Econ 2020a Math Review

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Agenda

1. Cover the six topics below;
2. Cover some mathematical notation that frequently appears in economics; and

The assumption here is that you've all been exposed to some optimization, either during math camp or in some other mathematics class. If that is not the case, then you should definitely brush up on Simon and Blume chapters 14-18. You don't have to know every detail within these chapters, but you should know the basics.

1 "If and only if" vs. "if"

Be careful when you see these expressions. The former denotes a "necessary and sufficient" condition, while the latter denotes a "necessary" or "sufficient" condition.

Example 1. A function $f : S \to \mathbb{R}$ [1]is a concave function on the set S *iff* $f(y) - f(x) \leq f'(x)(y - x)$, $\forall x, y \in S$.

Compare Example 1 with the following Example 2

Example 2. A function $g : \mathbb{R} \to \mathbb{R}$ is a homothetic function *if* it is a monotone transformation of a homogeneous function.

Why do we care? "[M]any economic principles, such as marginal rate of substitution equals the price ratio, or marginal revenue equals marginal costs are simply the first order *necessary* conditions of the corresponding maximization problem. Ideally, an economist would like such a rule to also be a *sufficient* condition guaranteeing that utility or profit is being maximized so that it can provide a guideline for economic behavior."[2]

[1] Unless noted otherwise, assume for the rest of this math review that $S \subset \mathbb{R}$

[2] Simon and Blume (1994, p.518)

2 Convex and Non-Convex Sets

Definition 3. Set S is a convex set **if and only if** $\forall x, y \in S$, and $\forall \alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in S$.

We can extend Definition3 to include sets $\Sigma \subseteq \mathbb{R}^L$. Using vector notation, recall the two bundles we saw in the first lecture, $\vec{x} = [x_1, ..., x_L]$ and $\vec{x}' = [x'_1, ..., x'_L]$, we can rewrite the definition above as follows:

Definition 4. Set Σ is a convex set **if and only if** $\forall \vec{x}, \vec{x}' \in \Sigma$, and $\forall \alpha \in [0,1]$, $\alpha\vec{x} + (1-\alpha)\vec{x}' \in \Sigma$.

- If $\Sigma \subseteq \mathbb{R}^2$, whether a set is convex is easy to see.
- We will soon be encountering convex sets in the form of the Walrasian budget set. Draw a two-dimensional picture and see for yourselves that the budget set (when we have two goods x, y with prices p, q and wealth w) is a convex set.

Why do we care? See Theorem 7.

3 Convex and Concave Functions

Definition 5. A function $f : S \to \mathbb{R}$ is a strictly convex function over the set S if $\forall x_1,\, x_2 \in S$ and $\forall \alpha \in (0,1)$, $f(\alpha x_1 + (1-\alpha)x_2) < \alpha f(x_1) + (1-\alpha)f(x_2)$.

- Note that if you change the $<$ to a $\leq$, then the function will be weakly convex.
- Draw a picture of a convex function and you will see that any line connecting two points on a convex function will lie *above* the function.

Definition 6. A function $f : S \to \mathbb{R}$ is a strictly concave function over the set S if $\forall x_1, x_2 \in S$ and $\forall \alpha \in (0,1)$, $f(\alpha x_1 + (1-\alpha)x_2) > \alpha f(x_1) + (1-\alpha)f(x_2)$.

- Note that if you change the $>$ to a $\geq$, then the function will be weakly convex.
- Draw a picture of a concave function and you will see that any line connecting two points on a concave function will lie *below* the function.
- Some alternative criteria for determining concave functions are as follows (notice the iff!!):
  - A function $f : S \to \mathbb{R}$ is a (weakly) concave function over S *iff* $f''(x) \leq 0$, $\forall x \in S$. We look at the second-order derivative.
  - A function $f : S \to \mathbb{R}$ is a concave function over S *iff* $f(y) - f(x) \leq f'(x)(y-x)$, $\forall x, y \in S$. Draw a picture.

- Some interesting properties that will be helpful for you to know regarding concave functions:

  Theorem 7. *If f is a concave function on an "open," convex set S, (or $f : S \to \mathbb{R}$ and $f''(\cdot) \leq 0$), then x^*s.t. $f'(x^*) = 0$ (or x^* is a critical point of f) is the global maximizer of f on S.*

  What does this mean? Let's forget about the "open" language in the theorem above and concentrate on the convex set S. Do you remember what else was convex? Can you see why this theorem makes our lives so much easier?

  Theorem 8. *Let $f_1...f_k$ be concave functions, each defined on the same convext subset U and let $a_1...a_k$be positive real numbers. Then any linear combination of these functions, $a_1f_1+...+a_kf_k$, is also a concave function.*

All of the definitions and theorems above can be generalized to higher dimensions, i.e. $\Sigma \subseteq \mathbb{R}^L$. See Simon and Blume ch. 21 for an extensive review.

4 Quasiconcave Functions

Definition 9. A function$f : S \to \mathbb{R}$ is a quasiconcave function if $\forall a \in \mathbb{R}$, the set $C_a^+ \equiv \{x \in S \,|\, f(x) \geq a\}$ is a convex set. Or in words, the upper level set of some number a, or the set of x where the function f takes values greater than or equal to a, is convex.

Confused? Let's take a look at some examples (draw each one and see if this helps):

- Leontieff utility functions of the form $Q(x, y) = min\{x, y\}$ with $x, y > 0$ are certainly not concave,[3] but they are quasiconcave.
- *Some* non-decreasing functions such as $f(x) = x^3$ may not be concave (note the signs of the second derivatives), but they are quasiconcave.

For our purposes, you should understand quasiconcavity as a "weak" form of concavity. This is true since all concave functions are quasiconcave (but the reverse is *not* correct).

Why do we care? We care about quasiconcavity because we like concavity. Concavity has the following great properties: 1) Theorem 7 (no need to check the last $n - m$ leading principal minors of the $(m + n) \times (n + m)$ bordered Hessian matrix or the relatives signs of the eigenvalues on the diagonalized Jacobian matrix!!!!); 2) Theorem 8; and 3) their upper level sets are convex, which means that they can represent convex preferences.[4]

[3] Test your understanding of concavity to see that Leontieff functions are not concave.

[4] See Simon and Blume (1994, p.517-27) for a great exposition on this topic.

However, concavity is a "cardinal" property, meaning that a monotonic transformation of a concave function may not result in a concave function, while quasiconcavity is an "ordinal" property, meaning that a monotonic transformation of a quasiconcave function will still be a quasiconcave function. In short, the above three properties are preserved if we monotonically transform a quasiconcave function (given certain restrictions that we won't discuss).

5 Homogeneity of Degree "k"

Definition 10. For any scalar k, a real-valued function f is homogeneous of degree k (hereinafter "hod$k$") if $f(\alpha\vec{x}) = \alpha^k f(\vec{x})$. If a function is hod$k$ in x_1, then $f(\alpha x_1, \vec{x}_{-1}) = \alpha^k f(x_1, \vec{x}_{-1})$ or in the two variable case, $f(\alpha x, y) = \alpha^k f(x, y)$.

Example 11. Walrasian demand is hod 0 in $\vec{p}$ and w. Assume that $x_i(\vec{p}, w) = \frac{4w+3p_j}{5p_i}$. What happens if you double/triple/quadruple... all of the terms? Nothing. Therefore, $x_i(\alpha\vec{p}, \alpha w) = \alpha^0 \times x_i(\vec{p}, w) = 1 \times x_i(\vec{p}, w)$. Do you see the intuition behind this?

Like concavity we really like homogeneous functions because it has some very convenient properties. One of which is that the marginal rate of substitution (or marginal rate of technical substitution if we're dealing with a production function) is constant along rays from the origin.

But again, like concavity, these great properties are not preserved after a monotonic transformation—i.e. they are cardinal, *not* ordinal. That's why we have homothetic functions (which we won't discuss here).[5]

6 Identities, "$\equiv$"

Definition 12. An identity is a definitional equation, expressing that two expressions equal. The statement is true for all values of the parameters.

$3y + 4 = 13$ is not an identity because it only holds for the parameter value, $y = 3$. But $2(y + 2) \equiv 2y + 4$ is an identity. Note that the Walrasian demand function is also an identity—i.e. $x_j(\vec{p}, w) \equiv \frac{4w}{5p_j}$ will hold for all values of $\vec{p}$ and w. We are often lazy and write identities as $=$.

Why do we care? If we have an identity, we can differentiate both sides and still have a true expression. Let's differentiate both sides of $3y + 4 = 13$ with regards to y. Then we get $3 = 0$, which is a contradiction. However, the following is true:

$$\frac{\partial x_j(\vec{p}, w)}{\partial w} = \frac{4}{5p_j}$$

.

[5] Again, Simon and Blume (1994, ch. 20) is a wonderful exposition of homogeneous and homothetic functions.

Example 13. Walras' Law. Let's differentiate both sides of the following identity: $\vec{p} \cdot \vec{x}(\vec{p}, w) \equiv w$,[6] where $\vec{p} \in \mathbb{R}^L$. We can rewrite the identity above as follows:

$$\begin{aligned} \vec{p} \cdot \vec{x}(\vec{p}, w) &\equiv w \\ p_1 x_1(\vec{p}, w) + p_2 x_2(\vec{p}, w) + ... + p_n x_n(\vec{p}, w) &\equiv w, \end{aligned}$$

[7] and if we differentiate both sides of the identity with regards to w,

$$\begin{aligned} p_1 \frac{\partial x_1(\vec{p}, w)}{\partial w} + p_2 \frac{\partial x_2(\vec{p}, w)}{\partial w} + ... + p_n \frac{\partial x_n(\vec{p}, w)}{\partial w} &\equiv 1 \\ \sum_{i=1}^{n} p_i \frac{\partial x_i(\vec{p}, w)}{\partial w} &\equiv 1 \\ D_w \vec{x}(\vec{p}, w) \times \vec{p} &\equiv 1. \end{aligned}$$

[8] The intrepretation of this is that the change in wealth will equal the change in expenditure, which we should expect if the demand function satisfies Walras' Law.

[6] Note that economists are terrible with notation. MWG writes this identity as follows:$p \cdot x(p, w) \equiv w$. Scalar and vectors aren't clear in MWG. You should get used to the notation that is in MWG (and for Chris and Elon's lectures), but I will (attempt to) clearly distinguish between scalars and vectors.

[7] Note that the LHS of the first identity is a "dot product" and not simply a multiplication.

[8] We don't use the dot product in the third identity since we're multiplying a row vector and a column vector (which results in a scalar).