# STABLE LIMIT CYCLES AS TUNABLE SIGNAL SOURCES

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## ABSTRACT

This paper presents a method for synthesizing audio signals from nonlinear dynamical systems exhibiting stable limit cycles, with control over frequency and amplitude independent of changes to the system's internal parameters. Using the van der Pol oscillator and the Brusselator as case studies, it is demonstrated how parameters are decoupled from frequency and amplitude by rescaling the angular frequency and normalizing amplitude extrema. Practical implementation considerations are discussed, as are the limits and challenges of this approach. The method's validity is confirmed experimentally and synthesis examples show the application of tunable nonlinear oscillator in sound design, including the generation of transients in FM synthesis by means of a van der Pol oscillator and a Supersaw oscillator bank based on the Brusselator.

## 1. INTRODUCTION

Nonlinear dynamical systems are systems whose evolution over time is governed by nonlinear differential equations. Unlike linear systems, nonlinear systems can exhibit periodic, quasiperiodic, and chaotic behaviors, making them attractive for the synthesis of complex waveforms. There is a vast amount of literature surrounding the use of nonlinear systems in sound synthesis (e.g. [1, 2, 3]), often with a particular focus on modeling nonlinear phenomena in musical instruments (e.g. [4, 5, 6]).

Stable limit cycles are one class of periodic solutions that can arise in such systems. Due to their inherent periodicity, limit cycles lend themselves as sources for pitched sound signals. Their appeal lies in their often rich harmonic content, the availability of parameters that alter timbre in complex ways, and the possibility of extending them to quasiperiodic or chaotic regimes using nonautonomous modifications. However, it appears that much of the existing literature treats them as incidental to the system's behavior rather than as a primary signal source for sound synthesis. Furthermore, little attention has been given to the challenge of making such systems tunable: in many nonlinear systems, parameters simultaneously influence the period, amplitude, offset, and harmonic structure of the limit cycle. As a result, frequency cannot be adjusted independently without distorting other aspects of the signal in undesirable ways [3].

This paper presents a conceptually simple method that enables the synthesis of antialiased and amplitude-normalized signals from limit cycles with arbitrarily chosen frequency and amplitude, both of which remain approximately stable even when the system's parameters vary over time. "Approximately" here refers to limitations introduced by numerical integration, convergence time, and approximations of certain properties of the limit cycle. The method is motivated in part by previous projects [7, 8, 9] that use limit cycles in sound synthesis, but lack tuning stability.



Section 2 describes systems that are well-suited to audio signal generation based on limit cycles and explores connections between their mathematical form and their resulting waveforms. Section 3 presents the proposed approach for synthesizing tunable, normalized, amplitude-stable signals from limit cycles, along with discussion of its prerequisites, limitations, and implementation considerations. Section 4 evaluates the proposed method and showcases experimental applications of the method in audio synthesis. Finally, Section 5 summarizes the main findings and outlines potential directions for future work.

## 2. SYSTEMS EXHIBITING LIMIT CYCLES

This section focuses on autonomous systems of the form

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \Phi_1(x, y),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \Phi_2(x, y),\end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ describe the system's state at time $t$, and $\Phi_1$ and $\Phi_2$ are nonlinear functions defining the system's dynamics. The state of the system at time $t$ corresponds to a point $(x(t), y(t))$ in phase space. Its evolution over time traces out a trajectory determined by the vector field defined by (1) and starting with initial condition $(x(0), y(0))$.

A closed orbit is a trajectory that repeats itself at some period $T$, such that $(x(t+T), y(t+T)) = (x(t), y(t))$ for all $t$. Such a closed orbit is called a stable limit cycle if it is isolated. In other words, it attracts nearby trajectories and is therefore the only closed orbit in its neighborhood.

In general, proving the existence of stable limit cycles is a nontrivial task [10]. This paper focuses on systems in which a stable limit cycle is proven to be globally unique by well-known theorems or sufficiently isolated, meaning that no other nearby attractors can be reached through small perturbations. While nonlinear systems can have multiple coexisting stable limit cycles (multistability), such cases are avoided here to sidestep the additional complexity associated with trajectory switching. Although multistable systems may offer interesting behavior for audio synthesis, as will be mentioned in Section 5, they are beyond the scope of this work.

### 2.1. Liénard Systems

Liénard systems are of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + f(x)\frac{\mathrm{d}x}{\mathrm{d}t} + g(x) = 0, \tag{2}$$

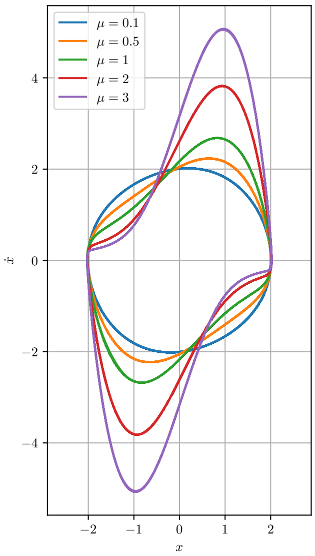


Figure 1: Limit cycles of the van der Pol oscillator for various $\mu$.

where $x$ is the system's displacement at time $t$, $f(x)$ is a damping function controlling how energy is injected into and dissipated from the system, and $g(x)$ is a restoring-force function controlling the system's tendency to return to equilibrium.

Liénard systems are of particular interest here because their waveforms and the harmonic structure of their limit cycles can be qualitatively inferred to some degree by inspecting the shapes of $f(x)$ and $g(x)$. Additionally, under certain conditions, systems of the form of (2) are guaranteed to exhibit a unique, stable limit cycle according to Liénard's theorem [10]. The original version of Liénard's theorem requires that $f(x)$ be even and $g(x)$ be odd, which ensures that the resulting limit cycle is point-symmetric, as can be visually verified by inspecting Figure 1, and that its time-domain waveform consists of only odd harmonics. More recent generalizations, such as those by Villari et al. [11] and Hayashi et al. [12], weaken these symmetry requirements.

#### 2.1.1. Case Study: Van der Pol Oscillator

The van der Pol oscillator, introduced in 1920 by Dutch physicist Balthasar van der Pol to model oscillating currents in triode circuits [13], is a classical example of a nonlinear system exhibiting a stable limit cycle. It fits the form of (2) with

$$\begin{aligned} f(x) &= -\mu\big(1 - x^2\big), \\ g(x) &= x, \end{aligned} \tag{3}$$

where $\mu \geq 0$ controls the system's nonlinear damping. For $\mu = 0$, the system reduces to a simple harmonic oscillator, which has infinitely many periodic solutions but no limit cycle. For $\mu > 0$, the damping function causes the system to inject energy if $|x| < 1$ and to dissipate energy if $|x| > 1$, giving rise to a limit cycle where these effects balance [10].

Since $f(x)$ is even and $g(x)$ is odd, we expect the resulting waveform to be symmetric and contain only odd harmonics. Figure 1 shows limit cycles for various $\mu$. Figure 2 displays the harmonics of a 100 Hz wave generated by the van der Pol oscillator and confirms that only odd harmonics are present.

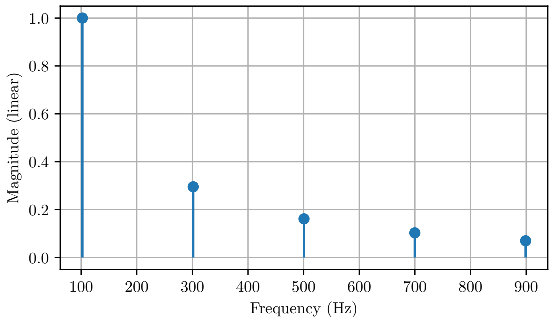


Figure 2: Harmonics of a 100 Hz wave generated by the van der Pol oscillator with $\mu = 5$.

By relaxing the symmetries of $f(x)$ or $g(x)$, we can modify (3) to produce even harmonics in addition to odd ones. Importantly, these modifications must still follow the conditions established in [12], which are omitted here for brevity. One of many possible modifications is

$$\begin{aligned} f(x) &= \mu\big(1 + x - x^2\big), \\ g(x) &= e^x - 1. \end{aligned} \tag{4}$$

Figure 3 shows this modified oscillator's harmonics.

### 2.2. Designed Systems

It is possible to design systems that have a unique, stable limit cycle by construction. This can be useful, for example, when allowing the damping parameter $\mu$ of the van der Pol oscillator to be dynamically modulated in a synthesizer application. While modulation introduces expressive possibilities in sound design, it also leads to a subtle but significant issue: For $\mu < 1$, the system converges to the limit cycle increasingly slowly as $\mu \to 0$, and for $\mu = 0$, it reduces to a simple harmonic oscillator, which has no limit cycle. This can cause the waveform's amplitude to temporarily or permanently become much too small or much too large.

One way to mitigate this, while approximately preserving the van der Pol oscillator's behavior at low $\mu$, is to design a system with a circular limit cycle and interpolate it with the original oscillator based on the value of $\mu$. A circular system can be defined in polar coordinates as

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= R - r, \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \omega, \end{aligned} \tag{5}$$

where $R$ is the limit cycle's radius, and $\omega$ is the angular frequency.

Transforming (5) to Cartesian coordinates yields

$$\begin{aligned} \frac{\mathrm{d}x_\mathrm{r}}{\mathrm{d}t} &= x_\mathrm{r}\left(\frac{R}{\sqrt{x_\mathrm{r}^2 + y_\mathrm{r}^2}} - 1\right) - \omega y_\mathrm{r}, \\ \frac{\mathrm{d}y_\mathrm{r}}{\mathrm{d}t} &= y_\mathrm{r}\left(\frac{R}{\sqrt{x_\mathrm{r}^2 + y_\mathrm{r}^2}} - 1\right) + \omega x_\mathrm{r}. \end{aligned} \tag{6}$$

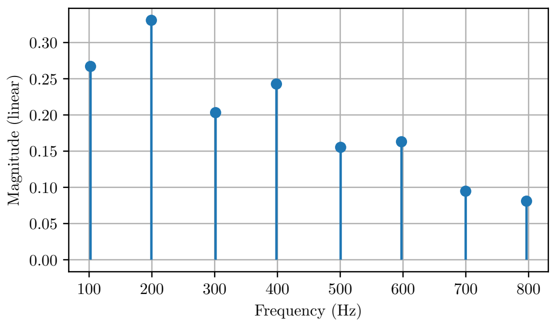


Figure 3: Harmonics of a 100 Hz wave generated by the asymmetrical Liénard system 4 with $\mu = 5$.

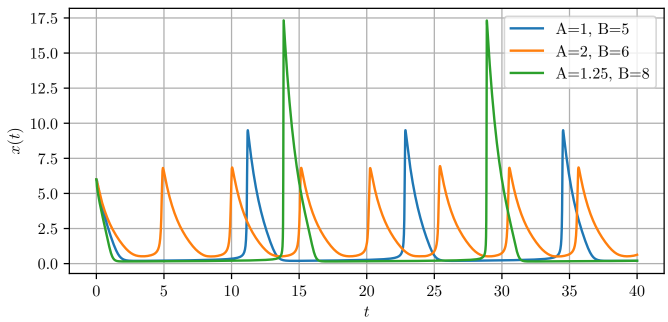


Figure 4: Time series of Brusselator solutions with different parameter choices.

Next, we recall the van der Pol equation from (2) and (3), rewrite it in two-dimensional form, and interpolate it with (6) using an interpolation factor $\tau$ to obtain

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \tau y - (1-\tau)\frac{\mathrm{d}x_{\mathrm{r}}}{\mathrm{d}t}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \tau\big(\mu(1-x^2)y - x\big) - (1-\tau)\frac{\mathrm{d}y_{\mathrm{r}}}{\mathrm{d}t}, \end{aligned} \tag{7}$$

where $\tau = \min(\mu, 1)$. For simplicity, we set $\omega = 1$ and $R = 2$, which is approximately equal to the amplitude of the van der Pol oscillator's limit cycle [14].

This serves as an example of a dynamical system designed to exhibit a unique stable limit cycle with an appropriate shape. More general approaches for constructing systems with arbitrarily shaped limit cycles are described by Pasandi et al. [15]. While beyond the scope of this paper, such methods offer powerful tools for custom oscillator design.

### 2.3. Other Systems

Beyond Liénard-type and explicitly designed systems, there are many other systems known to exhibit stable, unique, or sufficiently isolated limit cycles for certain parameter choices. Examples include the Rayleigh oscillator [16], the Duffing oscillator [17], and the Sel'kov glycolysis model [18].

#### *2.3.1. Case Study: Brusselator*

In this paper, we focus on the Brusselator, a theoretical model of autocatalytic chemical reactions presented by Prigogine and Lefever [19] in 1968. The system is governed by

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= A - (B+1)x + x^2 y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x(B - xy), \end{aligned} \tag{8}$$

where $A$ and $B$ are unitless parameters, originally representing concentrations of two reactants.

The Brusselator exhibits a unique, stable limit cycle if $B > 1 + A^2$ [20]. Unlike the van der Pol oscillator, it is not centered at the origin of the phase plane. In fact, its location is a function of both $A$ and $B$. Furthermore, while the van der Pol oscillator's limit cycle has an amplitude near 2 for all $\mu$ [14], the Brusselator's amplitude varies significantly for different $A$ and $B$.

As can be seen in Figure 4, the Brusselator produces waveforms resembling sawtooth waves to some degree.

## 3. SIGNAL SYNTHESIS

One factor limiting the practical use of limit cycle-based synthesis is that parameter changes typically alter the period and amplitude of the resulting signals in musically often inconvenient ways. In this section, we derive a conceptually simple method for synthesizing audio signals from limit cycles at a desired frequency and amplitude which remain stable under parameter modulation. Further, the method's limitations, and some implementation concerns are discussed.

### 3.1. Derivation

We begin by observing that the phase portrait of a simple harmonic oscillator is a circle whose radius corresponds to the desired signal amplitude. For the purpose of this derivation, we treat limit cycles as distorted and possibly translated perturbations of this circular trajectory. To simplify the analysis, we ignore convergence behavior and assume that the system's state is on the limit cycle and remains there at all times.

#### *3.1.1. Period Compensation*

For a sinusoid with unity amplitude and angular frequency $\omega$, we have

$$x(t) = \sin(\omega t), \tag{9}$$

where the angular frequency is traditionally defined as

$$\omega = \frac{2\pi}{T}. \tag{10}$$

Recognizing that the angular frequency is a ratio of the trajectory's angular period and the period corresponding to the desired frequency $f$, we can define a generalized angular frequency

$$\hat{\omega} = \frac{T_{\text{ang}}(\mathbf{p})}{T}, \tag{11}$$

where $T_{\text{ang}}$ is the trajectory's angular period as a function of parameter vector $\mathbf{p}$. This vector consists of all system parameters that affect the trajectory's geometry. For example, for the van der Pol oscillator, $\mathbf{p}_{\text{VDP}} = (\mu)$, and for the Brusselator $\mathbf{p}_{\text{B}} = (A, B)$.

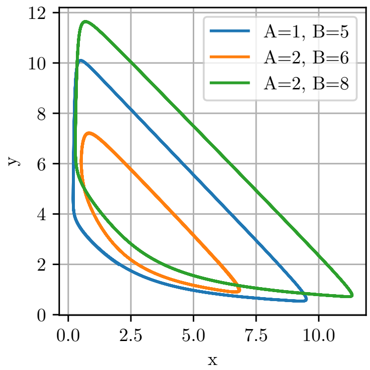


Figure 5: Limit cycles of the Brusselator for different parameters.

Since nonlinear systems are not generally solvable in closed form, we cannot expect a symbolic expression for $T_{\text{ang}}$. Instead, we will later discuss in Section 3.2.1 how to obtain approximations for $T_{\text{ang}}$ numerically by estimating it from the system's behavior.

#### 3.1.2. *Amplitude Normalization*

Unlike the simple harmonic oscillator, whose amplitude we assume to be normalized to 1, the amplitude of a limit cycle can vary significantly with changes in system parameters. For instance, as illustrated in Figure 5, the Brusselator's limit cycle not only shifts in location, which introduces a DC offset, but also changes in amplitude, which can result in clipping and unwanted amplitude fluctuations when parameters are modulated.

Assuming we know certain properties about the limit cycle, normalization terms are easily devised. The quantities we are interested in are the limit cycle's extrema

$$\begin{aligned}\gamma_{\text{min}} &= \min_{t\in[0,T_{\text{ang}}(\mathbf{p}))} \gamma(t),\\ \gamma_{\text{max}} &= \max_{t\in[0,T_{\text{ang}}(\mathbf{p}))} \gamma(t),\end{aligned} \tag{12}$$

where $\gamma$ is the projection of the system's limit cycle onto the output dimension, typically $x(t)$ or $y(t)$ in the two-dimensional case. Based on these extrema, the waveform is centered and scaled such that its amplitude extrema correspond to -1 and 1.

As with the angular period $T_{\text{ang}}$, these extrema are generally not available in closed form and must be estimated numerically, which will be discussed later in Section 3.2.1.

#### 3.1.3. *Equation for Tunable and Stable Signal Synthesis*

By devising normalization terms based on (12), using the generalized angular frequency (11), and using the projected limit cycle $\gamma$ as the source of the waveform, we arrive at the following expression for generating tunable, frequency- and amplitude-stable audio signals from nonlinear oscillators for arbitrary parameters $\mathbf{p}$:

$$x_{\text{tuned}}(t) = \underbrace{\frac{1}{\gamma_{\text{min}} - \gamma_{\text{max}}}}_{\substack{\text{amplitude}\\ \text{normalization}}} \big(\underbrace{\gamma_{\text{max}} + \gamma_{\text{min}} - 2}_{\text{waveform centering}}\, \underbrace{\gamma(\hat{\omega}t)}_{\text{oscillator}}\big). \tag{13}$$

### 3.2. Implementation Considerations

Equation (13) is idealized in the sense that it presumes the availability of the limit cycle, its angular period $T_{\text{ang}}$, and the extrema $\gamma_{\text{min}}$, $\gamma_{\text{max}}$. While these quantities are not analytically available for most nonlinear systems, they can be approximated in various ways. The limit cycle itself can be approximated by numerically integrating the system's differential equations. To reduce convergence time and avoid transients, initial conditions $(x(0), y(0))$ should be chosen close to the known limit cycle.

#### 3.2.1. *Offline Step Counting*

For certain well-studied systems, including the van der Pol oscillator and the Brusselator, analytical approximations for period [21, 22, 20] and amplitude [14] exist. However, these approximations are often insufficiently accurate. Instead, an offline approximation step can be applied, in which the system is numerically simulated for a grid of parameter vectors $\mathbf{p}$. For each configuration, the number of steps required to complete one full period is counted. Dividing this step count by the numerical step size yields an estimate of the angular period $T_{\text{ang}}(\mathbf{p})$. Similarly, $\gamma_{\text{min}}$ and $\gamma_{\text{max}}$ can be tracked simultaneously. These values are then stored in a lookup table.

An obvious drawback of this method is that it does not scale well with the number of parameters, as each new parameter introduces an additional dimension to the lookup space. This can be somewhat mitigated by sampling parameters that the limit cycle's geometry is less sensitive to at a lower resolution.

An interesting empirical observation emerged while constructing the lookup tables for the angular period: using the same numerical method for both step-counting and real-time synthesis led to the lowest tuning error, even though the use of highly accurate solvers and root-finding algorithms would be preferable in principle. It is not entirely clear why this phenomenon occurs but it is suspected that some inaccuracies inherent to the chosen numerical method might partially cancel out when used consistently in both the step counting and synthesis stages. However, when using the same method, decreasing the time step size during step counting does not appear to worsen tuning error.

Unlike the approximation of $T_{\text{ang}}$ by step counting, determining $\gamma_{\text{min}}$ and $\gamma_{\text{max}}$ benefits from the use of more accurate methods. If an extremum lies between discrete samples, coarse sampling may underestimate the maximum amplitude and result in audible clipping. Alternatively, substepping around extrema or using a "fudge factor" in the synthesizer can also mitigate this.

#### 3.2.2. *Sampling*

To realize the generalized angular frequency (11) in the solver, the time step size $h$ must be adapted to the desired tuning frequency $f$, sampling rate $f_{\text{s}}$, and angular period $T_{\text{ang}}$. Moreover, in highly nonlinear regimes, solvers may become numerically unstable, especially for high $f$. Conversely, at low $f$, aliasing artifacts may become audible. Both issues can be addressed by oversampling by a factor $K$, giving

$$h = \frac{f T_{\text{ang}}(\mathbf{p})}{K f_{\text{s}}}. \tag{14}$$

This produces $K$ intermediate samples per output sample, which are filtered using a $K^{\text{th}}$-band lowpass filter and downsampled by outputting every $K^{\text{th}}$ intermediate sample. The appropriate oversampling factor $K$ depends on the system, its parameters, the tar-

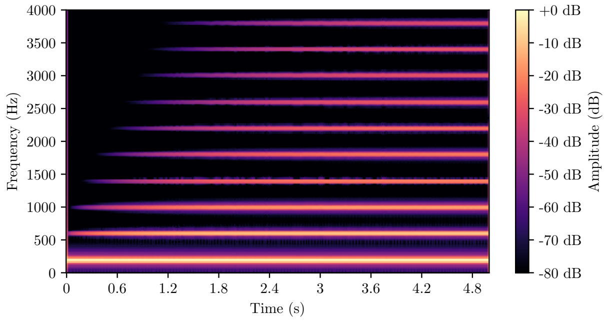


Figure 6: Spectrogram of the van der Pol oscillator with $f = 200\,\mathrm{Hz}$ and $\mu$ changing from 0 to 8.

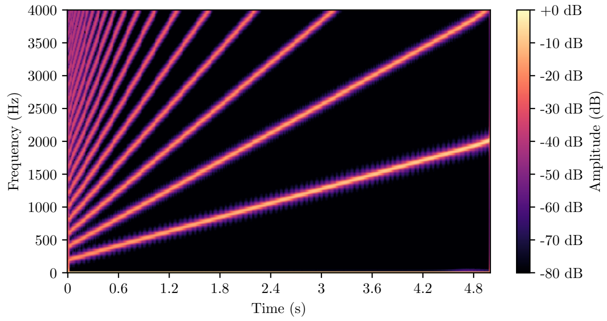


Figure 7: Spectrogram of the Brusselator with $A$ changing from 1 to 2.2, $B$ changing from 8 to 6, and $f$ changing from $200\,\mathrm{Hz}$ to $2000\,\mathrm{Hz}$.

get frequency range, and the level of tolerable aliasing, and should be determined empirically.

## 4. EVALUATION AND EXPERIMENTS

This section demonstrates the effectiveness of the method proposed and derived in Section 3 by showcasing its tuning stability and amplitude normalization under dynamic parameter changes. Additionally, two examples are presented illustrating the van der Pol oscillator and the Brusselator in sound design contexts.

### 4.1. Tuning Stability

Tuning stability was evaluated by synthesizing a continuous five-second tone using a van der Pol oscillator tuned to $200\,\mathrm{Hz}$. Over this duration, its tuning was not changed, while the damping parameter $\mu$ was linearly swept from 0 to 8. Figure 6 depicts the resulting spectrogram and shows no significant pitch instability.

### 4.2. Amplitude Normalization

As discussed in Section 2.3.1 and shown in Figure 5, the location and amplitude of the Brusselator's limit cycle vary with parameters $A$ and $B$. To test amplitude normalization, an audio signal was synthesized while modulating $A$ from 1 to 2.2 and $B$ from 8 to 6. Additionally, the oscillator's base frequency was swept from $200\,\mathrm{Hz}$ to $2000\,\mathrm{Hz}$. The spectrogram in Figure 7 shows consistent amplitude over the course of the modulation, and frequency linearly increasing without deviation. A minor DC component is present, as the signal is normalized rather than DC-debiased, which would ensure a mean amplitude of 0.

### 4.3. Van der Pol Frequency Modulation

Frequency modulation (FM) synthesis is traditionally performed using sinusoidal oscillators [23] due to their harmonic simplicity. However, the van der Pol oscillator approximates a simple harmonic oscillator at low values of $\mu$, especially in the modified variant introduced in Section 2.2, which retains a circular limit cycle even at $\mu = 0$. This makes it a promising candidate for FM synthesis, with the added benefit of enriching the spectrum controllably through the parameter $\mu$.

An FM synthesizer was constructed with two carriers, each driven by a separate modulator, all of which were sinusoidal. Parameter values were chosen to produce bell-like tones. In the first experiment, the carriers were replaced with van der Pol oscillators whose $\mu$ values were modulated by a decaying envelope, starting at $\mu = 2$ and decreasing to zero within $200\,\mathrm{ms}$. The excitation envelope was also shortened, and the base frequency slightly lowered. The resulting sounds had more energy in the transients, making them percussive and metallic. These transients emerged naturally from the nonlinear response of the oscillator depending on $\mu$.

In a second experiment, the modulators were also replaced by van der Pol oscillators with $\mu = 9$. This configuration produced distorted, "thumpy" metal impact sounds with a rougher spectral texture, showcasing the oscillator's potential for nonlinear modulation effects beyond purely sinusoidally-based FM synthesis.

### 4.4. Brusselator Supersaw

The Supersaw waveform, famously introduced by Roland's JP-8000 and JP-8080 synthesizers, consists of several detuned sawtooth oscillators playing at once. The Brusselator's limit cycle also exhibits a waveform resembling a sawtooth, as discussed in 2.3.1, suggesting its potential use in a Supersaw-style configuration.

To evaluate this potential, a synthesis setup modeled after Szabo's analysis of the original Supersaw [24] was implemented, consisting of seven detuned sawtooth oscillators. A second version of the patch replaced the sawtooth oscillators with Brusselators.

At parameter settings near $A = 2.2$, $B = 8$, which are the upper bounds supported by the implementation, the resulting sound closely matched that of the traditional Supersaw. As $B$ was reduced from 8 to 6, the overall timbre became softer, similar to the effect of a gradually lowering low-pass filter.

## 5. CONCLUSION

This work investigated the synthesis of audio signals from nonlinear dynamical systems exhibiting stable limit cycles. Two representative systems, the van der Pol oscillator and the Brusselator, were studied as case examples. A method was proposed that enables tuning the oscillator frequency and amplitude and keeping them stable independently of changes to the underlying system's parameters. This approach was derived from the geometry of the limit cycle and discussed with respect to its practical implementation, including normalization, lookup table generation, and oversampling. Experimental results confirmed the method's stability and effectiveness.

Synthesis examples demonstrated that tunable nonlinear oscillators lend themselves well to creative sound design. In an FM synthesis context, the van der Pol oscillator served as a rich transient generator for percussive and impact-like sounds. The Brusselator, whose waveform resembles a sawtooth for certain parameter ranges, was used in a Supersaw-style oscillator bank. For specific configurations, it was found to be similar to a standard sawtooth-based Supersaw, whereas certain choices of parameter values exhibited lowpass characteristics.

Several directions remain open for future exploration. One particularly interesting area is the use of multistable systems for polyphonic limit cycle oscillators in which each voice could correspond to another limit cycle. Additionally, it remains to be studied whether the the presented method changes the behavior of nonautonomous modifications of oscillators, such as external driving forces, feedback delay, or oscillator coupling.

The implementation, audio examples, and supplementary materials, including Pure Data objects, example patches, and Python scripts used in evaluation, are available at `https://wolframw.github.io/stable-limit-cycles/`.

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