Purdue University
West Lafayette
CS 182

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Assignment 5 Model Solutions

Due: Thursday, July 20, 2017, upload before 11:30pm

1) (20 pts.): How many times is function F called in each code segment given below? Clearly explain your answer and express the bounds in terms of n in big-O notation Assume $n = 2^k$.
Graded by: Ramya Vulimiri

Code Segment 1

for $i = 1$ **to** n^2 **do**
  for $j = 1$ **to** n **by** k **do**
    for $l = 1$ **to** j **do**
      $F(i, j)$

Code Segment 2

while $n > 10$ **do**
  i=1
  while $i < n$ **do**
    $F(i, k)$
    $i = i * k$
  $n = n/2$

Solution for Code segment 1: The outer loop goes from 1 to n^2, it is clear that this loop runs n^2 times. Next, let us calculate the total work done by the two inner loops. For each $j = 1, k, 2k, .., n$, the innermost loop calls $F(i, j)$ j times. Total work done

$$\begin{aligned}
&= n^2 * (1 + k + 2k + 3k + ... + n) \\
&= n^2 * (1 + \frac{\left\lfloor \frac{n}{k} \right\rfloor (n+1)}{2}) \\
&\le n^2 * (1 + \frac{\frac{n}{k}(n+1)}{2}) \\
&\le n^2 * (\frac{n^2 + n + 2k}{2k}) \\
&\le n^2 * (\frac{n^2 + n^2 + 2kn^2}{k}) \\
&\le c(\frac{n^4}{k}) \\
&= O\left(\frac{n^4}{\log_2 n}\right)
\end{aligned}$$

Solution for Code segment 2: Let j represent the number of times the outer loop has run. For $j = 1, 2, ..$, the value of $n = n, \frac{n}{2}, \frac{n}{2^2}, .. = 2^k, 2^{k-1}, .., 2^4$. The loop will stop when $n = 2^3 < 10$. In the inner loop, the value of i goes from $1, k^2, k^3, ..$ to current value of n (n_{cur}). Notice that the inner loop run $\log_k n_{cur}$ times. Total number of times $F(i,j)$ is called

$$
\begin{aligned}
&= \log_k n + \log_k \frac{n}{2} + \log_k \frac{n}{4} + .. + \log_k 2^4 \\
&= \log_k 2^k + \log_k 2^{k-1} + .. + \log_k 2^4 \\
&= \frac{\log_2 2^k + \log_2 2^{k-1} + .. + \log_2 2^4}{\log_2 k} \\
&= \frac{k + k - 1 + .. + 4}{\log_2 k} \\
&\leq \frac{k + k - 1 + .. + 4 + 3 + 2 + 1}{\log_2 k} \\
&= \frac{k(k+1)}{2\log_2 k} \leq \frac{k(k+k)}{2\log_2 k} = \frac{k^2}{\log_2 k} \\
&= O\left(\frac{(\log n)^2}{\log\log n}\right)
\end{aligned}
$$

2) (20 pts.): Find a phrase-structure grammar and construct the corresponding DFA for for each of these languages.

Graded by : Sneha Balasubramanian

1. the set of bitstrings consisting of an odd number of 0s followed by a final 1.

2. the set of bitstrings that have neither two consecutive 0s nor two consecutive 1s.

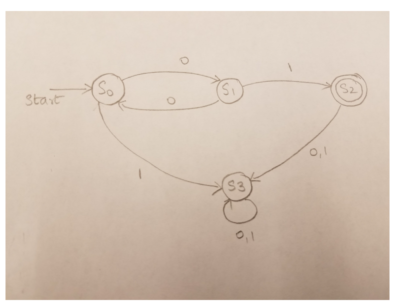
1. the grammar would be:

$G = (V, T, S, P)$

$V = S, A, 0, 1$

$T = 0, 1$

P $= S \to 0A, A \to 00A, A \to 1$



2. the grammar would be:

$G = (V, T, S, P)$

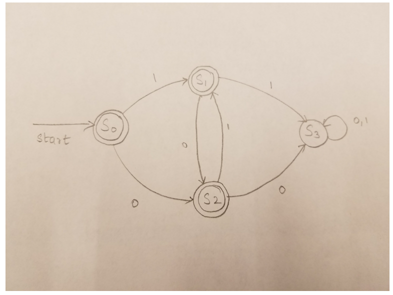
$V = (0, 1, A, B, S)$

$T = 0, 1$

P$= S \to \lambda, S \to 0, S \to 1, S \to 1A, S \to 0B, A \to 0, A \to 0B, B \to 1, B \to 1A$

3) (Answer the following questions and explain your answer):

1. (5 pts) The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter,



either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)

Graded by: Ramya Vulimiri

Solution:

For any variable name, there are 53 options for the 1st character (26 uppercase + 26 lowercase + 1 underscore) and 63 options for following characters, if any (26 uppercase + 26 lowercase + 1 underscore + 10 digits). The name can be of any length from 1 to 8. Thus, the number of possible variables $= 53+53*63+53*63^2+53*63^3+53*63^4+53*63^5+53*63^6+53*63^7 = 53\sum_{n=0}^{7} 63^n = 53*\frac{1-63^8}{1-63} \approx 2.12*10^{14}$.

2. (5 pts) How many positive integers less than $n = pq$, where p and q are prime, are relatively prime to n?

   Graded by: Ramya Vulimiri

   Solution:

   There are $n-1$ positive integers which are less than n. Let us consider how many of these are not relatively prime to n. All multiples of p which are less than n share a factor with n - p, $2p$, ..., $(q-1)p$. Similarly, all multiples of q - q, $2q$, ..., $(p-1)q$ share a factor of q with n. These two lists are disjoint, so we have $p-1+q-1 = p+q-2$ elements which are not relatively prime to n (These are the only exceptions since $n = pq$, with p and q both being prime).

   Therefore, there are $n-1-(p+q-2) = n-p-q+1 = pq-p-q+1 = (p-1)(q-1)$ positive integers less than and relatively prime to n. (This does not hold for the case that p = q because each multiple is counted twice. If $p = q$, then $n - \dfrac{n}{p} = p(p-1)$ numbers are relatively prime to n.).

3. (5 pts) Assume that no one has more than $200,000$ hairs on their head. In 2013, the population of Indianapolis was $852,866$. Show that at least 5 people in Indianapolis in 2013 must have had the exact same number of hairs on their heads.

   Graded by :Sneha Balasubramanian

   Solution:

   It says the maximum number of hair a person can have is $200,000$ which means there are $200,001$ different number of hairs a person can have. the population of Indianapolis is $852,866$. By pigeon hole principle, the population would be the object and the number of hairs would be the the boxes. thus,

   $\lceil\frac{852,866}{200,001}\rceil = \lceil 4.2 \rceil$ =5

   this shows that least 5 people in Indianapolis in 2013 must have had the exact same number of hairs on their heads.

4. (5 pts) How many permutations of the letters $ABCDEFGH$ contain

(a) the string ED?

(b) the string CDE?

(c) the strings BA and FGH?

(d) the strings AB, DE, and GH?

(e) the strings CAB and BED?

Graded by :Sneha Balasubramanian
Solution:
(a) The string ED:
the letter would be $[A, B, C, ED, F, G, H]$ which gives us $7! = 5040$
(b) the string CDE :
the letters would be $[A, B, CDE, F, G, H]$ which gives us $6! = 720$
(c) the strings BA and FGH :
The letters would be $[BA, C, D, E, FGH]$ which gives us $5! = 120$
(d) the strings AB, DE, and GH :
The letters would be $[AB, C, DE, F, GH]$ which gives us $5! = 120$
(e) the strings CAB and BED :
Since CAB ends with B and BED begins with B, any string with both will have CABED . Consider CABED a single element, which gives us $4! = 24$.

5. (5 pts) What is the coefficient of x^4y^8 in $(x+y)^{12}$.
Graded by :Sneha Balasubramanian
Solution:
by binomial expansion, the coefficient of $(x+y)^n$ is $\binom{n}{k}$ for each term being $x^{n-k}y^k$. thus, for the given term, the coefficient is $\binom{12}{8} = \frac{12!}{8!4!} = 495$
Thus 495 is the coefficient.

6. (5 points) Graded by Rashmi Soni
There are total 26 number of letters and we need to form a string with 6 letters; number of ways to fill the first letter is 26. Similarly, number of ways to fill second letter is 26.Thus, number of ways to form a string of 6 letters is equal to 26*26*26*26*26*26, i.e. 26^6 ways

7. (5 points) Graded by Rashmi Soni
We can first select any three random candidates out of the six different candidates to be placed on the ballot. Number of ways to do this is $\binom{6}{3}$. Now, we can have any permutations of these three chosen candidates on the ballot through 3! ways.Thus, total number of ways, in which the names of three candidates be printed on a ballot is :
$= \binom{6}{3}*3!$
$= \dfrac{6!}{3!3!}3!$
$= 6 * 5 * 4$
$= 120$

8. (5 points) Graded by Rashmi Soni

   We need to find the number of solutions for :

   $x_1 + x_2 + x_3 + x_4 + x_5 = 19$ when $x_i \geq 1$

   Re-writing this equation as :

   $x_1 + x_2 + x_3 + x_4 + x_5 = 14$ when $x_i \geq 0$

   The above equation states to find the number of ways 14 identical objects can be distributed among 5 distinct containers; which is given by :

   $=\binom{5+14-1}{14}$

   $=\binom{18}{14}$

   $= \dfrac{18!}{14!4!}$

   $= \dfrac{18 * 17 * 16 * 15}{4 * 3 * 2 * 1}$

   $= 3060$

9. (5 points) Graded by Rashmi Soni

   The bit string must contain exactly six 0s and fifteen 1s such that every 0 must be immediately followed by two 1s. So, let's say we pair up each 0 with two 1s such that 011 is treated as one object. Thus, if we make use of 6 0s to create such '011' objects, we will be left with 3 more 1s (as twelve 1s out of fifteen would be paired up with six 0s).So, the number of permutations of these 9 (six identical '011' objects + three 1s) elements are :

   $= \dfrac{9!}{6!3!}$

   $= \dfrac{9 * 8 * 7}{3 * 2 * 1}$

   $= 84$

10. (5 points) Graded by Rashmi Soni

    To find the number of strings that can be formed using the letters in ENGINEERING, let's first find the occurrences of each letter.

    letter 'E' is occurring 3 times

    letter 'N' is occurring 3 times

    letter 'G' is occurring 2 times

    letter 'I' is occurring 2 times

    letter 'R' is occurring 1 time

    There are 11 letters in total.Thus, the number of permutations of ENGINEERING can be given as :

    $= \dfrac{11!}{3!\ 3!\ 2!\ 2!\ 1!}$

    $= \dfrac{11 * 10 * 9 * 8 * 7 * 6 * 5 * 4}{3 * 2 * 1 * 2 * 1 * 2 * 1}$

    $= 277200$

11. (5 points) Graded by Rashmi Soni

We need to find the number of ways to deal 13-card hands to 4 players from a deck of 52 cards.

The number of ways to deal 13 cards out of 52 cards to the first player is $\binom{52}{13}$.

The number of ways to deal 13 cards out of remaining 39 cards to the second player is $\binom{39}{13}$.

The number of ways to deal 13 cards out of remaining 26 cards to the third player is $\binom{26}{13}$.

The number of ways to deal 13 cards out of remaining 13 cards to the fourth player is $\binom{13}{13}$.

Thus, the total number of ways to deal 13-card hands to 4 players is :

$=\binom{52}{13} * \binom{39}{13} * \binom{26}{13} * \binom{13}{13}$

$= \dfrac{52!}{13!\ 13!\ 13!\ 13!}$

$\approx 5.36X10^{28}$

12. (5 points) Graded by Rashmi Soni

We need to find the number of ways to place 5 distinct balls into 3 indistinguishable urns.

There is no closed form solution for this kind of problem.

We enumerate the possible arrangements and count ways to have each:

There are five different options to arrange 5 different balls into 3 identical urns :

1) Two urns are empty and one of the urns contains all the five balls or, (5,0,0) number of balls respectively in the three urns.

There is only **one** way to put five balls in one of the urns.

2) One of the urns is empty; (4,1,0) number of balls respectively in three urns or, (3,2,0) number of balls respectively in three urns.

Now, when we arrange for (4,1,0) case

There are $\binom{5}{4}$ = **5 ways** to put 4 balls in one of the urns and then put the fifth ball in any of the two remaining urns.

When we arrange for (3,2,0) case

There are $\binom{5}{3}$ = **10 ways** to put 3 balls in one of the urns and then put the remaining two balls in any of the two urns.

3) None of the urns are empty; (3,1,1) number of balls respectively in three urns or, (2,2,1) number of balls respectively in three urns.

Now, when we arrange for (3,1,1) case

There are $\binom{5}{3}$ = **10 ways** to put 3 balls in one of the urns and then distribute evenly the other two balls in the remaining urns.

When we arrange for (2,2,1) case

There are $\binom{5}{2}$ = 10 ways to put 2 balls in one of the urns and then $\binom{3}{2}$ = 3 ways to put two balls out of the remaining three in one of the urns and then put the fifth ball in the remaining urn. So, there should be 10x3 =30 total ways. But, we see that since two of the urns each contain two number of balls, so we have counted each pair of balls twice since the urns are identical. Thus, the total number of ways here will be =

$30/2 =$ **15 ways**

Thus, the total ways to put five balls into 3 identical urns are :
$= 1 + 5 + 10 + 10 + 15 = 41$ ways

Note: As you will have read on the course website, you need to submit your answers typed and as one PDF file named LastName.FirstName.5.PDF before the due date stated above. Follow the the template given at https://www.cs.purdue.edu/homes/amahmoo/cs182summer2017/homeworksolutiontemplate.pdf.