

# Section Notes 12

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## Agenda

1. Preferences over Two Alternatives
2. Preferences over more than Two Alternatives
3. Strategic Voting

For those of you who are interested in doing any type of work in welfare economics, political economy, or voting models, you should try to understand the proof of Arrow's Impossibility Theorem. MWG's proof in Section 21.C is the most intuitive for me, but there are several versions out there.

## 1 Preferences over Two Alternatives

Assume that there are two alternatives,  $x$  and  $y$ , both of which are elements of the set  $I$ ; agent  $i \in \{1, \dots, N\}$  has preferences as follows:

$$\alpha_i = \begin{cases} 1, & \text{if } x \succ_i y \\ 0, & \text{if } x \sim_i y \\ -1, & \text{if } x \prec_i y \end{cases}$$

Let  $A$  denote the set of all possible preferences or rankings over the set  $I$  for each agent. Therefore, the preference ordering  $\vec{\alpha} \in A^N$ .

Given the representation of individual preferences above, we can aggregate these preferences via majority voting as follows:

$$F(\vec{\alpha}) = \begin{cases} 1, & \text{if } \sum_{i=1}^N \alpha_i > 0 \\ 0, & \text{if } \sum_{i=1}^N \alpha_i = 0 \\ -1, & \text{if } \sum_{i=1}^N \alpha_i < 0 \end{cases}$$

**Theorem 1.1.** May's Theorem: *Given two alternatives, the majority voting social welfare function is the only social welfare function that satisfies the following conditions:*

1. Unrestricted domain:  $F : A^N \rightarrow \{-1, 0, 1\}$ ,  $\forall \vec{\alpha} \in A^N$ .
2. Anonymity (symmetry across agents):  $F(\vec{\alpha})$  only depends on the sums of  $\alpha_i$ 's equaling -1, 0, or 1. The identity of the individuals does not matter—*i.e.* each voter is identical.
3. Neutrality among alternatives: If we reversed every individual's preference, we would reverse society's preference:

$$F(\vec{\alpha}) = -F(-\vec{\alpha}).$$

Note that supermajorities are social welfare functions which also satisfy this condition.

4. Positive Responsiveness (monotonicity): Society does not have “thick indifference curves.”<sup>1</sup> A single change of preference is sufficient to break a tie. Further, preference shifts in favor of social preference will not change social preference. Formally, if  $\vec{\alpha} \in A^N$  s.t.  $F(\vec{\alpha}) \geq 0$ , then for  $\vec{\alpha}' \neq \vec{\alpha}$  where for some  $i$ ,  $\alpha'_i \geq \alpha_i$ ,  $F(\vec{\alpha}') = 1$ .

Given a majority voting social welfare function and two alternatives, is there an incentive for agent  $i$  to misrepresent her preferences? No. Voting your preference is a weakly dominant strategy.

## 2 Preferences over more than Two Alternatives<sup>2</sup>

Total of  $n \geq 3$  members on committee; all committee members have complete and transitive (weak) preferences over alternatives; complete information on the preferences of each committee member;<sup>3</sup> and the relevant social welfare function is a Borda count.

### Part (a)

Let's assume that committee members, denoted  $i$ , are elements of the set  $I$ . We also use the following notation for social preferences: if  $x$  is strictly preferred by society to  $y$ , then  $x \succ_S y$ , and if  $x$  is as good as  $y$  for society, then  $x \succeq_S y$ .<sup>4</sup>

A social welfare function is weakly Paretian if  $x \succ_i y$ ,  $\forall i$ , then  $x \succ_S y$ . A social welfare function is Paretian if  $x \succeq_i y$ ,  $\forall i$  and  $x \succ_i y$ , for some  $i$ , then  $x \succ_S y$ .<sup>5</sup>

<sup>1</sup>Recall that we had a similar concept in consumer theory. Can you recall which assumption got us to this result?

<sup>2</sup>2005 Final Exam Problem #5

<sup>3</sup>Note that if we have complete information of preferences, we could have strategic voting. However, this is assumed away in this problem.

<sup>4</sup>MWG uses  $xF_p(\succeq_1, \dots, \succeq_N)y$  to denote: society strictly prefers  $x$  to  $y$ ; and  $xF(\succeq_1, \dots, \succeq_N)y$  to denote:  $x$  is socially at least as good as  $y$ .

<sup>5</sup>Do you see the relation between this definition and Pareto efficiency?

## Part (b)

How are individual preferences aggregated using a Borda count? Let  $A$  be the set of all possible rankings over the set  $I$ . Then for each agent  $i$ ,  $\vec{\alpha}_i \in A$  is agent  $i$ 's ranking of each of the committee members in set  $I$ . To illustrate, assume that  $n = 3$  and agent 1's ranking  $\vec{\alpha}_1 = \{3, 1, 2\}$ . This is equivalent to  $3 \succ_1 1 \succ_1 2$  as the committee chair.

In a Borda count, for each agent  $i$ , we define  $c_i(x)$  which is equal to the rank of alternative  $x$ .<sup>6</sup> In the example above,  $c_1(3) = 1$ ;  $c_1(1) = 2$ ; and  $c_1(2) = 3$ . We can define this for all  $i \in I$ . Then the Borda count aggregates individual preferences such that alternatives  $x$  with a lower value of  $\sum_{i=1}^n c_i(x)$  is preferred by society.

Using the definition of weakly Paretian preferences above, given two alternatives  $x$  and  $y$ , assume that  $x \succ_i y$ ,  $\forall i \in I$ . This means that  $c_i(x) < c_i(y)$ ,  $\forall i \in I$ , which results in:

$$\sum_{i=1}^n c_i(x) < \sum_{i=1}^n c_i(y) \Rightarrow x \succ_S y$$

Therefore, we conclude that a Borda count is a weakly Paretian social welfare function.

The question then asks us to show that this is efficient. By efficient, we mean Pareto efficient. Given any two options,  $x$  and  $y$ , if all individuals prefer  $x$  to  $y$ , then  $x$  will be chosen by the social welfare function, which is the Pareto efficient result.

## Part (c)

To understand the definition of “non-dictatorial”<sup>7</sup> let's first define what it means for agent  $i$  to be a dictator.

**Definition 2.1.** An agent  $i$  is a dictator, if  $\forall \{x, y\} \subset I$  and  $\forall \vec{\alpha} \in A^n$ , we have that  $x$  is socially preferred to  $y$ , that is  $x \succ_S y$ , whenever  $x \succ_i y$ .

Therefore, a social welfare function is non-dictatorial iff no individual is a dictator.

The proof that a Borda count is non-dictatorial is by contradiction. Assume that agent  $i$  is a dictator with preferences  $x \succ_i y \succ_i z$ ; and all  $j \neq i \in I$  have preferences  $z \succ_j y \succ_j x$ . Consider  $\{x, z\} \subset I$ , the dictator prefers  $x$  to  $z$ . However,

$$\sum_{i=1}^n c_i(x) > \sum_{i=1}^n c_i(z) \Rightarrow z \succ_S x,$$

which contradicts the statement that agent  $i$  is a dictator.

<sup>6</sup>If we allow for indifference between two alternatives, then we take the average rank of the two indifferent alternatives.

<sup>7</sup>A weakening of May's symmetry / anonymity.

**Definition 2.2.** If agent  $i$ 's preferences are such that  $x \succ_i y$  for some  $\{x, y\}$  pairing, then there exists some  $\vec{\alpha} \in A^n$  such that  $y \succ_S x$ . Then agent  $i$  is not a dictator.

### Part (d)

The Independence of Irrelevant Alternatives (“IIA”) is defined formally as follows:

**Definition 2.3.** The social welfare function defined on the domain  $A^n$  satisfies IIA (or “pairwise independence” condition) if the social preference between any two alternatives  $\{x, y\} \subset I$  *depends only on the ordering of individual preferences over the same alternatives*. For any pair of alternatives  $\{x, y\} \subset I$ , and for any pair of preference orderings  $\vec{\alpha} \in A^n$  and  $\vec{\alpha}' \in A^n$  with the property that, for every  $i$ ,

$$x \succsim_i y \Leftrightarrow x \succsim'_i y$$

and

$$y \succsim_i x \Leftrightarrow y \succsim'_i x,$$

we have that

$$x \succsim_S y \Leftrightarrow x \succsim'_S y$$

and

$$y \succsim_S x \Leftrightarrow y \succsim'_S x.$$

In other words, if two sets of individual preference orderings have the same preference between  $x$  and  $y$  for all individuals, then the social preference between  $x$  and  $y$  must be the same.

Consider the following preferences  $\vec{\alpha}$  and  $\vec{\alpha}'$ :<sup>8</sup>

| $\vec{\alpha}$   |                  |                  | $\vec{\alpha}'$   |                   |                   |
|------------------|------------------|------------------|-------------------|-------------------|-------------------|
| $\vec{\alpha}_1$ | $\vec{\alpha}_2$ | $\vec{\alpha}_3$ | $\vec{\alpha}'_1$ | $\vec{\alpha}'_2$ | $\vec{\alpha}'_3$ |
| a                | b                | a                | a                 | b                 | a                 |
| b                | z                | b                | z                 | a                 | z                 |
| z                | a                | z                | b                 | z                 | b                 |

Notice that the individual preferences between alternatives “a” and “b” are the same for all agents. For preferences  $\vec{\alpha}$ , society is indifferent between “a” and “b” since:

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<sup>8</sup>I realize that this is a slight abuse of notation, since these aren't really vectors.

$$\sum_{i=1}^3 c_i(a|\vec{\alpha}) = 5 = \sum_{i=1}^3 c_i(b|\vec{\alpha}) \Rightarrow a \sim_S b$$

but for preferences  $\vec{\alpha}'$ , society prefers “a” to “b” since:

$$\sum_{i=1}^3 c_i(a|\vec{\alpha}') = 4 < \sum_{i=1}^3 c_i(b|\vec{\alpha}') = 7 \Rightarrow a \succ_S b$$

In short, because the rank of an alternative depends on the placement of every other alternative, a Borda count does not satisfy the IIA condition. Consider another example that illustrates IIA at work.

**Example 2.4.** Assume that there are two agents 1 and 2, and three alternatives a, b, and c. Suppose that the preferences of the first agent is given by:  $a \succ_1 b \succ_1 c$ , which we denote as “abc.” There are two preference orderings  $\vec{\alpha} = [abc, acb]$  and  $\vec{\alpha}' = [abc, bca]$ . Consider a social welfare function  $F : A^2 \rightarrow \{a, b, c\}$  which maps each of the preference orderings above as follows:  $F(\vec{\alpha} = [abc, acb]) = acb$  and  $F(\vec{\alpha}' = [abc, bca]) = abc$ .<sup>9</sup>

If we assume that the social welfare function  $F(\cdot)$  satisfies IIA and is weakly Paretian, then by IIA we have that any preference ordering in  $A$  which has  $b \succ_1 c$ ,  $c \succ_2 b$  results in  $c \succ_S b$  and any preference ordering in  $A^2$  which has  $a \succ_1 c$ ,  $c \succ_2 a$  results in  $a \succ_S c$ . Therefore, the social welfare function  $F(\cdot)$  must map the preference ordering  $\vec{\alpha}'' = [bac, cba]$  such that  $c \succ_S b$  and  $a \succ_S c$ .

## Part (e)

The problem has restricted the preferences that each agent  $i$  can hold such that each agent has “single-peaked” preferences. With single-peaked preferences we can define a Condorcet winner. Recall in May’s Theorem and in Arrow’s Impossibility Theorem (as well as Sen’s Impossibility Theorem) that we assumed the domain of the social welfare function was unrestricted. However, with single-peaked preferences, we’ve limited the domain of the social welfare function such that we can get rid of “cycling.”

As noted above, with single peaked preferences over a one-dimensional policy space (in this example, it’s age), we can apply the Median Voter Theorem.<sup>10</sup> The result is that the Condorcet winner is the most preferred policy of the median voter, which will be chosen by the social welfare function of simple majority rule.

<sup>9</sup>From MIT 14.773 Section Notes, Spring 2010.

<sup>10</sup>The proof of the MVT is very accessible, so for those interested I encourage you to review it.

### 3 Strategic Voting

An assumption that appears in the context of aggregating individual preferences via a social welfare function based on some form of voting is sincere voting. Of course, in some instances, sincere voting is endogenous to the model because it is a weakly dominant strategy. However, what happens if we allow for strategic voting?

Recall question 2 from the 2010 mid term. A similar example is as follows:

**Example 3.1.** The relevant voting rule or social welfare function is all vote for one option and drop the lowest performing option. Repeat until one option has been chosen. Assume that there exist 5 agents who have the following preferences:

| Agent $i$ : | 1 | 2 | 3 | 4 | 5 |
|-------------|---|---|---|---|---|
| 1st best    | A | A | B | B | C |
| 2nd best    | B | B | C | C | A |
| 3rd best    | C | C | A | A | B |

What is the SPNE?

If all agents vote sincerely, then in the second stage, alternative A faces off against B which results in A winning. However, if agents 3 and 4 vote for C in the first stage, then C vs A in the second stage which results in C winning. However, agents 1 and 2 have an incentive to deviate and vote for B in the first stage, which results in B vs C in the second stage resulting in B winning and no incentive to deviate.

Do you see the intuition behind this SPNE?

**5. (30 points):** Consider the election of a committee chair by its  $n$  members. Assume  $n$  is at least 3 and that the committee chair must be one of these  $n$  people. Each member has a complete and transitive (weak) preference ordering over possible choices for the chair.

The committee must express a ranking of all its members to the CEO of the company, who is not one of the committee members. The CEO will respect the preferences of the committee members if she can, but the CEO knows that some committee members will be promoted and transferred elsewhere, and hence may be unavailable for service as committee chair. Who is available and who is not is known only to the CEO and not to the members themselves.

The committee has adopted the rule that the ranking provided to the CEO will be the Borda count applied to the members' preferences. Since the preferences of the committee members are common knowledge to all the members (they know each other very well), it is straightforward to compute the Borda count for these preferences.

Throughout the problem you can assume that the social preference ordering is a Borda count based on the true preferences of each committee member.

**(a)** Define the concept that a social choice functional be Paretian.

**(b)** Show that the result of this procedure is Paretian, according to your definition, no matter what preferences are. Show that this procedure results in an efficient choice of the chair no matter what subset of members are available.

**(c)** Define the concept that a social choice functional be non-dictatorial. Show that this procedure is non-dictatorial according to your definition.

**(d)** Define the concept of Independence of Irrelevant Alternatives. Give an example of a sets of preferences (take  $n=3$ ) that demonstrate that this procedure does not satisfy Independence of Irrelevant Alternatives as you have defined it.

**(e)** Suppose that the ages of all committee members are known and that age is the only relevant factor in anyone's preference for who should be chair. Each member has a "most preferred age" for the chair. Each person's ranking of the committee members is decreasing in the absolute value of the difference between the member's most preferred age and that person's actual age. For the subset of preference that are possible under this assumption, reevaluate your answer to part d) above.