Section Notes 3

Wonbin Kang, Sam Richardson

February 10, 2011

Agenda

1. Finitely Repeated Games
2. Centipede Game with Multiple Types
3. Early Evaluations

1 Finitely Repeated Games

Normally, exposition of finitely repeated games use the Prisoners' Dilemma and the equilibrium concept is SPNE. However, let me introduce the following Proposition which applies to all repeated games:

Proposition 1.1. *If the stage game $\Gamma(I, \Sigma, \vec{u})$ has a unique NE, then the finitely repeated game of this stage game, with $t \in \{1, ..., T\}$ has a unique subgame perfect outcome: the NE of the stage game is played in every stage.*

Proposition 1.1 above states that if there exists a unique NE to the stage game, then there is a unique SPNE with NE being played in each stage game. Then a natural question to follow is if there are multiple NE in the stage game, does any combination of those NE *outcomes* constitute a SPNE? The answer to this question is yes.

The problem is that this isn't very interesting. What about non-NE play at some of the stage games in a finitely repeated game? For example, for a given SPNE, could we have "Cooperate" in some of the stage games of such SPNE of the finitely repeated Prisoners' Dilemma? [1]

The take-away is that if there exists a credible[2] threat or promise about future behavior (in the form of punishments which are NE in the stage game), then current behavior can be influenced. This means that a cooperative outcome can be achieved in certain finitely repeated games (even in games with 2 periods).

[1] In a 2 period Prisoners' Dilemma, we won't be able to sustain "Cooperate" as an SPNE. Can you think of a reason? HINT: It has to do with credibility in the final stage game. On the other hand, in an infinitely repeated Prisoners' Dilemma, "Cooperate" can be sustained in equilibrium despite the fact that the only NE of the stage game is [Defect, Defect].

[2] Recall that SPNE got rid of NE which were based on non-credible threats and promises.

1.1 Example 1

See Section Note 9 from Econ 2020a, U.S.-Soviet Union Twice Repeated Game.

1.2 Example 2[3]

Consider repeating the following stage game twice:

	L_2	M_2	R_2
L_1	$\underline{1}$,$\underline{1}$	$\underline{5}$,0	0,0
M_1	0,$\underline{5}$	4,4	0,0
R_1	0,0	0,0	$\underline{3}$,$\underline{3}$

The stage game has two NE. Therefore, a combination of the two NE outcomes to construct a SP outcome is trivial. The more interesting SPNE would have the non stage game equilibrium $[M_1,\, M_2]$ being played in the first stage of the game. Can we construct such a SPNE in the context of a twice repeated game? The answer is yes.

Consider the following strategy by each player i: "Play R_i in stage 2 if the outcome in stage 1 was $[M_i,\, M_j]$; play L_i, otherwise."[4] Based on this strategy, we can write the payoff matrix of the twice repeated game as follow:

	$L_2^{stage\,1}$	$M_2^{stage\,1}$	$R_2^{stage\,1}$
$L_1^{stage\,1}$	$\underline{2}$,$\underline{2}$	6,1	1,1
$M_1^{stage\,1}$	1,6	$\underline{7}$,$\underline{7}$	1,1
$R_1^{stage\,1}$	1,1	1,1	$\underline{4}$,$\underline{4}$

Note that the following strategy, which has a non NE action being played in the first stage, constitutes a SPNE outcome:

$$\left[M_1^{stage\,1},\, M_2^{stage\,1};\, R_1^{stage\,2},\, R_2^{stage\,2}\right]$$

In short, this is an example of when in a finitely repeated game of T stages and with multiple NE in the stage game, there can exist a SPNE outcome in which for some stage $t < T$ the outcome is not a NE of the stage game. The reason for this is that credible threats or promises about future behavior can influence current behavior.

[3]From Gibbons (1992, p. 85). Also note that I am being very careful of the terms I use—*i.e.* SPNE, SPNE outcome, NE outcome, *etc.*

[4]Note that this is very similar to the strategy considered in Example 1 above.

2 Centipede Game with Multiple Types

Sam and I will cover some parts of the centipede game, focusing on parts (a) and (b). The logic extends to parts (c) and (d). Note that there is going to be a lot of abuse of notation below.

Let the *ex ante* beliefs of Player 1 on Player 2's type be equal to λ and the conditional expectation at each move be as defined in the problem.

Part (a)

At $M = 3$, Player 1 holds the following beliefs of Player 2's type:

$$\begin{aligned} \Pr(Nice|M=3) &= \lambda_3 \\ \Pr(Mean|M=3) &= 1-\lambda_3 \end{aligned}$$

Given these beliefs, you can calculate the expected payoffs of player 1 depending on whether player 1 plays "Continue" or "Stop":

$$\begin{aligned} U_1(Continue) &= (1-\lambda_3)x + \lambda_3 y \\ U_1(Stop) &= z \end{aligned}$$

Comparing the two equations above should give you the beliefs required to sustain a PBE and the sequentially rational strategies of each player starting at $M = 3$.

Part (b)

Keep the following in mind for Part (b). Depending on Player 1's beliefs, nice Player 2's payoffs are:

$$U_2^{nice}(Continue|M=2) = \begin{cases} 100 & if\, \lambda_3 > \frac{1}{2} \\ 99 & if\, \lambda_3 < \frac{1}{2} \end{cases}$$

$$U_2^{nice}(Stop|M=2) = 98$$

Therefore, nice Player 2 has a dominant strategy at $M = 2$.

Part (b1)

The problem simplifies the analysis considerably by asking if a separating equilibrium exists for the two types of Player 2 and gives you the separating actions at $M = 2$. Since the nice type Player 2 plays "Continue" player 1's move at $M = 3$ will be to play "Continue". Does the mean type Player 2 have an incentive to deviate at $M = 3$?

Part (b2)

Note that we're now pooling on "Continue" at $M = 3$. Therefore, player 1's beliefs at $M = 3$ become central. Do you see that $\lambda = \lambda_3$?

If $\lambda = \lambda_3 > \frac{1}{2}$, we know from part (a) above that player 1 will play "Continue". What are the optimal actions at $M = 3$ for each type of Player 2? Do you see that both types will want to "Continue"?

If $\lambda = \lambda_3 < \frac{1}{2}$, we know from part (a) above that player 1 will play "Stop". What is the optimal action at $M = 3$ for the mean type of Player 2? Do you see that she will "Stop" at $M = 3$? What about the nice type Player 2?

If $\lambda = \lambda_3 = \frac{1}{2}$, we know from part (a) above that player 1 will play mix (let's write this as: $[pContinue, (1-p)Stop]$). Since "Continue" is a dominant strategy for the nice type Player 2, we again only need to focus on the mean type.

$$\begin{aligned} U_2^{mean}\left(Continue|M=2\right) &= p101 + (1-p)99 \\ U_2^{mean}\left(Stop|M=2\right) &= 100 \end{aligned}$$

which gives you conditions on Player 1's mixing.

Part (b3)

For the values of $\lambda < \frac{1}{2}$,we know that there isn't a pooling equilibrium on "Continue"[5]; and we know that the two types can't pool on "Stop". We know that there isn't a separating equilibrium where the nice type plays "Continue" and the mean type plays "Stop"[6]; and we know that the converse separating equilibrium can't exist as well. So the question is asking you whether there is a partial pooling equilibrium.

In words, the partial pooling equilibrium will have the nice type playing "Continue" and the mean type mixing (again let's write this as: $[qContinue, (1-q)Stop]$). In the partial pooling equilibrium, Player 1 is also mixing (let's write this as: $[pContinue, (1-p)Stop]$). To figure out the strategy of Player 1 at $M = 3$, we need to again calculate λ_3, but because of the partial pooling characteristic of the equilibrium, Player 1's beliefs will NOT equal the *ex ante* distribution of nice/mean types. However, since we know the equilibrium strategy of both types of Player 2, we can calculate Player 1's beliefs via Baye's Rule:[7]

$$\lambda_3 = \frac{1\cdot\lambda}{1\cdot\lambda + q\cdot(1-\lambda)} = \frac{\lambda}{\lambda + q\cdot(1-\lambda)}$$

[5] Per (b1) above.

[6] Per (b2) above.

[7] To be explicit:

$$\Pr\left(nice|M=3\right) = \frac{\Pr\left(M=3|nice\right)\cdot\Pr\left(nice\right)}{\Pr\left(M=3|nice\right)\cdot\Pr\left(nice\right) + \Pr\left(M=3|mean\right)\cdot\Pr\left(mean\right)}$$

where you should interpret $M = 3$ as Player 2 playing continue at $M = 2$, since this is the only way that you could have arrived at $M = 3$.

For Player 1 to mix between her two strategies it must be the case the $\lambda_3 = \frac{1}{2}$, which gives you the mean type Player 2's mixing strategyy $[q,\, 1-q]$. Now what about Player 1's mixing strategy $[p,\, 1-p]$? We actually solved for this above.

3 Early Evaluations