Purdue University
West Lafayette
CS 182

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Assignment 3 Model Solutions

1) (20 pts.) Do Exercises 40 and 50 of Section 5.1 (page 331).

Graded by : Sneha Balasubramanian

Exercise 40:

Let P(n) $\rightarrow$ ($A_1 \cap A_2 \cap ... \cap A_n$) $\cup$ B = ($A_1 \cup$ B) $\cap$ ($A_2 \cup$ B) $\cap ... \cap$ ($A_n \cup$ B)

Basic step:

For n=1,

P(1) = (A_1) $\cup$ B = $A_1 \cup$ B. This is a trivial case. so we consider n=2 as our base case

For n=2,

P(2) = ($A_1 \cap A_2$) $\cup$ B = ($A_1 \cup$ B) $\cap$ ($A_2 \cup$ B) which is true.

Assumption step:

For n=k,

Let us assume P(k) $\rightarrow$ ($A_1 \cap A_2 \cap ... \cap A_k$) $\cup$ B = ($A_1 \cup$ B) $\cap$ ($A_2 \cup$ B) $\cap ... \cap$ ($A_k \cup$ B) is true.

Inductive step:

We shall prove that P(k+1) is true. (ie)

P(k+1) $\rightarrow$ ($A_1 \cap A_2 \cap ... \cap A_{k+1}$) $\cup$ B = ($A_1 \cup$ B) $\cap$ ($A_2 \cup$ B) $\cap ... \cap$ ($A_{k+1} \cup$ B)

Let us consider C= ($A_1 \cap A_2 \cap ... \cap A_k$)

thus, P(k+1) $\rightarrow$ ($A_1 \cap A_2 \cap ... \cap A_k \cap A_{k+1}$) $\cup$ B = (C $\cap A_{k+1}$) $\cup$ B

From the basic step, (C $\cap A_{k+1}$) $\cup$ B = (C $\cup$ B) $\cap$ ($A_{k+1} \cup$ B) is true.

(C $\cup$ B) $\cap$ ($A_{k+1} \cup$ B) = (($A_1 \cap A_2 \cap ... \cap A_k$) $\cup$ B) $\cap$ ($A_{k+1} \cup$ B)

(($A_1 \cap A_2 \cap ... \cap A_k$) $\cup$ B) $\cap$ ($A_{k+1} \cup$ B) =(($A_1 \cup$ B) $\cap$ ($A_2 \cup$ B) $\cap ... \cap$ ($A_k \cup$ B)) $\cap$ ($A_{k+1} \cup$ B)

($A_1 \cap A_2 \cap ... \cap A_k \cap$ (A_{k+1}) $\cup$ B = ($A_1 \cup$ B) $\cap$ ($A_2 \cup$ B) $\cap ... \cap$ ($A_k \cup$ B) $\cap$ ($A_{k+1} \cup$ B)

this proves P(k) $\rightarrow$ P(k+1), making it true for any value of n.

Exercise 50:

the basic step is wrong in this problem where n=1. this case is fails.

$$\sum_{i=1}^{n} i = \frac{(n + \frac{1}{2})^2}{2}$$

for n=1,

$$\sum_{i=1}^{1} i = \frac{(1+\frac{1}{2})^2}{2}$$

$=$

$$\frac{9}{8}$$

$1 \neq \frac{9}{8}$

thus this induction proof is wrong.

2) (20 pts.) Do Exercises 38 and 40 of Section 5.4 (page 371).

Graded by : Sneha Balasubrmanian

Exercise 38:

let λ be the empty string and let the concatenation of strings w_1 and w_2 be $w_1.w_2$.

procedure concatenation(w: string, i: copies of string w)

λ := ""

if i=0 **then return** λ

else if i=1 **then return** w

else w.concatenation(w, i-1)

Exercise 40:

We have to prove that algorithm concatenation(w,i) is true for all values of i.

Basic step:

For i=0,

The algorithm will return λ meaning there are zero copies of w.

For i=1,

The algorithm will return w indicating there is one copy of w.

Assumption step:

We assume for n=k, the algorithm concatenation(w,k) will return w^k which gives k copies of string w.

Induction step:

For n=k+1,

the algorithm concatenation(w,k+1) will return w.concatenation(w,k).

We know that concatenation(w,k) will return w^k .Thus, w.concatenation(w,k) will return w.w^k = w^{k+1} which gives k+1 copies of w.

Since concatenation(w,k+1) returns the correct result if concatenation(w,k) is true, this recursive is correct for finding w^i.

3) (20 pts.) Graded by : Ramya Vulimiri

Let P(n) be the statement that given n students in a class, you can find the popular student (if there is one) with at most 3(n - 1) questions.

Basis Step: You can show either P(1) or P(2). If n=1, then that student is the popular student, needing 0 questions.

$0 \leq 3(1-1) = 0 \Rightarrow$ P(1) holds. If n=2, say there are two students A and B. To find if there is a popular student, ask A if he follows B and ask B if he follows A - 2 questions.

$2 \leq 3(2-1) = 3 \Rightarrow$ P(2) holds.

Inductive Step: Assume P(k) for an arbitrary k $\geq$ 2, that the popular student in a class of k students can be found with 3(k-1) questions. Show P(k) $\rightarrow$ P(k+1), where P(k+1) := given k+1 students, you can find the popular student (if there is one) with at most 3(k) questions.

Let us pick two random students A and B from (k+1) students and ask A if (s)he follows B. Either A follows B, in which case A is not a popular student or A doesn't follow B, in which case B is not a popular student. In either case, one student is eliminated as popular.

Without loss of generality, let us say A is the student who is not popular. In the remaining k people, the popular student can be found with 3(k-1) questions (inductive hypothesis).

Case 1: If there is no popular student among k students, then there are no popular students among k+1 students.

No. of questions needed $= 1 + 3(k-1) = 3k - 2 \leq 3k$.

Case 2: If there is a popular student, let's call him P, we need to determine if P is still popular among k+1 students. For that, we ask 2 questions, if P follows A and if A follows P.

No. of questions needed $= 1 + 3(k-1) + 2 = 3k - 3 + 3 \leq 3k$.

From case 1 and 2, P(k+1) holds. Therefore, P(n) holds for $n \geq 1$.

4) (15 pts.) Graded by : Ramya Vulimiri

Let $P(n)$ be the proposition that $\displaystyle\sum_{i=0}^{n-1} \frac{i}{2^i} = 2 - \frac{n+1}{2^{n-1}}$. We must do two things to prove that $P(n)$ is true for n = 1, 2, 3,.... Namely,we must show that P(1) is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for k = 1, 2, 3, ...

BASIS STEP: $P(1)$ is true, because LHS $= \sum_{i=0}^{0} \frac{i}{2^i} = \frac{0}{1} = 0$ and RHS $= 2 - \frac{1+1}{2^0} = 2 - 2 = 0$.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$\sum_{i=0}^{k-1} \frac{i}{2^i} = 2 - \frac{k+1}{2^{k-1}}$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$\sum_{i=0}^{(k+1)-1} \frac{i}{2^i} = 2 - \frac{(k+1)+1}{2^{(k+1)-1}}$$

$$
\begin{aligned}
\sum_{i=0}^{(k+1)-1} \frac{i}{2^i} = \sum_{i=0}^{k} \frac{i}{2^i} = \sum_{i=0}^{k-1} \frac{i}{2^i} + \frac{k}{2^k} \\
&= 2 - \frac{k+1}{2^{k-1}} + \frac{k}{2^k} \text{ (Applying IH)} \\
&= 2 - \frac{2(k+1)-k}{2^k} \\
&= 2 - \frac{(k+1)+1}{2^{(k+1)-1}}
\end{aligned}
$$

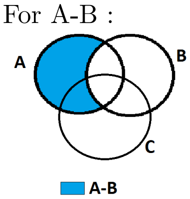
This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step. We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n.

5) (15 points) Graded By : Rashmi Soni

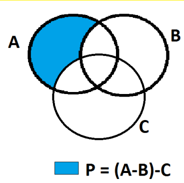
Given, the set P = (A-B)-C and Q = (A-C)-(B-C)

1) Drawing the venn diagram

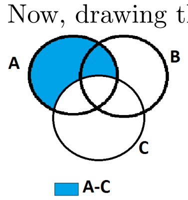
For A-B :



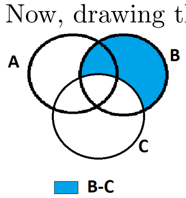
So,the venn diagram for P=(A-B)-C :



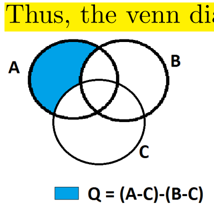
Now, drawing the venn diagram for A-C :



Now, drawing the venn diagram for B-C :



Thus, the venn diagram for Q = (A-C)-(B-C) is :



We can see that the venn diagrams of sets P and Q are equal. Thus, P=Q.

2)Constructing membership table

A	B	C	A - B	P = (A - B) - C	A - C	B - C	Q = (A - C) - (B - C)
1	1	1	0	0	0	0	0
1	1	0	0	0	1	1	0
1	0	1	1	0	0	0	0
1	0	0	1	1	1	0	1
0	1	1	0	0	0	0	0
0	1	0	0	0	0	1	0
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

So, we can conclude from the membership table that P = Q.

3)Using set identities with set builder notation

We can write,

A-B = $\{x|x \in A \wedge x \notin B\}$

So, (A-B)-C = $\{x|x \in (A - B) \wedge x \notin C\}$

$= \{x|(x \in A \wedge x \notin B) \wedge x \notin C\}$

Now,

A-C = $\{x|x \in A \wedge x \notin C\}$

B-C = $\{x|x \in B \wedge x \notin C\}$

Thus, we can write

(A-C)-(B-C) = $\{x|x \in (A - C) \wedge x \notin (B - C)\}$

$= \{x|(x \in A \wedge x \notin C) \wedge \neg(x \in B \wedge x \notin C)\}$

$= \{x|(x \in A \wedge x \notin C) \wedge (x \notin B \vee x \in C)\}$

$= \{x|((x \in A \wedge x \notin C) \wedge x \notin B) \vee ((x \in A \wedge x \notin C) \wedge x \in C)\}$

$= \{x|((x \in A \wedge x \notin C) \wedge x \notin B) \vee (x \in A \wedge (x \notin C \wedge x \in C))\}$

$= \{x|((x \in A \wedge x \notin C) \wedge x \notin B) \vee (x \in A \wedge \emptyset)\}$

$= \{x|((x \in A \wedge x \notin C) \wedge x \notin B) \vee (\emptyset)\}$

$= \{x|((x \in A \wedge x \notin B) \wedge x \notin C)\}$

Thus,

(A-B)-C = (A-C)-(B-C)

6) (10 points) Graded By : Rashmi Soni

If A,B, and C are sets, whether the symmetric difference is associative: A $\oplus$ (B $\oplus$ C) = (A $\oplus$ B) $\oplus$ C

Constructing the membership table:

A	B	C	$B \oplus C$	$A \oplus (B \oplus C)$	$A \oplus B$	$(A \oplus B) \oplus C$
1	1	1	0	1	0	1
1	1	0	1	0	0	0
1	0	1	1	0	1	0
1	0	0	0	1	1	1
0	1	1	0	0	1	0
0	1	0	1	1	1	1
0	0	1	1	1	0	1
0	0	0	0	0	0	0

From the membership table we see that the column values for $(A \oplus (B \oplus C))$ and $((A \oplus B) \oplus C)$ are equal, so $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.