	$p = 1, \ldots, n/2 - 1$
E	1	1	1	1	$\ldots 2 \ldots$	$\chi^{(i)}$ is equal to the dimension of $D^{(i)}$
R_k, R_{n-k}	1	1	$(-1)^k$	$(-1)^k$	$\ldots 2\cos\frac{2\pi kp}{n}\ldots$	This gives $(\frac{1}{2}n - 1)$ classes of 2 members each as $k = 1, \ldots, n/2 - 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$R_{\frac{1}{2}n}$	1	1	$(-1)^{\frac{1}{2}n}$	$(-1)^{\frac{1}{2}n}$	$\ldots 2(-1)^p \ldots$	
All P_{2k}	1	-1	1	-1	$\ldots 0 \ldots$	This gives one class of $\frac{1}{2}n$ members as k runs from 0 to $\frac{1}{2}n - 1$
All P_{2k+1}	1	-1	-1	1	$\ldots 0 \ldots$	Ditto

TABLE 2

Character table for the group D_n with n odd

Class	Characters			Comments
	$\chi^{(1)}$	$\chi^{(2)}$	$\ldots\chi^{(2+p)}\ldots$	$p = 1, \ldots, \frac{1}{2}(n - 1)$
E	1	1	$\ldots 2 \ldots$	$\chi^{(i)}$ is equal to the dimension of $D^{(i)}$
R_k, R_{n-k}	1	1	$\ldots 2\cos\frac{2\pi pk}{n}\ldots$	This gives $\frac{1}{2}(n - 1)$ classes of 2 members each as $k = 1, \ldots, \frac{1}{2}(n - 1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
All P_k	1	-1	$\ldots 0 \ldots$	This gives one class of n members as k runs from 0 to $(n - 1)$

the original choice of basis were changed to the normal coordinates then D would change to say $\bar{D}$. Each new matrix $\bar{\mathbf{D}}(S)$ would be related by a similarity transformation to the corresponding old one $\mathbf{D}(S)$ and these would have equal traces: i.e., the same character. Representations such as D and $\bar{D}$ whose differences correspond only to a change of basis are called "equivalent" and are not usually regarded as being distinct. When an equality sign is used between representation symbols it means that they are equivalent. When a statement of the type $A = B \oplus C$ is used it means that if matrices $\mathbf{A}(S)$, $\mathbf{B}(S)$ and $\mathbf{C}(S)$ correspond to the same element S of D_n in three representations A, B and C, respectively, then $\mathbf{A}(S)$ is equal to

$$\begin{pmatrix} \mathbf{B}(S) & 0 \\ 0 & \mathbf{C}(S) \end{pmatrix}$$

after some similarity transformation.

The original choice of generalized displacements which leads to D is related by a basis transformation to the normal coordinates of the system. Thus D is equivalent to a representation with the normal coordinates as basis. Now by virtue of the rule given at the end of section 3 this last representation is completely reduced provided there are no accidental degeneracies. Thus D can be broken down into the irreducible representation $D^{(i)}$ of D_n and one can therefore write

$$D = \sum_{i=1}^{l} \oplus\, c_i D^{(i)}, \qquad l = \begin{Bmatrix} \tfrac{1}{2}n + 3 & n \text{ even} \\ \tfrac{1}{2}(n+3) & n \text{ odd} \end{Bmatrix}. \tag{11}$$

The c_i are integers greater than or equal to zero. They must be found if full use is to be made of the results summarized in section 3.

If $\chi_k^{(i)}$ is the character of the elements in class k of the group D_n in the ith irreducible representation then the following orthogonality property is satisfied [1]:

$$\sum_k q_k \chi_k^{(i)*} \chi_k^{(j)} = 2n\delta_{i,j},$$

where q_k is the number of elements in class k and $2n$ is the order of the group. Using equation (11) in conjunction with the orthogonality property gives

$$c_i = \frac{1}{2n} \sum_k q_k \chi_k^{(i)*} \chi_k, \tag{12}$$

where χ_k is the character of class k in the representation D of D_n. The q_k and $\chi_k^{(i)}$ are known from Tables 1 and 2 so in order to perform the decomposition of D it remains only to find the χ_k. To do this the various matrices in D must be displayed and their traces evaluated.

Consider first the identity operator E. This gives

$$\boldsymbol{\eta} \to \boldsymbol{\eta}' = \mathbf{E}\boldsymbol{\eta} = \boldsymbol{\eta}.$$

Thus $\mathbf{E} = \mathbf{1}_{2n \times 2n}$ and so $\chi_1 = \operatorname{Tr} \mathbf{1}_{2n \times 2n} = 2n$. Similarly under the operation R, which is defined in section 4,

$$\boldsymbol{\eta} \to \boldsymbol{\eta}' = \mathbf{R}\boldsymbol{\eta},$$

where, from the geometry of the polygon,

$$\mathbf{R} = \begin{pmatrix} 0 & 0 & . & . & 0 & \mathbf{r} \\ \mathbf{r} & 0 & . & . & 0 & 0 \\ 0 & \mathbf{r} & . & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & \mathbf{r} & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} \cos\dfrac{2\pi}{n} & -\sin\dfrac{2\pi}{n} \\ \sin\dfrac{2\pi}{n} & \cos\dfrac{2\pi}{n} \end{pmatrix}. \tag{13}$$

Thus $\chi_2 = \operatorname{Tr}(\mathbf{R}) = 0$. Similarly $\chi_k = 0$ for all $k \neq 1$.

Hence

$$\chi_k = 2n\delta_{k,1}. \tag{14}$$

By utilising Tables 1 and 2 and equation (14), the c_i can now be obtained from equation (12). Substituting these values into equation (11) then gives

$$D = \begin{Bmatrix} D^{(1)} \oplus D^{(2)} \oplus 2D^{(3)} \oplus \ldots \oplus 2D^{[2+\frac{1}{2}(n-1)]}, & n \text{ odd} \\ D^{(1)} \oplus D^{(2)} \oplus D^{(3)} \oplus D^{(4)} \oplus 2D^{(5)} \oplus \ldots \oplus 2D^{[4+(\frac{1}{2}n-1)]}, & n \text{ even} \end{Bmatrix}. \tag{15}$$

6. THE CASE OF n ODD

Table 2 shows that, if n is odd, all the $D^{(i)}$ are two-dimensional apart from $D^{(1)}$ and $D^{(2)}$ which are one-dimensional. Thus, from equation (15) and the rule given in section 3 it follows at once that the vibration spectrum consists of two singlets, corresponding to $D^{(1)}$ and $D^{(2)}$, and $2n-2$ degenerate doublets. There may be extra "accidental" degeneracies but these have nothing to do with the symmetries contained in the group D_n. In normal coordinates, then, from equations (3), (8) and (15),

$$\mathbf{V} = \begin{array}{c} \left(\begin{array}{cccccccc} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & 0 & \\ & & \lambda_{31} & & & & & \\ & & & \lambda_{31} & & & & \\ & & & & \lambda_{32} & & & \\ & & & & & \lambda_{32} & & \\ & 0 & & & & & \lambda_{41} & \\ & & & & & & & \ddots \\ & & & & & & & \lambda_{(\frac{1}{2}[n+3])2} \end{array}\right) \end{array} \begin{array}{l} D^{(1)} \\ D^{(2)} \\ \left.\begin{array}{l} \\ \end{array}\right\} D^{(3)} \\ \left.\begin{array}{l} \\ \end{array}\right\} D^{(3)}. \\ \cdot \\ \cdot \\ D^{[\frac{1}{2}(n+3)]} \end{array} \tag{16}$$

Indicated on the right-hand side of the diagonal matrix of eigenvalues are the irreducible representations according to which the eigenvectors corresponding to the various eigenvalues transform. These have been used as the basis for a new labelling scheme for the eigenvalues. This scheme puts in the non-accidental degeneracies automatically. It should be noticed that when an irreducible representation appears twice in equation (15) the two corresponding degenerate doublets are not necessarily mutually degenerate. In other words there are two separate doublets rather than a quadruplet.

Consider $\mathbf{D}(S)\mathbf{V}$ where S is any element of the group D_n. In terms of normal coordinates,

$$\mathbf{D}(S)\mathbf{V} = \begin{pmatrix} \lambda_1\mathbf{D}^{(1)}(S) & & & & & 0 \\ & \lambda_2\mathbf{D}^{(2)}(S) & & & & \\ & & \lambda_{31}\mathbf{D}^{(3)}(S) & & & \\ & & & \lambda_{32}\mathbf{D}^{(3)}(S) & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_{[\frac{1}{2}(n+3)]2}\mathbf{D}^{[\frac{1}{2}(n+3)]}(S) \end{pmatrix}.$$

Since the normal coordinates are not known this result is of no use until it is observed that the trace of the matrix must be invariant under coordinate transformations. Thus, in any coordinate system,

$$\mathrm{Tr}\,[\mathbf{D}(S)\mathbf{V}] = \lambda_1\chi_k{}^{(1)} + \lambda_2\chi_k{}^{(2)} + \sum_{p=1}^{\frac{1}{2}(n-1)} [\lambda_{(p+2)1} + \lambda_{(p+2)2}]\,\chi_k{}^{(p+2)},$$

where k is the class to which S belongs. Similarly it can be shown that

$$\mathrm{Tr}\,[\mathbf{D}(S)\mathbf{V}^2] = \lambda_1{}^2\chi_k{}^{(1)} + \lambda_2{}^2\chi_k{}^{(2)} + \sum_{p=1}^{\frac{1}{2}(n-1)} [\lambda^2_{(p+2)1} + \lambda^2_{(p+2)2}]\,\chi_k{}^{(p+2)}.$$

The left-hand sides of these equations can be evaluated directly from equations (4), (13) and similar equations for the other elements of D_n. The right-hand sides will of course be the same for all S in a given class k. Thus,

$$2n\delta_{k,0} - n\cos\frac{2\pi}{n}\,\delta_{k,1} + 4\sin^2\left(\frac{\pi}{n}\right)\delta_{k,\frac{1}{2}(n+1)}$$

$$= \lambda_1\chi_k{}^{(1)} + \lambda_2\chi_k{}^{(2)} + \sum_{p=1}^{\frac{1}{2}(n-1)}[\lambda_{(p+2)1} + \lambda_{(p+2)2}]\,\chi_k{}^{(p+2)}, \tag{17a}$$

where $k = 0, 1, \ldots, \frac{1}{2}(n+1)$. Also

$$n\left(5 + \cos\frac{4\pi}{n}\right)\delta_{k,0} + \tfrac{3}{4}\delta_{n,3}\,\delta_{k,1} - 4n\delta_{k,1}\cos\frac{2\pi}{n}$$

$$+ n\cos^2\left(\frac{2\pi}{n}\right)\delta_{k,2}\,(1-\delta_{n,3}) + 16\sin^4\left(\frac{\pi}{n}\right)\delta_{k,\frac{1}{2}(n+1)}$$

$$= \lambda_1{}^2\chi_k{}^{(1)} + \lambda_2{}^2\chi_k{}^{(2)} + \sum_{p=1}^{\frac{1}{2}(n-1)}[\lambda^2_{(p+2)1} + \lambda^2_{(p+2)2}]\,\chi_k{}^{(p+2)}, \tag{17b}$$

where $k = 0, 1, \ldots, \frac{1}{2}(n+1)$.

It should be noted that here $k = \frac{1}{2}(n+1)$ is being taken as the class of all P_k. Since the $\chi_k{}^{(i)}$ are all known from Table 2 equations (17a) and (17b) provide $n+3$ simultaneous equations in the $n+1$ unknown values of λ. The solution of these equations by the usual methods is, however, a tedious business. It is much easier to project out the various eigenvalues by means of the orthogonality condition for the characters which were quoted in section 5.

Consider first the set of equations (17a). Multiply each term in each of the equations by $q_k\chi_k{}^{(i)}$ for arbitrary i in the range $1 \leqslant i \leqslant \frac{1}{2}(n+3)$ and with the appropriate value of k. Add up all the resulting equations and collect up terms with common values of λ on the right hand side. Using the orthogonality condition and dividing throughout by $2n$ then gives

$$\chi_0{}^{(i)} - \cos\frac{2\pi}{n}\,\chi_1{}^{(i)} + 2\sin^2\left(\frac{\pi}{n}\right)\chi^{(i)}_{\frac{1}{2}(n+1)} = \lambda_1\delta_{i,1} + \lambda_2\delta_{i,2}$$

$$+ \sum_{p=1}^{\frac{1}{2}(n-1)}[\lambda_{(p+2)1} + \lambda_{(p+2)2}]\,\delta_{i,(p+2)}, \qquad i = 1, 2, \ldots, \tfrac{1}{2}(n+3).$$

Hence, by using Table 2 one finds

$$\lambda_1 = 4\sin^2\left(\frac{\pi}{n}\right), \qquad \lambda_2 = 0$$

and

$$\lambda_{(p+2)1} + \lambda_{(p+2)2} = 2\left[1 - \cos\frac{2\pi}{n}\cos\frac{2\pi p}{n}\right], \qquad p = 1, \ldots, \tfrac{1}{2}(n-1).$$

Similarly from equation (17b) one obtains

$$\lambda_1{}^2 = 16\sin^4\left(\frac{\pi}{n}\right), \qquad \lambda_2{}^2 = 0$$

and

$$\lambda^2_{(p+2)1} + \lambda^2_{(p+2)2} = 4\left[1 - \cos\frac{2\pi}{n}\cos\frac{2\pi p}{n}\right]^2, \qquad p = 1, \ldots, \tfrac{1}{2}(n-1).$$

Comparing the two sets of results shows that one of each pair $\lambda_{(p+2)1}$ and $\lambda_{(p+2)2}$ must be zero. Here the second of each pair is chosen to be zero. This involves no loss of generality.

Since the eigenvalues λ are simply related to the angular frequencies of the normal modes by equation (6) the spectrum of in-plane vibrations for odd values of n is now fully determined including the degeneracy structure. It is summarized in Table 3.

TABLE 3
Spectrum for the regular polygon with n odd

Singlets	Doublets
$\lambda_1 = 4\sin^2\left(\frac{\pi}{n}\right)$	$\lambda_{(2+p)1} = 2\left[1 - \cos\frac{2\pi}{n}\cos\frac{2\pi p}{n}\right]$
$\lambda_2 = 0$	and $\lambda_{(2+p)2} = 0$, where $p = 1,..,\frac{1}{2}(n-1)$

A check on the calculations can now be made by comparing the predictions of Table 3 in the special case of $n = 3$ with the well known spectrum for in-plane vibrations of an equilateral triangle of identical springs and particles. From Table 3, $\lambda_1 = 3$ (singlet), $\lambda_2 = 0$ (singlet), $\lambda_{31} = 3/2$ (doublet) and $\lambda_{32} = 0$ (doublet). Thus $\lambda = 3$ (once), 3/2 (twice) and 0 (three times) in agreement with the standard result [6].

7. THE CASE OF n EVEN

From Table 1 it follows that all the $D^{(i)}$ are two-dimensional apart from the cases with $i = 1,2,3$ and 4 which are one-dimensional. Thus, from equation (15), the spectrum will consist of four singlets and $2n - 4$ doublets, apart from further possible "accidental" degeneracies. Proceeding in a similar manner to the case of n odd but referring to Table 1 rather than Table 2 shows that the singlets are given by

$$\lambda_1 = 4\sin^2\left(\frac{\pi}{n}\right), \quad \lambda_2 = 0, \quad \lambda_3 = 4\cos^2\left(\frac{\pi}{n}\right) \quad \text{and} \quad \lambda_4 = 0.$$

The doublets are given by

$$\left.\begin{cases} \lambda_{(p+4)1} = 2\left[1 - \cos\frac{2\pi}{n}\cos\frac{2\pi p}{n}\right] \\ \lambda_{(p+4)2} = 0 \end{cases}\right\}, \qquad p = 1, 2, \ldots, \tfrac{1}{2}n - 1.$$

For completeness the vibration spectrum for n even is summarized in Table 4.

TABLE 4
Spectrum for the regular polygon with n even

Singlets	Doublets
$\lambda_1 = 4\sin^2\left(\frac{\pi}{n}\right)$	$\lambda_{(4+p)1} = 2\left[1 - \cos\frac{2\pi}{n}\cos\frac{2\pi p}{n}\right]$
$\lambda_2 = \lambda_4 = 0$	$\lambda_{(4+p)2} = 0$
$\lambda_3 = 4\cos^2\left(\frac{\pi}{n}\right)$	where $p = 1, 2, \ldots, \frac{1}{2}n - 1$

8. DISCUSSION

In this paper a specific vibration problem has been considered and, by applying group theoretical arguments, two different types of information have been obtained. Firstly, purely qualitative information has been obtained—the degeneracy structure of the spectrum. The technique for doing this is well known in the study of molecular vibrations. Secondly, a less familiar method was employed in order to obtain quantitative information—the actual frequencies in the spectrum. The acquisition of this quantitative information, however, required much more manipulative effort than did that of the qualitative type.

It is possible to obtain qualitative information other than degeneracy structures in vibration problems by means of group theoretical arguments. Perhaps the best known is the question of whether or not a degenerate multiplet will split when the system is subjected to a perturbation which may itself possess some of the symmetries of the system. For an example of such an application see reference [3].

It is the author's view that the obtaining of the qualitative type of information is probably the most useful application of group theory to vibration problems because it is obtained with such relative ease. Nevertheless the fact that quantitative information can be obtained by group arguments in the specific type of problem studied in detail in the present paper is a clear indication that group theory is of far wider potential value in vibration studies than is often supposed.

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