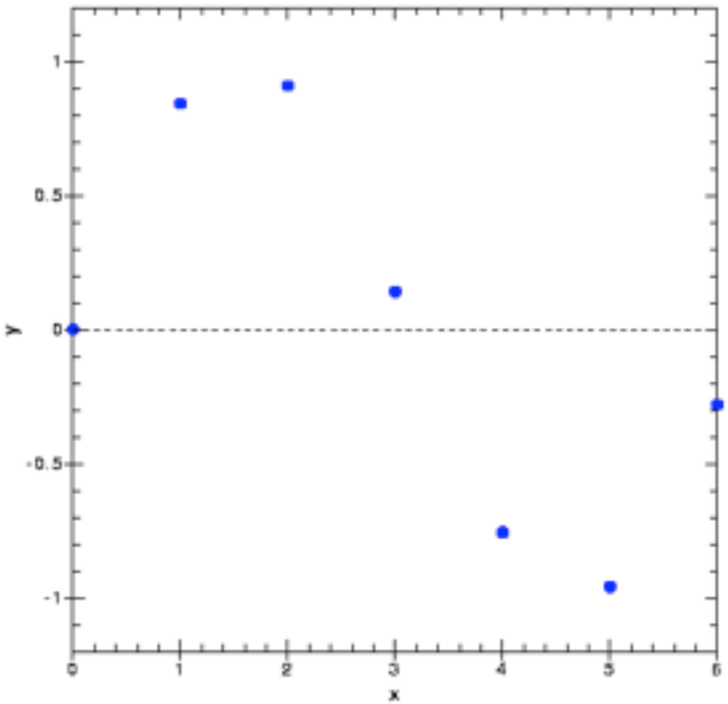
Interpolation

Let's assume we have a set of measurements:

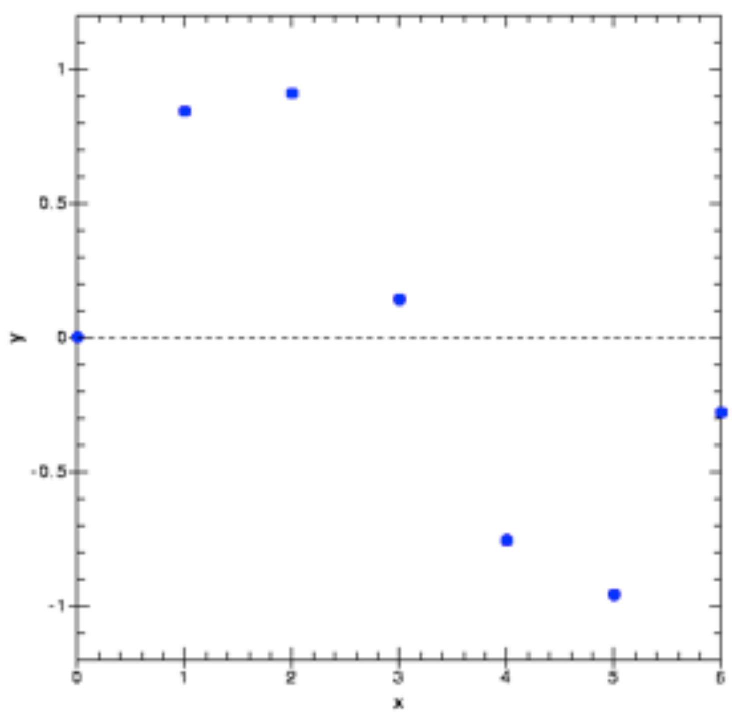
x	*y*
0	*0*
1	*0.8415*
2	*0.9093*
3	*0.1411*
4	*-0.7568*
5	*-0.9589*
6	*-0.2794*



Interpolation

Interpolation is a ***method to approximate a function****, f, at a* ***desired location, x****, given a set of* ***n+1 support points*** *(x and y values)*

$$f(x_i) = y_i \quad \text{for } 0 \le \text{i} \le \text{n}$$



Interpolation

Interpolation is a ***method to approximate a function****, f, at a* ***desired location, x****, given a set of* ***n+1 support points*** *(x and y values)*

$$f(x_i) = y_i \quad \text{for } 0 \le \text{i} \le \text{n}$$

Let's consider a ***family of functions****, Φ, that is characterized by* ***n+1 parameters****, c_0, c_1,….., c_n and our variable x*

$$\phi(c_0, c_1, \ldots, c_n, x)$$

Interpolation *tries to determine the* ***coefficients*** *c_i, so that*

$$\phi(c_0, c_1, \ldots, c_n, x_i) = y_i \quad \text{for } 0 \le \text{i} \le \text{n}$$

Types of Interpolation

$$\phi(c_0, c_1, \dots, c_n, x_i) = y_i \quad \text{for } 0 \le i \le n$$

Linear Interpolation: *The function* Φ *depends linearly on the* c_i*'s*

1. ***Polynominal:*** $\Phi = c_0 + c_1 x + \ldots + c_n x^n$

2. ***Splines:*** *nonlocal interpolation*

3. ***Trigonometric:*** $\Phi = c_0 + c_1 e^{ix} + \ldots + c_n e^{inx}$

   (Fourier analysis, Fast Fourier Transforms)

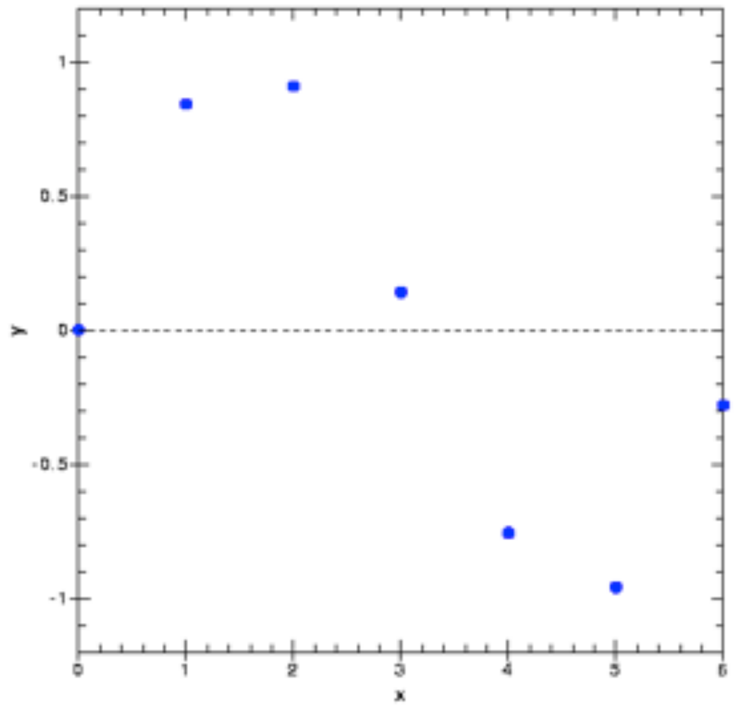
Nonlinear Interpolation: Φ *depends nonlinearly on the* c_i*'s*

1. ***Rational Functions:*** $\phi = \dfrac{p_\mu(x)}{q_\nu(x)} = \dfrac{c_0 + c_1 x + \ldots + c_\mu x^\mu}{d_0 + d_1 x + \ldots + d_\nu x^\nu}$

Rational functions are a good choice if the function that we want to interpolate has a pole in the region of interest.

Polynominal Interpolation

Linear Interpolation



*A **linear function** interpolating between the first two values can be written as:*

$$P_{0,1}(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

Polynominal Interpolation

$$P_{0,1}(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

*We can easily see that the function $P_{0,1}$ is of **degree 1** and satisfies*

$$P_{0,1}(x_0) = y_0 \quad \text{and} \quad P_{0,1}(x_1) = y_1$$

We can rewrite this as:

$$P_{0,1}(x) = y_0 L_0^{(1)}(x) + y_1 L_1^{(1)}(x)$$

Polynominal Interpolation

$$P_{0,1}(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

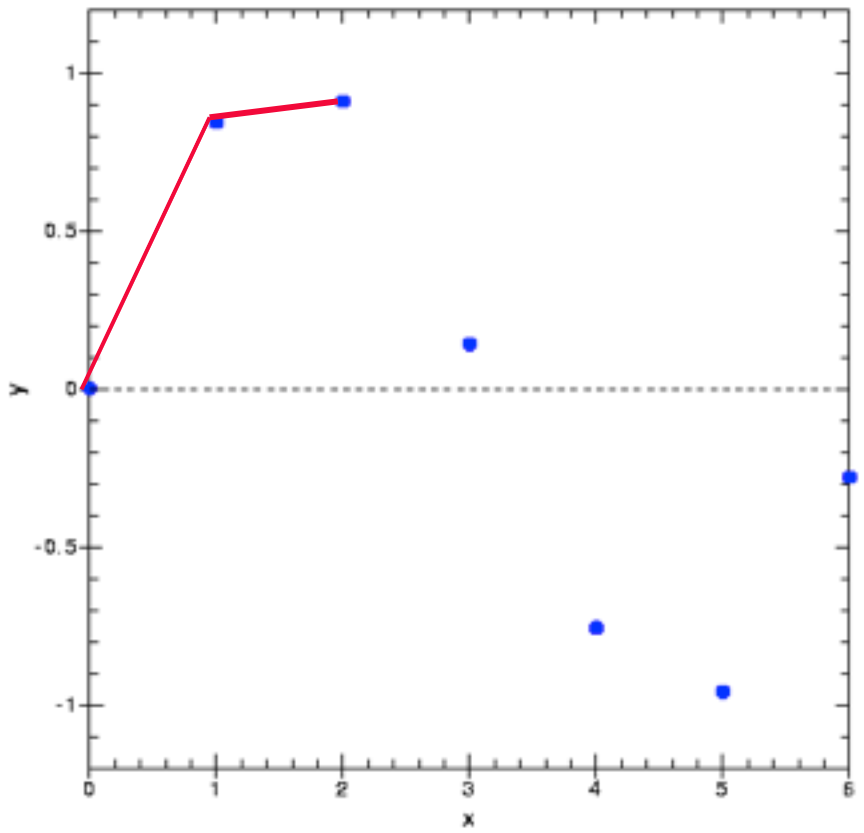
$$L_0^{(1)} = \frac{x - x_1}{x_0 - x_1} \qquad L_1^{(1)} = \frac{x - x_0}{x_1 - x_0}$$

The functions $L_0^{(1)}$ and $L_1^{(1)}$ are examples of ***Lagrange interpolating polynominals.*** *They have the property*

$$L_0^{(1)}(x_0) = 1, \qquad L_0^{(1)}(x_1) = 0, \qquad L_1^{(1)}(x_0) = 0, \qquad L_1^{(1)}(x_0) = 1,$$

$$\Rightarrow \quad L_i^{(1)}(x_j) = \delta_{i,j}$$

Polynominal Interpolation



*The **slope** between the 1st and 2nd and between the 2nd and 3rd data point **dramatically changes.***

Polynominal Interpolation

A better interpolation might be obtained by taking the ***change in the slope*** *into account by selecting* ***polynominals with curvature***

A polynominal of ***degree 2*** *that passes through the points 0, 1, and 2 can be written as:*

$$P_{0,1,2}(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$L_0^{(2)}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$L_1^{(2)}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \qquad \Rightarrow \qquad L_i^{(2)}(x_j) = \delta_{i,j}$$

$$L_2^{(2)}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Polynominal Interpolation

Back to our example:

x	y
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794

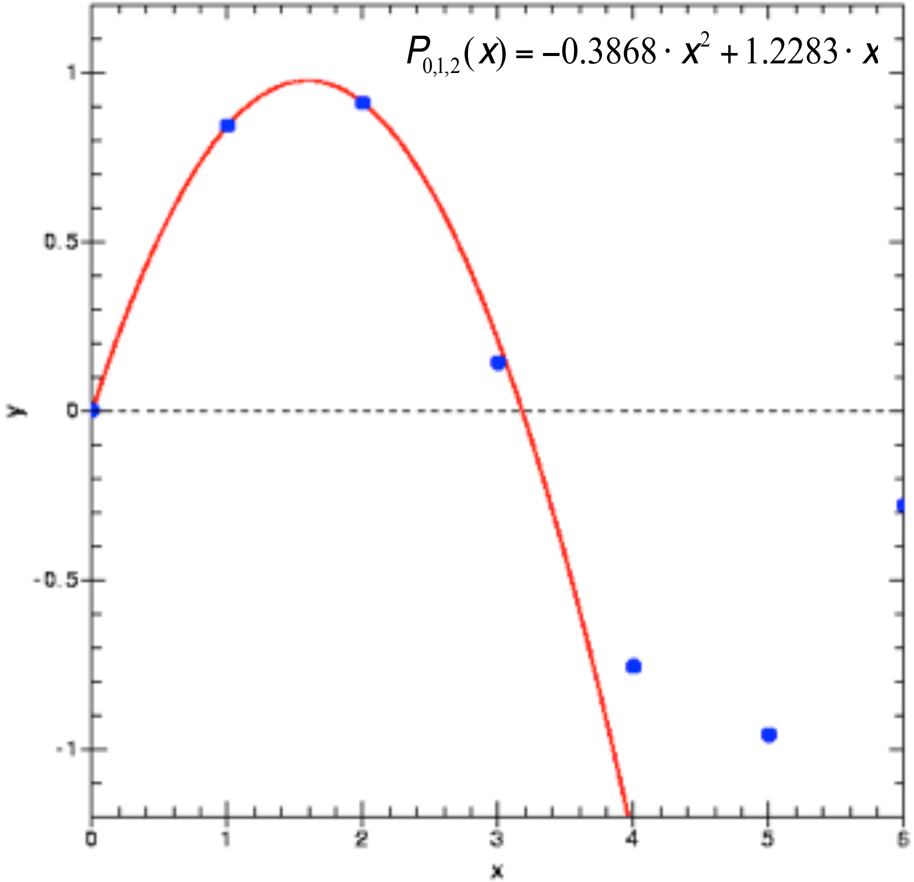
$$\boxed{P_{0,1,2}(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}$$

$$P_{0,1,2}(x) = 0 \cdot \frac{(x-1)(x-2)}{(0-1)(1-2)} + 0.8415 \cdot \frac{(x-0)(x-2)}{(1-0)(1-2)} + 0.9093 \frac{(x-0)(x-1)}{(2-0)(2-1)}$$

$$P_{0,1,2}(x) = -0.8415 \cdot \left(x^2 - 2x\right) + 0.9093 \cdot \frac{\left(x^2 - x\right)}{2} = -0.3868 \cdot x^2 + 1.2283 \cdot x$$

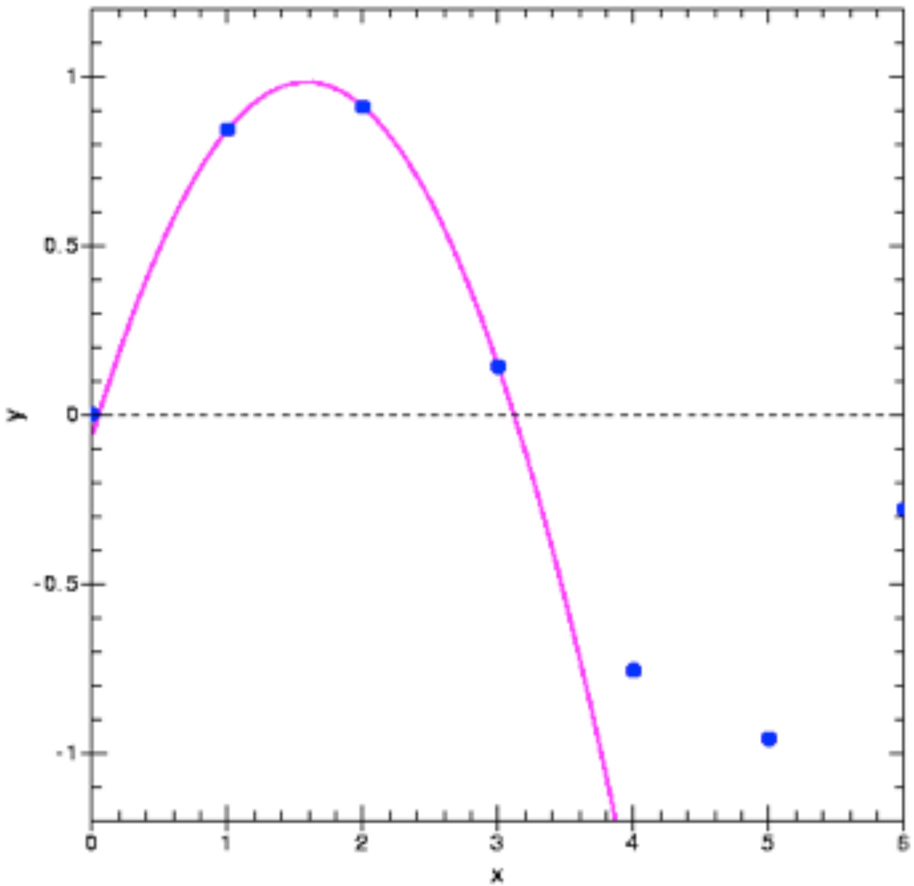
Polynominal Interpolation

A Parabola through the first three points



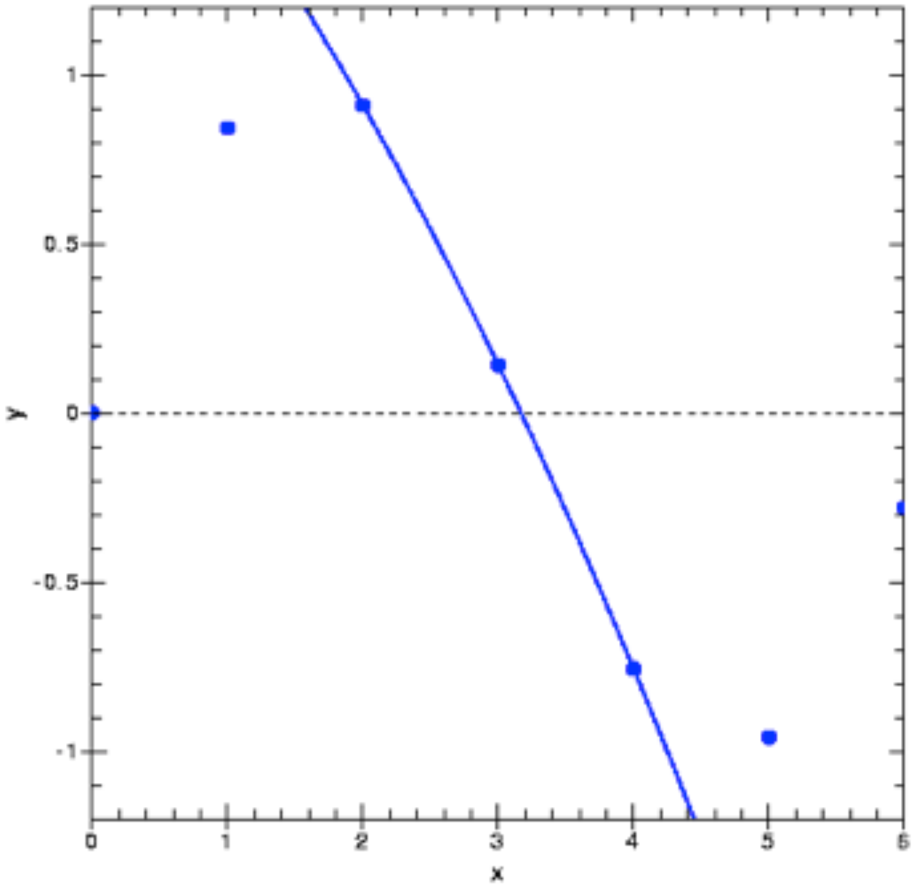
Polynominal Interpolation

A Parabola through point 1, 2, and 3



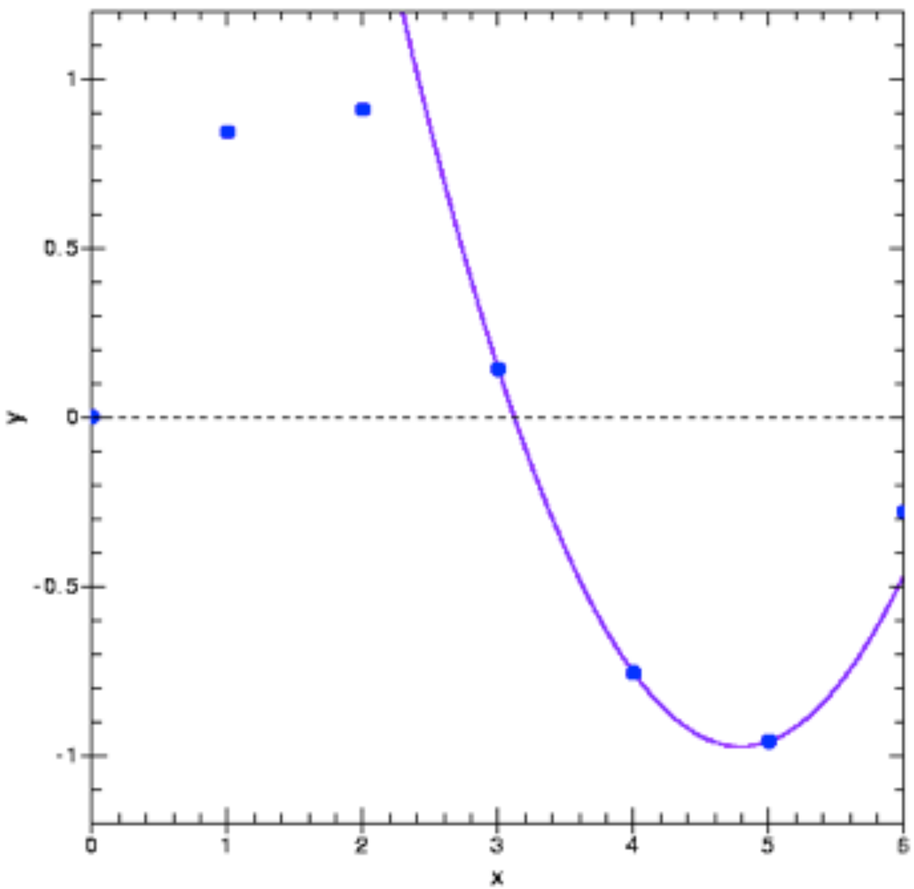
Polynominal Interpolation

A Parabola through points 2, 3, and 4



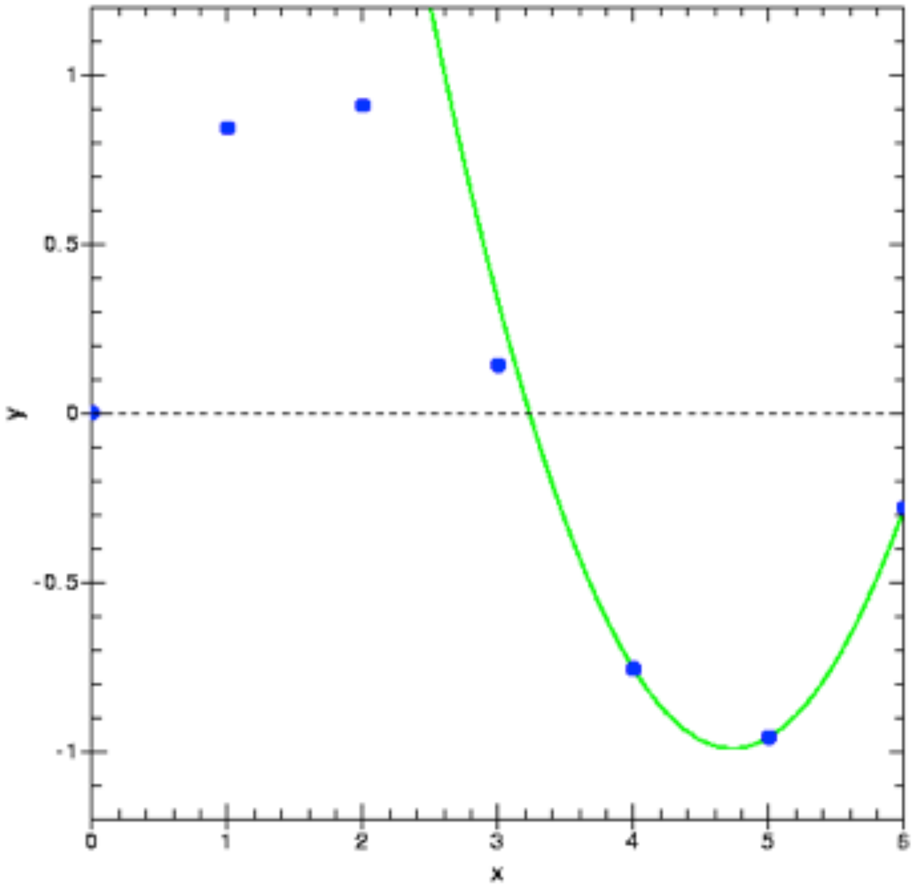
Polynominal Interpolation

A Parabola through points 3, 4, and 5



Polynominal Interpolation

A Parabola through the final three points



Polynominal Interpolation

Let's include all n+1 points, which will result in a polynominal of degree n.

➡ *Construct this polynominal from* ***higher degree Lagrange interpolation polynominals:***

So far our Lagrange polynominals looked like:

First Order

$$L_0^{(1)} = \frac{x - x_1}{x_0 - x_1}$$

Second Order

$$L_0^{(2)}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

***n*^th^ Order:**

$$L_i^{(n)}(x) = \prod_{j=0,\, j\neq i}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) \quad \Rightarrow \quad L_i^{(n)}(x_j) = \delta_{i,j}$$

Polynominal Interpolation

The higher order Lagrange Polynominals are

$$L_i^{(n)}(x) = \prod_{j=0,\, j\neq i}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) \quad \Rightarrow \quad L_i^{(n)}(x_j) = \delta_{i,j}$$

*The final polynominal that passes through **all n+1 points** is than gives as:*

$$\boxed{\Rightarrow \quad \mathrm{P}_{0,\ldots,\mathrm{n}}(x) = \sum_{i=0}^{n} L_i^{(n)}(x)\, y_i}$$

Polynominal Interpolation

$$\boxed{\Rightarrow \quad \mathrm{P}_{0,\ldots,\mathrm{n}}(x) = \sum_{i=0}^{n} L_i^{(n)}(x)\, y_i} \qquad L_i^{(n)}(x) = \prod_{j=0,\, j\neq i}^{n} \left(\frac{x - x_j}{x_i - x_j} \right)$$

*For our example with the **seven** points*

$$P_{0,1,..6}(x) = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)(x-x_6)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)(x_0-x_6)}$$

$$+\, y_1 \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)(x-x_6)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)(x_1-x_6)}$$

$$+ \quad \ldots$$

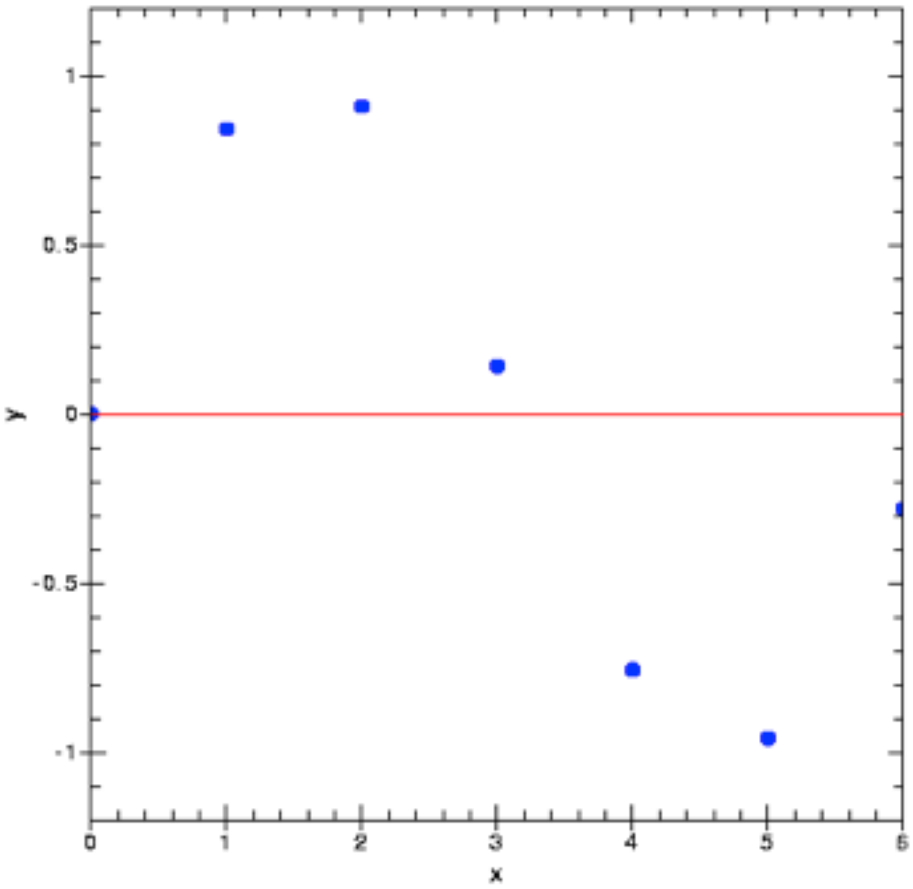
$$+\, y_6 \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_6-x_0)(x_6-x_1)(x_6-x_2)(x_6-x_3)(x_6-x_4)(x_6-x_5)}$$

First Term

$$P_{0,1,..6}(x) = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)(x-x_6)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)(x_0-x_6)}$$

Back to our example:

x	*y*
0	*0*
1	*0.8415*
2	*0.9093*
3	*0.1411*
4	*-0.7568*
5	*-0.9589*
6	*-0.2794*

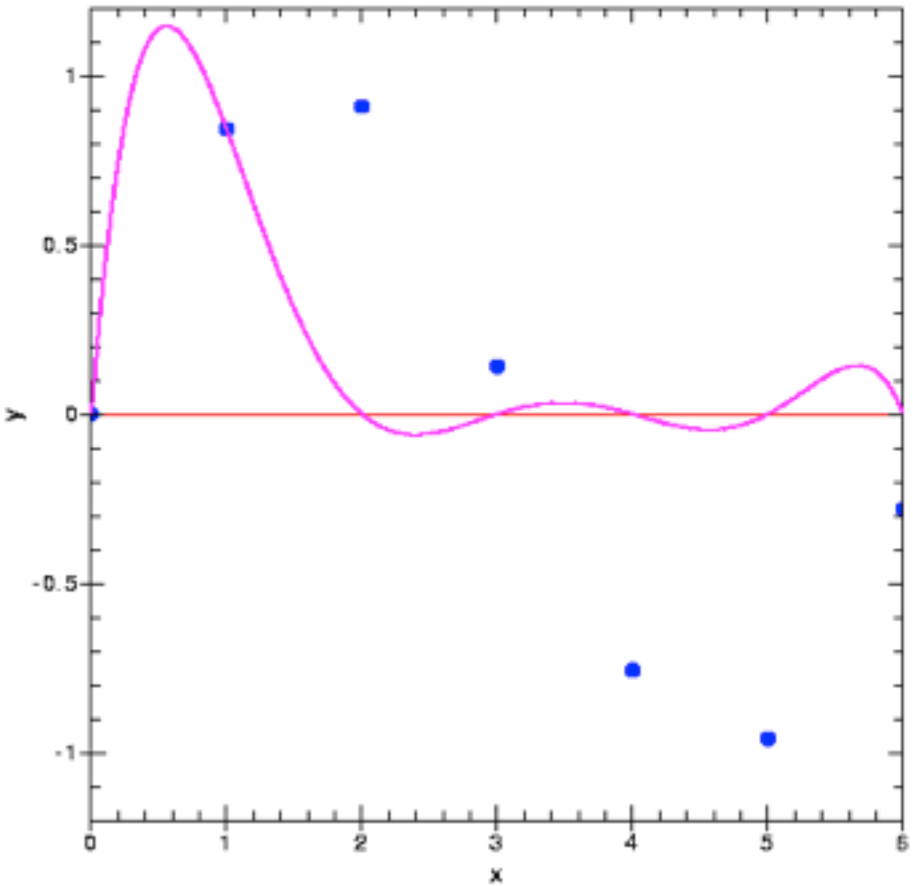


Interpolation

Back to our example:

x	y
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794

2nd Term in Series

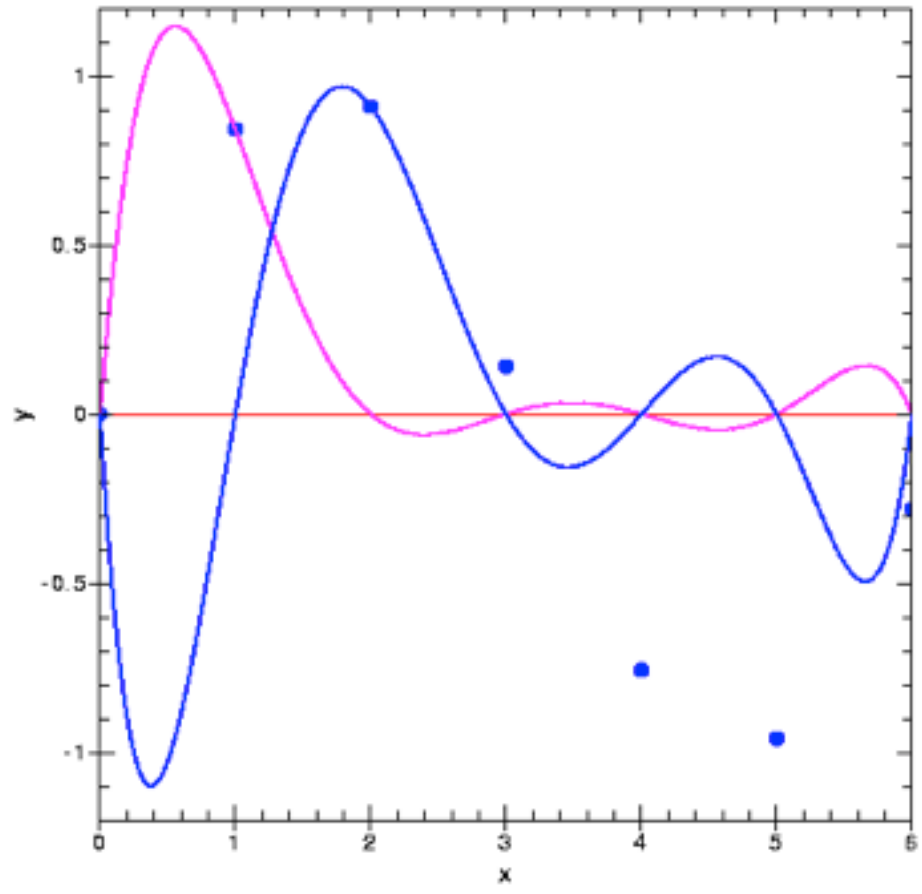


Interpolation

Back to our example:

x	*y*
0	*0*
1	*0.8415*
2	*0.9093*
3	*0.1411*
4	*-0.7568*
5	*-0.9589*
6	*-0.2794*

3rd Term in Series

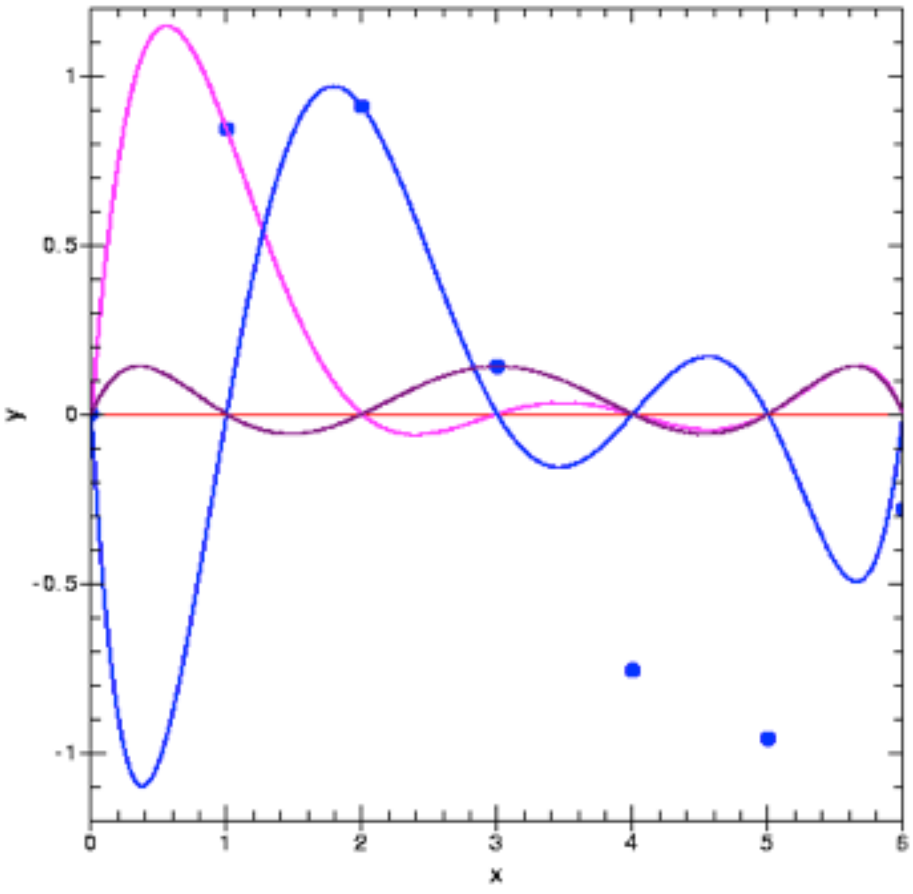


Interpolation

Back to our example:

x	y
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794

4th Term in Series

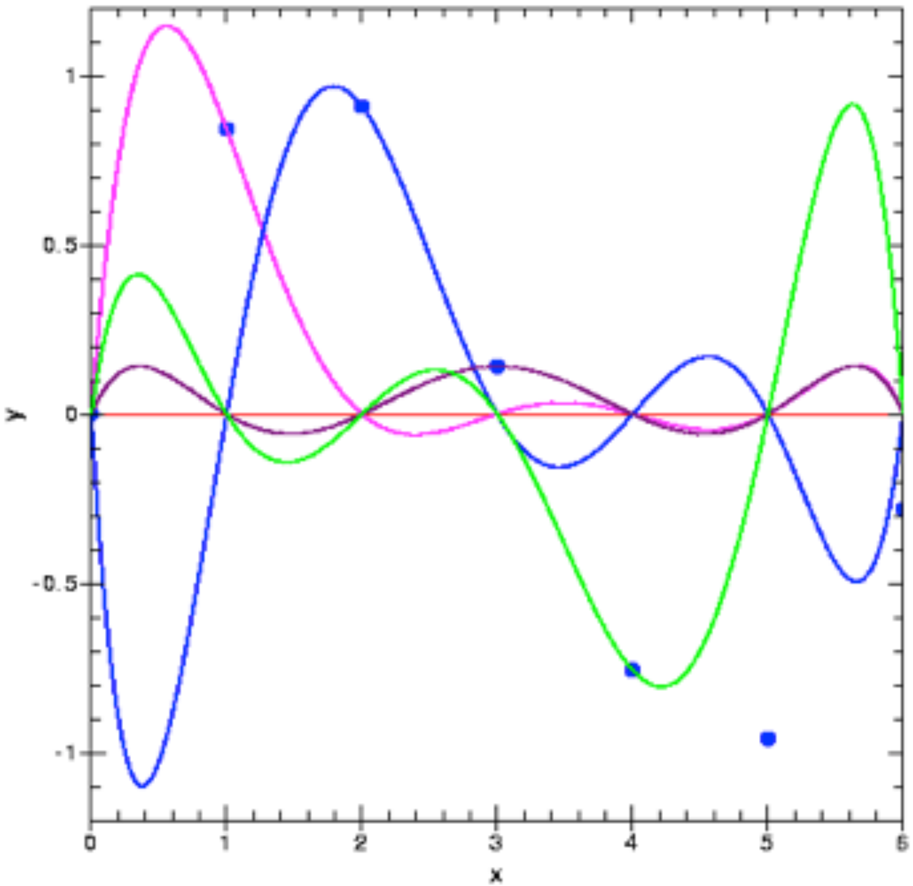


Interpolation

Back to our example:

x	*y*
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794

5th Term in Series

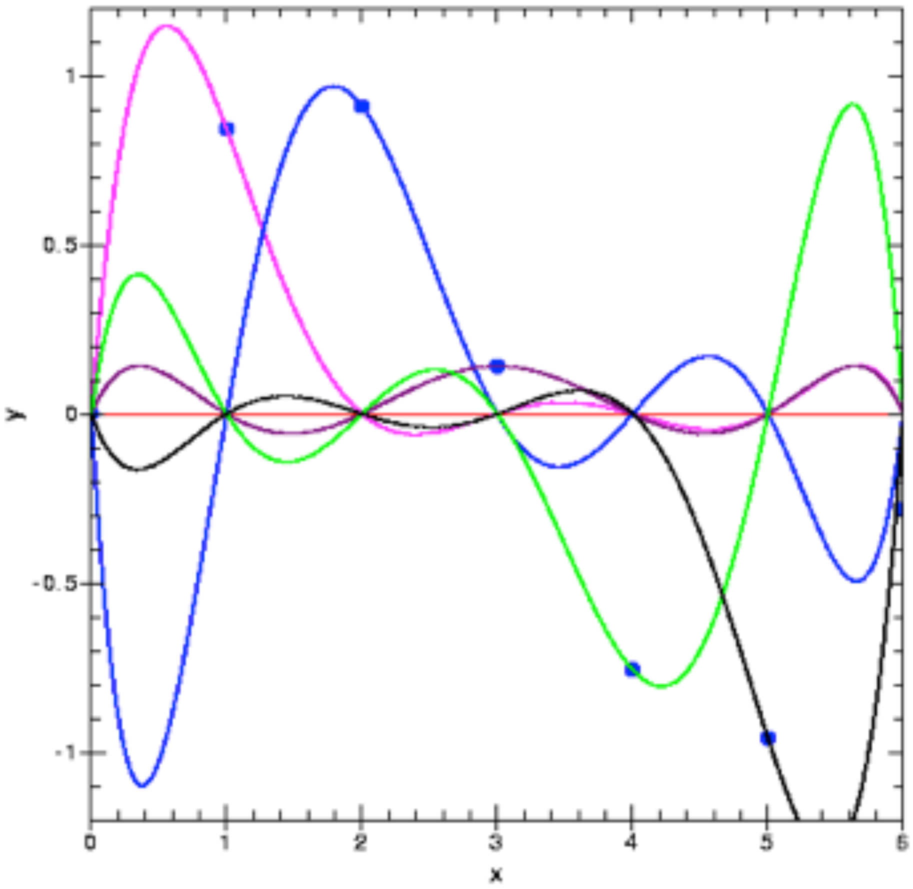


Interpolation

Back to our example:

x	y
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794

6th Term in Series

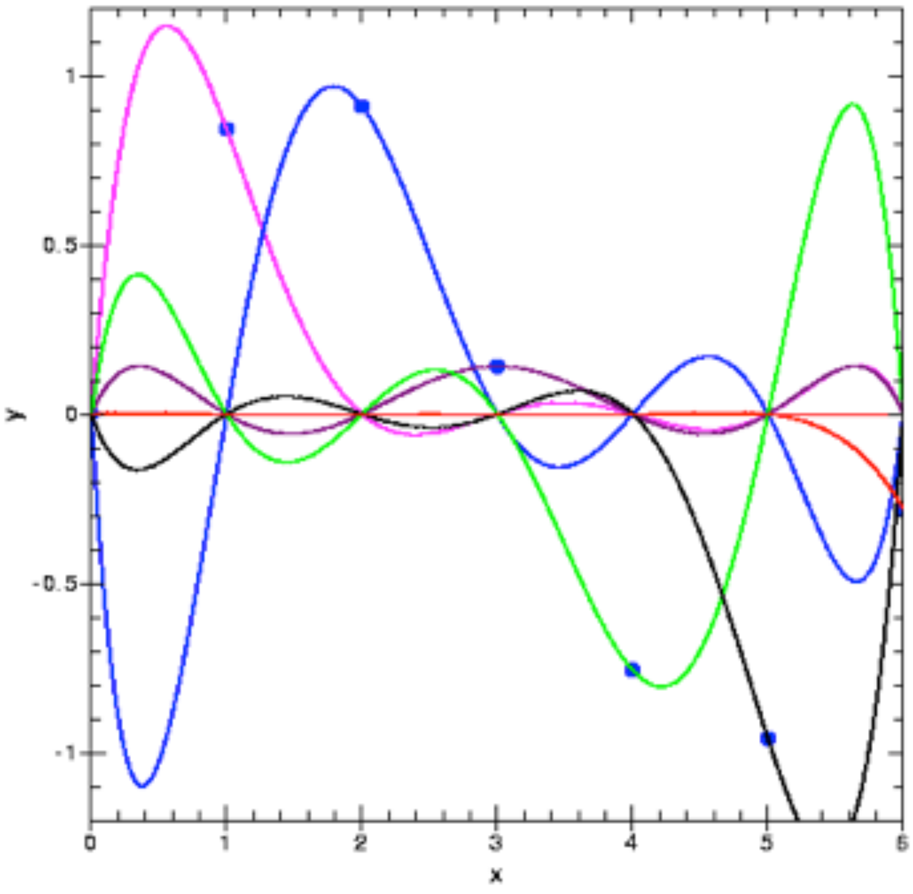


Interpolation

Back to our example:

x	*y*
0	*0*
1	*0.8415*
2	*0.9093*
3	*0.1411*
4	*-0.7568*
5	*-0.9589*
6	*-0.2794*

7th Term in Series



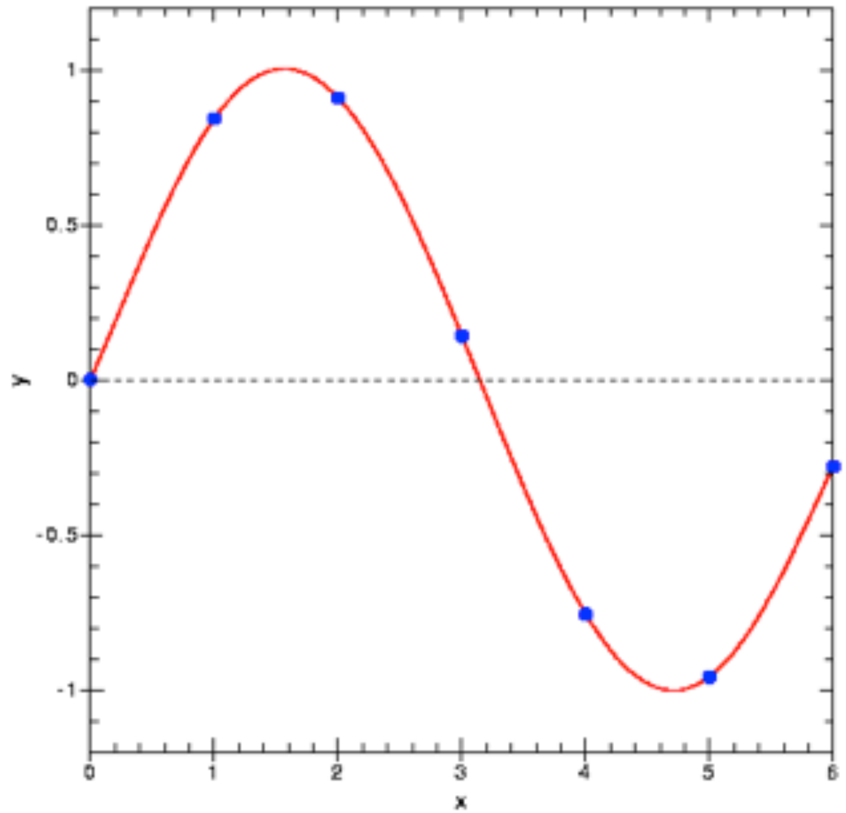
Interpolation

Back to our example:

x	y
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794

$$\boxed{\Rightarrow \quad P_{0,\ldots,n}(x) = \sum_{i=0}^{n} L_i^{(n)}(x)\, y_i}$$

All Terms added together



$$P(x) = -\,0.0001521*x^6 - 0.003130*x^5 + 0.07321*x^4 - 0.3577*x^3 + 0.2255*x^2 + 0.9038*x$$

Existence and Uniqueness

*For **n+1 arbitrary support points** (x_i, y_i) where $x_0 < x_1 < \ldots < x_n$, there **exists** a **unique** n-th order polynominal which satisfies*

$$P(x_i) = y_i \quad \text{for} \quad 0 \le i \le n$$

***Existence:** Construct the Lagrange polynominals as before*

$$L_i^{(n)}(x) = \prod_{j=0,\, j \ne i}^{n} \left(\frac{x - x_j}{x_i - x_j} \right)$$

If we construct our polynominal as: $\mathrm{P}(x) = \sum_{i=0}^{n} L_i^{(n)}(x)\, y_i$

than this polynominal has in fact n-th order and satisfies

$$P(x_i) = y_i \quad \text{for} \quad 0 \le i \le n$$

Existence and Uniqueness

Uniqueness: *(show by method of contradiction)*

Let's assume that $\boldsymbol{P_1}$ *and* $\boldsymbol{P_2}$ *are* ***n-th order polynominals*** *that* ***both*** *satisfy* $P(x_i) = y_i$ *for* $0 \le i \le n$.

If we construct the polynominal $\boldsymbol{P_3 = P_1 - P_2}$ *, than* P_3 *is* ***at most*** *an* ***n-th order polynominal*** *with at* ***most n distinct roots*** $\boldsymbol{z_i}$

$$P_3(z_i) = 0 \quad \text{with} \quad i \le n \qquad \textit{(From complete factorization)}$$

However, $\quad P_3(x_i) = P_1(x_i) - P_2(x_i) = 0 \;$ for the $n+1$ x_i

Consequently, our assumption is wrong and P_3 *can only be equal 0*

---> $P_1 = P_2$