Data modeling: CSCI E-106

Inferences in Regression and Correlation Analysis

Summary

The linear regression function with one predictor variable:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \quad i = 1, \ldots, n$$

- Y_i: the value of response variable in the ith trial
- β_0, β_1: parameters
- X_i: a known constant; the value of the predictor variable in the ith trial
- ε_i: random error term; $E\{\varepsilon_i\} = 0$; $\sigma^2(\varepsilon_i) = \sigma^2$; uncorrelated ($\sigma\{\varepsilon_i, \varepsilon_j\} = 0$, $i \neq j$)

How to obtain the estimators b_0, b_1 ?

$$Q = \sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_i)^2 \quad \Rightarrow \quad b_1 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$$

$$b_0 = \frac{1}{n}\left(\sum Y_i - b_1 \sum X_i\right) = \bar{Y} - b_1 \bar{X}$$

1. The sum of the residuals is zero:

$$\sum e_i = 0$$

2. The sum of observed values of Y_i equals the sum of the fitted values : $\hat{Y}_i$

$$\sum Y_i = \sum \hat{Y}_i$$

3. The sum of the weighted residuals is zero when the residual in the ith trial is weighted by the level of the predictor variable in the ith trail.

$$\sum X_i e_i = 0$$

And consequently

$$\sum \hat{Y}_i e_i = 0$$

4. The regression line always goes through the point $(\bar{X}, \bar{Y})$

Example: Toluca Data

```
> toluca.reg<-lm(formula = workhrs ~ lotsize,data=toluca_data)
> summary(toluca.reg)

Call:
lm(formula = workhrs ~ lotsize, data = toluca_data)

Residuals:
    Min      1Q  Median      3Q     Max 
-83.876 -34.088  -5.982  38.826 103.528 

Coefficients:
            Estimate Std. Error t value       Pr(>|t|)    
(Intercept)   62.366     26.177   2.382         0.0259 *  
lotsize        3.570      0.347  10.290 0.000000000445 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 48.82 on 23 degrees of freedom
Multiple R-squared:  0.8215,	Adjusted R-squared:  0.8138 
F-statistic: 105.9 on 1 and 23 DF,  p-value: 0.000000000004449
```

Regression Model Coefficients
$Y_i=62.366+\beta_1 X_i$

Standard Errors
$S(\beta_0)=26.177$ and $S(\beta_1)=0.347$

T Test Values
$t=\frac{\beta_0}{S(\beta_0)}$ and $t=\frac{\beta_1}{S(\beta_1)}$

P values for T Test

Use 0.05 significance (or 95% confidence level) for T test

Estimate of σ

n-p=25-2=23

R Square: %of Y explained by the model

F test for the model(is the model significant)

R Code Syntax

Simple Linear Regression: In R, the primary tool for linear regression is the lm() function, which stands for **linear model**.

The general structure uses a **formula** lm(y ~ x) where y is your dependent variable and x is your independent variable

 # Simple Linear Regression (one predictor)

```
model <- lm(y ~ x, data = my_data)
```

 # Regression model without any variable (just intercept)

```
model <- lm(y ~ -1, data = my_data)
```

 # Multiple Linear Regression (multiple predictors)

```
model <- lm(y ~ x1 + x2 + x3, data = my_data)
```

 # Use a dot to include all other columns as predictors

```
model <- lm(y ~ ., data = my_data)
```

R Code Syntax, *cont'd*

 # model coefficients (one predictor)

coef(model)

 # model detailed information including statistical tests and parameters

summary(model)

 # Diagnostic Plots

par(mfrow=c(2,3)) # sets up a 2-by-3 grid for plotting

- par() changes global graphical parameters in R.
- mfrow = c(2, 3) tells R to divide the plotting window into 2 rows and 3 columns.
- After running this, the next six plots you create will fill the grid row by row

plot(model)

 # All information related to regression model

names(model)   # shows all components stored inside the model object

R Code Syntax, *cont'd*

 # All information related to regression model

names(model)   # shows all components stored inside the model object

You'll usually see something like:

```
> names(model)
 [1] "coefficients"  "residuals"     "effects"       "rank"          "fitted.values" "assign"        "qr"            "df.residual"   "xlevels"
[10] "call"          "terms"         "model"
```

What is most useful for us?

name	R code	Explanation
coefficients	model$coefficients	Shows the model coefficients
residuals	model$residuals	Shows the residuals (errors)
fitted.values	model$fitted.values	Shows the fitted values (predictions)

Exploring Model Results

```
> names(toluca.reg)
 [1] "coefficients"  "residuals"     "effects"       "rank"          "fitted.values" "assign"
 [7] "qr"            "df.residual"   "xlevels"       "call"          "terms"         "model"
```

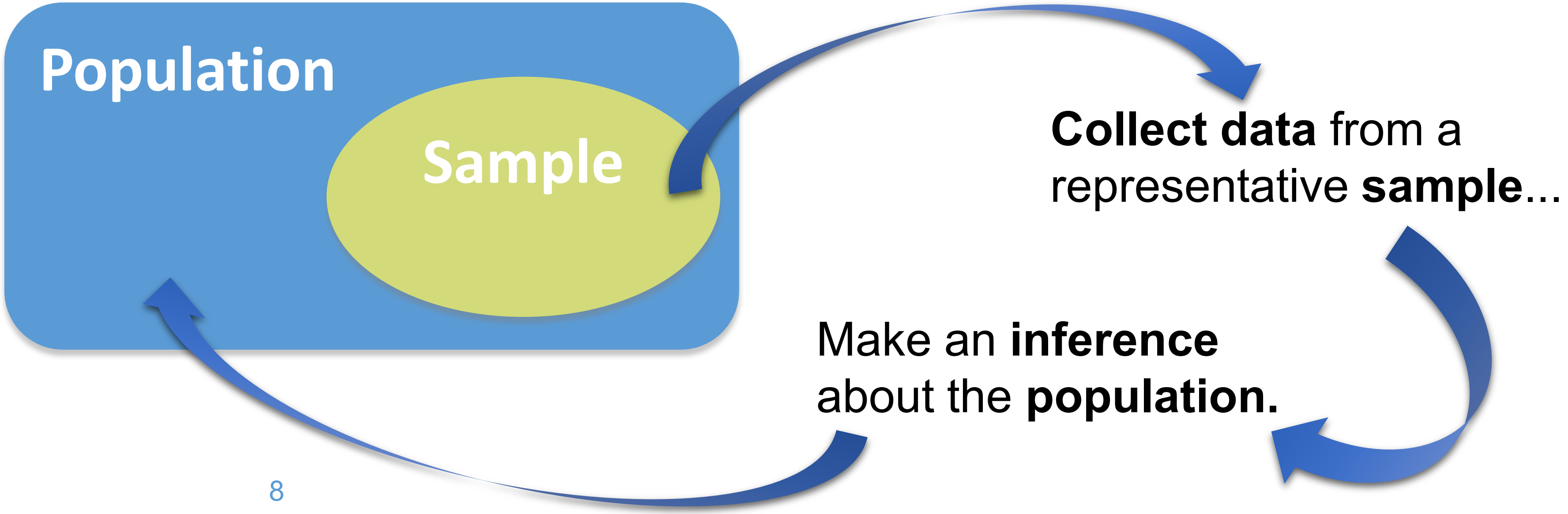
A brief description of these components is provided in the following list:

- coefficients: A named vector of coefficients
- residuals: The residuals, that is, response minus fitted values
- effects: Returns (orthogonal) effects from a fitted model, usually a linear model
- rank: The numeric rank of the fitted linear model
- fitted.values: The fitted mean values
- assing: An integer vector with an entry for each column in the matrix, giving the term in the formula that gave rise to the column
- qr: Computes the QR decomposition of a matrix
- df.residual: The residual degrees of freedom
- xlevels: A record of the levels of the factors used in fitting
- call: Represents the call used to build the model
- terms: The terms object used
- model: The model frame used

Statistical Inference

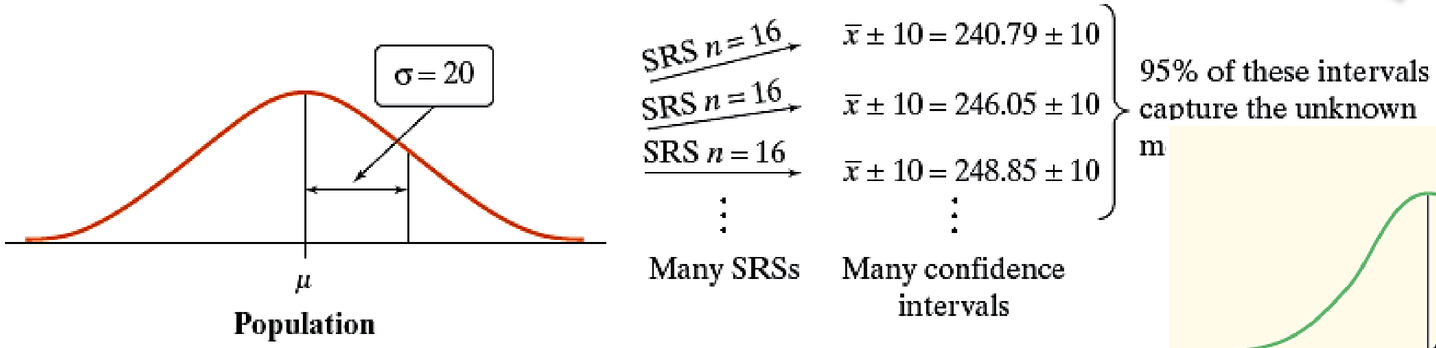
After we have selected a sample, we know the responses of the individuals in the sample. However, the reason for taking the sample is to infer from that data some conclusion about the wider population represented by the sample.

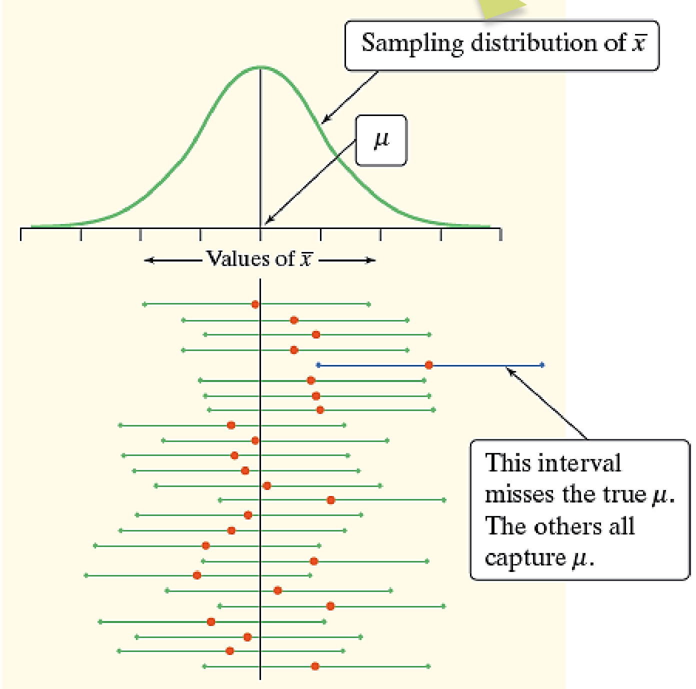
Statistical inference provides methods for drawing conclusions about a population from sample data.



Confidence Level

The confidence level, or *C*, is the overall capture rate if the method is used many times. The sample mean will vary from sample to sample, but when we use the method "*estimate ± margin of error*" to get each interval, $C \times 100\%$ of all intervals capture the unknown population mean μ. (**Note:** By convention, *C* will be expressed as a decimal, so it will always be between 0 and 1.)





Interpreting a Confidence Level

To say that we are 95% *confident* is shorthand for "95% of all possible samples of a given size from this population will yield intervals that capture the unknown parameter."

Confidence Interval

The Big Idea: The sampling distribution of $\bar{x}$ tells us how close to μ the sample mean $\bar{x}$ is likely to be. All confidence intervals we construct will have a form similar to this:

$$\textbf{estimate} \pm \textbf{margin of error}$$

A **level C confidence interval** for a parameter has two parts:

- An **interval** calculated from the data, which has the form:

  estimate ± margin of error

- A **confidence level *C*,** where *C* is the probability that the interval will capture the true parameter value in repeated samples. In other words, the confidence level is the success rate for the method.

We usually choose a confidence level of 90% or higher because we want to be quite sure of our conclusions. The most common confidence level is 95%.

The Margin of Error

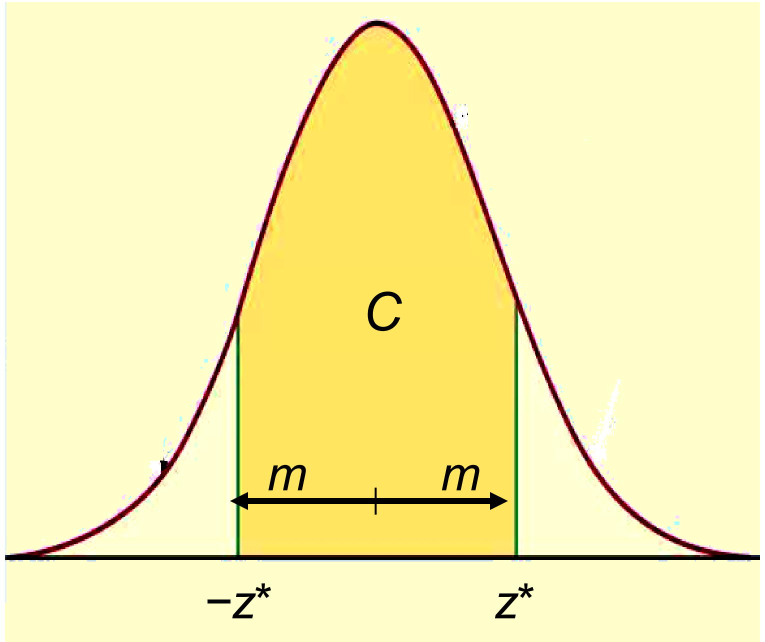
The confidence level *C* determines the value of *z** (in Table D).

The margin of error also depends on *z**.

$$m = z^* \sigma/\sqrt{n}$$

Higher confidence ***C*** implies a larger margin of error ***m*** (thus less precision in our estimates).

A lower confidence level ***C*** produces a smaller margin of error ***m*** (thus better precision in our estimates).



Tests for a Population Mean

z TEST FOR A POPULATION MEAN

Draw an SRS of size n from a Normal population that has unknown mean μ and known standard deviation σ. To test the null hypothesis that μ has a specified value,

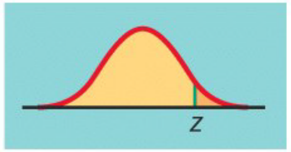
$$H_0\colon \mu = \mu_0$$

calculate the **one-sample z statistic**

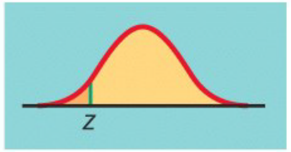
$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

In terms of a variable Z having the standard Normal distribution, the P-value for a test of H_0 against

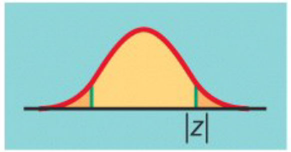
$H_a\colon \mu > \mu_0$ is $P(Z \geq z)$



$H_a\colon \mu < \mu_0$ is $P(Z \leq z)$



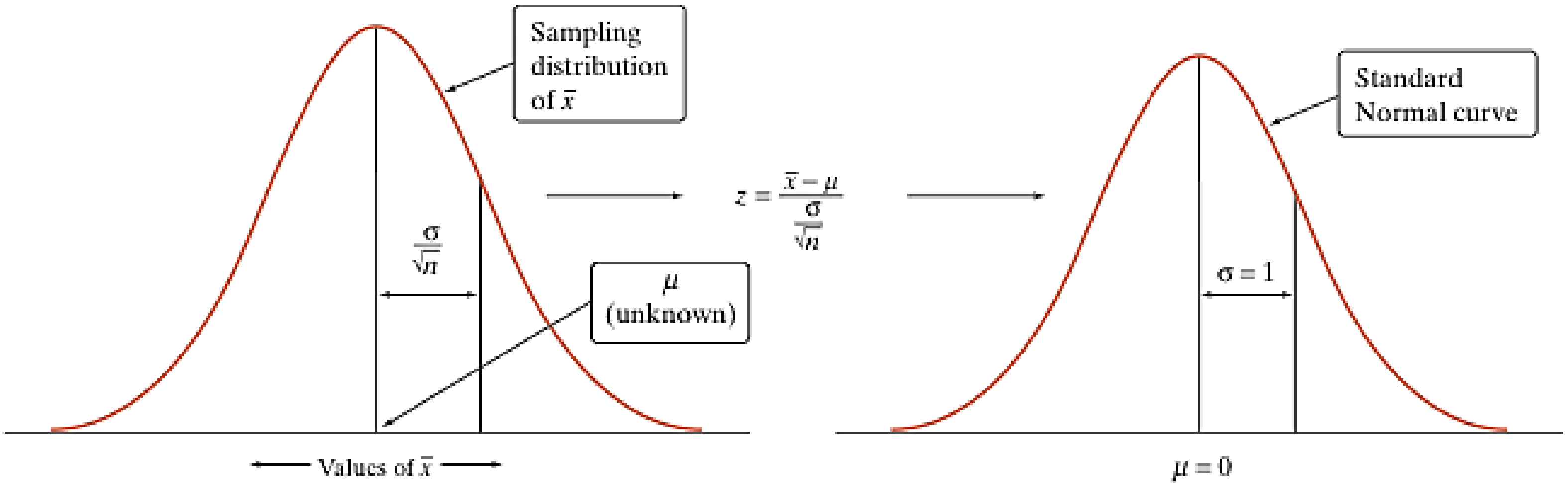
$H_a\colon \mu \neq \mu_0$ is $2P(Z \geq |z|)$



The t Distributions

When the sampling distribution of $\bar{x}$ is close to Normal, we can find probabilities involving $\bar{x}$ by standardizing:

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$



When we don't know σ, we can estimate it using the sample standard deviation s_x. What happens when we standardize?

$$?? = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

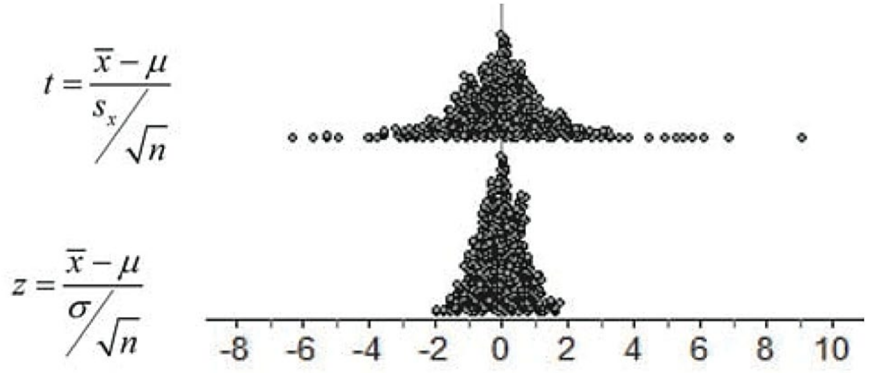
This new statistic does *not* have a Normal distribution!

The t Distributions

When we standardize based on the sample standard deviation s_x, our statistic has a new distribution called a ***t* distribution**.

The t distribution has a shape similar to that of the standard Normal curve in that *it is symmetric with a single peak at 0.*

However, it differs from the Normal curve in that has *more area in the tails*.



Like any standardized statistic, t tells us how far $\bar{x}$ is from its mean μ in standard deviation units.

However, there is a different t distribution for each sample size, specified by its **degrees of freedom (*df*)**.

The *t* Distributions

When we perform inference about a population mean μ using a *t* distribution, the appropriate degrees of freedom are found by subtracting 1 from the sample size *n*, making *df* = *n* – 1.

The *t* Distributions: Degrees of Freedom

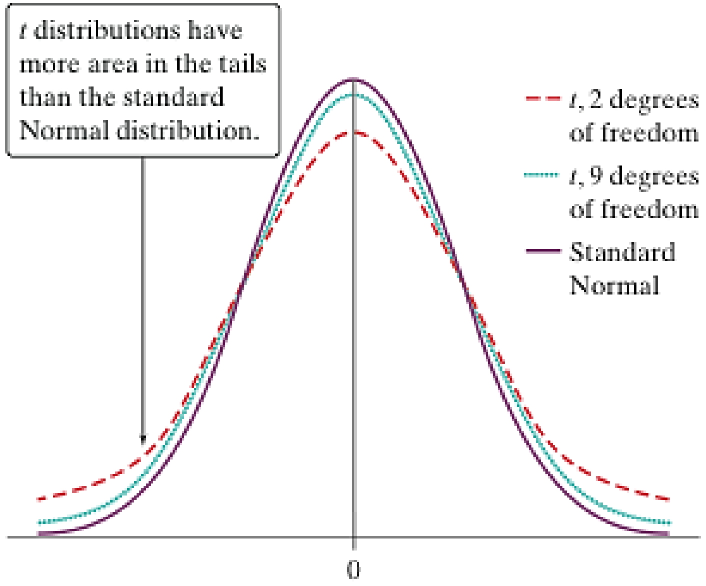
Draw an SRS of size *n* from a large population that has a Normal distribution with mean μ and standard deviation σ. The **one-sample *t* statistic**

$$t = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

has the ***t* distribution** with **degrees of freedom** *df* = *n* – 1.

The *t* Distributions

When comparing the density curves of the standard Normal distribution and *t* distributions, several facts are apparent:



- ✓ The density curves of the *t* distributions are similar in shape to the standard Normal curve.
- ✓ The spread of the *t* distributions is a bit larger than that of the standard Normal distribution.
- ✓ The *t* distributions have more probability in the tails and less in the center than does the standard Normal.
- ✓ As the degrees of freedom increase, the *t* density curve becomes ever closer to the standard Normal curve.

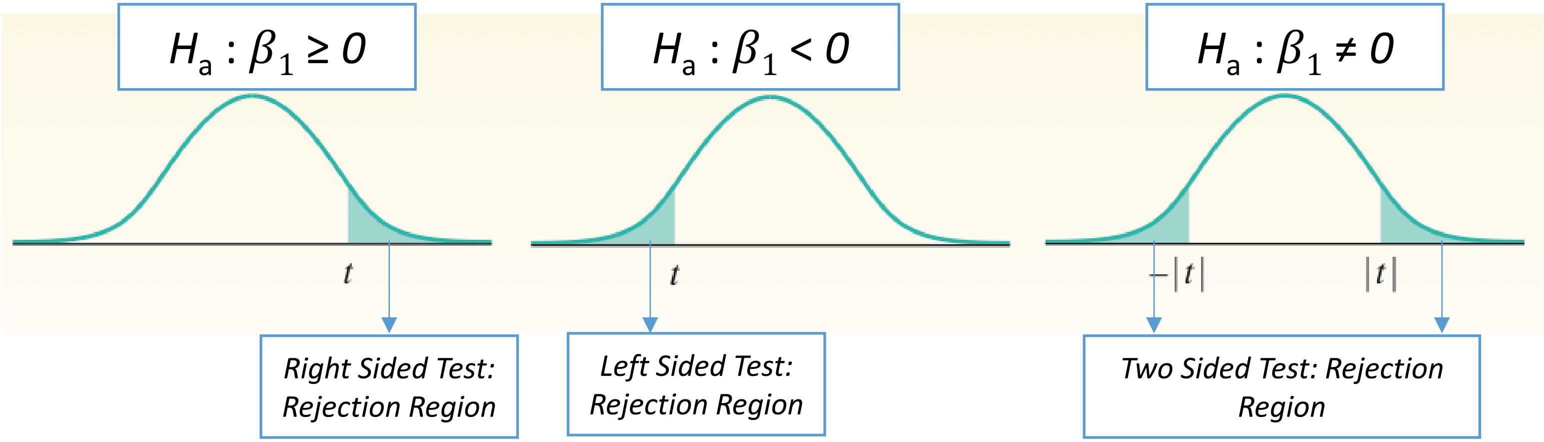
We can use Table D in the back of the book to determine critical values *t** for *t* distributions with different degrees of freedom.

The One-Sample t Test

To test the hypothesis $H_0 : \beta_1 = 0$, compute the one-sample t statistic:

$$t = \frac{b_1}{s(b_1)}$$

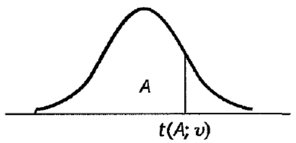
Find the *P*-value by calculating the probability (at degree of freedom = $n - 2$) of getting a t statistic this large or larger *in the direction specified by the alternative hypothesis H_a.*



T TABLE

TABLE B.2 Percentiles of the *t* Distribution.

Entry is $t(A;\nu)$ where $P\{t(\nu) \le t(A;\nu)\} = A$



ν	.60	.70	.80	.85	.90	.95	.975
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.537	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

TABLE B.2 (*concluded*) Percentiles of the *t* Distribution.

ν	.98	.985	.99	.9925	.995	.9975	.9995
1	15.895	21.205	31.821	42.434	63.657	127.322	636.590
2	4.849	5.643	6.965	8.073	9.925	14.089	31.598
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.849
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.690
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.659
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.291

Inferences in Regression and Correlation Analysis

Inferences concerning:

- the regression parameters β_0 and β_1
- interval estimation of β_0 and β_1 tests about them
- interval estimation of E (Y) of the probability distribution of Y for given X
- prediction intervals of a new observation Y
- confidence bands for the regression line

Normal Error Regression Model

Assume that the normal error regression model is applicable:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

- β_0 and β_1 are parameters
- X_i are known constants
- $\varepsilon_i \sim N(0, \sigma^2)$: are independent
- Y_i: independently, normally distributed
- A linear combination of independent normal random variables is normally distributed

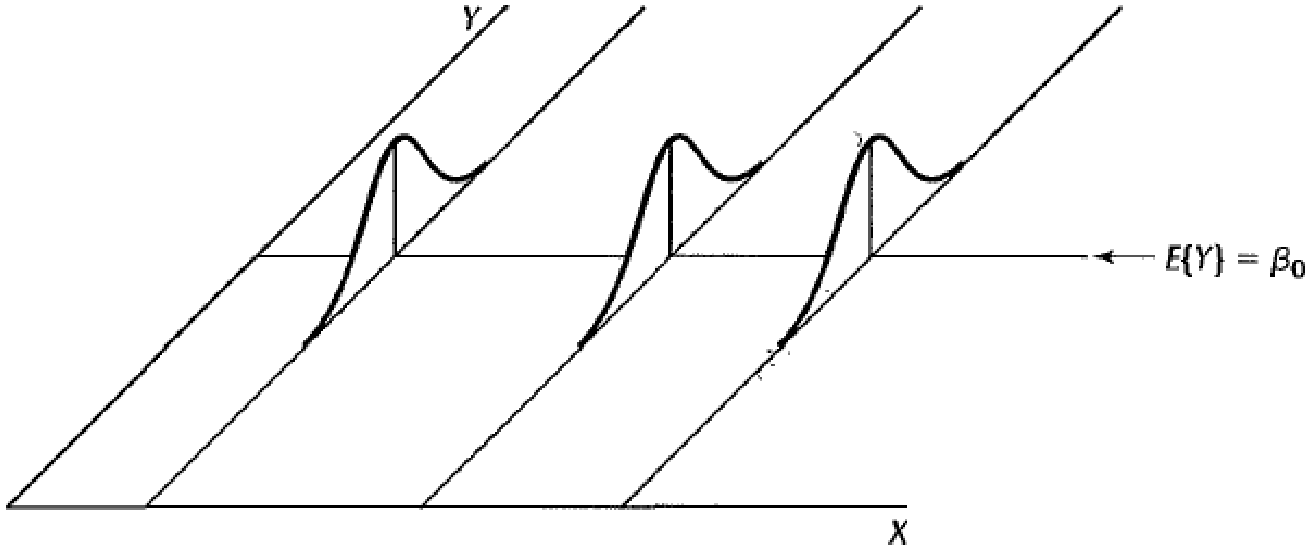
Inferences about the slope β_1 of the regression line

- Market research analyst studying the relation between sales *(Y)* and advertising expenditures *(X)* may wish to obtain an interval estimate of β_1, because it will provide information as to how many additional sales dollars, on the average, are generated by an additional dollar of advertising expenditure.
- At times, tests concerning β_1 are of interest, particularly one of the form:

$$H_0 : \beta_1 = 0;$$

$$H_a : \beta_1 \neq 0.$$

Inferences about the slope β_1 of the regression line, cont'd



- $\boldsymbol{\beta_1 = 0} \Rightarrow$ no linear association between Y and X
- The regression line is horizontal. The means of Y: $E\{Y\} = \beta_0$.
- The probability distribution of Y are identical at all levels of X

Sampling Distribution of b_1

- The sample distribution of b_1 refers to the different values of b_1 that would be obtained with repeated sampling when the levels of the predictor variable X are held constant from sample to sample.

$$b_1 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$$

- For normal error regression model, the sampling distribution of b_1 is normal, with mean and variance

$$E\{b_1\} = \beta_1$$

$$\sigma^2\{b_1\} = \frac{\sigma^2}{\sum(X_i - \bar{X})^2}$$

Sampling Distribution of b_1, *cont'd*

- b_1 as Linear Combination of the Y_i
- It can be shown that b_1 can be expressed as follows:

$$b_1 = \sum k_i Y_i$$

where:

$$k_i = \frac{X_i - \bar{X}}{\sum(X_i - \bar{X})^2}$$

The coefficients k_i have a number of interesting properties that will be used later:

$$\sum k_i = 0 \tag{2.5}$$

$$\sum k_i X_i = 1 \tag{2.6}$$

$$\sum k_i^2 = \frac{1}{\sum(X_i - \bar{X})^2} \tag{2.7}$$

Sampling Distribution of b_1 , *cont'd*

- Mean. The unbiasedness of the point estimator b_1 stated earlier in the Gauss-Markov theorem (1.11), is easy to show:

$$E\{b_1\} = E\left\{\sum k_i Y_i\right\} = \sum k_i E\{Y_i\} = \sum k_i(\beta_0 + \beta_1 X_i)$$
$$= \beta_0 \sum k_i + \beta_1 \sum k_i X_i$$

By the equations 2.5 and 2.6 in the previous slide, we then obtain $E\{b_1\} = \beta_1$.

- Variance. The variance of b_1 can be derived readily. We need to remember that the Y_i are independent random variables, each with variance σ^2 , and that the k_i are constants. Hence, we obtain:

$$\sigma^2\{b_1\} = \sigma^2\left\{\sum k_i Y_i\right\} = \sum k_i^2 \sigma^2\{Y_i\}$$
$$= \sum k_i^2 \sigma^2 = \sigma^2 \sum k_i^2$$
$$= \sigma^2 \frac{1}{\sum(X_i - \bar{X})^2}$$

Sampling Distribution of b_1 , *cont'd*

- Estimated Variance. We can estimate the variance of the sampling distribution of b_1 :

$$\sigma^2\{b_1\} = \frac{\sigma^2}{\sum(X_i - \bar{X})^2}$$

- by replacing the parameter σ^2 with *MSE,* the unbiased estimator of σ^2:

$$s^2\{b_1\} = \frac{MSE}{\sum(X_i - \bar{X})^2}$$

- The point estimator $s^2\{b_1\}$ is an unbiased estimator of $\sigma^2 \{b_1\}$. Taking the positive square root, we obtain $s\{b_1\}$, the point estimator of $\sigma\{b_1\}$.

Theorem 1:

- The estimator b_1 has minimum variance among all unbiased linear estimators of:

$$\widehat{\beta_1} = \sum c_i Y_i$$

Where c_i are arbitrary constants which holds $\sum c_i = 0$; $\sum c_i X_i = 1$

We now prove this. Since b_1 is required to be unbiased, the following must hold:

$$E\{\hat{\beta}_1\} = E\{\sum c_i Y_i\} = \sum c_i E(Yi) = \beta_1 \text{ then}$$

$$\sum c_i E(Yi) = \sum c_i E(\beta_0 + \beta_1 X_i) = \beta_0 \sum c_i + \beta_1 \sum c_i X_i$$

By using conditions above $\sum c_i = 0$; $\sum c_i X_i = 1$, it follows that

$$E\{\hat{\beta}_1\} = \beta_1$$

$$\sigma^2\{\hat{\beta}_1\} = \sum c_i^2 \sigma^2\{Y_i\} = \sigma^2 \sum c_i^2$$

Theorem 1, cont'd

- We now prove that b_1 has minimum variance among all unbiased linear estimators of:

$$\widehat{\beta_1} = \sum c_i Y_i$$

under the conditions that $\sum c_i = 0; \sum c_i X_i = 1$

$$\sigma^2\{\hat{\beta}_1\} = \sigma^2\{\sum c_i Y_i\} = \sum c_i^2 \sigma^2(Yi)$$
$$= \sum c_i^2{}_i \sigma^2(Yi) = \sum c_i^2 \sigma^2 = \sigma^2 \sum c_i^2$$

Let us define $c_i = d_i + k_i$, where k_i are the least squares constants (remember from slide 20 that $\sigma^2\{b_1\} = \sigma^2 \sum k_i^2$), now we can write that

$$\sigma^2\{\hat{\beta}_1\} = \sigma^2 \sum c_i^2 = \sigma^2 \sum (di + ki)2 = \sigma^2(\sum d_i^2 + \sum k_i^2 + \sum d_i\, k_i) = \sigma^2 \sum d_i^2 + \sigma^2 \sum k_i^2 + \sum d_i\, k_i$$
$$= \sigma^2 \sum d_i^2 + \sigma^2\{b_1\} + \sum d_i\, k_i$$

Theorem 1, cont'd

- We will show that $\sum d_i \, k_i = 0$

$$\sum k_i d_i = \sum k_i (c_i - k_i)$$
$$= \sum c_i k_i - \sum k_i^2$$
$$= \sum c_i \left[\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} \right] - \frac{1}{\sum (X_i - \bar{X})^2}$$
$$= \frac{\sum c_i X_i - \bar{X} \sum c_i}{\sum (X_i - \bar{X})^2} - \frac{1}{\sum (X_i - \bar{X})^2} = 0$$

- $\sigma^2\{\hat{\beta}_1\} = \sigma^2 \sum d_i^2 + \sigma^2\{b_1\} + \sum d_i \, k_i = \sigma^2 \sum d_i^2 + \sigma^2\{b_1\}$

- Note that the smallest values of $\sum d_i^2$ is 0, it occurs if all d_i=0, which implies that $c_i = k_i$. Thus, this proves the theorem.

Sampling Distribution of $(b_1 - \boldsymbol{\beta}_1)/s\{b_1\}$

- Since b_1, is normally distributed, we know that the standardized statistic $\frac{b_1-\beta_1}{\sigma(b_1)}$ is a standard normal variable.
- $\sigma(b_1)$ is estimated by $s(b_1)$.

$\boldsymbol{\frac{b_1-\beta_1}{s(b_1)}}$ **is distributed as t(n-2), t distribution with n-2 degree of freedom.**

Proof:

Reminder:

Z_i are independent identical distributed (iid) with ~N(0,1):

- $\sum_{i=1}^{n} Z_i$ ~ N(0,n)
- $Z_i^2 \sim \chi^2(1)$ and $\sum_{i=1}^{n} Z_i^2 \sim \chi^2\ (n)$

The pdf of a Normal distribution N(μ, σ^2)

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

The pdf of a Normal distribution N(0,1)

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

Sampling Distribution of *$(b_1 - \boldsymbol{\beta}_1)/s\{b_1\}$, cont'd*

Chi-square Distribution

$$f(x|p) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2}, \quad 0 < x < \infty,$$

Where Γ is the gamma function.

$\mathrm{t(v)} = \frac{Z}{\sqrt{\frac{\chi^2\,(v)}{v}}}$ is the t-distribution with v degree of freedom, where Z is the standard normal distribution and $\chi^2\ (v)$ is the chi square distribution with v degree of freedom.

Sampling Distribution of *$(b_1 - \boldsymbol{\beta}_1)/s\{b_1\}$, cont'd*

- First we $need\ to\ rewrite$ $\frac{b_1-\beta_1}{s(b_1)}$
- $\frac{b_1-\beta_1}{s(b_1)} * \frac{\sigma(b_1)}{\sigma(b_1)} = \frac{b_1-\beta_1}{\sigma(b_1)} * \frac{\sigma(b_1)}{s(b_1)} = \frac{b_1-\beta_1}{\sigma(b_1)} \div \frac{s(b_1)}{\sigma(b_1)} = Z \div \frac{s(b_1)}{\sigma(b_1)}$

$$\frac{s^2\{b_1\}}{\sigma^2\{b_1\}} = \frac{\dfrac{MSE}{\sum(X_i-\bar{X})^2}}{\dfrac{\sigma^2}{\sum(X_i-\bar{X})^2}} = \frac{MSE}{\sigma^2} = \frac{\dfrac{SSE}{n-2}}{\sigma^2}$$

$$= \frac{SSE}{\sigma^2(n-2)} \sim \frac{\chi^2(n-2)}{n-2}$$

- *Then,* $\frac{b_1-\beta_1}{s\{b_1\}} \sim \frac{z}{\sqrt{\frac{\chi^2(n-2)}{n-2}}}$ *is distributed with t distribution with n-2 degree of freedom*

Hypothesis testing for $\boldsymbol{\beta}_1$

(1-$\alpha/2$)% confidence interval for $\boldsymbol{\beta_1}$ is

$$b_1 \pm t(1 - \alpha/2; n - 2)s(b_1) = [b_1 - t(1 - \alpha/2; n - 2)s(b_1)\ ,\ b_1 + t(1 - \alpha/2; n - 2)s(b_1)]$$

Example: Toluca Company example

$n = 25$	$\bar{X} = 70.00$
$b_0 = 62.37$	$b_1 = 3.5702$
$\hat{Y} = 62.37 + 3.5702X$	$SSE = 54{,}825$
$\sum(X_i - \bar{X})^2 = 19{,}800$	$MSE = 2{,}384$
$\sum(Y_i - \bar{Y})^2 = 307{,}203$	

The regression equation is
Y = 62.4 + 3.57 X

Predictor	Coef	Stdev	t-ratio	p
Constant	62.37	26.18	2.38	0.026
X	3.5702	0.3470	10.29	0.000

s = 48.82 R-sq = 82.2% R-sq(adj) = 81.4%

Analysis of Variance

SOURCE	DF	SS	MS	F	p
Regression	1	252378	252378	105.88	0.000
Error	23	54825	2384		
Total	24	307203			

Inferences about the slope β_0 of the regression line

- There are only infrequent occasions when we wish to make inferences concerning β_0, the intercept of the regression line. These occur when the scope of the model includes X = 0 times.
- The point estimator b_0 was shown in Chapter as follow:

$$b_0 = \bar{Y} - b_1 \bar{X}$$

- The sampling distribution of b_0 is normal, with mean and variance:

$$E\{b_0\} = \beta_0$$

$$\sigma^2\{b_0\} = \sigma^2 \left[\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2} \right]$$

Inferences about the slope β_0 of the regression line, cont'd

- The normality of the sampling distribution of b_0 follows because b_0 , like b_1 , is a linear combination of the observations Yi
- The results for the mean and variance of the sampling distribution of b_0 can be obtained in similar fashion as those for b_1
- An estimator of a σ^2(b_0) is obtained by replacing a σ^2 by its point estimator MSE:

$$s^2\{b_0\} = MSE\left[\frac{1}{n} + \frac{\bar{X}^2}{\sum(X_i - \bar{X})^2}\right]$$

- The positive square root, *s(b_o),* is an estimator of σ(b_0)

Sampling Distribution of $(b_o - \boldsymbol{\beta_0})/s(b_o)$

- Similar to $b_{1,}$ the sampling distributions follows a t distribution with n-2 degree of freedom.

$$\frac{b_o - \beta_0}{s(b_0)} \sim t(n\text{-}2)$$

- (1-$\alpha/2$)% confidence interval for $\boldsymbol{\beta_0}$ is

$$b_0 \pm t(1 - \alpha/2;n - 2)s(b_0) = [b_0 - t(1 - \alpha/2;n - 2)s(b_0)\ ,\ b_0 + t(1 - \alpha/2;n - 2)s(b_0)]$$

Example: Toluca Example

Long Way	Using build in Function

Long Way

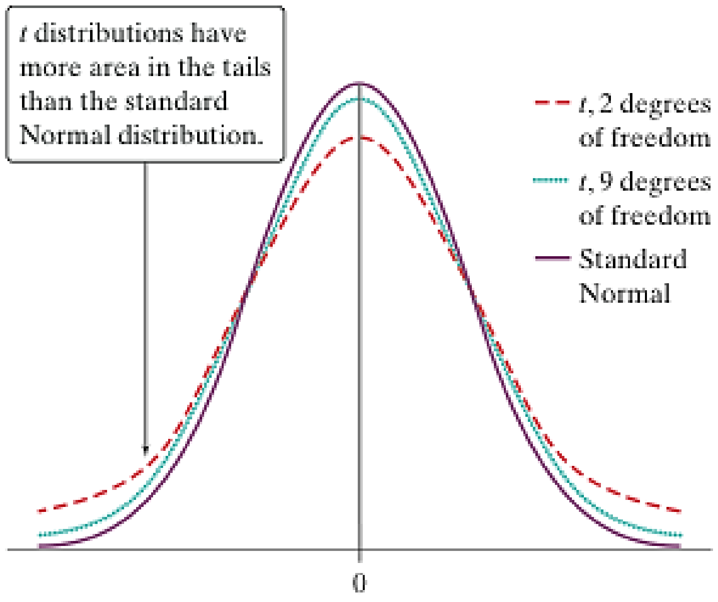
```
n<-length(lotsize)
alpha<-0.05
tdf<-qt(1-alpha/2,n-2)
Sxx<-sum((lotsize-mean(lotsize))^2)
b1<-sum((lotsize-mean(lotsize))*(workhrs-mean(workhrs)))/Sxx
b0<-mean(workhrs)-b1*mean(lotsize)
SSE<-sum((workhrs-(b0+b1*lotsize))^2)
MSE<-SSE/(n-2)
sb1<-sqrt(MSE/sum((lotsize-mean(lotsize))^2)) #s{b1}
conf_b1<-c(b1-tdf*sb1,b1+tdf*sb1)
conf_b1
sb0<-sqrt(MSE*(1/n+mean(lotsize)^2/sum((lotsize-
mean(lotsize))^2)))
conf_bo<-c(b0-tdf*sb0,b0+tdf*sb0)
conf_bo
```

Using build in Function

```
toluca.reg<-lm(workhrs~ lotsize)
summary(toluca.reg)
 confint(toluca.reg)
```

Departures from Normality

- Even if the distribution of Y are far from normal, the estimators b_0 and b_1 generally have the property of asymptotic normality-their distributions approach normality under very general conditions as the sample size increases. Thus, with sufficiently large samples, the confidence intervals and decision rules given earlier still apply even if the probability distributions of Y depart far from normality. For large samples, the t value is, of course, replaced by the z value for the standard normal distribution.



Interval Estimation of $E\{Y_h\}$

- Let X_h denote the level of X for which we wish to estimate the mean response.

- X_h may be a value which occurred in the sample, or it may be some other value of the predictor variable within the scope of the model.

- The mean response when X = X_h is denoted by $E\{Y_h\}$.

- Formula gives us the point estimator Y_h of $E\{Y_h\}$:
  - $Y_h = b_o + b_l X_h$

- What is the sampling distribution of Y_h?

Sampling Distribution of $\hat{Y}_h$

- For normal error regression model, the sampling distribution of $\hat{Y}_h$ is normal, with mean and variance:

$$E\{\hat{Y}_h\} = E\{Y_h\}$$

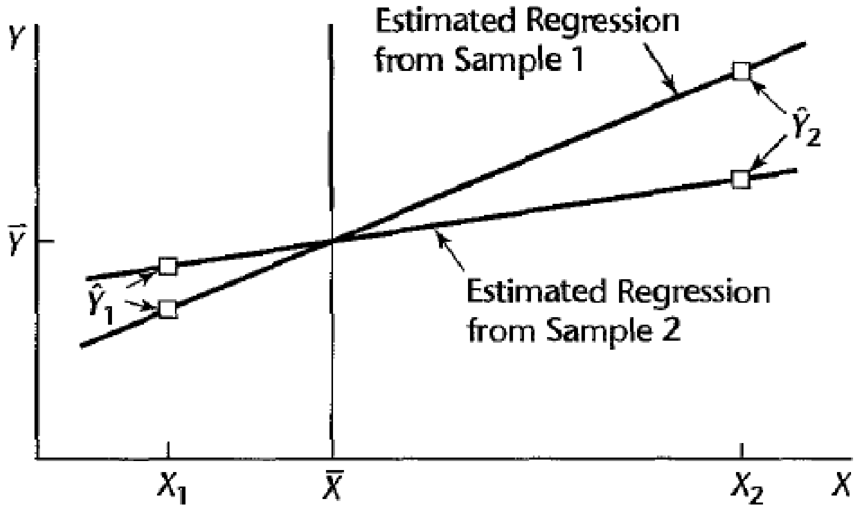
$$\sigma^2\{\hat{Y}_h\} = \sigma^2\left[\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right]$$

- Normality: The normality of the sampling distribution of Y_h follows directly from the fact that $\hat{Y}_{h,}$ like b_o and b_1 , is a linear combination of the observations Y_i

- Mean. Note from that Y_h is an unbiased estimator of $E\{Y_h\}$. To prove this, we proceed as follows:

$$E\{\hat{Y}_h\} = E\{b_0 + b_1 X_h\} = E\{b_0\} + X_h E\{b_1\} = \beta_0 + \beta_1 X_h$$

Sampling Distribution of $\hat{Y}h$, cont'd



- The two regression lines are assumed to go through the same $(\bar{X}, \bar{Y})$ point to isolate the effect of interest, namely, the effect of variation in the estimated slope b_1 from sample to sample.
- Note that at X_1 near $\bar{X}$, the fitted values $\hat{Y}_1$ for the two sample regression lines are close to each other.
- At X_2 which is far from $\bar{X}$, the fitted values $\hat{Y}_2$ differ substantially

- Thus, variation in the slope b_1 from sample to sample has a much more pronounced effect on $\hat{Y}_h$ for X levels far from the mean X than for X levels near X.
- Hence, the variation in the $\hat{Y}_h$ values from sample to sample will be greater when X_h is far from the mean than when X_h is near the mean.

Sampling Distribution of $\hat{Y}h$, cont'd

- When *MSE* is substituted for σ^2, we obtain $s^2\{\hat{Y}_h\}$, the estimated variance of $\hat{Y}_h$:

$$s^2\{\hat{Y}_h\} = MSE\left[\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right]$$

- The estimated standard deviation of $\hat{Y}_h$ is then $s\{\hat{Y}_h\}$, the positive square root of $s^2\{\hat{Y}_h\}$.

Sampling Distribution of $\widehat{Y}h$, cont'd

Derivation of $s^2\{\hat{Y}_h\}$

- $Y_h = b_o + b_1 X_h$ and $b_o = \bar{Y} - b_1 \bar{X}$ and then
- $Y_h = \bar{Y} - b_1 \bar{X} + b_1 X_h = \bar{Y} + b_1(X_h - \bar{X})$
- $\sigma^2(Y_h) = \sigma^2(\bar{Y} + b_1(X_h - \bar{X})) = \sigma^2(\bar{Y}) + \sigma^2(b_1)(X_h - \bar{X})^2 + 2(Xh\_\bar{X})\ {}^2\sigma^2(\bar{Y}, b_1)$
- $= \frac{\sigma^2}{n} + (X_h - \bar{X})^2 \frac{\sigma^2}{\sum(X_i - \bar{X})^2} + 0$
- $= \sigma^2\left(\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right)$

Sampling Distribution of $(\hat{Y}_h - E(\hat{Y}_h))/s(\hat{Y}_h)$,

$\frac{\hat{Y}_h - E(\hat{Y}h)}{s(\hat{Y}_h)}$ ~ t distribution with n-2 df

(1-α/2)% Confidence Interval for *E{$\hat{Y}_h$}*

$$\hat{Y}_h \pm t(1 - \alpha/2;n - 2)s(\hat{Y}_h) = [\hat{Y}_h - t(1 - \alpha/2;n - 2)s(\hat{Y}_h) , \hat{Y}_h + t(1 - \alpha/2;n - 2)s(\hat{Y}_h)]$$

Example

- Returning to the Toluca Company example, let us find a 90 percent confidence interval for E $\{Y_h\}$ when the lot size is X_h = 65 units.

$$\hat{Y}_h = 62.37 + 3.5702(65) = 294.4$$

$$s^2\{\hat{Y}_h\} = 2{,}384\left[\frac{1}{25} + \frac{(65 - 70.00)^2}{19{,}800}\right] = 98.37$$

$$s\{\hat{Y}_h\} = 9.918$$

- For a 90 percent confidence coefficient, we require t(.95; 23) = 1.714. Hence, our confidence interval with confidence coefficient .90 is

$$294.4 - 1.714(9.918) \leq E\{Y_h\} \leq 294.4 + 1.714(9.918)$$

$$277.4 \leq E\{Y_h\} \leq 311.4$$

- We conclude with confidence coefficient .90 that the mean number of work hours required when lots of 65 units are produced is somewhere between 277.4 and 311.4 hours.

R Function - predict

```
> predict(toluca.reg,data.frame(lotsize=65),se.fit=TRUE,interval="confidence")
$fit
      fit      lwr      upr
1 294.429 273.9129 314.9451

$se.fit
[1] 9.917579

$df
[1] 23

$residual.scale
[1] 48.82331
```

$s(\hat{Y}_h)$

$n-2$

$\sqrt{MSE}$

Example

- Let us find a 90 percent confidence interval for E $\{Y_h\}$ when the lot size is X_h = 100 units.

$$\hat{Y}_h = 62.37 + 3.5702(100) = 419.4$$

$$s^2\{\hat{Y}_h\} = 2{,}384\left[\frac{1}{25} + \frac{(100 - 70.00)^2}{19{,}800}\right] = 203.72$$

$$s\{\hat{Y}_h\} = 14.27$$

$$t(.95; 23) = 1.714$$

- our confidence interval with confidence coefficient .90 is

$$419.4 - 1.714(14.27) \leq E\{Y_h\} \leq 419.4 + 1.714(14.27)$$

$$394.9 \leq E\{Y_h\} \leq 443.9$$

- Note that this confidence interval is somewhat wider than that from the previous example, since the X_h level here (X_h = 100) is substantially farther from the mean $\bar{X}$ = 70.0 than the X_h level from the previous example (X_h = 65).

Prediction of New Observation

- The new observation on Y to be predicted is viewed as the result of a new trial, independent of the trials on which the regression analysis is based.
- We denote the level of X for the new trial as X_h and the new observation on Y as $Y_{h(new)}$.
- **It is assumed that the underlying regression model applicable for the basic sample data continues to be appropriate for the new observation !**
- We predict an individual outcome drawn from the distribution of Y.

Example: College Admissions

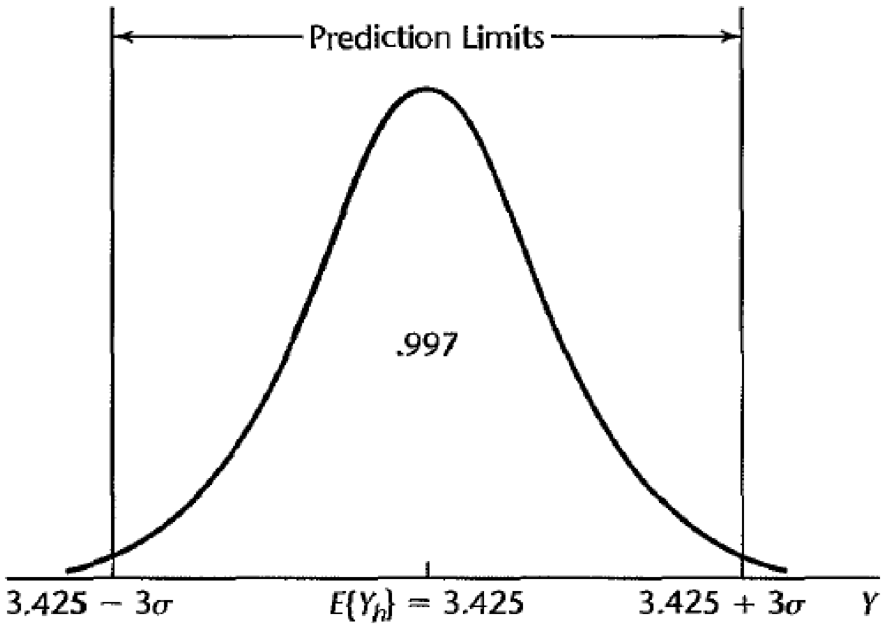
- Suppose that in the college admissions example the relevant parameters of the regression model are known to be

$$\beta_0 = .10 \qquad \beta_1 = .95$$

$$E\{Y\} = .10 + .95X$$

$$\sigma = .12$$

- The admissions officer is considering an applicant whose high school GPA is $X_h = 3.5$.
- The mean college GPA for students whose high school average is 3.5 is:
- $E\{Y_h\} = .10 + .95(3.5) = 3.425$
- $E\{Y_h\} \pm 3\sigma$:
    - $3.425 \pm 3(0.12) \Rightarrow 3.065 \leq Y_{h(new)} \leq 3.785$



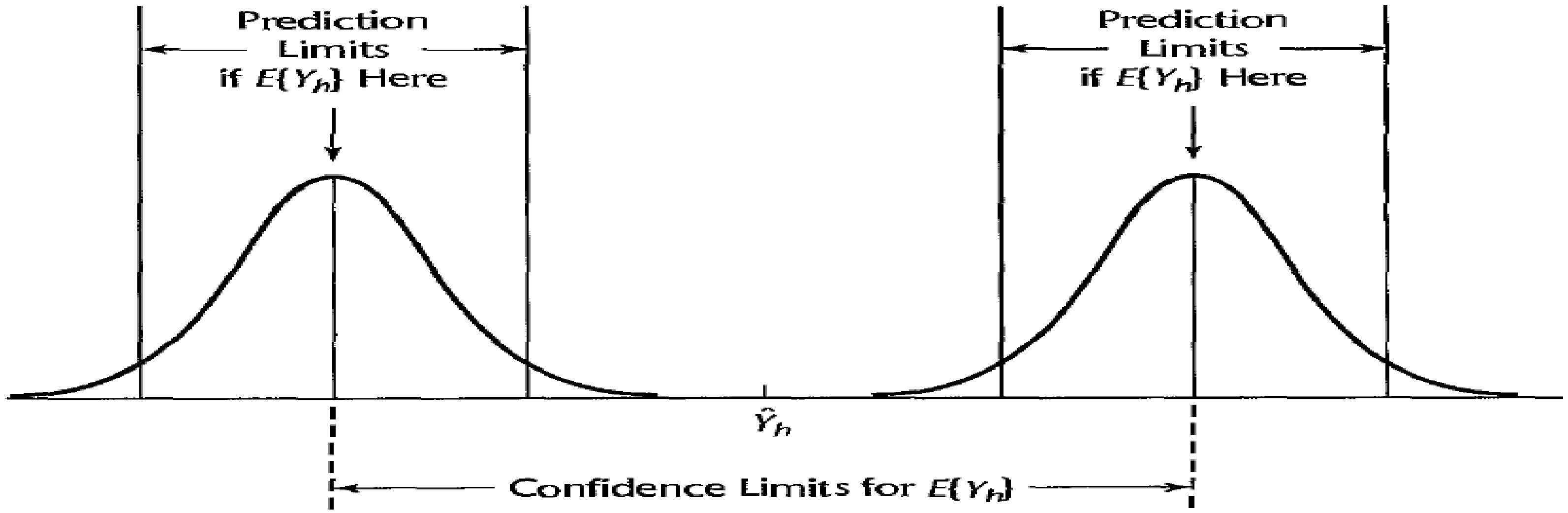
Probability Distribution of Y when $X_h = 3.5$

Prediction of New Observation, cont'd

- The basic idea of a prediction interval is to choose a range in the distribution of Y wherein most of the observations will fall, and then to declare that the next observation will fall in this range.
- When the regression parameters of normal error regression model are known, the 1 − α prediction limits for $Y_{h(new)}$ are:

$$E\{Y_h\} \pm z(1 - \alpha/2)\sigma$$

Prediction Interval for $Y_{h(new)}$ when Parameters Unknown



- Figure shows the prediction limits for each of the two probability distributions of *Y* presented there. Since we cannot be certain of the location of the distribution of *Y,* prediction limits for $Y_{h(new)}$ must take account of two elements

  1. Variation in possible location of the distribution of *Y* .
  2. Variation within the probability distribution of *Y.*

Prediction Interval for $Y_{h(new)}$ when Parameters Unknown, cont'd

- Prediction limits for a new observation $Y_{h(new)}$ at a given level Xh are obtained by means of the following theorem:

- $\frac{Y_{h(new)} - \hat{Y}_h}{S(pred)}$ ~ *t(n-2) distribution*

- The 1 − α prediction limits for $Y_{h(new)}$:

  - $\hat{Y}h \pm$ t(1 − α/2; n − 2)s{pred}

- The difference may be viewed as the prediction error, with $\hat{Y}_h$ serving as the best point estimate of the value of the new observation $Y_{h(new)}$

- $\sigma^2\{pred\}$: the variance of the prediction error

  - $\sigma^2\{\text{pred}\} = \sigma^2\{Y_{h(new)} - \hat{Y}_h\} = \sigma^2 + \sigma^2\{\hat{Y}_h\}$

Prediction Interval for $Y_{h(new)}$ when Parameters Unknown, cont'd

- σ^2\{pred\} has two components:
  - The variance of the distribution of Y at $X = X_h$; σ^2
  - The variance of the sampling distribution of $\hat{Y}_h$; $\sigma^2\{\hat{Y}_h\}$

- An unbiased estimator of σ^2\{pred\}

$$s^2(\text{pred}) = MSE + s^2(\hat{Y}_h) = MSE + \text{MSE}\left(\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum (X_i - \bar{X})^2}\right)$$

$$= \text{MSE}\left[1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum (X_i - \bar{X})^2}\right]$$

Example: Toluca Company

- Toluca Company: $X_h = 100$
- 90 percent prediction interval: $t(0.95; 23) = 1.714$

  - $\hat{Y}_h = 419.4$ $\quad s^2\{\hat{Y}_h\} = 203.72$ $\quad MSE = 2,384$
  - $\Rightarrow s^2(\text{pred}) = 2,384 + 203.72 = 2,587.72$
  - $s(\text{pred}) = 50.87$
- The 90 percent prediction interval for $Y_{h(new)}$:

  - $332.2 \leq Y_{h(new)} \leq 506.6$

Example: Toluca Company - Rcode

- toluca.reg <-lm(workhrs ~ lotsize , data= toluca_data)
- Xh<-data.frame(lotsize=100)
- fitnew<-predict(toluca.reg,Xh,se.fit=T, interval="prediction", level=0.9)
- s2pred<-fitnew$se.fit^2+fitnew$residual.scale^2
- c(s2pred,sqrt(s2pred))

OR

- predict(toluca.reg,Xh,interval="prediction",level=0.90)

Example: Toluca Company, cont'd

- This prediction interval is rather wide and may not be useful for planning worker requirements for the next lot.
- The interval can still be useful for control purposes.
  - If the actual work hours fall outside the prediction limits $\Rightarrow$ some alerts may have occurred a change in the production process

- Toluca Company: The C.I. for $Y_{h(new)}$ is wider than the C.I. for $E\{Y_h\}$:
- predicting the work hours required for a new lot
- encounter the variability in $\hat{Y}_h$ from sample to sample as well as the lot-to-lot variation within the probability distribution of Y
- The prediction interval is wider the further X_h is from $\bar{X}$

Confidence-Band for Regression Line

- At times we would like to obtain a confidence band for the entire regression line E{Y} $=\beta_0 + \beta_1 X$
- This band enables us to see the region in which the entire regression line lies.
- The Working-Hotelling $1 - \alpha$ confidence band for the regression line for regression model has the following two boundary values at any level X_h :

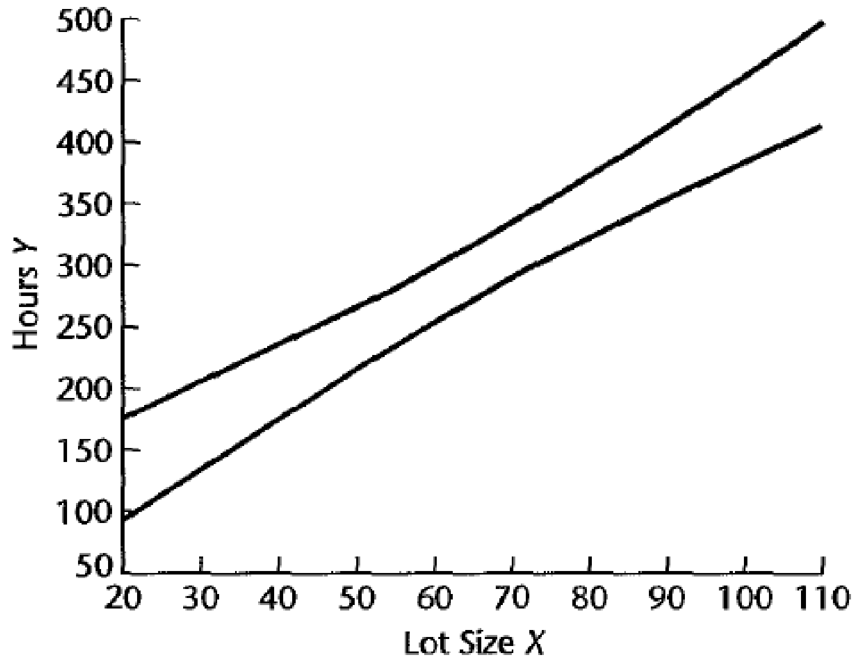
$$\hat{Y}_h \pm W\, s(\hat{Y}_h)$$

- Where $W^2 = 2F(1 - \alpha; 2, n - 2)$ and F is the F distribution with 2 and n-2 degree of freedom.

Example: Toluca Data

- Toluca Company: $X_h = 100$
- $\hat{Y}_h = 419.4$; $s^2\{Y_h\} = 203.72$; MSE = 2, 384;

- $W^2 = 2F(1 - \alpha; 2, n - 2) = 2F(.90; 2, 23) = 2(2.549) = 5.098$
- W = 2.258
- Hence, the boundary values of the confidence band for the regression line at $X_h = 100$ are
- $419.4 \pm 2.258(14.27)$, and the confidence band is:
- $387.2 \leq \beta_0 + \beta_1 Xh \leq 451.6$ for $X_h = 100$



Confidence Band for the regression line

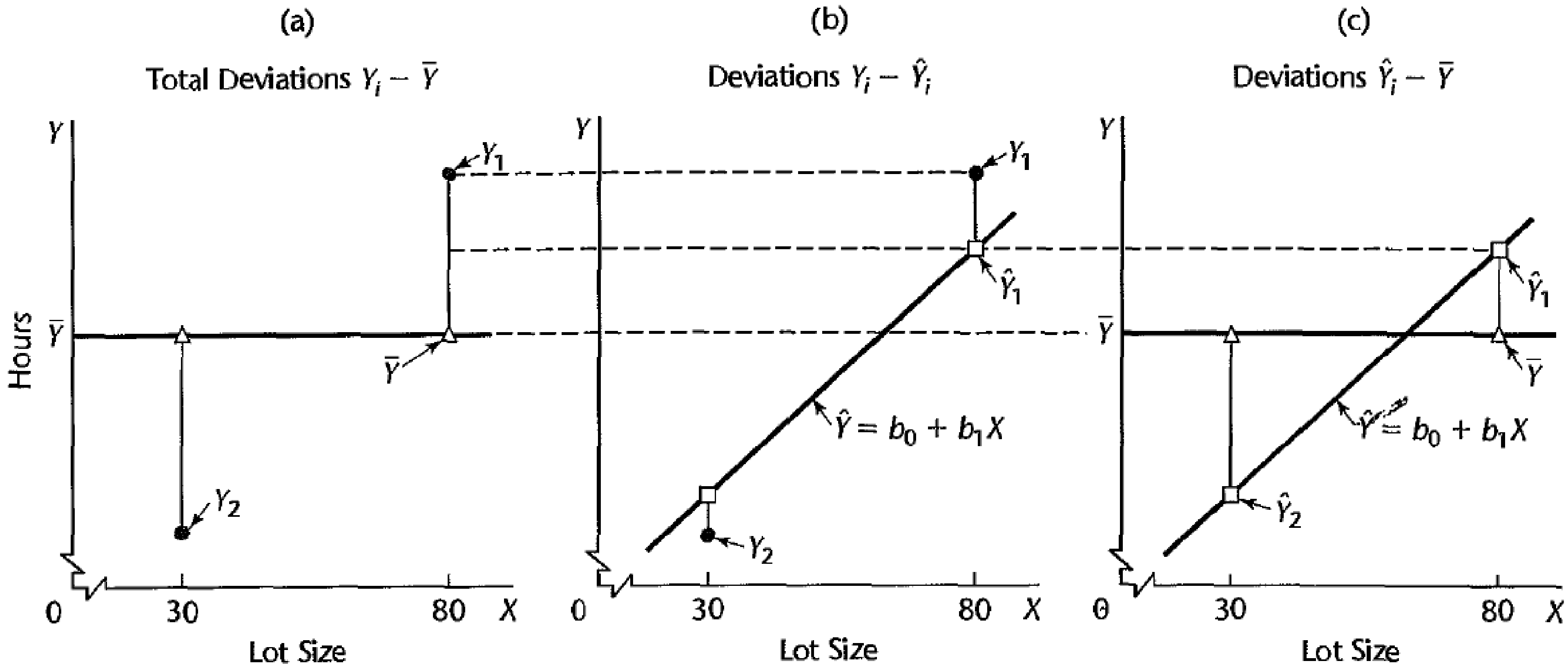
Partition of Total Sum of Squares

- The analysis of variance approach is based on the partitioning of sums of squares and degrees of freedom associated with Y .

- The variation is measured: the deviations of the Y_i around their mean $\bar{Y}$:

$$Y_i - \bar{Y}$$

Partition of Total Sum of Squares, cont'd



Partition of Total Sum of Squares, cont'd

- Total variation: (SSTO or SST: total sum of squares)

$$\text{SST} = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

- Y_i are the same ⇒ SST = 0
- The greater the variation among the Y_i, the larger is SST.

- Total Error: (*SSE*: Sum of Squares of Error)

$$\text{SSE} = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

- Y_i fall on the fitted regression line ⇒ SSE = 0
- The greater the variation of the Y_i around the fitted regression line, the larger is SSE .

Partition of Total Sum of Squares, cont'd

- Total Regression: (*SSR:* Regression Sum of Squares)

$$SSR = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$$

- The regression line is horizontal ⇒ SSR = 0, otherwise SSR > 0
- a measure associated with the regression line
- The larger SSR is in relation to SST, the greater is the effect of the regression relation in accounting for the total variation in the Yi observations.

Partition of Total Sum of Squares, cont'd

- The total deviation:

$$\underbrace{Y_i - \bar{Y}}_{\substack{\text{Total}\\ \text{deviation}}} = \underbrace{\hat{Y}_i - \bar{Y}}_{\substack{\text{Deviation}\\ \text{of fitted}\\ \text{regression}\\ \text{value}\\ \text{around mean}}} + \underbrace{Y_i - \hat{Y}_i}_{\substack{\text{Deviation}\\ \text{around}\\ \text{fitted}\\ \text{regression}\\ \text{line}}}$$

- Two components:
  1. The deviation of the fitted value $\hat{Y}_i$ around the mean $\bar{Y}$.
  2. The deviation of the observation Y_i around the fitted regression line.

Partition of Total Sum of Squares, cont'd

$$\underbrace{\sum (Y_i - \bar{Y})^2}_{SSTO} = \underbrace{\sum (\hat{Y}_i - \bar{Y})^2}_{SSR} + \underbrace{\sum (Y_i - \hat{Y})^2}_{SSE}$$

$$2\sum (\hat{Y}_i - \bar{Y})(Y_i - \hat{Y}_i) = 0$$

$$SSR = b_1^2 \sum (X_i - \bar{X})^2$$

Source of Variation	Degree of Freedom (df)	Explanation
SSR	1	β_0 *and* β_1 are estimated, due to constraint we lose 1 df.
SSE	n - 2	β_0 *and* β_1 are estimated, as such we lose 2 df
SST	n – 1	$\sum_{i=1}^{n}(Y_i - \bar{Y})$ = 0 , as such we lose 1 df

MEAN SQUARES

- Mean Square (MS) is Sum of Square (SS) divided by the degree of freedom (df)

SS	df	MS
SSR	1	$MSR = \frac{SSR}{1}$ (regression mean square)
SSE	$n-2$	$MSE = \frac{SSE}{n-2}$ (error mean square)

Analysis of Variance (ANOVA) Table

Source of Variation	SS	df	MS	$E\{MS\}$
Regression	$SSR = \sum(\hat{Y}_i - \bar{Y})^2$	1	$MSR = \frac{SSR}{1}$	$\sigma^2 + \beta_1^2 \sum(X_i - \bar{X})^2$
Error	$SSE = \sum(Y_i - \hat{Y}_i)^2$	$n-2$	$MSE = \frac{SSE}{n-2}$	σ^2
Total	$SSTO = \sum(Y_i - \bar{Y})^2$	$n-1$		

ANOVA TABLE, cont'd

$$SST = \sum(Y_i - \bar{Y})^2 = \sum Y_i{}^2 - n\bar{Y}^2$$

- In the modified ANOVA table, the total uncorrected sum of squares, denoted by SSTOU, is defined as:

$$SSTOU = \sum Y_i{}^2$$

- SS(correction for mean) = $n\bar{Y}^2$

Modified ANOVA Table

Source of Variation	SS	df	MS
Regression	$SSR = \sum(\hat{Y}_i - \bar{Y})^2$	1	$MSR = \frac{SSR}{1}$
Error	$SSE = \sum(Y_i - \hat{Y}_i)^2$	$n-2$	$MSE = \frac{SSE}{n-2}$
Total	$SSTO = \sum(Y_i - \bar{Y})^2$	$n-1$	
Correction for mean	$SS(\text{correctionformean}) = n\bar{Y}^2$	1	
Total, uncorrected	$SSTOU = \sum Y_i^2$	n	

Expected Mean Squares

$$E\{MSE\} = \sigma^2$$

$$E\{MSR\} = \sigma^2 + \beta_1^2 \sum (X_i - \bar{X})^2$$

- MSE is an unbiased estimator of σ^2. Implication:
  - The mean of the sampling distribution of MSE is σ^2;
  - when $\beta_1 = 0$, the mean of the sampling distribution of MSR is σ^2;
  - when $\beta_1 \neq 0$, $E\{MSR\} > E\{MSE\}$

F test of $\beta_1 = 0$ vs. $\beta_1 \neq 0$

- The analysis of variance approach provides us with a battery highly useful tests for regression models.
- For the simple linear regression case, the ANOVA provides us with a test:

$$H_0 : \beta_1 = 0$$
$$H_a : \beta_1 \neq 0$$

- Test Statistics

$$F^* = \frac{MSR}{MSE}$$

- large values of $F^* \Rightarrow H_a$;
- values of F^* near 1 $\Rightarrow H_0$;

F test of $\beta_1 = 0$ vs. $\beta_1 \neq 0$, cont'd

Cochran's theorem

If all n observations Y_i come from the same normal distribution with mean μ and variance σ^2, and $SSTO$ is decomposed into k sums of squares SS_r, each with degrees of freedom df_r, then the SS_r/σ^2 terms are independent χ^2 variables with df_r degrees of freedom if

$$\sum_{r=1}^{k} df_r = n - 1$$

F test of $\beta_1 = 0$ vs. $\beta_1 \neq 0$

Property

If $\beta_1 = 0$ so that all Y_i have the same mean $\mu = \beta_0$ and the same variance σ^2, SSE/σ^2 and SSR/σ^2 are independent χ^2 variables.

- When H_0 holds:

$$F^* = \frac{\frac{SSR}{\sigma^2}}{1} \div \frac{\frac{SSE}{\sigma^2}}{n-2} = \frac{MSR}{MSE} \sim \frac{\chi^2(1)}{1} \div \frac{\chi^2(n-2)}{n-2} \sim F(1, n-2)$$

- When H_a holds, F^* follows the noncentral F distribution.

F test of $\beta_1 = 0$ vs. $\beta_1 \neq 0$, cont'd

$F^* \sim F(1, n-2)$

The decision rule: α=Type I error

Decision

$$\text{If} F^* \leq F(1-\alpha; 1, n-2), \text{conclude} H_0;$$
$$\text{If} F^* > F(1-\alpha; 1, n-2), \text{conclude} H_a;$$

where $F(1-\alpha; 1, n-2)$ is the $(1-\alpha)100$ percentile of the approximate F distribution.

Example: Toluca Data

Two-Sided Test

$H_0 : \beta_1 = \beta_{10}$

$$\text{If } |t^*| = \frac{b_1 - \beta_{10}}{s\{b_1\}} \leq t(1 - \alpha/2; n-2), \text{ conclude } H_0$$

$$\text{If } |t^*| \frac{b_1 - \beta_{10}}{s\{b_1\}} > t(1 - \alpha/2; n-2), \text{ conclude } H_a$$

Example: Toluca Data, cont'd

Using the F test

$$H_0 : \beta_1 = \beta_{10} = 0$$
$$H_a : \beta_1 \neq \beta_{10} = 0$$

$\alpha = 0.05;\ n = 26;\ F(0.95; 1, 23) = 4.28$

$$\text{If}\, F^* \leq 4.28, \text{conclude}\, H_0$$

We have

$$F^* = \frac{MSR}{MSE} = \frac{252,378}{2,384} = 105.9$$

What is the conclusion?

Equivalence of *F* test and *t* Test

For a given α level, the F test of

$$H_0 : \beta_1 = 0 \quad H_a : \beta_1 \neq 0$$

is equivalence algebraically to the two-tailed t test.

Decision

$$F^* = \frac{b_1^2}{s^2\{b_1\}} = \left(\frac{b_1}{s\{b_1\}}\right)^2 = (t^*)^2$$

$$[t(1 - \alpha/2; n-2)]^2 = F(1 - \alpha; 1, n-2)$$

t test: two-tailed; F test: one-tailed;

General Linear Test Approach

1. Fit the full model and $SSE(F)$
2. Fit the reduced model under H_0 and $SSE(R)$
3. Use test statistic and decision rule

Full Model

The *full* or *unrestricted model*:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

$SSE(F)$:

$$SSE(F) = \sum \left[Y_i - (b_0 + b_1 X_i) \right]^2 = \sum (Y_i - \hat{Y}_i)^2 = SSE$$

Reduced Model

Hypothesis:

$$H_0 : \beta_1 = 0 \quad H_a : \beta_1 \neq 0$$

The *reduced* or *restricted model* when H_0 holds:

$$Y_i = \beta_0 + \varepsilon_i$$

$SSE(R)$:

$$SSE(R) = \sum \left[Y_i - (b_0) \right]^2 = \sum (Y_i - \bar{Y})^2 = SSTO$$

Test Statistics

$$SSE(F) \leq SSE(R)$$

- The more parameters are in the model, the better one can fit the data and the smaller are the deviations around the fitted regression function.

Test Statistics, cont'd

- When $SSE(F)$ is not much less than $SSE(R)$, using the full model does not account for much more of the variability of the Y_i than does the reduced model.

- $\Rightarrow$ Suggest that the reduced model is adequate i.e., H_0 holds.
- When $SSE(F)$ is close to $SSE(R)$, the variation of the observations around the fitted regression function for the full model is almost as great as the variation around the fitted regression function for the reduced model.

Test Statistics, cont'd

- A small difference $SSE(R) - SSE(F)$ suggests that H_0 holds.
- $\Leftrightarrow$ A large difference suggests that H_a holds.
- Test Statistic: a function of $SSE(R) - SSE(F)$:

$$F^* = \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} \sim F\text{distribution}$$

  - when H_0 holds. Decision rule:

  - If $F* \leq F(1 - \alpha; df_R - df_F, df_F)$, conclude H_0
  - If $F* > F(1 - \alpha; df_R - df_F, df_F)$, conclude H_a

Test Statistics, cont'd

For testing whether or not $\beta_1 = 0$, we have

$$SSE(R) = SSTO \qquad SSE(F) = SSE$$

$$df_R = n - 1 \qquad df_F = n - 2$$

$$\Rightarrow F^* = \frac{MSR}{MSE}$$

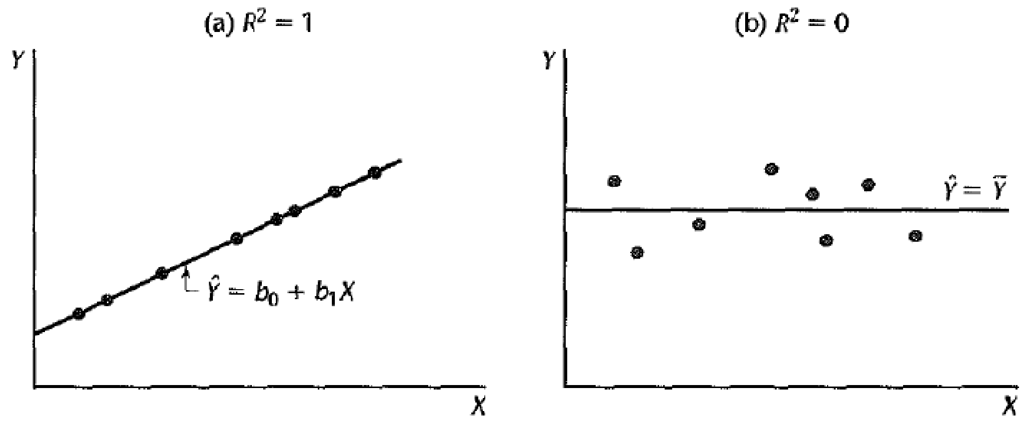
Coefficient of Determination

- SST (or SSTO) is a measure of the uncertainty in predicting Y when X is not considered.
- SSE measures the variation in the Y_i when a regression model utilizing the predictor variable X is employed.
- A natural measure of the effect of X in reducing the variation in Y is to express the reduction in variation (SST − SSE = SSR) as a proportion of the total variation:

$$R^2=\frac{SSR}{SST}=1-\frac{SSE}{SST}$$

Coefficient of Determination, cont'd

- $R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$ and given $SST \geq SSR \Rightarrow 0 \leq R^2 \leq 1$



$R^2 = 0$: There is no linear association between *X* and *Y* in the sample data, and the predictor variable *X* is of no help in reducing the variation in Y_i with linear regression.

$R^2 = 1$: The closer it is to 1, the greater is said to be the degree of linear association between *X* and *Y*.

Example: Toluca Company, cont'd

- fitreg<-lm(Hrs~Size,data=toluca)
- anova(fitreg)
- summary(fitreg)

```
> anova(fitreg)
Analysis of Variance Table

Response: Hrs
          Df Sum Sq Mean Sq F value    Pr(>F)
Size       1 252378  252378  105.88 4.449e-10 ***
Residuals 23  54825    2384
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example: Toluca Company, cont’d

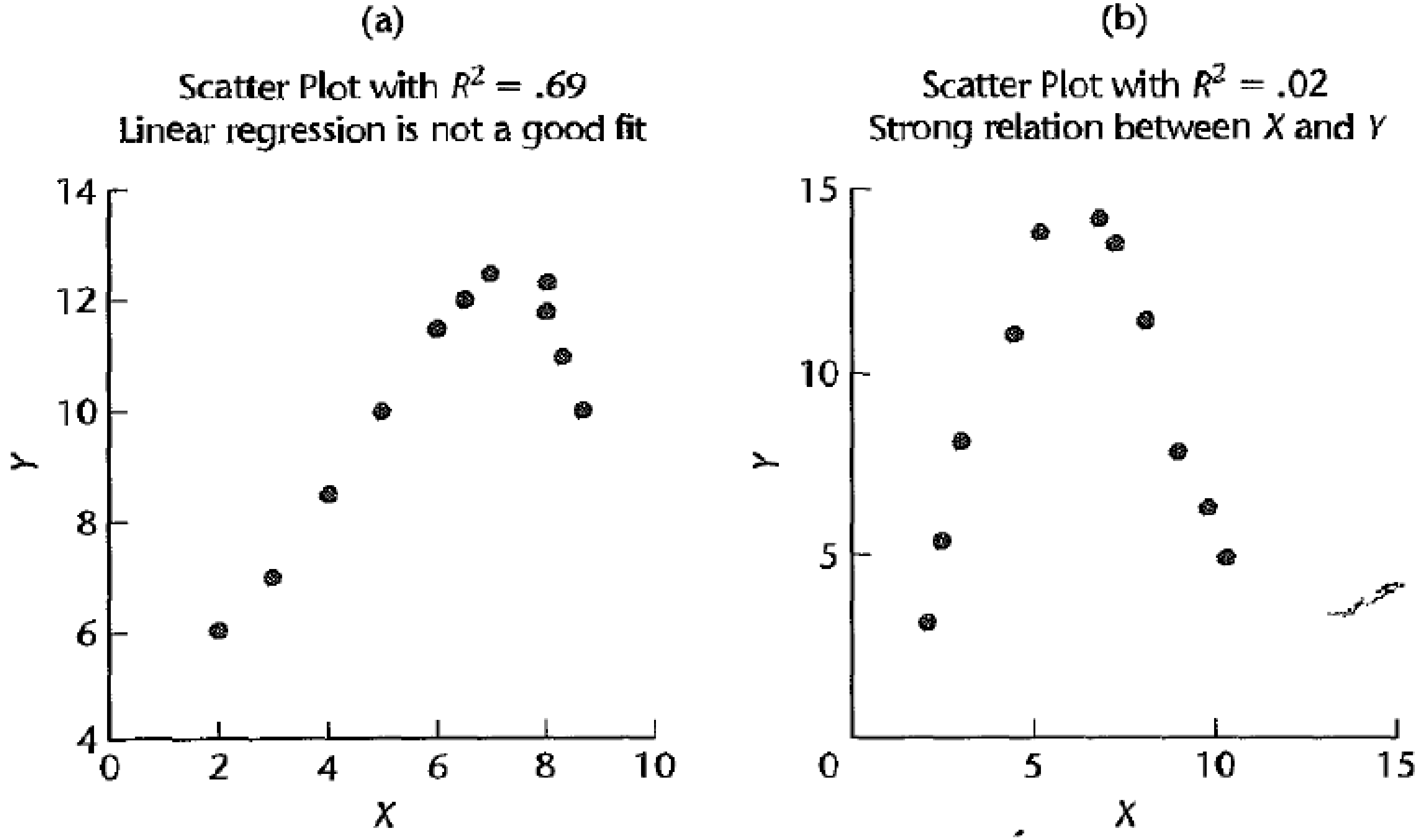
```
> summary(fitreg)

Call:
lm(formula = Hrs ~ Size, data = toluca)

Residuals:
    Min      1Q  Median      3Q     Max
-83.876 -34.088  -5.982  38.826 103.528

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   62.366     26.177   2.382   0.0259 *
Size           3.570      0.347  10.290 4.45e-10 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 48.82 on 23 degrees of freedom
Multiple R-squared: 0.8215,     Adjusted R-squared: 0.8138
F-statistic: 105.9 on 1 and 23 DF,  p-value: 4.449e-10
```

- The variation in work hours is reduced by 82.2% when lot size is considered.

Coefficient of Determination, cont'd

- R^2 is widely used for describing the usefulness of a regression model.
- Serious misunderstanding:
  - A high R^2 indicates that useful predictions can be made. (Not necessarily correct. Ex: $X_h = 100$)
  - A high R^2 indicated that the estimated regression line is a good fit. (Not necessarily correct. Curvilinear)
  - A R^2 near 0 indicated that X and Y are not related. (Not necessarily correct. Curvilinear)

Coefficient of Determination, cont'd



Coefficient of Correlation

- Measure of linear association between Y and X when both Y and X are random is the coefficient of correlation. This measure is the signed square root of R^2:

$$r = \pm\sqrt{R^2}$$

- A plus or minus sign is attached to this measure according to whether the slope of the fitted regression line is positive or negative. Thus, the range of r is: $-1 \leq r \leq 1$.

Example:

- For the Toluca Company example, we obtained $R^2 = .822$.
- Treating X as a random variable, the correlation coefficient here is:
- $r = \pm\sqrt{.822} = .907$
- The plus sign is affixed since b_1 is positive.