Convergence Analysis of Hopfield Neural Networks

T. Ensari[1], M.Erkan Yüksel[2]

[1] Istanbul University, Istanbul/Turkey, ensari@istanbul.edu.tr
[2] Harran University, Şanlıurfa/Turkey, erkan.yuksel@harran.edu.tr

***Abstract*—In this paper, we analyze the convergence and stability properties of Hopfield Neural Networks (HNN). The global convergence and asymptotic stability of HNN have successful various applications in computing and optimization problems. After determining the mathematical model of the network, we do some analysis on the model. This analysis base on Lyapunov Stability Theorem. Firstly, we define positive definite energy function for the system, then investigate the first derivative of this function. According to the Lyapunov Stability Theorem, the first derivative of function must be negative. We do some mathematical processes while proofing the theorem. At the end, we give a sufficient condition in the Theorem-1 for HNN which guarantees the convergence of the system. We give the definition of Lyapunov Stability Theorem and use it for the proof steps. Finally, we give some simulation results for the globally asymptotically stable system. It can be easily observed that the equilibrium point for the system goes to zero. Derived conditions are also supported by the previous results in the literature and the analyses contribute new sufficient condition to the literature.**

***Keywords*—Hopfield Neural Networks, Lyapunov Theorem, Stability Analysis.**

I. INTRODUCTION

THE Hopfield Neural Networks (HNN) was proposed in1982, by John J. Hopfield [1]. Neural networks research has faced many ups and downs in its history. The idea of creating a network of neurons got a boost when McCulloch and Pitts presented their model of the artificial neuron laying the foundations [10]. J. Hopfield brought his skills in physics to the world of neurobiology as part of a larger effort to better understand how the brain thinks [10].

The optimization ability of Hopfield networks has been studied for many years. Applications of Hopfield neural networks have been extended to many fields including pattern classification, parallel computing, associative memories, and especially to solving some optimization problems [1-3]. The global convergence and asymptotic stability of Hopfield neural networks are known to be bases of successful applications of networks in various computing and recognition tasks. The obtained results not only generalize the existing results, but also provide a theoretical foundation of performance analysis and new applications of the Hopfield networks [11].

The aim of this article is analyzing the convergence and stability properties of Hopfield Neural Networks (HNN). According to the network parameters the system will behave in different ways. Therefore, if we establish any sufficient condition about convergence criteria related with network parameters then we can apply it any optimization problems or different kinds of engineering problems.

The globally convergent dynamics implies that every trajectory of the network can converge to some equilibrium state, so that, when used as an associative memory, every state in the underlying space can serve as a key to recovering certain stored memory and therefore the state space is totally covered by distinct basins of the stored memories [12].

II. STABILITY ANALYSIS

The HNN model introduced by Hopfield in (1) has been studied in various applications, especially in optimization problems. The dynamical behavior of the model proposed by Hopfield is described by the following form of ordinary differential equations:

$$\frac{du_i(t)}{dt} = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) + I_i, \; j = 1,2,...,n$$

which stated in the vector-matrix form as follows:

$$\frac{du(t)}{dt} = -Au(t) + Wf(u(t)) + I \qquad (1)$$

where $u = [u_1, u_2, ..., u_n]^T$ is the neuron state vector, $A = diag(a_i)$ is a positive diagonal matrix, $W = (w_{ij})_{nxn}$ is the interconnection matrix representing the weight coefficients of the neurons, $I = [I_1, I_2, ..., I_n]^T$ is the constant external input vector and the $g(u) = [g_1(u_1), g_2(u_2), ..., g_n(u_n)]^T$ denotes the neuron activation functions.

We use Lyapunov Theorem as a tool for convergence analysis for HNN. According to Lyapunov Theorem, the system is globally asymptotically stable if the equilibrium point converges to zero. Firstly, positive definite energy function is described for the designed system. Some definitions and main stability results will be given in the below.

Definition (Lyapunov Theorem):
There exists a positive definite function V (x, t) and its partial derivatives $\dot{V}$ and $V(0,t)=0$, following statements are true

- If $\dot{V}\le 0$, then x_e is stable
- If $\dot{V}<0$, then x_e is asymptotically stable
- If $\dot{V}<0$ and $\lim_{\|x\|\to\infty} V=\infty$, then x_e is globally asymptotically stable.

(Here, x_e is an equilibrium point and Lyapunov positive definite function can be defined as an energy function.)

Firstly, we have shifted the equilibrium point to zero with this operator:

$$x(\cdot)=u(\cdot)-u^*$$

Let activation functions satisfy this inequality which is related boundary condition:

$$\left|f_j(x_j(t))\right|\le\sigma_j\left|x_j(t)\right| \qquad \forall x_j(t)\in R,\ j=1,2,\ldots,n$$

The following Lyapunov functional will be used for the stability analysis:

$$V(x(t))=x^T(t)Ax(t)+\sum_{i=1}^{n}d_i\int_0^{x_i}f_i(s)ds$$

where D is a positive diagonal matrix. Taking time derivative of the functional along the trajectories of system (1):

$$\begin{aligned}\dot{V}(x(t))=&-2x^T(t)A^2x(t)+2x^T(t)AWf(x(t))\\&-f^T(x(t))DAx(t)+f^T(x(t))DWf(x(t))\end{aligned}$$

We can write the following inequality:

$$-x^T(t)A^2x(t)+2x^T(t)AWf(x(t))$$

$$\le f^T(x(t))W^TWf(x(t))$$

Using this inequality in $\dot{V}(x(t))$ yields,

$$\begin{aligned}\dot{V}(x(t))\le&-x^T(t)A^2x(t)+f^T(x(t))W^TWf(x(t))\\&-f^T(x(t))DA\Sigma^{-1}f(x(t))+f^T(x(t))DWf(x(t))\end{aligned}$$

$$\begin{aligned}\dot{V}(x(t))=&-x^T(t)A^2x(t)\\&+f^T(x(t))(W^TW-DA\Sigma^{-1}+DW)f(x(t))\end{aligned}$$

$$\dot{V}(x(t))=-x^T(t)A^2x(t)+f^T(x(t))(\Omega)f(x(t))$$

Since, $-x^T(t)A^2x(t)<0,\quad \forall x_t\ne 0$ and for $\Omega=W^TW-DA\Sigma^{-1}+DW<0$ then it can be easily seen that,

$$\dot{V}(x(t))\le 0,\quad \forall f(x(t))\ne 0$$

Therefore, we have proved that $\dot{V}(x(t))=0$ if and only if $f(x(t))=x(t)=0$, otherwise $\dot{V}(x(t))\le 0$. $V(x(t))$ is also radially unbounded, that is $V(x(t))\to\infty$ as $\|x\|\to\infty$. Therefore, according to Lyapunov Theorem the origin of system (1) is globally asymptotically stable.

Now, we can give the following theorem about convergence analysis of HNN:

Theorem 1:

Suppose that in HNN system (1), an activation function satisfies

$$\left|f_j(x_j(t)\right|\le\sigma_j\left|x_j(t)\right| \qquad \forall x_j(t))\in R,\quad j=1,2,\ldots,n$$

and let $\Sigma=diag(\sigma_i>0)$. Under these assumptions if there exist a positive diagonal matrix D such that,

$$\Omega=W^TW-DA\Sigma^{-1}+DW<0$$

then, the origin of the system (1) is globally asymptotically stable.

III. Results

We have obtained a sufficient condition for global asymptotic stability of HNN and now we will give the simulation results with MATLAB with two examples. Different type activation functions and numerical values tested for the network. These are shown in Example 1 and Example 2. It can be easily shown from Figure 1 and Figure 2 that, equilibrium points goes to zero while time goes to infinity.
Example 1:
Let assume these numerical values for the two dimensional HNN.

$f(x(t))=0.5(|x+1|-|x-1|)$ is sigmoid activation function (Piecewise activation function) and A= [1 0 ; 0 1], W= [-1/4 1/4 ; -1/4 -1/4] , initial conditions are [0.8 ; -0.4] for the MATLAB simulation.

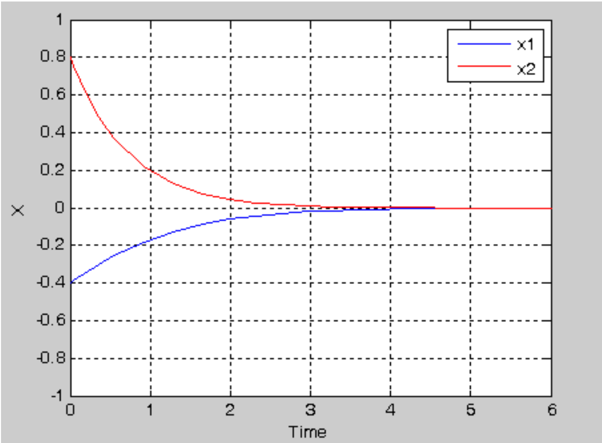

Figure 1: State responses for the Hopfield Neural Networks (HNN)

It can be easily seen that the equilibrium point of the system goes to zero, so it means that the system is globally asymptotically stable.

Example 2:
In this example, let assume these numerical values again for the two dimensional HNN.

$f(x(t)) = \tanh(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$ is sigmoid activation function (Tangent hyperbolic function) and A= [3 0 ; 0 3], W= [-1/2 1/2 ; -1/4 -1/4] , initial conditions are [0.5 ; -0.9] for the MATLAB simulation.

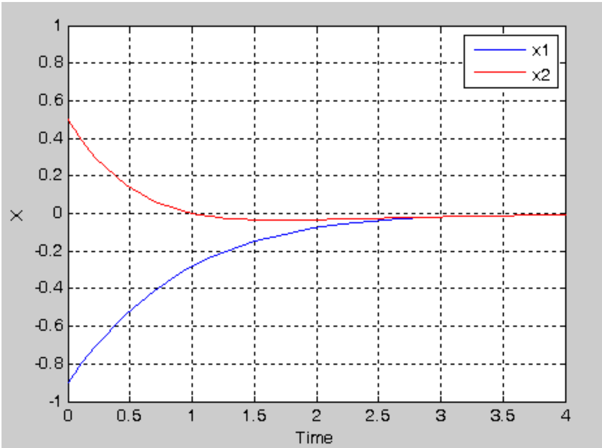

Figure 2: State responses for the Hopfield Neural Networks (HNN)

Again, in this simulation the equilibrium point goes to zero which means that the system is globally asymptotically stable.

IV. Conclusion

We have studied stability and convergence properties of Hopfield neural networks. Our analysis base on Lyapunov Stability Thereom. A sufficient stability criteria has been derived by employing Lyapunov functional. It has been shown that the system converges to stable equilibrium point and it is globally asymptotically stable. Numerical examples have been given for designed network with MATLAB and these simulations have sufficient results for the system and the study contribute new sufficient condition to the literature.

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