An Alternative Approach

Construct an approximation for f(x), namely g(x), by imposing the following conditions:

1. $g'(x)$ is a smoothly varying function between two support points and is continuous at the support points

2. $g''(x)$ is continuous at the support points

This results a cubic (third order) polynominal in each interval while enforcing the above conditions.

---> Cubic Spline

Cubic Splines

Verify that number of imposed conditions equals the number of available coefficients:

There are n+1 support points (knots) → n subintervals.

On each subinterval we shell have a different cubic polynominal. Since each cubic polynominal has four coefficients

→ total of 4n coefficients available.

Within each interval cubic polynominal must go through two points
→ 2n conditions.

The 1st and 2nd derivative must be continuous at n-1 interior points
→ 2(n-1) conditions

The missing two conditions need to be provided as boundary conditions for the 1st or 2nd derivatives at the end points.

Types of Cubic Splines

Examples of Boundary conditions:

1. ***"Natural" Spline:*** $g_0''(x_0) = 0 \qquad g_n''(x_n) = 0$

   No curvature at the endpoints
   → equivalent to assuming that the end cubics approach linearity at their extremities.

2. ***"Clamped" Spline:*** $g_0'(x_0) = f'(x_0) \qquad g_n'(x_n) = f'(x_n)$

   Specify the 1st derivative of the interpolating function

3. ***Assume that*** $g_0''(x_0) = g_1''(\mathrm{x}_1) \qquad g_n''(x_n) = g_{n-1}''(\mathrm{x}_{n-1})$

   Equivalent to assuming that the end cubics approach parabolas at their extremeties.

Example

Find a natural Cubic Spline that passes through the points

P(-1) = 1 ***P(0)=2*** ***P(1)=-1***

We need to find two cubic polynominals

$$g(x) = \begin{cases} ax^3 + bx^2 + cx + d & x \in [-1,0] \\ ex^3 + fx^2 + gx + h & x \in [0,+1] \end{cases}$$

From the interpolation condition, we have

$$d = 2 \quad h = 2$$

and

$$-a+b-c = -1 \qquad e+f+g = -3$$

Example

The first derivative is:

$$g'(x) = \begin{cases} 3ax^2 + 2bx + c & x \in [-1,0] \\ 3ex^2 + 2fx + g & x \in [0,+1] \end{cases}$$

The continuity of g' gives us: $\boldsymbol{c = g}$

The second derivative is:

$$g''(x) = \begin{cases} 6ax + 2b & x \in [-1,0] \\ 6ex + 2f & x \in [0,+1] \end{cases}$$

The continuity of g'' gives us: $\boldsymbol{b = f}$

The boundary condition on g'' (natural spline) gives us:

$$3a = b \quad \text{and} \quad 3e = -f$$

Example

The interpolation condition:

$$d = 2$$
$$h = 2$$
$$-a+b-c = -1$$
$$e+f+g = -3$$

The continuity of g' gives us: $c = g$

The continuity of g'' gives us: $b = f$

The boundary condition:

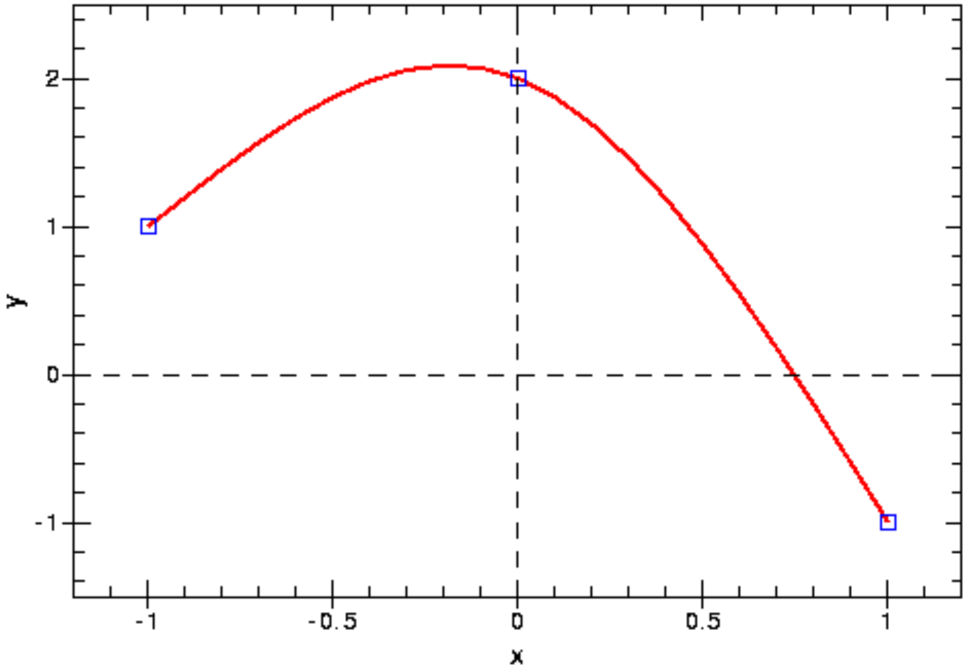
$$3a = b$$
$$3e = -f$$

$$\longrightarrow \boxed{\begin{array}{llll} a = -1 & b = -3 & c = -1 & d = 2 \\ e = 1 & f = -3 & g = -1 & h = 2 \end{array}}$$

Example

Solving for the 8 coefficients give us:

$$g(x) = \begin{cases} -x^3 - 3x^2 - x + 2 & x \in [-1,0] \\ +x^3 - 3x^2 - x + 2 & x \in [0,+1] \end{cases}$$



Algorithm for Cubic Splines

Recall our Assumptions:

***Cubic Splines** construct an approximation for f(x), namely g(x), by imposing the following conditions:*

1. *$g'(x)$ is a smoothly varying function between two support points and is continuous at the support points*

2. *$g''(x)$ is continuous at the support points*

Algorithm for Cubic Splines

We have assumed that the 2nd derivative is continuous.

Therefore, the numbers

$$z_i = g''(x_i) \qquad (1 \le i \le n-1)$$

are unambiguously defined (we just don't know them yet).

Let's just assume that the z_i were known:

Since g(x) is a cubic in any interval $[x_i, x_{i+1}]$

→ 2nd derivative g''(x) is a linear polynominal in that interval.

$$g_i''(x) = g''(x_i)\frac{(x_{i+1} - x)}{x_{i+1} - x_i} + g''(x_{i+1})\frac{(x - x_i)}{x_{i+1} - x_i}$$

Algorithm for Cubic Splines

$$g_i''(x) = g''(x_i)\frac{(x_{i+1}-x)}{x_{i+1}-x_i} + g''(x_{i+1})\frac{(x-x_i)}{x_{i+1}-x_i}$$

Let's integrate this:

$$g_i'(x) = g''(x_i)\left(-\frac{1}{2}\right)\frac{(x_{i+1}-x)^2}{x_{i+1}-x_i} \quad + g''(x_{i+1})\left(+\frac{1}{2}\right)\frac{(x-x_i)^2}{x_{i+1}-x_i} \quad + c_i$$

Let's integrate again:

$$g_i(x) = g''(x_i)\left(\frac{1}{6}\right)\frac{(x_{i+1}-x)^3}{x_{i+1}-x_i} + g''(x_{i+1})\left(\frac{1}{6}\right)\frac{(x-x_i)^3}{x_{i+1}-x_i} + c_i x + d_i$$

Algorithm for Cubic Splines

$$g_i(x) = g''(x_i)\left(\frac{1}{6}\right)\frac{(x_{i+1}-x)^3}{x_{i+1}-x_i} + g''(x_{i+1})\left(\frac{1}{6}\right)\frac{(x-x_i)^3}{x_{i+1}-x_i} + c_i x + d_i$$

$$= g''(x_i)\left(\frac{1}{6}\right)\frac{(x_{i+1}-x)^3}{x_{i+1}-x_i} + g''(x_{i+1})\left(\frac{1}{6}\right)\frac{(x-x_i)^3}{x_{i+1}-x_i}$$

$$+C_i(x-x_i) + D_i(x_{i+1}-x)$$

Adjusted the integration constants

The C_i and D_i can be found by the interpolation condition

$$g_i(x_i) = y_i \quad \text{and} \quad g_i(x_{i+1}) = y_{i+1}$$

Algorithm for Cubic Splines

For $g_i(x_i) = y_i$: $y_i = g''(x_i)\left(\frac{1}{6}\right)\frac{(x_{i+1} - x_i)^3}{x_{i+1} - x_i} + D_i(x_{i+1} - x_i)$

$$\Rightarrow \quad D_i = \frac{y_i}{x_{i+1} - x_i} - g''(x_i)\frac{x_{i+1} - x_i}{6}$$

and similar for C_i:

$$\Rightarrow \quad C_i = \frac{y_{i+1}}{x_{i+1} - x_i} - g''(x_{i+1})\frac{x_{i+1} - x_i}{6}$$

Algorithm for Cubic Splines

Let's insert the C_i and D_i back in our original equation

$$g_i(x) = g''(x_i)\left(\frac{1}{6}\right)\frac{(x_{i+1}-x)^3}{x_{i+1}-x_i} + g''(x_{i+1})\left(\frac{1}{6}\right)\frac{(x-x_i)^3}{x_{i+1}-x_i}$$

$$+C_i(x-x_i) + D_i(x_{i+1}-x)$$

$$= g''(x_i)\left(\frac{1}{6}\right)\frac{(x_{i+1}-x)^3}{x_{i+1}-x_i} + g''(x_{i+1})\left(\frac{1}{6}\right)\frac{(x-x_i)^3}{x_{i+1}-x_i}$$

$$+\left(\frac{y_{i+1}}{x_{i+1}-x_i} - g''(x_{i+1})\frac{x_{i+1}-x_i}{6}\right)(x-x_i)$$

$$+\left(\frac{y_i}{x_{i+1}-x_i} - g''(x_i)\frac{x_{i+1}-x_i}{6}\right)(x_{i+1}-x)$$

Algorithm for Cubic Splines

$$g_i(x) = A_i(x)y_i + B_i(x)y_{i+1} + C_i y_i'' + D_i y_{i+1}''$$

$$A_i(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} \qquad B_i(x) = \frac{x_i - x}{x_i - x_{i+1}}$$

$$C_i(x) = \frac{1}{6}\frac{\left(x - x_{i+1}\right)^3}{\left(x_i - x_{i+1}\right)} - \frac{1}{6}\left(x_i - x_{i+1}\right)\left(x - x_{i+1}\right) = \frac{A_i^3 - A_i}{6}\left(x_i - x_{i+1}\right)^2$$

$$D_i(x) = \frac{1}{6}\frac{\left(x_i - x\right)^3}{\left(x_i - x_{i+1}\right)} - \frac{1}{6}\left(x_i - x_{i+1}\right)\left(x_i - x\right) = \frac{B_i^3 - B_i}{6}\left(x_i - x_{i+1}\right)^2$$

Algorithm for Cubic Splines

What is left to be done is finding the second derivatives

Algorithm for Cubic Splines

One condition that we have not used yet is the continuity of the 1st derivative of g(x).

At the interior knots x_i, we must have

$$g_{i-1}'(x_i) = g_i'(x_i) \quad \text{for } i = 1,\ldots,n-1$$

From our equation for g(x) (previous slide), we find:

$$g_i'(x) = \frac{y_{i+1}''}{2(x_{i+1}-x_i)}(x-x_i)^2 - \frac{y_i''}{2(x_{i+1}-x_i)}(x_{i+1}-x)^2$$

$$+\frac{y_{i+1}}{(x_{i+1}-x_i)} - \frac{y_i}{(x_{i+1}-x_i)} - \frac{(x_{i+1}-x_i)}{6}y_{i+1}'' + \frac{(x_{i+1}-x_i)}{6}y_i''$$

Algorithm for Cubic Splines

$$g_i'(x) = \frac{y_{i+1}''}{2(x_{i+1} - x_i)}(x - x_i)^2 - \frac{y_i''}{2(x_{i+1} - x_i)}(x_{i+1} - x)^2$$

$$+ \frac{y_{i+1}}{(x_{i+1} - x_i)} - \frac{y_i}{(x_{i+1} - x_i)} - \frac{(x_{i+1} - x_i)}{6} y_{i+1}'' + \frac{(x_{i+1} - x_i)}{6} y_i''$$

At $x = x_i$:

$$g_i'(x_i) = -\frac{y_i''}{2(x_{i+1} - x_i)}(x_{i+1} - x_i)^2 \quad + \frac{1}{(x_{i+1} - x_i)}(y_{i+1} - y_i)$$

$$-\frac{(x_{i+1} - x_i)}{6} y_{i+1}'' \quad + \frac{(x_{i+1} - x_i)}{6} y_i''$$

$$g_i'(x_i) = -\frac{(x_{i+1} - x_i)}{6} y_{i+1}'' - \frac{y_i''}{3}(x_{i+1} - x_i) + \frac{1}{(x_{i+1} - x_i)}(y_{i+1} - y_i)$$

Algorithm for Cubic Splines

$$g_i'(x_i) = -\frac{(x_{i+1} - x_i)}{6} y_{i+1}'' - \frac{y_i''}{3}(x_{i+1} - x_i) + \frac{1}{(x_{i+1} - x_i)}(y_{i+1} - y_i)$$

$$g_i'(x_i) = -\frac{\Delta x_i}{6} y_{i+1}'' - \frac{\Delta x_i}{3} y_i'' + \frac{1}{\Delta x_i}\Delta y_i$$

With $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_i = y_{i+1} - y_i$

Similar for g'$_{i-1}$(x$_i$)

$$g_{i-1}'(x_i) = \frac{\Delta x_{i-1}}{6} y_{i-1}'' + \frac{\Delta x_{i-1}}{3} y_i'' + \frac{1}{\Delta x_{i-1}}\Delta y_{i-1}$$

Algorithm for Cubic Splines

The continuity of g'(x) requires

$$g_{i-1}'(x_i) = g_i'(x_i) \quad \text{for i = 1,....n - 1}$$

$$-\frac{\Delta x_i}{6} y_{i+1}'' - \frac{\Delta x_i}{3} y_i'' + \frac{1}{\Delta x_i} \Delta y_i = \frac{\Delta x_{i-1}}{6} y_{i-1}'' + \frac{\Delta x_{i-1}}{3} y_i'' + \frac{1}{\Delta x_{i-1}} \Delta y_{i-1}$$

$$-\frac{\Delta x_{i-1}}{6} y_{i-1}'' - \frac{\Delta x_i}{3} y_i'' - \frac{\Delta x_{i-1}}{3} y_i'' - \frac{\Delta x_i}{6} y_{i+1}'' = -\frac{1}{\Delta x_i} \Delta y_i + \frac{1}{\Delta x_{i-1}} \Delta y_{i-1}$$

$$\Delta x_{i-1} y_{i-1}'' + 2\Delta x_i y_i'' + 2\Delta x_{i-1} y_i'' + \Delta x_i y_{i+1}'' = 6\left(\frac{1}{\Delta x_i} \Delta y_i - \frac{1}{\Delta x_{i-1}} \Delta y_{i-1}\right)$$

$$\Delta x_{i-1} y_{i-1}'' + 2\left(\Delta x_i + \Delta x_{i-1}\right) y_i'' + \Delta x_i y_{i+1}'' = 6\left(\frac{\Delta y_i}{\Delta x_i} - \frac{\Delta y_{i-1}}{\Delta x_{i-1}}\right)$$

Algorithm for Cubic Splines

$$\Delta x_{i-1} y_{i-1}'' + 2\left(\Delta x_i + \Delta x_{i-1}\right) y_i'' + \Delta x_i y_{i+1}'' = 6\left(\frac{\Delta y_i}{\Delta x_i} - \frac{\Delta y_{i-1}}{\Delta x_{i-1}}\right)$$

Let's organize the y_i'' in a vector and the coefficients in a matrix

$$\begin{pmatrix} ? & ? & & & & & & & & \\ a_1 & b_1 & c_1 & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & a_{i-1} & b_{i-1} & c_{i-1} & & & & & \\ & & & a_i & b_i & c_i & & & & \\ & & & & a_{i+1} & b_{i+1} & c_{i+1} & & & \\ & & & & & \cdots & \cdots & \cdots & & \\ & & & & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & & & & ? & ? \end{pmatrix} \bullet \begin{pmatrix} y_0'' \\ y_1'' \\ \cdots \\ y_{i-1}'' \\ y_i'' \\ y_{i+1}'' \\ \cdots \\ y_{n-1}'' \\ y_n'' \end{pmatrix} = \begin{pmatrix} ? \\ r_1 \\ \cdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \cdots \\ r_{n-1} \\ ? \end{pmatrix}$$

Algorithm for Cubic Splines

$$\Delta x_{i-1} y_{i-1}'' + 2\left(\Delta x_i + \Delta x_{i-1}\right) y_i'' + \Delta x_i y_{i+1}'' = 6\left(\frac{\Delta y_i}{\Delta x_i} - \frac{\Delta y_{i-1}}{\Delta x_{i-1}}\right)$$

$$\begin{pmatrix} ? & ? & & & & & & & \\ a_1 & b_1 & c_1 & & & & & & \\ & \cdots & \cdots & \cdots & & & & & \\ & & a_{i-1} & b_{i-1} & c_{i-1} & & & & \\ & & & a_i & b_i & c_i & & & \\ & & & & a_{i+1} & b_{i+1} & c_{i+1} & & \\ & & & & & \cdots & \cdots & \cdots & \\ & & & & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & & & & ? & ? \end{pmatrix} \bullet \begin{pmatrix} y_0'' \\ y_1'' \\ \cdots \\ y_{i-1}'' \\ y_i'' \\ y_{i+1}'' \\ \cdots \\ y_{n-1}'' \\ y_n'' \end{pmatrix} = \begin{pmatrix} ? \\ r_1 \\ \cdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \cdots \\ r_{n-1} \\ ? \end{pmatrix}$$

$$a_i = x_i - x_{i-1} \quad (i = 1, \ldots, n-1) \qquad b_i = 2(x_{i+1} - x_{i-1}) \quad (i = 1, \ldots, n-1)$$

$$c_i = x_{i+1} - x_i \quad (i = 1, \ldots, n-1) \qquad r_i = 6\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right) \quad (i = 1, \ldots, n-1)$$

Boundary Condition for Natural Splines

Boundary conditions for a natural spline: $y_0'' = y_n'' = 0$

$$\begin{pmatrix} b_1 & c_1 & & & & & \\ \cdots & \cdots & \cdots & & & & \\ & a_{i-1} & b_{i-1} & c_{i-1} & & & \\ & & a_i & b_i & c_i & & \\ & & & a_{i+1} & b_{i+1} & c_{i+1} & \\ & & & & \cdots & \cdots & \cdots \\ & & & & & a_{n-1} & b_{n-1} \end{pmatrix} \bullet \begin{pmatrix} y_1'' \\ \cdots \\ y_{i-1}'' \\ y_i'' \\ y_{i+1}'' \\ \cdots \\ y_{n-1}'' \end{pmatrix} = \begin{pmatrix} r_1 \\ \cdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \cdots \\ r_{n-1} \end{pmatrix}$$

$$a_i = x_i - x_{i-1} \quad (i = 2,\ldots,n-1) \qquad b_i = 2(x_{i+1} - x_{i-1}) \quad (i = 1,\ldots,n-1)$$

$$c_i = x_{i+1} - x_i \quad (i = 1,\ldots,n-2) \qquad r_i = 6\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right) \quad (i = 1,\ldots,n-1)$$

Boundary Condition for Clamped Splines

Boundary conditions for a clamped spline: y_0' and y_n' given

We had found before:

$$g_i'(x_i) = -\frac{\Delta x_i}{6} y_{i+1}'' - \frac{\Delta x_i}{3} y_i'' + \frac{1}{\Delta x_i} \Delta y_i$$

At $x_i = x_0$

$$y_0' = -\frac{\Delta x_0}{6} y_1'' - \frac{\Delta x_0}{3} y_0'' + \frac{1}{\Delta x_0} \Delta y_0$$

$$\Rightarrow \quad 2\Delta x_0 y_0'' + \Delta x_0 y_1'' = 6\left(\frac{\Delta y_0}{\Delta x_0} - y_0'\right) \quad \Rightarrow \quad b_0 y_0'' + c_0 y_1'' = r_0$$

Similar at $x_i = x_n$

$$\Rightarrow \quad \Delta x_{n-1} y_{n-1}'' + 2\Delta x_n y_n'' = 6\left(\frac{\Delta y_n}{\Delta x_n} - y_n'\right) \quad \Rightarrow \quad a_n y_{n-1}'' + b_n y_n'' = r_n$$

Boundary Condition for Clamped Splines

$$\begin{pmatrix} b_0 & c_0 & & & & & \\ \dots & \dots & \dots & & & & \\ & a_{i-1} & b_{i-1} & c_{i-1} & & & \\ & & a_i & b_i & c_i & & \\ & & & a_{i+1} & b_{i+1} & c_{i+1} & \\ & & & & \dots & \dots & \dots \\ & & & & & a_n & b_n \end{pmatrix} \bullet \begin{pmatrix} y_0'' \\ \dots \\ y_{i-1}'' \\ y_i'' \\ y_{i+1}'' \\ \dots \\ y_n'' \end{pmatrix} = \begin{pmatrix} r_0 \\ \dots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \dots \\ r_n \end{pmatrix}$$

$$b_0 = 2(x_1 - x_0) \qquad c_0 = x_1 - x_0 \qquad r_0 = 6\left(\frac{y_1 - y_0}{x_1 - x_0} - y_0'\right)$$

$$a_n = x_n - x_{n-1} \qquad b_n = 2(x_n - x_{n-1}) \qquad r_n = 6\left(y_n' - \frac{y_n - y_{n-1}}{x_n - x_{n-1}}\right)$$

Natural Spline:

$$\begin{pmatrix} b_1 & c_1 & & & & & & \\ \cdots & \cdots & \cdots & & & & & \\ & a_{i-1} & b_{i-1} & c_{i-1} & & & & \\ & & a_i & b_i & c_i & & & \\ & & & a_{i+1} & b_{i+1} & c_{i+1} & & \\ & & & & \cdots & \cdots & \cdots \\ & & & & & a_{n-1} & b_{n-1} \end{pmatrix} \bullet \begin{pmatrix} y_1'' \\ \cdots \\ y_{i-1}'' \\ y_i'' \\ y_{i+1}'' \\ \cdots \\ y_{n-1}'' \end{pmatrix} = \begin{pmatrix} r_1 \\ \cdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \cdots \\ r_{n-1} \end{pmatrix}$$

Clamped Spline:

$$\begin{pmatrix} b_0 & c_0 & & & & & & \\ \cdots & \cdots & \cdots & & & & & \\ & a_{i-1} & b_{i-1} & c_{i-1} & & & & \\ & & a_i & b_i & c_i & & & \\ & & & a_{i+1} & b_{i+1} & c_{i+1} & & \\ & & & & \cdots & \cdots & \cdots \\ & & & & & a_n & b_n \end{pmatrix} \bullet \begin{pmatrix} y_0'' \\ \cdots \\ y_{i-1}'' \\ y_i'' \\ y_{i+1}'' \\ \cdots \\ y_n'' \end{pmatrix} = \begin{pmatrix} r_0 \\ \cdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \cdots \\ r_n \end{pmatrix}$$