Introduction to Linear Models

FGCZ Protein Informatics Training

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linear models (lm)

Overview

- What is a linear regression model?
- How to estimate coefficients
- What are contrasts
- How to determine the error of the coefficients, test statistics and p-values
- What are interactions in linear models
- Example: Yeast data with batches
- limma - Empirical Bayes
- Benchmarking
- Conclusions

What is a linear regression model?

By a model, we mean a mathematical representation about the process that generated our observed data. That is, how an outcome of interest Y is related to one or more predictor X variables.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$$

Linear models, or regression models, trace the the distribution of the dependent variable Y – or some characteristic of the distribution (the mean) – as a function of the independent variables X.I

- simple linear regression - one explanatory variable;
- multiple linear regression - more than one explanatory variable

https://uvastatlab.github.io/phdplus/linearmodel.html

lm intro

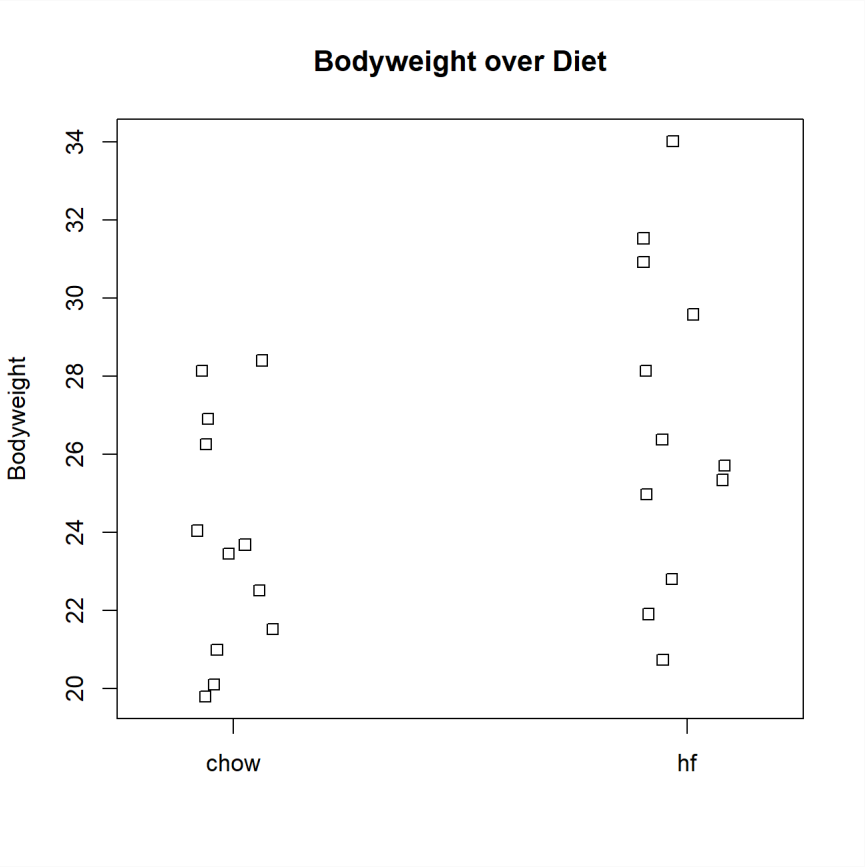
```
dat ← read.csv("femaleMiceWeights.csv")
head(dat)
```

```
##   Diet Bodyweight
## 1 chow      21.51
## 2 chow      28.14
## 3 chow      24.04
## 4 chow      23.45
## 5 chow      23.68
## 6 chow      19.79
```

```
table(dat$Diet)
```

```
##
## chow   hf
##   12   12
```

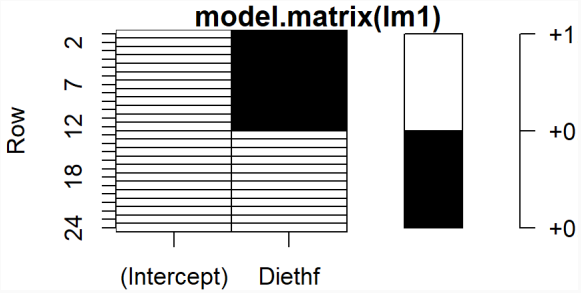
```
stripchart(Bodyweight ~ Diet, # < formula interface
           data= dat,
           vertical=TRUE,
           method="jitter",
           main="Bodyweight over Diet")
```



lm intro

```r
# compute group means
meansum ←
  dat %>%
  group_by(Diet) %>%
  summarise(mean = mean(Bodyweight))
# use linear model
lm1 ← lm(Bodyweight ~ Diet,
         data = dat)
coefs ← coef(lm1)
```

$$y = b_0 X_0 + b_1 X_1 + \epsilon$$



group means

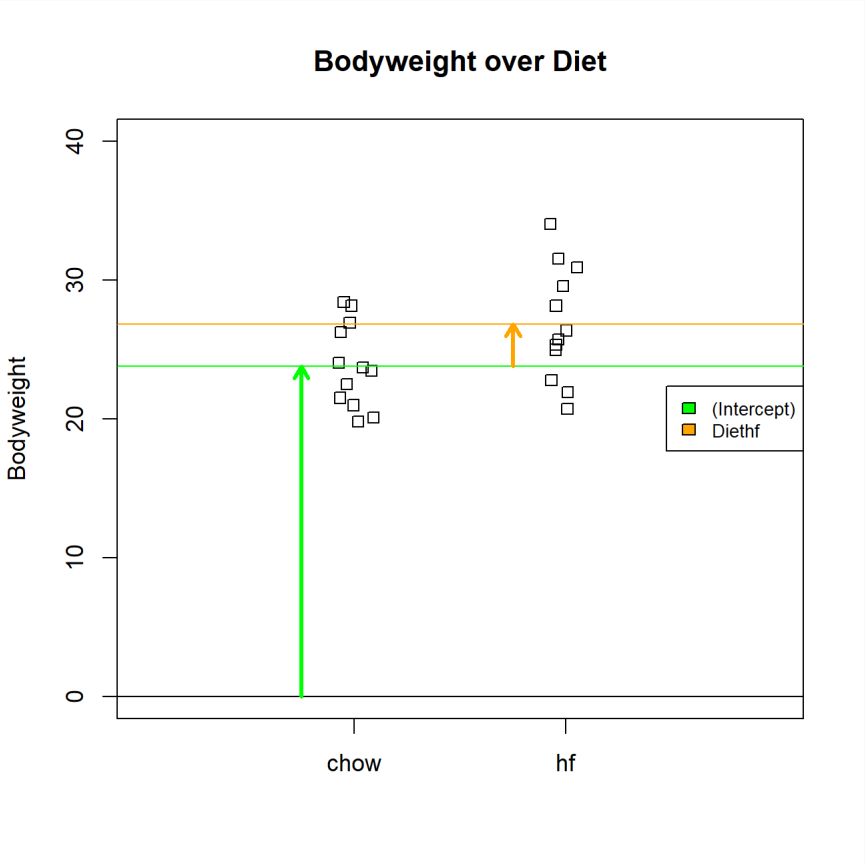
Diet	mean
chow	23.81333
hf	26.83417

coefficients

	x
(Intercept)	23.813333
Diethf	3.020833

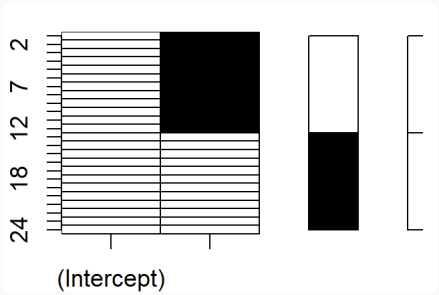
lm intro - examin the coefficients

```
stripchart(Bodyweight ~ Diet,
           data = dat , vertical=TRUE,
           method="jitter",
           main="Bodyweight over Diet",
           ylim=c(0,40), xlim=c(0,3))
a ← -0.25; lgth ← .1
abline(h=0)
arrows(1+a,0,1+a,coefs[1],lwd=3,
       col="green",length=lgth)
abline(h=coefs[1],col="green")
arrows(2+a,coefs[1],2+a,coefs[1]+coefs[2],
       lwd=3,col="orange",length=lgth)
abline(h=coefs[1]+coefs[2],col="orange")
legend("right",names(coefs),
       fill=c("green","orange"),
       cex=.75,bg="white")
```



lm intro - determining the coefficients

```
Y ← dat$Bodyweight
X ← model.matrix(lm1)
par(mar = c(2,2,1,1))
plot(X, col=c("black", "white"), main="")
```




```
beta ← solve(t(X) %*% X) %*% (t(X) %*% Y)
epsilon ← Y - t(beta) %*% t(X)
beta
```

```
##                  [,1]
## (Intercept) 23.813333
## Diethf       3.020833
```

$$\beta = (X^T X)^{-1}(X^T Y)$$

β minimizes

$$\sum (Y - X\beta)^2 = (Y - X\beta)(Y - X\beta)^T$$

.

predicting Y

$$\hat{Y} = X\beta = b_0 X_0 + b_1 X_1$$

residues

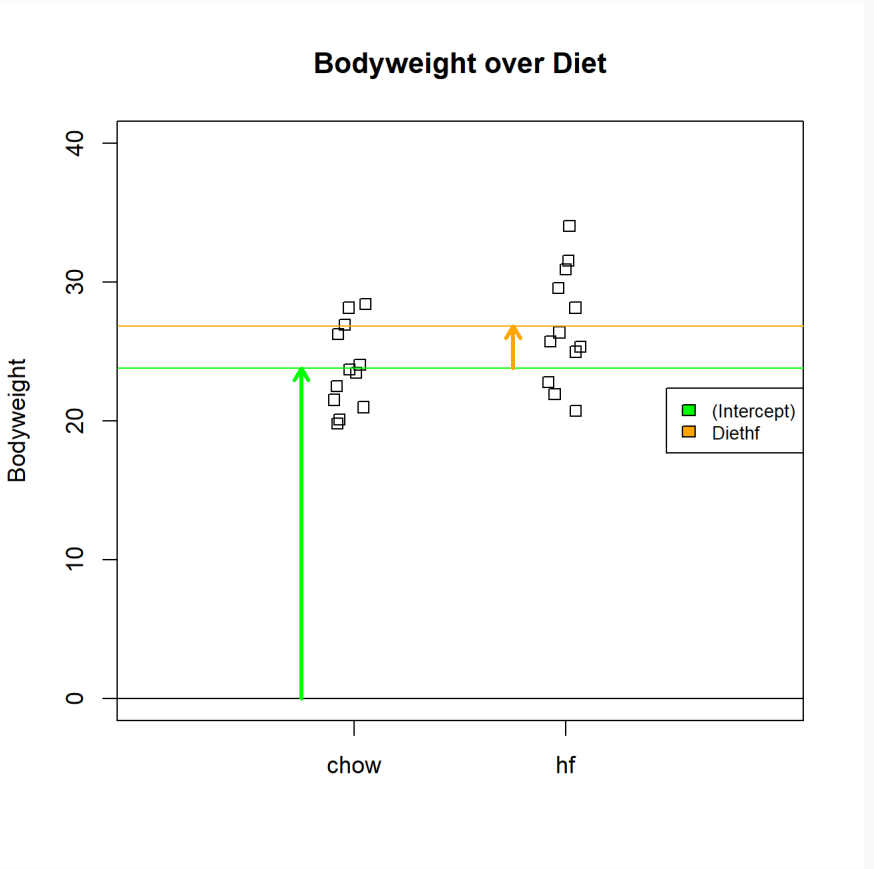
$$e = Y - X\beta$$

lm intro - group mean

```
linfct ← rbind(
  chow = c(1, 0),
  hf = c(1, 1 )
  )
linfct %*% coef(lm1)
```

```
##          [,1]
## chow 23.81333
## hf   26.83417
```

$$Y_{chow} = b_0 \cdot 1 + b_1 \cdot 0$$
$$Y_{hf} = b_0 \cdot 1 + b_1 \cdot 1$$



lm intro - contrasts

A contrast is a linear combination of variables (parameters or statistics) whose __coefficients add up to zero, allowing comparison of different treatments.

$$Y_{chow-hf} = (1) \cdot Y_c + (-1) \cdot Y_h$$

```
contrasts ← rbind(
  "chow - hf" =
    1 * linfct["chow",] + -1 * linfct["hf",]
)
contrasts
```

```
##           [,1] [,2]
## chow - hf    0   -1
```

```
contrasts %*% coef(lm1)
```

```
##                [,1]
## chow - hf -3.020833
```

lm intro - LSE standard error

```
epsilon  ← resid(lm1)
sigma ←
  sqrt(sum(epsilon ^2)/
         (length(epsilon ) -
            length(coef(lm1))) )

X ← model.matrix(lm1)
solve(t(X) %*% X) * sigma^2
```

```
##             (Intercept)    Diethf
## (Intercept)    1.080255 -1.080255
## Diethf        -1.080255  2.160510
```

```
vcov(lm1)
```

```
##             (Intercept)    Diethf
## (Intercept)    1.080255 -1.080255
## Diethf        -1.080255  2.160510
```

$$
\begin{aligned}
var(\hat{\beta}) &= var((X^\top X)^{-1} X^\top Y) \\
&= \cdots \\
&= \sigma^2 (X^\top X)^{-1} \\
with \quad \sigma^2 &= \sum e^2/(n-p)
\end{aligned}
$$

LSE = least square estimate

lm intro - computing the test statistic

```
std.error ← sqrt(diag(
  linfct %*%
  vcov(lm1) %*%
  t(linfct)))
t.statistic ←
  linfct%*%coef(lm1)/std.error
t.statistic
```

```
##          [,1]
## chow 22.91168
## hf   25.81814
```

```
std.error ← sqrt(diag(
  contrasts %*%
    vcov(lm1) %*%
    t(contrasts)))
t.statistic ←
  contrasts %*% coef(lm1) / std.error
t.statistic
```

```
##               [,1]
## chow - hf -2.055174
```

$$t_i = \frac{\beta_i}{se(\beta_i)}$$

```
head(linfct)
```

```
##      [,1] [,2]
## chow    1    0
## hf      1    1
```

```
head(contrasts)
```

```
##           [,1] [,2]
## chow - hf    0   -1
```

lm intro - getting the p-values

```
lfq ←
  prolfqua::my_contrast( lm1,
  rbind(linfct, contrasts)) %>%
  dplyr::select(lhs, estimate,
                std.error,
                statistic, p.value) %>%
  mutate(p.value =
           round(p.value,digits=3))
```

prolfqua

	lhs	estimate	std.error	statistic	p.value
chow	chow	23.81	1.04	22.91	0.00
hf	hf	26.83	1.04	25.82	0.00
chow - hf	chow - hf	-3.02	1.47	-2.06	0.05

The p-value is a function of the t-statistics and the degrees of freedom.

lm intro - getting the p-values (adjusted)

```
library(multcomp)
multcomp ← summary(
  multcomp::glht(lm1, rbind(linfct, contrasts))) %>%
  broom::tidy() %>%
  dplyr::select(contrast, estimate,
                std.error, statistic ,
                adj.p.value)
```

multcomp

contrast	estimate	std.error	statistic	adj.p.value
chow	23.81	1.04	22.91	0.00
hf	26.83	1.04	25.82	0.00
chow - hf	-3.02	1.47	-2.06	0.12

Testing of many (multiple) contrasts, requires adjusting of p-values.

lm intro - multiple linear regression

- repsonse viariable μ friction coefficient
- explanatory viariables:
  - leg with levels: L1 L2 L3 L4
  - movement type with levels: push and pull

https://www.nature.com/articles/srep01101

lm intro - multiple linear regression

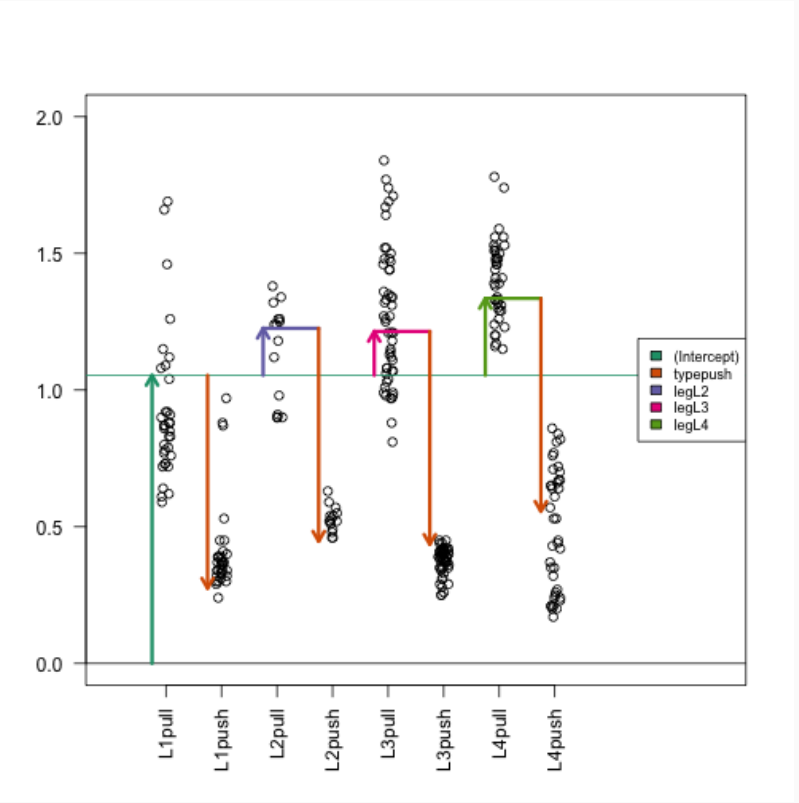
Data and Model with 2 factors

```
spider ← read.csv("spider_wolff_gorb_2013.csv"
                  , skip = 1)
table(spider$leg, spider$type)
```

```
##
##      pull push
##   L1   34   34
##   L2   15   15
##   L3   52   52
##   L4   40   40
```

```
noI ← lm(friction ~ type + leg, data = spider)
coef(noI)
```

```
## (Intercept)    typepush       legL2       legL3       legL4
##   1.0539153  -0.7790071   0.1719216   0.1604921   0.2813382
```



lm intro - interactions

Think of interaction effects as an **"it depends"** effect.
Q : "Do you prefer ketchup or chocolate
sauce on your food?"

A : "**It depends** on the type of food!"

You cannot answer the question without **more** information about the other variable in the interaction term.

https://statisticsbyjim.com/regression/interaction-effects/

lm intro - interactions

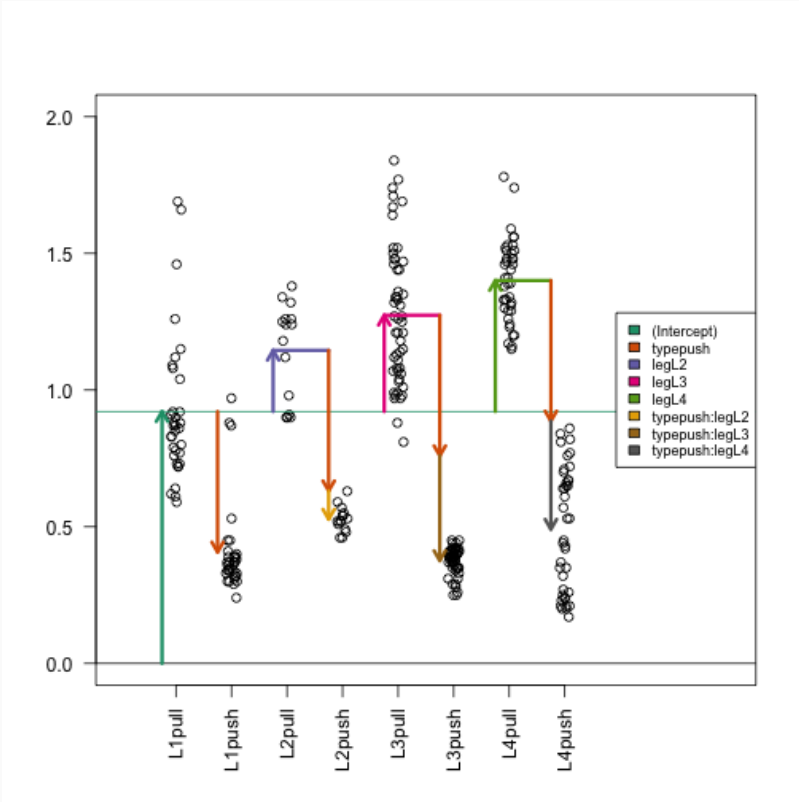
Model with interactions

```
withI ← lm(friction ~ type + leg + type:leg,
           data = spider)
an ← anova(withI)
broom::tidy(an)[1:3,c("term", "p.value")]
```

```
## # A tibble: 3 x 2
##   term       p.value
##   <chr>        <dbl>
## 1 type     2.75e-101
## 2 leg      2.97e- 15
## 3 type:leg 2.26e- 11
```

```
coef(withI)
```

```
##    (Intercept)       typepush          legL2          legL3          legL4 typepush:legL2
##      0.9214706     -0.5141176      0.2238627      0.3523756      0.4792794     -0.1038824
## typepush:legL3 typepush:legL4
##     -0.3837670     -0.3958824
```



Yeast analysis - batches

- Condition : enthanol and glucose
- Batch: p2691 (12 to 16 March 2018) and p2370 (March 2017)

R linear model:

```
lm(normalizedIntensity ~ Condition + Batch + Condition:Batch, data = proteinData)
```

And we are going to compute the following contrasts ($\log_2(FC)$):

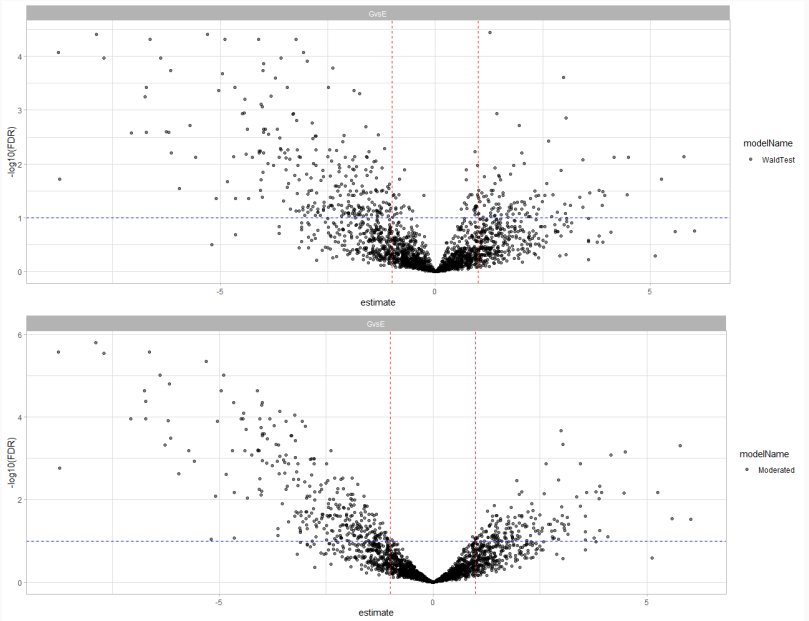
$$\begin{aligned}
\text{fc}_{glucose-ethanol} \\
\text{fc}_{p2370-p2691} \\
\text{fc}_{glucose:p2370-ethanol:p2370} \\
\text{fc}_{glucose:p2691-ethanol:p2691} \\
\text{fc}_{interaction} &= \text{fc}_{glucose:p2370-ethanol:p2370} - \text{fc}_{glucose:p2691-ethanol:p2691}
\end{aligned}$$

For details see : https://wolski.github.io/prolfqua/articles/Modelling2Factors.html

limma - Empirical Bayes

Both volcano plots are generated from same data.

What is the difference between the two Volcanos?



https://wolski.github.io/prolfqua/articles/Comparing2Groups.html

limma - Empircial Bayes

- In a mass spectrometric LFQ experiment, we measure hundreds of proteins in parallel.
- Also, the analysis has a parallel structure, and we fit the same linear model to all protein.
- These measurements are correlated.

- Potentially we can transfer information from the measurement of one peptide/protein to the other.
- The empirical Bayes approach is used to improve the test statistic to test the null hypothesis $H_0 : \beta_{pj} = 0$.

"Linear models and empirical bayes methods for assessing differential expression in microarray experiments" Gordon K. Smyth

limma - Empirical Bayes

Define the moderated t-statistic by:

$$\tilde{t}_{pj} = \frac{\hat{\beta}_{pj}}{\tilde{s}_p \sqrt{v_{pj}}}$$

with p protein index j parameter index, v element of the variance covariance matrix, $\hat{\beta}$ model parameter, $\tilde{s}$ posterior standard error.

The posterior values shrink the observed variances towards the prior values with the degree of shrinkage depending on the relative sizes of the observed and prior degrees of freedom.

$$\tilde{s}_p^2 = E(\sigma^2 | s_p^2) = \frac{d_0 s_0^2 + d_p s_p^2}{d_0 + d_p}$$

where d are the degrees of freedom.

This statistic represents a hybrid classical/Bayes approach in which the posterior variance \(\\) has been substituted into the classical t-statistic in place of the usual sample variance.

limma - Empirical Bayes

$$\tilde{s}_p^2 = E(\sigma^2|s_p^2) = \frac{d_0 s_0^2 + d_p s_p^2}{d_0 + d_p}$$

For $d_p << d_0$, $\tilde{s} \rightarrow s_0$
For $d_p >> d_0$, $\tilde{s} \rightarrow s_p$

s_0 is the same for all proteins in and experiment.

What happens with the $cor(T, \log_2 FC)$ for $d_p \rightarrow 0$ sample sizes? Where T is the t-statistics and $\log_2 FC$ is the difference between samples.

Hint: $T \propto log_2 FC / \tilde{s}$

```
d_0 = 4; s2_0 = 2;
s2_p = 6;
d_p = 4;
(d_0*s2_0 + d_p*s2_p)/(d_0 + d_p)
```

```
## [1] 4
```

```
d_p = 8
(d_0*s2_0 + d_p*s2_p)/(d_0 + d_p)
```

```
## [1] 4.666667
```

```
d_p = 12
(d_0*s2_0 + d_p*s2_p)/(d_0 + d_p)
```

```
## [1] 5
```

```
d_p = 1
(d_0*s2_0 + d_p*s2_p)/(d_0 + d_p)
```

```
## [1] 2.8
```

What we learned so far

For each protein we obtain

- $\log_2 FC$
- T-statistic
- p-values
- FDR

What are each of these measures is good for?

- $2^{\log_2 FC}$ and 95% confidence intervals - have a biological interpretation.

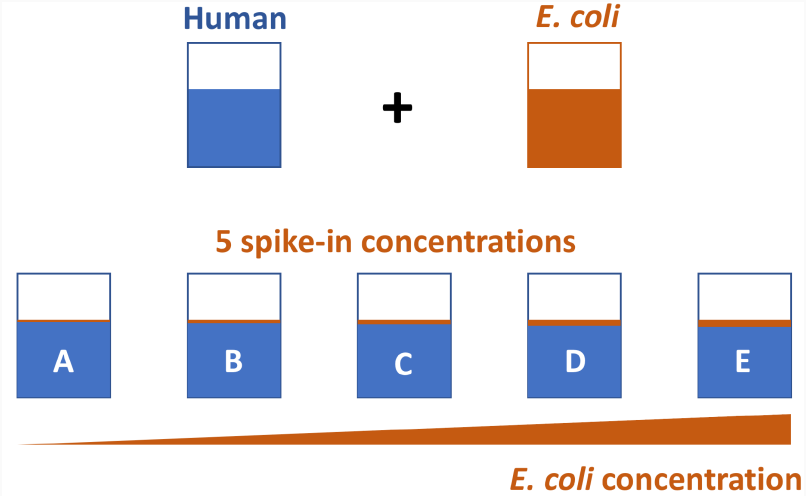
- FDR can be used to select proteins for follow up experiments.
- $\log_2 FC$, T-statistic, p-value and FDR can be used to rank proteins for GSEA

Is the FDR estimate correct? What measure is best to rank proteins for GSEA?

Benchmarking - The Ionstar dataset

Table: All possible pairs of E. coli concentrations with the expected fold-changes.

c1	c2	fc
7.5	9.0	1.20
6.0	7.5	1.25
4.5	6.0	1.33
3.0	4.5	1.50
6.0	9.0	1.50
4.5	7.5	1.67
3.0	6.0	2.00
4.5	9.0	2.00
3.0	7.5	2.50
3.0	9.0	3.00



- Human proteins $\beta = 0$
- E. Coli $\beta \neq 0$

https://www.pnas.org/content/115/21/E4767

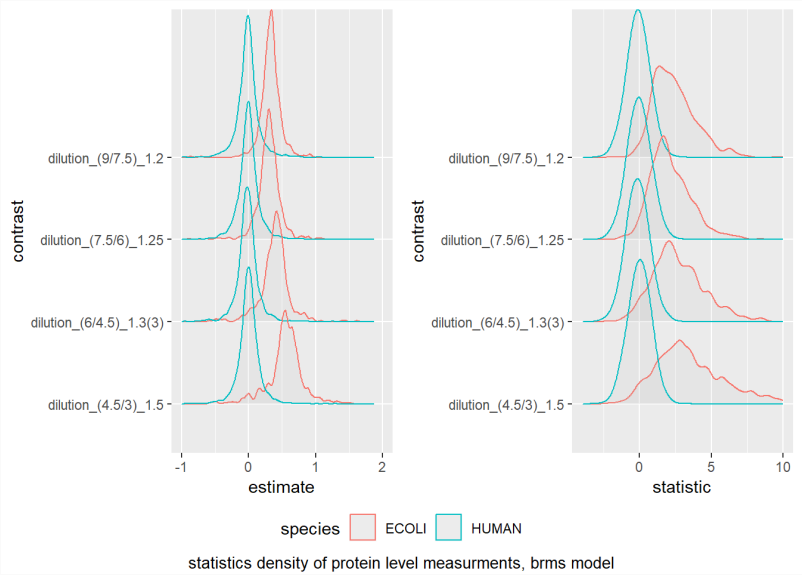
Benchmarking

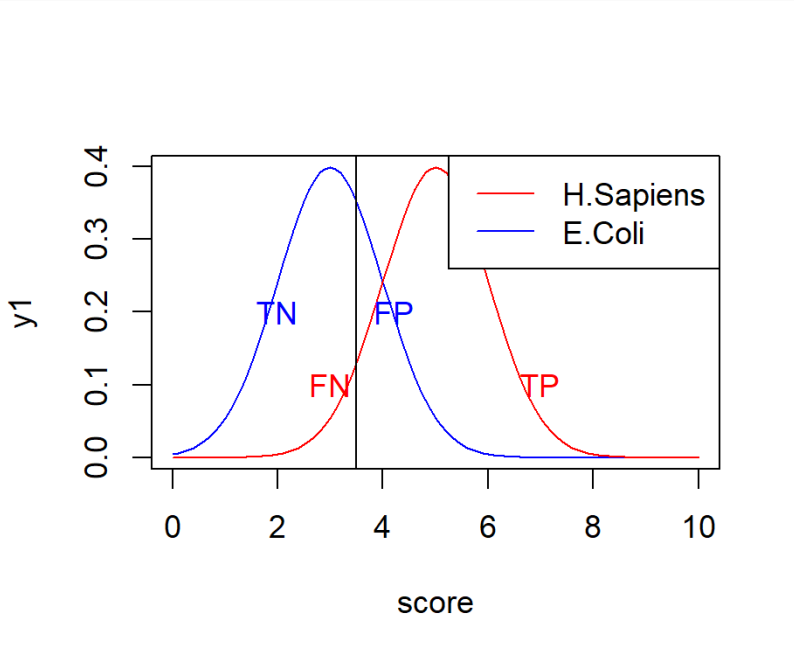
Table: Confusion matrix, TP - true positive, FP - false positive, FN - false negative, P - all positive cases (all E. coli proteins), N - all negative cases (all H. sapiens proteins), m- all proteins.

Prediction \ Truth	**E.coli**	**H.sapiens**	**Total**
beta != 0	TP	FP	R
beta = 0	FN	TN	
Total	P	N	m

$$TPR = \frac{TP}{TP + FN} = \frac{TP}{P}$$
$$FPR = \frac{FP}{FP + TN} = \frac{FP}{N}$$
$$FDP = \frac{FP}{TP + FP} = \frac{FP}{R}$$

Benchmarking





Compute the confusion matrix for each value of the score.

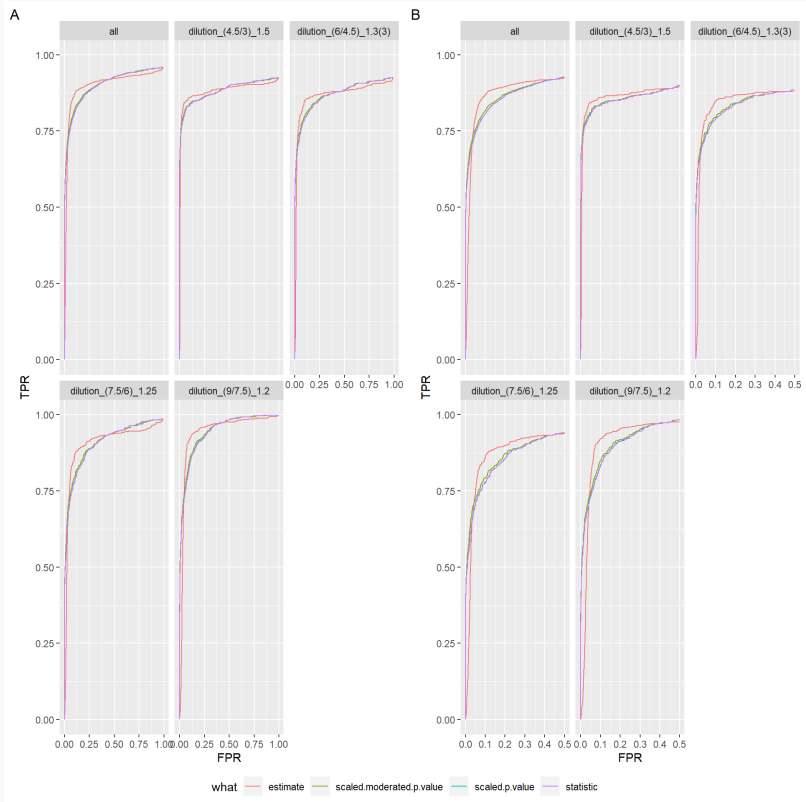
Benchmarking - ROC curve

By plotting the TPR versus the FPR we obtain
the **receiver operator characteristic** curve (ROC curve).
The **area under the curve** (AUC) or
partial areas under the curve (pAUC)
at various values of the FPR,
are further measures of performance.

Greater Area under the ROC is better.
$AUC = 1$ perfect separation
of E.Coli and H.sapiens porteins.

Conclusion:

log_2FC < T-statistic < sign$(log_2FC) \times$ p-value = FDR

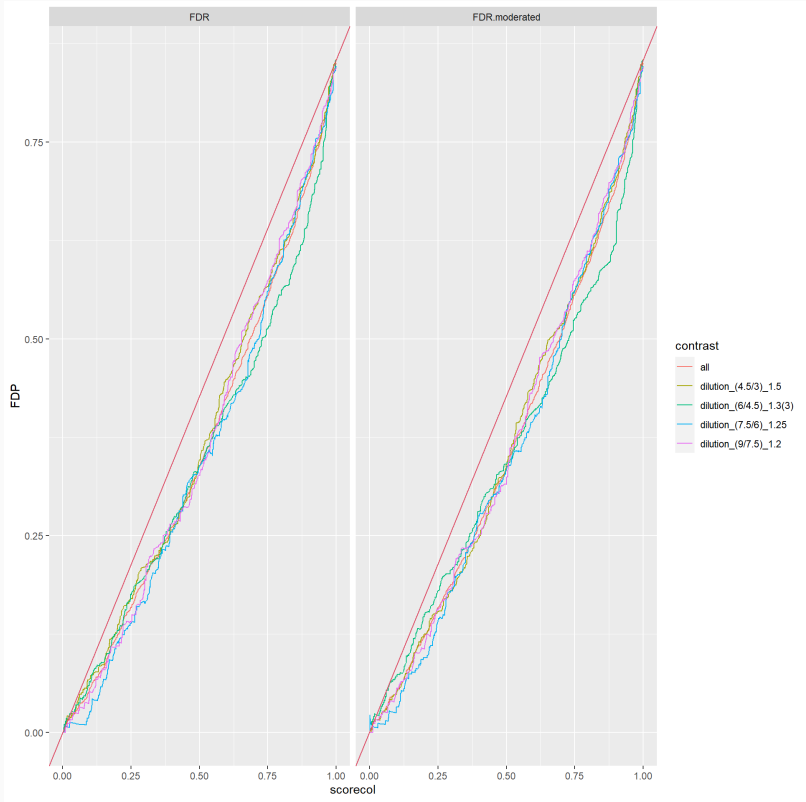


ROC curves

Benchmarking - FPR vs FDP

Does the **false discovery estimate** (FDR)
obtained to from the statistical model
matches the false discovery proportion (FDP)?

The FDR is the expected value of the FDP.
The FDR should be an unbiased estimate of the FDP.
By plotting the FDR against the FDP we can see how well
the FDR approximates the FDP.



FDR vs FDP curves

Conclusions

- Linear models allow to
  - estimate fold changes between condition using contrasts
  - but also test differences of fold changes (interactions).
  - run ANOVA analysis
- If you model more than two conditions
  - Problems because of missing data are more prominent
    (no observations in one of the conditions.)
- p-value moderation improves the protein/peptide variance estimates, the t-statistics and p-values
- Benchmark data is used to test analysis pipelines

Other Software

Other software for modelling fold changes used in Proteomics:

Using linear models

- **limma** - Ritchie, Smyth at al. 2015 PMID: 25605792
- **MSStats** https://www.bioconductor.org/packages/release/bioc/html/MSstats.html
- **ROPECA** Suomi and Elo 2017 PMID: 28724900
- **MSqRob** - Geomine, Gevaert and Clement 2016 PMID: 26566788

Other models

- **mapDIA** - Teo, Kim et al. 2016 PMID: PMID: 26381204
- **tirqler** - https://github.com/statisticalbiotechnology/triqler