Chapter 1

Topology

1.1. Topological Spaces

1.1.1 Definition: *Topology*
A topology $\mathcal{T}$ on a set X is a collection of open sets such that

1. X and $\emptyset$ are open.
2. An arbitrary union of open sets is open.
3. A finite intersection of open sets is open.

1.1.2 Example: *Discrete Topology*
For any set X, the discrete topology is given by $\mathcal{T} = \{X, \emptyset\}$. This is a topology straight from (1) and since $X \cup \emptyset = X$, and $X \cap \emptyset = \emptyset$.

1.1.3 Definition: *Neighbourhood*
Let X be a topological space, and $x \in X$. The neighbourhood of X is a set $A \subset X$ containing x such that there exists an open set $U \subset A$ which also contains x.

1.1.4 Definition: *Open ball in R*
Let $\epsilon > 0$, and let $x \in \mathbb{R}$. An open ball $B_\epsilon(x)$ in $\mathbb{R}$ is the set

$$B_\epsilon(x) = \{y \in \mathbb{R} \mid |x - y| < \epsilon\}$$

In one dimension, the open ball is simply the interval $(x - \epsilon, x + \epsilon)$. In two dimensions, the open ball is the geometric sphere with radius "almost" equal to epsilon.

The need for a cohesive language to

1.1.5 Definition: *Interior*
Let X be a topological space. For any $A \subset X$, the interior of A, denoted $\operatorname{Int} A$ or A°, is the set of all points $a \in A$ such that there exists an open set $U \subset A$ that contains a.

$$A^\circ = \overline{A} \setminus A$$

1.1.6 Definition: *Limit Point*
Let X be a topological space, and let $A \subset X$. A limit point of A is a point $x \in X$ such that any neighbourhood B of x intersects with A; that is, $A \cap B \neq \emptyset$.

1.1.7 Definition: *Closure*
Let X be a topological space, and let $A \subset X$. The closure of A, denoted $\operatorname{Cl} A$ or $\overline{A}$, is the set of all limit points of A.

1.1.8 Definition: *Boundary*
Let X be a topological space, and let $A \subset X$. The boundary of A, denoted $\operatorname{Bd} A$ or ∂A, is the set of limit points of A that are not in A. In terms of closure, the boundary can be formulated as

$$\partial A = \overline{A} \setminus A$$

1.1.9 DEFINITION: *Continuous Function*
Let X, Y be topological spaces. A function $f : X \to Y$ is continuous at a point $y \in Y$ if for every neighbourhood U of y, the set $f^{-1}(U)$ is open in X.

1.1.10 DEFINITION: *Homeomorphism*
Let X, Y be topological spaces. The space X is homeomorphic to Y if there exists a function $f : X \to Y$ such for that every open set $U \in Y$, $f^{-1}(U)$ is open in X, and for every open set $V \in X$, $f(V)$ is open in Y.

1.1.11 DEFINITION: *Connectedness*
Let X be a topological space. If there are two open sets U, V such that $U \cup V = X$, then X is said to be separated. Otherwise, X is connected.

1.1.12 DEFINITION: *Open Cover*
Let X be a topological space. An open cover of X is a collection of open sets $\{U_\alpha\}_{a \in A}$ such that

$$X \subset \bigcap_{\alpha \in A} U_\alpha$$

1.1.13 DEFINITION: *Compactness*
Let X be a topological space. The space X is compact if for every open cover $\{U_\alpha\}_{\alpha \in A}$ of X, there exists a finite subcover $\{V_i\}_{i \in I}$ such that

$$X \subset \{V_i\}_{i \in I}$$

1.2. Metric Spaces

1.2.1 DEFINITION: *Metric Space*
Let X be a topological space. The space X is said to be a metric space if, for all $x, y \in X$, the function $d(x, y) : X \times X \to [0, \infty]$ satisfies the following properties:

1. (Non-negativity) $d(x, y) \geq 0$; $d(x, y) = 0$ if and only if $x = y$
2. (Symmetry) $d(x, y) = d(y, x)$
3. (Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$

We write (X, d) to denote that X is a metric space with the metric d.

1.2.2 REMARK: *Euclidean Metric*
On the topological space $\mathbb{R}^n$, we define the Euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$d_{l2}(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{0.5}$$

The Euclidean metric is a metric, since (1) follows trivially from the

$$\begin{aligned} d_{l2}(x, y) + d_{l2}(y, z) &= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} \\ &= ||x - y|| + ||y - z|| \\ &\geq ||x - z|| \end{aligned}$$

1.2.3 Remark: *Taxi-cab Metric*
The taxi-cab metric

$$\sum_{i=1}^{n} |x_i - y_i|$$

Triangle Inequality:

$$d_{l1}(x,y) + d_{l1}(y,z) = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| \qquad\qquad = dw$$

1.2.4 Remark: *Discrete Metric*
The discrete metric is given by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Chapter 2

Integration in one dimension

2.0.5 Remark: *What is integration?*
Roughly speaking, integration is the process of finding the measure of a function's range. Measure in this context refers to a generalization of the notions of length, area and volume in higher dimensions - that is, measure assigns a real number to every measurable set. The integral of a measurable set is an assignment of a real number

2.0.6 Remark: *Defining integration in one dimension*
The purpose of defining integration in one-dimension is strictly pedagogical.

2.0.7 Remark: *Measurable sets*
bnn

2.0.8 Definition: *Rectangle*
In $\mathbb{R}^2$, a rectangle is a set of the form $\{(x, y) \mid x, y \in [0, 1]\}$.

2.0.9 Definition: *Axiomatic construction of area*
Let $\mathbb{M}$ be a class of measurable sets. We define the set function $a : \mathbb{M} \to \mathbb{R}$ as the *area* function, which, for all $S, T \in \mathbb{M}$, obeys the following properties:

1. (Choice of scale)
2. (Non-negativity) $a(S) \geq 0$.
3. (Additive property)

2.0.10 Definition: *Partition*
A partition P of an interval $[a, b]$ is a finite collection of points $t_0, t_1, \ldots, t_n$ such that

$$a = t_0 < t_1 < \cdots < t_n = b$$

Chapter 3

LINEAR ALGEBRA

3.1. Vectors

3.1.1 DEFINITION: *Vector Space*
A vector space is a set $\mathbb{V}$, endowed with a field of scalars F, (denoted $\mathbb{V}(F)$), such that

1. (Associativity) $(u+v)+w = u+(v+w)$
2. (Commutativity) $u+v = v+u$
3. (Distributivity) $a(u+v) = au+av$ and $(a+b)v = av+bv$
4. (Additive identity) There exists $0 \in V$ such that $0+x = x$
5. (Multiplicative identity) There exists $1 \in V$ such that $1 \cdot x = x$.
6. (Additive inverse) For each $x \in V$, there exists an $(-x) \in V$, called the inverse of x, such that $x+(-x) = 0$.

3.1.2 NOTATION: *Vector spaces over fields*
Every vector space has scalars which are elements of a field F, usually $\mathbb{R}$ or $\mathbb{C}$. We say a vector space $\mathbb{V}$ is *over* a field F to denote that the field of scalars is F, and use the notation $\mathbb{V}(\mathbb{F})$. A vector space over $\mathbb{R}$ is called a real vector space, and a vector space over $\mathbb{C}$ is called a complex vector space.

3.1.3 THEOREM: $\mathbb{R}^n$ *is a vector space*
Let $\mathbb{R}^n$ be the set of all n-tuples of the form $v = (v_1, \dots, v_n)$, where $v_1, \dots, v_n \in \mathbb{R}$. Addition is defined as $u+v = (u_1+v_1, \dots, u_n+v_n)$. Scalar multiplication, for every $a \in \mathbb{R}$, is defined as $av = (av_1, \dots, av_n)$. Then the set $\mathbb{R}^n$ is a vector space.

PROOF: Let u, v, w be elements of $\mathbb{R}^n$ such that $u = (u_1, \dots, u_n)$ and so forth. We check every axiom of vector space in order to ensure that $\mathbb{R}^n$ is a vector space. (1 - Associativity):

$$\begin{aligned}
(u+v)+w &= ((u_1, \dots, u_n) + (v_1, \dots, v_n)) + (w_1, \dots, w_n) \\
&= (u_1+v_1, \dots, u_n+v_n) + (w_1, \dots, w_n) \\
&= (u_1+v_1+w_1, \dots, u_n+v_n w_n) \\
&= (u_1, \dots, u_n) + (v_1+w_1, \dots, v_n+w_n) \\
&= (u_1, \dots, u_n) + ((v_1, \dots, v_n) + (w_1, \dots, w_n)) \\
&= u+(v+w)
\end{aligned}$$

(2 - Commutativity):

$$u+v = (u_1+v_1, \dots, u_n+v_n) = (v_1+u_1, \dots, v_n+u_n) = v+u$$

(3 - Distributivity):

$$\begin{aligned}
a(u+v) &= a(u_1+v_1, \dots, u_n+v_n) \\
&= (a(u_1+v_1), \dots, a(u_n+v_n)) \\
&= (au_1+av_1, \dots, au_n+av_n) \\
&= (au_1, \dots, au_n) + (av_1, \dots, av_n) \\
&= a(u_1, \dots, u_n) + a(v_1, \dots, v_n) \\
&= au+av
\end{aligned}$$

and

$$\begin{aligned}(a+b)v &= ((a+b)v_1, \dots, (a+b)v_n) \\ &= (av_1 + bv_1, \dots, av_n + bv_n) \\ &= (av_1, \dots av_n) + (bv_1, \dots, bv_n) \\ &= a(v_1, \dots, v_n) + b(v_1, \dots, v_n) \\ &= av + bv\end{aligned}$$

(4 - Additive Identity):

$$0 + v = (0, \dots, 0) + (v_1, \dots, v_n) = (v_1, \dots, v_n) = v = (v_1, \dots, v_n) + (0, \dots, 0) = v + 0$$

□

3.1.4 THEOREM: *Multiplicative inverse*
Let $v \in V$. There exists a vector $V^{-1} \in V$ such that

$$v \cdot v^{-1} = 0 = v^{-1}$$

3.1.5 REMARK: *Linearity*
dsd

3.1.6 DEFINITION: *Inner Product*
An inner product $\langle \cdot, \cdot \rangle$ on a vector space V

1. $\langle a(x+y), z \rangle = a\langle x, z \rangle + a\langle y, z \rangle$
2. $\langle a, b \rangle \geq 0$
3. $\overline{\langle x, y \rangle} = \langle y, x \rangle$

3.1.7 EXAMPLE: *Dot Product*
Let a, b be two vectors in $\mathbb{R}^n$. The dot product of a and b is defined as

$$\sum_{i=1}^{n} a_i b_i$$

3.1.8 THEOREM: *Cauchy-Schwartz Inequality*

$$(A \cdot B)^2 \leq (A \cdot A)(B \cdot B)$$

PROOF: If either A or B is the zero vector, then the two sides are equal. If neither A nor B is the zero vector, then

□

3.1.9 REMARK: *A suitable definition of orthogonality*
In Euclidean 2-space, two lines are orthogonal (perpendicular) when their angle of intersection is 90°. To generalize this notion of orthogonality to higher dimensions

3.1.10 DEFINITION: *Orthogonal vectors*
Two vectors x, y are called orthogonal if and only if $a \cdot b = 0$.

3.1.11 REMARK: *Constructing an orthogonal*
dnds

3.1.12 DEFINITION: *Cross Product*
Let $a, b \in \mathbb{V}(\mathbb{R}^3)$. The cross product $a \times b$ is defined as

$$a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

3.1.13 THEOREM: *The cross product is orthogonal to both its vectors*
Let $a, b \in \mathbb{V}(\mathbb{R}^n)$. Then

$$a \cdot (a \times b) = 0 = b \cdot (a \times b)$$

PROOF:

$$\begin{aligned}
a \cdot (a \times b) &= (a_1, a_2, a_3) \cdot (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \\
&= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) \\
&= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1 \\
&= 0
\end{aligned}$$

Likewise,

$$\begin{aligned}
b \cdot (a \times b) &= (b_1, b_2, b_3) \cdot (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \\
&= b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) \\
&= a_2b_1b_3 - a_3b_1b_2 + a_3b_1b_2 - a_1b_2b_3 + a_1b_2b_3 - a_2b_1b_3 \\
&= 0
\end{aligned}$$

□

Chapter 4

Analytic Geometry

4.0.14 Remark: *Synthetic versus analytic geomtry*
The origins of synthetic geometry predate to first ever axiomatic treatment of mathematics

4.0.15 Remark: *The natural setting of vector spaces*
dfjf

4.1. Lines

4.1.1 Remark:
dsd

4.1.2 Definition: *Line*
Let P be a point, let A be a non-zero vector. The set of all points $P + tA$, where $t \in \mathbb{R}$, is called a line through P parallel to A. This is denoted

$$L(P; A) = \{P + tA | t \in \mathbb{R}\}$$

4.1.3 Theorem: *Two*
Two parallel lines

4.1.4 Definition: *Plane*
Let P be a point, let $x, y \in \mathbb{V}(\mathbb{R}^n)$ be linearly independent. A plane in $\mathbb{V}(\mathbb{R}^n)$ is defined as

$$M = \{P + sA + tB \mid s, t \in \mathbb{R}\}$$

Chapter 5

Calculus of Vector-Valued Functions

5.1. Algebriac Properties of Vector-valued Functions

5.1.1 Definition: *Vector-valued functions*
A function f whose domain is $\mathbb{R}$ and whose co-domain is a vector n-space.

5.1.2 Example: *A line in Euclidean space*
The line through a point P with a direction vector A is the range of the vector-valued function $X(t) = P + tA$.

5.1.3 Definition: *I*
f $f : \mathbb{R} \to \mathbb{V}(\mathbb{R}^n)$ is a vector valued function, we define the following

$$lim_{x\to a} f(t) = (lim_{x\to a} f_1(t), \ldots, lim_{x\to a} f_n(t))$$
$$f'(t) = (f_1'(t), \ldots, f_n'(t))$$
$$\int_a^b f(t) \frac{\mathrm{d}}{\mathrm{d}t} = (\int_a^b f_1(t) \frac{\mathrm{d}}{\mathrm{d}t}, \ldots, \int_a^b f_n(t) \frac{\mathrm{d}}{\mathrm{d}t})$$

Chapter 6

Polynomial Approximations

6.1. Taylor Polynomials

Chapter 7

Differentiation

7.1. Limits and Continuity

7.1.1 Notation: *Scalar and Vector Fields*
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. The function is called a scalar field (also called a scalar-valued function) if $n \geq 1$ and $m = 1$. On the other hand, the function is called a vector field (also called a vector-valued function) if $n \geq 1$ and $m > 1$.

7.1.2 Notation: *Components of a vector field*
For a vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$, the *components* of f are

$$f = (f_1(x), f_2(x), \ldots, f_m(x))$$

7.1.3 Theorem: *Continuity and components of a vector field*
The vector field $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at a point $a \in \mathbb{R}^n$ if and only if the components of f, $(f_1(x), f_2(x), \ldots, f_m(x))$ are all continuous at a.

Proof: $(\Rightarrow)$ Let $e_1, \ldots, e_m$ be the unit vectors in $\mathbb{R}^m$. Then $f_k(x)$ is given by

$$f_k(x) = f(x) \cdot e_k$$

which, by the limit laws, means that $f_k(x)$ is continuous at a.

$(\Leftarrow)$ Since

$$f(x) = \sum_{i=1}^{m} f_i(x) \cdot e_i$$

then by the limit laws, $f(x)$ is continuous at a. $\square$

Chapter 8

INTEGRATION

8.1. Riemann Integral

8.1.1 DEFINITION: *Rectangle in R^n*
We define a rectangle Q in $\mathbb{R}^n$ to be the Cartesian product of intervals $[a_i, b_i]$:

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

8.1.2 NOTATION: *The width and volume of a rectangle*
We define the *width* of a rectangle in $\mathbb{R}^n$ to be the maximum of the numbers $a_1 - b_1, \ldots, a_n - b_n$. The *volume* of the rectangle is simply

$$v(Q) = (a_1 - b_1)(a_2 - b_2) \ldots (a_n - b_n)$$

8.1.3 DEFINITION: *Partition*
A partition P of an interval $[a, b]$ is a finite collection of points $t_0, t_1, \ldots, t_n$ such that

$$a = t_0 < t_1 < \cdots < t_n = b$$

In the higher-dimensional case, a partition P of a rectangle Q is an n-tuple $(P_1, \ldots, P_n)$ such that P_i is a partition for the interval $[a_i, b_i]$.