Section Notes 2

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Clarifications from Class

Recall from class, homothetic preferences are preferences such that a bundle of goods $\vec{x} \sim \vec{x}'$ is equivalent to $\alpha\vec{x} \sim \alpha\vec{x}'$ where α is a positive constant.

With a Walrasian demand function of the form $x^*(\vec{p}, w)$ represents homothetic preferences if the following holds:

$$x^*(\vec{p}, \alpha w) = \alpha x^*(\vec{p}, w),$$

which leads to the wealth expansion curve that we saw in class.

Now what is the relationship between homothetic preferences and utility functions? As Chris said in class, homothetic preferences are represented by a homogeneous of degree 1 utility function—*i.e.* $u(\alpha\vec{x}) = \alpha u(\vec{x})$. An example of such a utility function is the Cobb-Douglas which we see in section today.

Recall that once you see a utility function, you might want to transform the function such that the function is easier to manipulate *and the property that wealth expansion paths are linear is preserved.* Chris touched on this in class, but really didn't get into this, but FYI, the resulting transformed function is a homothetic function. The definition of a homothetic function is:

Definition 1. Homothetic functions: Function $u : \mathbb{R}^L \to \mathbb{R}^1$is homothetic if it is a monotone transformation of a homogeneous function.[1]

A useful way to think about this definitions is that homothetic functions are to homogeneous functions what quasiconcavity is to concavity. In short, they are properties of functions (and imply certain properties of functions) that are preserved under monotonic transformation. Therefore, since the definition of a homogeneous of degree k function is that $f(\alpha\vec{x}) = \alpha^k f(\vec{x})$, this means that a homothetic function will also have this property. This is what Chris was getting at in the lecture.

Definition 2. Numeraire good: Numeraire goods are goods, usually with a fixed price of "1," used when only the relative prices of goods are relevant (as

[1]As I wrote in the math review notes, a great exposition of homothetic and homogeneous functions can be found in Ch. 20 of Simon and Blume (1994).

in the case of utility maximization and most of the problems that you will encounter in Ec 2020a). Suppose we have a set X of designer suits for which an economic agent has preferences that can be represented by a quasiconcave and continuous utility function (which means that the consumer's preference over the designer suit set is convex). Let's assume for simplicity that $X \equiv \{Zegna, Versace, Dolce\}$. The agent knows the market prices of each good which we can denote in the form of a price vector:

$$\vec{p'} = [p_{Zegna}, p_{Versace}, p_{Dolce}],$$

but since we usually only care about relative prices in our model of consumer behavior, we could write the price vector as follows:

$$\vec{q'} = \left[1, \frac{p_{Versace}}{p_{Zegna}}, \frac{p_{Dolce}}{p_{Zegna}}\right] = [1, q_{Versace}, q_{Dolce}].$$

Then we would call Zegna the *numeraire* good, since we can express the other goods in set X in terms of Zegna—*e.g.* Versace is worth 0.9 Zegna and Dolce is worth 1.15 Zegnas. That's why the price of the numeraire is usually set to 1.

Agenda

1. Utility Maximization Problem (hereinafter "UMP") with Cobb-Douglas utility function
2. UMP with quasi-linear utility function (using the Kuhn-Tucker method)

1 UMP with Cobb-Douglas utility functions

Problem 3. How much x_1, x_2 should I consume given wealth w; prices p_1, p_2; and a utility function such that $u(x_1, x_2) = x_1 x_2^2$?

1. What is the objective? Maximize utility. A related question might be: what is the objective function?
2. What are the choice variables (the endogenous variables)? What are the parameters (exogenous / state variables)?
3. What are the constraints? The budget constraint and non-negativity constraints on the choice/endogenous variables.
4. How do we set up the problem? Once set up, how do we solve? How do we interpret the answer? Can we use pictures like the ones we saw in Section Notes 1 to help us?

1.1 Setting up the UMP

$$\max_{x_1 x_2 \geq \geq 0} u(x_1, x_2) = x_1 x_2^2 \tag{1}$$

[2] s.t.

$$p_1 x_1 + p_2 x_2 - w \leq 0. \tag{2}$$

Do we like the form of equation 1? We could work with this form, but recall from the readings and from the math review notes that utility is an *ordinal* concept. Therefore, the preferences of the consumer who is solving Problem 3 can be represented by not only $u(x_1, x_2) = x_1 x_2^2$, but also any monotonic transformation of this equation.

Let's transform the objective function in equation 1 so that we can write the UMP as follows:

$$\max_{x_1 x_2 \geq \geq 0} \ln u(x_1, x_2) = \ln x_1 + 2 \ln x_2 \tag{3}$$

s.t.

$$p_1 x_1 + p_2 x_2 - w \leq 0 \tag{4}$$

Note that the budget constraint in equation 2 will hold with equality in our utility maximization problem.

1.2 Solving the UMP

Solving the UMP requires use of the Lagrangian. In Econ 2020a, we use the Lagrangian in a mechancial manner. Think of the Lagrangian as a clever way of turning a constrained optimization problem into an unconstrained optimization problem.[3] Consider the following two choice variables and two inequality constraint problem:

$$\max_{x_1 x_2} f(x_1, x_2)$$

s.t.

[2]Is the objective function a concave function? For a precise answer, we'd have to look at signs of the two principal minors of the two-dimensional Hessian. Is it a quasiconcave function? What is the implication? See Section 3.2.2 of the Miller Notes and the math review notes on quasiconcavity.

[3]See Simon and Blume (1994, chs. 17, 18) for an extensive review of unconstrained and constrained optimization.

$$
\begin{aligned}
g_1(x_1, x_2) &\leq b_1, \\
g_2(x_1, x_2) &\leq b_2
\end{aligned}
$$

In general, we write the Lagrangian for the above problem as follows:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = f(x_1, x_2) - \lambda_1[g_1(x_1, x_2) - b_1] - \lambda_2[g_2(x_1, x_2) - b_2].$$

Try the more general case where there are n choice variables and m inequality constraints.

In Problem 3, since we have two endogenous variables and one constraint, the Lagrangian is as follows:

$$\mathcal{L}(x_1, x_2) = \ln x_1 + 2 \ln x_2 - \lambda[p_1 x_1 + p_2 x_2 - w]. \tag{5}$$

Before we start differentiating the Lagrangian, we have to make a key assumption: the critical points of the Lagrangian in equation 5, which are defined as x_1^*, x_2^*, are greater than zero—*i.e.* the critical points are internal solutions, and not boundary/corner solutions. After making this assumption, we start differentiating and we find the following *first order conditions* (FOCs):[4]

$$
\begin{aligned}
\frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial x_1} &= \frac{1}{x_1^*} - \lambda^* p_1 = 0 \Rightarrow \lambda^* = \frac{1}{p_1 x_1^*} \\
\frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial x_2} &= \frac{2}{x_2^*} - \lambda^* p_2 = 0 \Rightarrow \lambda^* = \frac{2}{p_2 x_2^*} \\
\frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial \lambda} &= w - p_1 x_1 - p_2 x_2 = 0.
\end{aligned}
$$

We have a system of three equations in three unknowns, x_1^*, x_2^*, λ^*. If we solve this system of equations, we find that:

$$x_1^*(p_1, p_2, w) = \frac{1}{3}\left(\frac{w}{p_1}\right)$$

$$x_2^*(p_1, p_2, w) = \frac{2}{3}\left(\frac{w}{p_2}\right)$$

$$\lambda^*(p_1, p_2, w) = \frac{1}{w}.$$

[4]You don't need to know this, but if there aren't any assumptions on the functional form of the objective function, the first order conditions are *necessary* conditions and not *sufficient* conditions for an optimum.

1.3 Interpretation

1. Check the demands for good 1 and good 2 and see if they are homogeneous of degree zero.

2. Note that $p_1 x_1^* = \frac{1}{3}w$ and $p_2 x_2^* = \frac{2}{3}w$. What does this mean? It means that total expenditure on good $x_1$will be equal to $\frac{1}{3}$ of wealth and total expenditure on good x_2 will be equal to $\frac{2}{3}$ of wealth. These numbers correspond to the powers on the original objective function.

3. Let's do some comparative statics:[5]

$$
\begin{aligned}
\frac{\partial x_1^*}{\partial p_1} &= -\frac{w}{3p_1^2} < 0 \\
\frac{\partial x_1^*}{\partial p_2} &= 0 \\
\frac{\partial x_1^*}{\partial w} &= \frac{1}{3p_1} > 0.
\end{aligned}
$$

2 UMP with quasi-linear utility functions

When should you use the Kuhn-Tucker (hereinafter "K-T") conditions? When dealing with UMPs, the answer is almost always! If you're dealing with an optimization problem that has non-negativity constraints on its arguments and there is a chance that the global optimum will occur at zero for one of the arguments, you should use K-T. With that said, many of our problem sets and/or exam questions assume the existence of interior solutions (non-zero maximizers) or are set up in such a way that the optimum will always take place at an interior solution.

An example is helpful in showing you why we need to analyze the K-T conditions. Consider a function $u : X \to \mathbb{R}^1$, where $X \subset \mathbb{R}^1$. Let's also assume that $u(x) = -(x+2)^2$.[6] We know that $u(x)$ is maximized at $x = -2$, which is "interior" to the restricted domain. But what if we restricted $x \in [0, +\infty)$? Then the value of the endogenous variable that maximizes the function $u : X \to \mathbb{R}^1$ is equal to $x^* = 0$, which is at the "corner" of the restricted domain. We need to consider these *corner* solutions when dealing with multivariable functions as well.

What is the bottom line? Draw a picture and check to see if we might be looking at an optimum occurring at a corner. Recall the graphs that we looked at in the previous section notes. Also, if you solve the Lagrangian (in our "normal" way) and discover that the objective function is maximized at negative values, then it's a good bet that you should be using the K-T conditions.

[5] Recall that we can do comparative statics because the FOCs are identities and not just equations.

[6] Do the preferences represented by this one-dimensional utility function satisfy strong monotonicity? How about monotonicity? How about local non-satiation?

Now let's look at a UMP where we would have to consider corner solutions

Problem 4. How much x_1, x_2 should I consume given wealth w; prices p_1, p_2; and a utility function such that $u(x_1, x_2) = x_1 + \sqrt{x_2}$?

2.1 Setting up the UMP

$$\max_{x_1 x_2} u(x_1, x_2) = x_1 + \sqrt{x_2} \tag{6}$$

[7] s.t.

$$\begin{aligned} p_1x_1 + p_2x_2 - w &\leq 0. && (7)\\ -x_1 &\leq 0 && (8)\\ -x_2 &\leq 0 && (9) \end{aligned}$$

Can we make any monotonic transformation that will simplify the problem for us? I can't think of anything with quasi-linear utility functions, but if you find a simplification, please let me and the entire class know.

2.2 Solving the UMP

For Problem 4, the Lagrangian is as follows:

$$\mathcal{L}(x_1, x_2) = x_1 + \sqrt{x_2} - \lambda[p_1x_1 + p_2x_2 - w].$$

We now have to find the *Kuhn-Tucker first order conditions*, which include the following inequalities:

$$\frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial x_1} = 1 - \lambda^* p_1 \leq 0 \Rightarrow \lambda^* p_1 \geq 1 \tag{10}$$

$$\frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial x_2} = \frac{1}{2\sqrt{x_2^*}} - \lambda^* p_2 \leq 0 \Rightarrow \lambda^* p_2 \geq \frac{1}{2\sqrt{x_2^*}} \tag{11}$$

$$\frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial \lambda} = w - p_1x_1 - p_2x_2 \geq 0, \tag{12}$$

and the following *complimentary slackness* conditions:

$$x_1^* \frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial x_1} = x_1^*(1 - \lambda^* p_1) = 0 \tag{13}$$

[7] Is this objective function concave? Again we'd have to look at the principal minors of the two dimensional Hessian (whatever that means), but recall one of the theorems that I discussed in the Math Review notes—linear combinations of concave functions are what?

$$x_2^* \frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial x_2} = x_2^* \left(\frac{1}{2\sqrt{x_2^*}} - \lambda^* p_2 \right) = 0 \tag{14}$$

$$\lambda^* \frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial \lambda} = \lambda^*(w - p_1 x_1 - p_2 x_2) = 0. \tag{15}$$

Now what do we do? Recall why we're doing this. It's because we want to check whether one or both of the two endogenous variables x_1^*, x_2^* take on the value zero. We're worried about the non-negativity constraints. So we know that there are going to be four different combinations that we need to consider:

2.2.1 Optimal values satisfy $x_1^* = x_2^* = 0$

This is trivial since from equations 10 and 15, it must be that $w = 0$. Do you see why? If an economic agent has no wealth than the maximum utility will be equal to zero at $x_1^* = x_2^* = 0$. Further, check out the conditions 11 and 14. Do you see that an indeterminancy arises? We don't want that.

2.2.2 Optimal values satisfy $x_1^* > 0$, $x_2^* = 0$ (Border Solution)

$>$From equation 13, it must be that $\lambda^* p_1 = 1$, and from equation 15 $x_1^* = \frac{w}{p_1}$. But look at inequality 11, at $x_2^* = 0$: $\frac{1}{2\sqrt{0}} - \lambda^* p_2^* \leq 0$. Notice that we have a zero in the denominator of one of the arguments in the LHS. Therefore, we have a contradiction because of the indeterminancy of the LHS of the inequality.

For those of you who care, this is telling us that an economic agent with the above utility function would never consume zero of the non-numeraire good, because at $x_2^* = 0$, the marginal utility of good 2 is infinite, or

$$\lim_{x_2 \to 0} \frac{\partial u(x_1, x_2)}{\partial x_2} = \infty,$$

which is one element of the famed Inada conditions in neoclassical growth models. Basically, the concavity of the non-numeraire good utility function leads to the first graph that Chris put on the board where the marginal utility of the non-numeraire good was greater than the marginal utility of the numeraire good.

2.2.3 Optimal values satisfy $x_1^* = 0, x_2^* > 0$ (Border Solution)

$>$From equation 14, we know that $\lambda^* p_2 = \frac{1}{2\sqrt{x_2^*}}$, which means that $\lambda^* \neq 0$. From equation 15, $x_2^* = \frac{w}{p_2}$. Plug this into the equation, $\lambda^* p_2 = \frac{1}{2\sqrt{x_2^*}}$, and we have that $\lambda^* = \frac{1}{2\sqrt{p_2 w}}$. We need to check our answers against the only inequality that is left for us to check: inequality 10. If we plug our solution for λ^* into inequality 10, then we find that our solution will hold if $p_1 \geq 2\sqrt{p_2 w}$.

2.2.4 Optimal values satisfy $x_1^* > 0$, $x_2^* > 0$ (Interior Solution)

We know from equation 13 that $\lambda^* = \frac{1}{p_1}$ and from equation 14 that $x_2^* = \frac{1}{4(\lambda^* p_2)^2} = \frac{p_1^2}{4p_2^2}$. Note that since we now know that λ^* cannot be negative, it must be the case that $x_1^* = \frac{w}{p_1} - \frac{p_1}{4p_2}$ ($\because$ equation 15).

So which of the solutions is the "correct" one. First, look at Section 2.2.4 and notice that $x_2^* = \frac{p_1^2}{4p_2^2} > 0$[8] However, for $x_1^* = \frac{w}{p_1} - \frac{p_1}{4p_2}$, this will have values greater than zero (as we assumed) if $\frac{w}{p_1} > \frac{p_1}{4p_2} \Rightarrow 4p_2 w > p_1^2 \Rightarrow 0 < p_1 < 2\sqrt{p_2 w}$). Therefore, we now see that the solution to this problem depends on the relative values of good 1 and good 2, and the value of the agent's wealth.

2.3 Interpretation

1. Check that the demands for good 1 and good 2 are h.o.d. zero. You should find that all of the solutions in 2.2.3 and 2.2.4 are h.o.d. zero.

2. Let's do some comparative statics for the solutions to 2.2.3, where $x_1^* = 0$, $x_2^* = \frac{w}{p_2}$. Since the comparative statics with regard to good 1 is trivial, let's jump right to good 2.

$$\begin{aligned}
\frac{\partial x_2^*}{\partial p_1} &= 0 \\
\frac{\partial x_2^*}{\partial p_2} &= -\frac{w}{p_2^2} < 0 \\
\frac{\partial x_2^*}{\partial w} &= \frac{1}{p_2} > 0
\end{aligned}$$

3. Now let's do some comparative statics for the solutions to 2.2.4, where $x_1^* = \frac{w}{p_1} - \frac{p_1}{4p_2}, x_2^* = \frac{p_1^2}{4p_2^2}$.

$$\begin{aligned}
\frac{\partial x_1^*}{\partial p_1} &= -\frac{w}{p_1^2} - \frac{1}{4p_2} < 0 \\
\frac{\partial x_1^*}{\partial p_2} &= +\frac{p_1}{4p_2^2} > 0 \\
\frac{\partial x_1^*}{\partial w} &= \frac{1}{p_1} > 0
\end{aligned}$$

[8] Of course, we have to assume that prices are greater than zero.

$$
\begin{aligned}
\frac{\partial x_2^*}{\partial p_1} &= 2\frac{p_1}{4p_2^2} \\
\frac{\partial x_2^*}{\partial p_2} &= -2\frac{p_1^2}{4p_2^3} < 0 \\
\frac{\partial x_2^*}{\partial w} &= 0
\end{aligned}
$$

Notice that there isn't a wealth effect for the non-numeraire good (good 2), which will be very important as we move on toward partial/general equilibrium models.