

PROBLEM SET 2
STAT 221

WON I. LEE

1. POSTERIOR AND LIKELIHOOD

The joint posterior of the model is given by the following:

$$p(\mu, \sigma^2, \log \theta | Y, w) \propto p(Y | \mu, \sigma^2, \log \theta, w) p(\mu, \sigma^2, \log \theta | w)$$

where we treat w as constant and fixed (thus conditioned on) throughout the calculation. We have:

$$p(Y | \mu, \sigma^2, \log \theta, w) = \prod_{j=1}^J \prod_{n=1}^N p(Y_{jn} | \mu, \sigma^2, \log \theta_j, w_j) = \prod_{j=1}^J \prod_{n=1}^N \text{Pois}(Y_{jn} | w_j \theta_j)$$

and:

$$p(\mu, \sigma^2, \log \theta | w) = p(\log \theta | \mu, \sigma^2, w) p(\mu, \sigma^2 | w) \propto \frac{1}{\sigma^2} \prod_{j=1}^J \mathcal{N}(\log \theta_j | \mu, \sigma^2)$$

Putting these together, we have:

$$p(\mu, \sigma^2, \log \theta | Y, w) \propto \frac{1}{\sigma^2} \prod_{j=1}^J \mathcal{N}(\log \theta_j | \mu, \sigma^2) \prod_{n=1}^N \text{Pois}(Y_{jn} | w_j \theta_j)$$

The conditional posterior is given by:

$$p(\log \theta_j | Y, w, \mu, \sigma^2) \propto p(Y_{j\cdot} | \log \theta_j, Y_{-j\cdot}, w, \mu, \sigma^2) p(\log \theta_j | Y_{-j\cdot}, w, \mu, \sigma^2)$$

where $Y_{j\cdot}$ indicates Y_{j1}, \dots, Y_{jN} and $Y_{-j\cdot}$ indicates all other values of Y (i.e. all other columns). We note that by the presumed i.i.d. assumptions:

$$p(Y_{j\cdot} | \log \theta_j, Y_{-j\cdot}, w, \mu, \sigma^2) = p(Y_{j\cdot} | \log \theta_j, w) = \prod_{n=1}^N \text{Pois}(Y_{jn} | w_j \theta_j)$$

and:

$$p(\log \theta_j | Y_{-j\cdot}, w, \mu, \sigma^2) = p(\log \theta_j | \mu, \sigma^2) = \mathcal{N}(\log \theta_j | \mu, \sigma^2)$$

so putting this together:

$$p(\log \theta_j | Y, w, \mu, \sigma^2) \propto \mathcal{N}(\log \theta_j | \mu, \sigma^2) \prod_{n=1}^N \text{Pois}(Y_{jn} | w_j \theta_j)$$

The conditional posterior is unimodal, as verified by both analytical observation and numerical visualization. That is, we substituted reasonable values for the parameters and computed the posterior numerically at grid-points of $\log \theta_j$ to verify

that the posterior is unimodal. It is also log-concave, since taking derivatives twice of the log-posterior according to $\log \theta_j$ yields:

$$\frac{\partial^2}{\partial \theta_j^2} \log p(\log \theta_j | Y_{-j, \cdot}, w, \mu, \sigma^2) = -\sigma^{-2} - nw_j \theta_j < 0$$

2. SIMULATE DATA

3. EVALUATE COVERAGE (SIMPLE)

4. EVALUATE COVERAGE WITH EXPOSURE WEIGHTS

5. EVALUATE COVERAGE WITH MISSPECIFICATION

6. INTERPRETATION AND RESULTS