Gauss Elimination and LU Decomposition Example

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System of equations	Associated matrices

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$$2x_1 - 4x_2 + 2x_3 = 6 \qquad (1)$$
$$4x_1 - 9x_2 + 7x_3 = 20 \qquad (2)$$
$$2x_1 + x_2 + 3x_3 = 14 \qquad (3)$$

Associated matrices

$$\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 4 & -9 & 7 \\ 2 & 1 & 3 \end{bmatrix}}_{A}$$

(Step 1)

(2') = (2) - (1)× 2; (1')=(1); (3')=(3)

$$2x_1 - 4x_2 + 2x_3 = 6 \qquad (1')$$
$$-x_2 + 3x_3 = 8 \qquad (2')$$
$$2x_1 + x_2 + 3x_3 = 14 \qquad (3')$$

(2) = 2×(1) + 1×(2'); (1)=(1'); (3)=(3')

$$\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 4 & -9 & 7 \\ 2 & 1 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}}_{A'}$$

(Step 2)

(3") = (3') - (1')× 1; (1")=(1'); (2")=(2')

(3') = 1×(1') + 1×(3"); (1')=(1"); (2')=(2")

$$\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}}_{A'} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 5 & 1 \end{bmatrix}}_{A''}$$

$$2x_1 - 4x_2 + 2x_3 = 6 \qquad (1")$$
$$-x_2 + 3x_3 = 8 \qquad (2")$$
$$5x_2 + x_3 = 8 \qquad (3")$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 5 & 1 \end{bmatrix}}_{A''}$$

$$(\text{Step 3})$$

$(3''') = (3'') - (2'') \times (-5); \ (1''')=(1''); \ (2''')=(2'')$ | $(3'') = -5\times(2'') + \times(3'''); \ (1'')=(1'''); \ (2'')=(2''')$

$$2x_1 - 4x_2 + 2x_3 = 6 \qquad (1''')$$
$$-x_2 + 3x_3 = 8 \qquad (2''')$$
$$16x_3 = 48 \qquad (3''')$$

$$\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 5 & 1 \end{bmatrix}}_{A''} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}}_{L_3} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix}}_{A'''}$$

Thus

$$A = L_1 A'$$
$$\ = L_1 L_2 A''$$
$$\ = L_1 L_2 L_3 A'''$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -5 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix}}_{U}.$$