Section Notes 5

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Agenda

1. Risk Aversion and Measuring Risk Aversion
2. Summary of Risk Attitudes
3. Investment Example (like Insurance example in class)
4. Math Appendix (Second Order Taylor Expansion and Implicit Function Theorem)

1 Risk Aversion and Measuring Risk Aversion

Definition 1. An economic agent is risk averse if for any lottery over money $F(x)$, the degenerate lottery that yields the amount $\int x dF(x)$ with certainty is at least as preferred as the lottery $F(\cdot)$ itself. If the agent is indifferent between the two lotteries, he is risk neutral.

More formally, for a given Bernoulli utility function $u(\cdot)$, an agent is risk averse, iff:

$$\int u(x)dF(x) \leq u\left(\int x dF(x)\right),$$

for all $F(\cdot)$; or in the discrete case when dealing with a lottery $L = [p_1 ... p_n]$,

$$\sum_{i=1}^{n} p_i u(x_i) \leq u\left(\sum_{i=1}^{n} p_i x_i\right),$$

[1]where p_i is equal to the probability of event x_i occurring; or

$$E\left[u(x)\right] \leq u\left(E\left[x\right]\right).$$

[1]Note that we can replace all instances of $\int x dF(x)$, which is the expected value of a continuous random variable, with $\sum_{i=1}^{n} p_i x_i$, which is the expected value of a discrete random variable.

If you're having trouble understanding *Jensen's Inequality*, try drawing a concave utility function and then check the inequalities above. How do the equations above change if an agent is *strictly* risk averse?

1.1 Arrow-Pratt Measures

Definition 2. Given a C^2 Bernoulli utility function $u(\cdot)$ for money, the Arrow-Pratt coefficient of *absolute* risk aversion at the point x^* is defined as:

$$r_A(x^*) = -\frac{u''(x^*)}{u'(x^*)}.$$

Given a C^2 Bernoulli utility function $u(\cdot)$ for money, the Arrow-Pratt coefficient of *relative* risk aversion at the point x^* is defined as:

$$r_R(x^*) = -x^* \cdot \frac{u''(x^*)}{u'(x^*)}.$$

Consider the measurement of relative risk aversion for two utility functions which are often used:

Example 3. The Constant Relative Risk Aversion ("CRRA") utility function of the form: $u(x) = \ln x$

$$\begin{aligned} r_A(x^*) &= -\frac{u''(x^*)}{u'(x^*)} = \frac{1}{x} \\ r_R(x^*) &= -x^* \cdot \frac{u''(x^*)}{u'(x^*)} = 1 \end{aligned}$$

So a CRRA utility function has decreasing absolute risk aversion, but a constant relative risk aversion.

Example 4. The Constant Absolute Risk Aversion ("CARA") utility function of the form: $u(x) = -\exp\{-\alpha x\}$

$$\begin{aligned} r_A(x^*) &= -\frac{u''(x^*)}{u'(x^*)} = a \\ r_R(x^*) &= -x^* \cdot \frac{u''(x^*)}{u'(x^*)} = ax \end{aligned}$$

So a CARA utility function has constant absolute risk aversion, but an increasing relative risk aversion.

1.2 Certainty Equivalent

Definition 5. The CE of a money lottery, $F(\cdot)$, is equal to the amount of money for which the individual is indifferent between the lottery and the certain amount CE, or $c(F,u)$:

$$u(CE) = \int u(x)dF(x) = \int u(x)f(x)dx,$$

or in the discrete case,

$$u(CE) = \sum_{i=1}^{n} p_i u(x_i) = U(L)$$

[2]

If an agent is risk averse, then $CE \leq \int x dF(x) = E(x)$. Let's draw a picture to see that this is in fact the case.

Example 6. Consider a Bernoulli utility function, $u(x) = x^{\frac{2}{3}}$; a lottery $L = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$; and event $x = [0, 1000, 8000]$.

So, $U(L) = \frac{1}{3} \cdot 0^{\frac{2}{3}} + \frac{1}{3} \cdot 1000^{\frac{2}{3}} + \frac{1}{3} \cdot 8000^{\frac{2}{3}} = \frac{500}{3}$; and the expected value of this lottery given the events is, $E(x) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1000 + \frac{1}{3} \cdot 8000 = 3000$. From the definition, we know that CE will be such that:

$$\begin{aligned} u(CE) &= CE^{\frac{2}{3}} = \frac{500}{3} \\ \therefore \quad CE &\approx 2152. \end{aligned}$$

Notice that this is less than the expected value of the lottery given the events.

One final note on CE. Can you see the intuition behind the fact that the more risk averse you are, the lower is CE. To use the terms in MWG, if the utility function $u_1$denotes the preferences of an agent *more* risk averse than an agent with the utility function u_2, it will be the case that:

$$c(F, u_1) \leq c(F, u_2) \tag{1}$$

1.3 Risk Premium[3]

Definition 7. The risk premium is equal to the amount of money a risk averse agent would be willing to pay to remove all of the risk in his payoff. Therefore, the risk premium is calculated as:

$$\pi(F, u) = E(X) - CE.$$

[2]Notice that I used the capital U to denote the expected utility of the lottery L which is a vector of probabilities, $[p_1, ..., p_n]$, over the events $x_1, ..., x_n$.

[3]Don't confuse this with the *probability* premium, which is a different concept (but using very similar principles to calculate).

In our example above, the risk premium $\pi(F,u) = 3000 - 2152 = 848$.

Notice that since CE$\leq \int x dF(x) = E(x)$ if an agent is risk averse, it must be the case that $\pi(F,u) > 0$ if an agent is risk averse. Further, because of weak inequality 1, if the utility function $u_1$denotes the preferences of an agent *more* risk averse than an agent with the utility function u_2, it will be the case that:

$$\pi(F,u_1) \geq \pi(F,u_2)$$

Example 8. Now what if the agent in Example 6 had \$10,000 (in our previous example, the agent started with \$0). Then we can set up the problem as follows: Consider a Bernoulli utility function, $u(x) = x^{\frac{2}{3}}$; a lottery $L = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$; and event $x = [10000, 11000; 18000]$.

So, $U(L) = \frac{1}{3} \cdot 10000^{\frac{2}{3}} + \frac{1}{3} \cdot 11000^{\frac{2}{3}} + \frac{1}{3} \cdot 18000^{\frac{2}{3}} = 548.53$; and the expected value of this lottery given the events is, $E(x) = \frac{1}{3} \cdot 10000 + \frac{1}{3} \cdot 11000 + \frac{1}{3} \cdot 18000 = 13000$. From the definition, we know that CE will be such that:

$$\begin{aligned} u(CE) &= CE^{\frac{2}{3}} = 548.53 \\ \therefore\ CE &\approx 12,857 \\ \pi &\approx 143. \end{aligned}$$

Compare the risk premiums of Examples 6 and 8. Do you see that as your wealth increase, you become less risk averse.

2 Summary of Risk Attitudes

Here is a quick dirty exposition of risk attitudes and their characteristics:

Strictly Risk Averse	Risk Neutral	Strictly Risk Loving
$u(x)$ strictly concave	$u(x)$ linear	$u(x)$ strictly convex
$E[u(x)] < u(E[X])$	$E[u(x)] = u(E[X])$	$E[u(x)] > u(E[X])$
$c(F,u) < E[X]$	$c(F,u) = E[X]$	$c(F,u) > E[X]$
$\pi(F,u) > 0$	$\pi(F,u) = 0$	$\pi(F,u) < 0$
$r_A > 0$, $r_R > 0$	$r_A = 0$, $r_R = 0$	$r_A < 0$, $r_R < 0$

3 Investment Example (from Miller Notes 180-182)[4]

Example 9. Consumer has utility function $u(\cdot)$ s.t. $u'(\cdot) > 0$ and $u''(\cdot) < 0$ and initial wealth w. The decision that the consumer must make is how much

[4]Sam may go through a previous mid term, but I thought that it would be more helpful to see an abstract version of what Chris did in class in the context of insurance. Please go through your notes from class; Miller Notes 178-182; and the 2004 Midterm, Problem 3.

of her wealth to invest in a riskless asset and how much to invest in a risky asset that has a return of zero with probability equal to π and rx dollars with probability $1-\pi$. Let x be the number of dollars the investor invests in the risky asset. In the alternative, the investor could invest in a riskless asset which has a return of 1.

Do you see what the choice variable is? Can you see the connection between this problem and the insurance problem?

As you should all very well know by now, half of the battle is setting up the correct optimization problem, which in this case is:

$$\max_x \pi \cdot u(w-x) + (1-\pi) \cdot u(w-x+rx). \tag{2}$$

s.t.

$$x \in [0, w]$$

The first order conditions are:

$$\begin{aligned} -\pi \cdot u'(w-x^*) + (1-\pi)(r-1) \cdot u'(w-x^*+rx^*) &\leq 0\, if\, x^* = 0 \\ &= 0\, if\, x^* \in (0, w) \quad (3) \\ &\geq 0\, if\, x^* = w. \end{aligned}$$

Notice that we have two possible corner solutions. Now let's consider an instance where the investor does not invest in a risky asset at all, or $x^* = 0$. From the FOCs above we know that this means:

$$\begin{aligned} -\pi \cdot u'(w-x^*) + (1-\pi)(r-1) \cdot u'(w-x^*+rx^*) &\leq 0 \\ u'(w)\{-\pi + (1-\pi)(r-1)\} &\leq 0 \\ -\pi + (1-\pi)(r-1) &\leq 0\, (\because u'(\cdot) > 0) \\ (1-\pi)r &\leq 1. \quad (4) \end{aligned}$$

We interpret this as meaning that the only time that it is optimal for an investor not to invest in a risky asset is when the expected return on the risky asset is less than the return on the safe asset. Trivially, inequality 4 is telling us not to invest in the risky asset if its expected return is less than the expected return on the riskless asset.

But now what if there exists a unique interior solution which satisfies equation 3 of the FOCs (assuming that the utility function $u(\cdot)$ is well-behaved)?[5]Because of equation 3 above, we can solve for x^* as a function of the parameter w. Then

[5]By well-behaved, we're usually talking about the function being strictly quasi-concave (or strictly concave) and satisfying the Inada conditions. See Section Note 3 if you don't know what the Inada conditions are.

we can do some comparative statics and see how the optimal x^* varies as you change one of the parameters. Let's denote $x^* = x(w)$ and plug into equation 3. Then:

$$-\pi \cdot u'(w - x(w)) + (1-\pi)(r-1) \cdot u'(w - x(w) + rx(w)) = 0. \tag{5}$$

Notice that equation 5 is an implicit function, where we can use the Implicit Function Theorem ("IFT").[6]

Let $F(x,w) = -\pi \cdot u'(w-x) + (1-\pi)(r-1) \cdot u'(w-x+rx) = 0$, and apply the IFT.

$$x'(w) = \frac{dx^*}{dw} = -\frac{\frac{\partial F(x,w)}{\partial w}}{\frac{\partial F(x,w)}{\partial x}} = \frac{-\{-\pi u''(w-x) + (1-\pi)(r-1)u''(w-x+rx)\}}{\pi u''(w-x(w)) + (1-\pi)(r-1)^2 u''(w-x+rx)}. \tag{6}$$

[7] We already know that the denominator is negative. Further, the numerator of equation 6 will be negative if:

$$\begin{aligned}
\pi u''(w-x) &< (1-\pi)(r-1)u''(w-x+rx) \\
\frac{u''(w-x)}{u''(w-x+rx)} &> \frac{(1-\pi)(r-1)}{\pi} \\
\frac{u''(w-x)}{u''(w-x+rx)} \cdot \frac{u'(w-x+rx)}{u'(w-x)} &> \frac{(1-\pi)(r-1)}{\pi} \cdot \frac{u'(w-x+rx)}{u'(w-x)} = 1.
\end{aligned}$$

Therefore, you will increase the share of your wealth in the risky asset as your wealth increase if:

$$\frac{u''(w-x)}{u'(w-x)} < \frac{u''(w-x+rx)}{u'(w-x+rx)} \Rightarrow -\frac{u''(w-x)}{u'(w-x)} > -\frac{u''(w-x+rx)}{u'(w-x+rx)},$$

which can be interpreted as the agent having decreasing absolute risk aversion in wealth.

As a final step, compare equation 2 with the following maximization problem from the insurance problem:

$$\max_{\alpha} \pi \cdot u(w - D + (1-q)x) + (1-\pi) \cdot u(w - xq).$$

Also consider the comparative statics that Chris attempted during class. See how it would've been much easier if we used the Implicit Function Theorem.

[6] An explicit function is function of the form: $y = G(\vec{x})$, and an implicit function is a function of the form: $F(\vec{x}, y) = a$. Consider the following functions: $4x + 2y = 5$ and $y^2 - 5xy + 4x^2 = 0$. For more on the IFT, see Simon and Blume (1994, ch.15) and the appendix to these notes.

[7] For all of the comparative statics that Chris did in class for the Insurance model, we can use the IMF to simplify the comparative statics.

4 Math Appendix

4.1 Second Order Taylor Expansion

Compare the following to the First Order Linear Approximation we used to estimate the Hicksian in the previous section:

$$
\begin{aligned}
f(x) &\approx \tilde{f}(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2!} f''(x^*)(x - x^*)^2 \\
\tilde{f}(x) &= f(x^*) + f'(x^*)\Delta x + \frac{1}{2!} f''(x^*)\Delta x^2
\end{aligned}
$$

4.2 Implicit Function Theorem

In its simplest form, the IFT is as follows:

Theorem 10. *Let $G(x, y)$ be a C^1 function on a ball about (x_0, y_0) in $\mathbb{R}^2$. Suppose that $G(x, y) = c$, then if $\frac{\partial G(x_0, y_0)}{\partial y} \neq 0$, there exists a function $y(x)$ defined on an interval I "in the neighborhood" of the point x_0 such that:*

$$
y'(x_0) = \frac{dy(x_0)}{dx} = -\frac{\frac{\partial G(x_0, y_0)}{\partial x}}{\frac{\partial G(x_0, y_0}{\partial y}}
$$