Section Notes 4

Wonbin Kang

February 18, 2010

Agenda

1. Trembling Hand Perfect Nash Equilibrium ("THPNE")
2. Bayesian Nash Equilibrium

1 THPNE

By now you should know that in certain games, you will find a number of NEs. Going forward, we're going to be "refining" the set of NE or using new equilibrium concepts to give us better predictions in games. THPNE will be the first refinement we consider. Let's look at the following example:

Example 1.1.

	L	M	R
U	4,-4	2,4	0,-3
D	5,3	2,2	-1,1

$\Rightarrow$

	L	M
U	4,-4	$\underline{2},\underline{4}$
D	$\underline{5},\underline{3}$	2,2

Remember, when you first see a game, the first thing you should do is check to see if there is a strictly *dominated* strategy. In the game above, we can simplify the 2×3 matrix into the 2×2 matrix since "R" is a strictly dominated strategy. It's strictly dominated by player 2's mixed strategy where player 2 plays $\vec{p} = \left[\frac{1}{2}, \frac{1}{2}, 0\right]$, among other mixed strategies. In the reduced form game, there are two NE: (U,M) and (D,L). What we want to think about is which of the two NEs is "better." We're going to use THPNE to solve for the "better" equilibrium.[1]

1.1 Definition of THPNE

Let's start with a couple of definitions:

[1] Elon called this the equilibrium selection process and the "better" equilibrium as the self-enforcing equilibrium.

Definition 1.2. A strategy σ_i is a *completely mixed strategy* if it puts positive probability on all of player i's pure strategies. A strategy profile σ is completely mixed if all players are playing a completely mixed strategy.

To prevent confusion, let's assume that $i \in \{1, ..., I\}$, then:

Definition 1.3. A NE strategy profile σ is trembling hand perfect *if there exists* a sequence of completely mixed strategy profiles $\{\sigma_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \sigma_n = \sigma$ and such that for all players i, σ_i is a best response to σ_{-in} for a large enough n.

1.2 Examples of THPNE

Let's look at a few examples to clarify the definition above:

For Example 1.1 above, consider the NE where player 1 plays "U" and player 2 plays "M". From the definition of THP, we need to consider completely mixed strategy *profiles*, so let's assume that player 1 is playing a completely mixed strategy where he places $1-\epsilon$ probability on "U" and ϵ probability on "D". Also let's assume that player 2 is playing a completely mixed strategy, denoted σ_2^* where she places $1-\epsilon$ probability on "M" and ϵ probability on "L".

Note the following:

1. We can write $\epsilon = \frac{1}{n}$ such that $n \to \infty$ is equivalent to $\epsilon \to 0$. So we do have a sequence of strategy profiles of the form in the definition.

2. Note that as $\epsilon \to 0$, the strategy profiles converge to the NE under consideration, [U,M].

Now let's look at the expected utilities of player 1 given the completely mixed strategy of player 2:

$$\begin{aligned} U_1(Up; \sigma_2^*) &= 4\epsilon + 2(1-\epsilon) = 2 + 2\epsilon \\ U_1(Down; \sigma_2^*) &= 5\epsilon + 2(1-\epsilon) = 2 + 3\epsilon \end{aligned}$$

Because we're looking at values of ϵ arbitrarily close to zero (but not equal to zero, because we're considering completely mixed strategy profiles), $U_1(Down; \sigma_2^*) > U_1(Up; \sigma_2^*)$, since $3\epsilon > 2\epsilon$. Therefore, we conclude that [U,M] is *not* trembling hand perfect.

Note that the definition of THPNE states that there is *no* sequence of completely mixed strategies $\{\sigma_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \sigma_n = \sigma$ and such that for all players i, σ_i is a best response to σ_{-in} for a large enough n. We didn't do that above, but for purposes of this class it is sufficient.

Example 1.4. [2]

	L	C	R
U	3,1	1,0	4,0
M	1,3	2,4	1,2
D	0,1	1,1	2,2

$\Rightarrow$

	L	C	R
U	3,1	1,0	4,0
M	1,3	2,4	1,2

[2] From Practice Problem 4.2.

$\Rightarrow$

	L	C
U	3,1	1,0
M	1,3	2,4

Iterated elimination of strictly dominated strategies gets us to the 2×2 matrix, where we have 2 pure strategy NE: [U,L] and [M,C]; and a mixed strategy NE where $\vec{p}^* = [\Pr(U), \Pr(M)] = \left[\frac{1}{2}, \frac{1}{2}\right]$ and $\vec{p}^* = [\Pr(L), \Pr(C)] = \left[\frac{1}{3}, \frac{2}{3}\right]$.

Check whether the two PSNE are THP.

Let's check to see whether the MSNE is THP.

1. We need to come up with a completely mixed strategy for each player that at its limit is equal to the strategy being played in our equilibrium of interest. Consider the following:

$$\sigma_{1\epsilon} = \left[\frac{1}{2} - \epsilon,\ \frac{1}{2} - \epsilon,\ 2\epsilon\right];\ \sigma_{2\epsilon} = \left[\frac{1}{3} - \epsilon,\ \frac{2}{3} + \frac{\epsilon}{4},\ \frac{3}{4}\epsilon\right]$$

2. Now we calculated the expected utilities for each of the players. Let's start with player 1:

$$\begin{aligned} U_1(U) &= (1 - 3\epsilon) + \left(\frac{2}{3} + \frac{\epsilon}{4}\right) + (3\epsilon) = \frac{5}{3} + \frac{\epsilon}{4} \\ U_1(M) &= \left(\frac{1}{3} - \epsilon\right) + \left(\frac{4}{3} + \frac{\epsilon}{2}\right) + \left(\frac{3}{4} \cdot \epsilon\right) = \frac{5}{3} + \frac{\epsilon}{4}. \end{aligned}$$

Since $U_1(M) = U_1(U)$ as $\epsilon \to 0$, we've found a sequence of completely mixed strategies which converge to the NE and for which the mixed strategy is a best response.

3. Try this for player 2 and you should see that the same holds.

$$\begin{aligned} U_2(L) &= \left(\frac{1}{2} - \epsilon\right) + \left(\frac{3}{2} - 3\epsilon\right) + (2\epsilon) = 2 - 2\epsilon \\ U_2(C) &= 0 + (2 - 4\epsilon) + (2\epsilon) = 2 - 2\epsilon. \end{aligned}$$

Example 1.5. From class:

	L	M	R
Up	4, 4	1, 4	0, -3
Down	5, 3	2, 2	-999999, -2

Notice that "R" is strictly dominated by "M" and iterated dominance outcome (NE) is [D, L].

1. Assume that the probabilities of player 2's completely mixed strategy is equal to $\vec{q} = [q_L, q_M, q_R]$. Then we can solve for player 1's expected values

of playing “Up” and “Down.”

$$\begin{aligned} U_1(Up) &= 4q_L + q_M \\ U_1(Down) &= 5q_L + 2q_M - 999999q_R \\ \therefore U_1(Down) - U_1(Up) &= q_L + q_M - 999,999q_R = 1 - 1000000q_R \end{aligned}$$

2. For player 1, if $q_R \leq \frac{1}{1,000,000}$ then “Down” is a best response for player 1. Any sequence of strategies for player 2, which has as its limit the strategy “L”, has the property that eventually $\Pr(R) < \frac{1}{1,000,000}$. Therefore, [D,L] is THPNE.

1.3 Main Results of THPNE

1. All finite games have at least one THPNE.
2. Any completely mixed NE is THPNE. Let the sequence of strategies be equal to the mixed strategy candidate, then the definition of THPNE is satisfied.
3. If $\sigma^*$is a THPNE, then no player is putting positive probability on a weakly dominated strategy.
4. In a two player game, if $\sigma^*$is a NE not involving a weakly dominated strategy, then σ^* is THP (this doesn't hold if you have $I > 2$).

2 Bayes-Nash Equilibrium

Recall that one of the assumptions of the games that we've looked at was *complete information*, the players knew the payoff functions of the other players. Now let's consider the case where the parties are unaware of their opponents' payoffs because the parties are of different types, and the players are only aware of the probability distribution of types. Consider the following examples and its extension.

Example 2.1. Player 1 is a graduate student who has strategies work (“W”) or shirk (“S”). Player 2 is a professor who must decide whether to guide or ignore the graduate student. Let's assume that the graduate student can be of two types: motivated or lazy. The probability that the graduate student is motivated is equal to $m \in (0, 1)$. The payoffs for each type are:

Motivated:

	G	I
W	8,5	4,-1
S	2,-1	2,0

Lazy:

	G	I
W	3,5	-1,-1
S	2,-1	2,0

. Draw the extensive form of this game and the Bayesian Normal Form.

1. The extended form can be found in Appendix A-1. Note that we need the move by “Nature” to draw this game.

2. The Bayesian Normal Form is as follows:[3]

	G	I
WW	3+5m, 5	-1+5m, -1
WS	2+6m, -1+6m	2+2m, -m
SW	3-m, 5-6m	-1+3m, -1+m
SS	2,-1	2,0

(a) For player 1 (the graduate student) "SS" is strictly dominated by "WS".

(b) For player 1 (the graduate student) "SW" is strictly dominated by "WW".

(c) If we underline best responses, we see that [WW,G] is BNE. Further, if $-1+6m > -m \Rightarrow m > \frac{1}{7}$, then [WW,G] is the unique BNE. Otherwise, we have two BNEs: [WW,G] and [WS, I].

(d) Assuming that $m \leq \frac{1}{7}$, which means that there are two BNEs, what is the mixed strategy BNE? Let the probability of player 1 playing "WW" equal p and the probability of player 2 playing "I" equal q. We won't be putting positive probability on the strictly dominated strategies, and so no need to check against any remaining pure strategies.

i. For player 1:

$$\begin{aligned} U_1(WW) &= (3+5m)\cdot(1-q)+(-1+5m)\cdot q = 3+5m-4q \\ U_1(WS) &= (2+6m)\cdot(1-q)+(2+2m)\cdot q = 2+6m-4mq \\ &\Rightarrow \; q = \frac{1-m}{4-4m} = \frac{1}{4} \end{aligned}$$

ii. For player 2:

$$\begin{aligned} U_2(G) &= 5p+(1-p)\cdot(-1+6m) = 6p-1+6m-6mp \\ U_2(I) &= -p+(1-p)\cdot(-m) = -p-m+mp \\ &\Rightarrow \; p = \frac{1-7m}{7-7m} \end{aligned}$$

Example 2.2. [Free-riding] There are two players $i = 1,2$; denote player 1's effort level x and player 2's effort level y; and total profit for the two players is $\Pi(x,y) = 4(x+y+kxy)$, where $k \in \left[0, \frac{1}{2}\right]$. The effort cost of player 1 and player 2 are, respectively, $c(x) = x^2$ and $c(y) = y^2$. Each player chooses their level of effort simultaneously. What is the NE of this game?

[3] To reiterate my notation, [WS] means that the "motivated" type will "Work" and the "lazy" type will "Shirk".

Recall the way we solved the Cournot model. We set up the maximization problems and solve for $i = 1$:

$$U_1(x, y) = 2(x + y + kxy) - x^2 \Rightarrow^{FOC} \frac{\partial U_1}{\partial x} = 2(1 + ky) - 2x = 0$$

Since we have a symmetric problem, we have the following equations:

$$\begin{aligned} x^* &= 1 + ky^* \quad (1) \\ y^* &= 1 + kx^* \quad (2) \end{aligned}$$

, which results in the following NE:[4]

$$x^* = y^* = \frac{1}{1-k} \quad (3)$$

We can do one interesting comparative static on the equilibrium identified in 3

$$\frac{dx^*}{dk} = \frac{dy^*}{dk} = \frac{1}{(1-k)^2} > 0.$$

What does this mean? Note that k is the complementary increase in profit if both players increase their effort level.

Example 2.3. Same set up as Example 2.1, except now suppose that player 1 doesn't know (or observe) the payoffs of player 2. All she knows is the distribution of player 2's payoffs: there is 50% chance that player 2's payoff is $U_2^{high}(x, y) = 2(x + y + kxy) - 2y^2$ and a 50% chance that his payoff is $U_2^{low}(x, y) = 2(x+y+kxy) - y^2$, where the "high" and "low" superscripts denote high cost vs low cost type.

The strategy space of each player are: $S_1 \equiv [0, \infty)$ and $S_2 \equiv \left\{S_2^{low}, S_2^{high}\right\} = \{[0.\infty), [0, \infty)\}$.

Solving for each of player 2's best responses, we have:

$$\begin{aligned} y^*_{low} &= \arg\max_y 2(x + y + kxy) - y^2 = 1 + kx \quad (4) \\ y^*_{high} &= \arg\max_y 2(x + y + kxy) - 2y^2 = \frac{1 + kx}{2} \quad (5) \end{aligned}$$

Given these payoffs for player 2's two types, player 1 will maximize her expected utility to solve for x^*:

[4] An interesting exercise might be to see what the equilibrium effort levels are when we only have one player or when we have two players but player 1 moves first and player 2 moves after observing player 1's move.

$$
\begin{aligned}
x^* &= \arg\max_x \left[\frac{1}{2}\left\{2(x + y^*_{low} + kxy^*_{low}) - x^2\right\} + \frac{1}{2}\left\{2(x + y^*_{high} + kxy^*_{high}) - x^2\right\}\right] \\
&= \arg\max_x \left[2x + y^*_{low} + y^*_{high} + kxy^*_{low}kxy^*_{high} - x^2\right].
\end{aligned}
$$

FOCs (don't worry about K-T Conditions) result in the following:

$$
\frac{\partial U_1(x, y^*_{low}, y^*_{high})}{\partial x}|_{x=x^*} \quad = \quad 2 + ky^*_{low} + ky^*_{high} - 2x^* = 0
$$

$$
\therefore x^* = 1 + k \cdot \frac{(y^*_{low} + y^*_{high})}{2}. \tag{6}
$$

We have three equations and three unknowns (Equations 4, 5, and 6), and if we solve we get the following Bayes-Nash equilibrium:

$$
x^* = \frac{4+3k}{4-3k^2};\ y^*_{low} = 1 + k \cdot \frac{4+3k}{4-3k^2};\ y^*_{high} = \frac{1}{2}\left\{1 + k \cdot \frac{4+3k}{4-3k^2}\right\}.
$$

We tried some comparative statics in the original game. Try them now and see what results you get.

Appendix Section Note ~~8~~4