Section Notes 5

Wonbin Kang

Feb 25, 2010

Agenda

1. Perfect Bayesian Equilibrium
2. PBE Examples
3. Signaling Examples

1 Perfect Bayesian Equilibrium ("PBE")

1.1 PBE in Context

To put things into perspective, let's look at the equilibrium concepts we've covered. We've looked at NE in static games of complete information; SPNE in dynamic games of complete and perfect information; BNE in static games of incomplete information; and PBE in dynamic games of incomplete information. As emphasized throughout section, each new equilibrium concept is a way to "refine" or "strengthen" our original NE concept. However, it is important to note that "the stronger equilibrium concept differs from the weaker concept only for richer games, not for simpler games."[1]

For example, if asked to consider whether a strategy profile is PBE in a *static* game with incomplete information, you'll discover that PBE is equivalent to BNE. However, PBE strengthens BNE in the context of a *dynamic* game with incomplete information. In short, PBE refines BNE much like SPNE refines NE. So we know that all PBE are BNE, but all BNE are not PBE. Just as SPNE requires strategies to be a NE for the entire game and each *proper subgame*, PBE requires strategies to be BNE for the entire game and for each and every *continuation* game.

PBE is actually even richer because we can consider it a refinement of not only BNE, but also SPNE. PBE does so by analyzing each players' beliefs (as in BNE), but in a dynamic setting. Further, the definition of PBE forces us to consider strategy profiles and beliefs which are reasonable on the equilibrium path (or off).

[1] Gibbons (1992, p. 173)

1.2 Definition of PBE

A PBE consists of a strategy profile (σ) and system of beliefs (μ). Some texts call the pairing of the two, $[\sigma, \mu]$ a behavioral strategy profile. For a candidate behavioral strategy profile to be PBE, it must satisfy the following conditions:

1. At each information set, the player who moves must assign a belief, or probability μ_k to *each* node within the information set. For a singleton information set, the player's belief puts probability one on the single decision node.

2. Given the beliefs at each information set and other players' subsequent strategies, the player who moves at the information set must select an action that is optimal—*i.e.* all players must best-respond given their beliefs and the strategies of the other players. In other words, given each player's beliefs, their strategies must be *sequentially rational*.

3. Beliefs at all information sets *on the equilibrium path* are determined by Bayes' Rule and the players' equilibrium strategies. So the probability that a player is at decision node x given a certain history of play h and given the strategy profile σ is:

$$\Pr(x|h, \sigma) = \frac{\Pr(x \cap h|\sigma)}{\Pr(h|\sigma)}$$

4. [Beliefs at all information sets *off the equilibrium path* are determined by Bayes' Rule and the players' equilibrium strategies *where possible*.][2]

We solve for PBE using the guess and check method:

- Proposing a candidate behavioral strategy profile $[\sigma, \mu]$ and check the behavioral strategy for one of the players.
- Calculate players' beliefs based on the profile above.
- Calculate the BR of other players conditional on these calculated beliefs.
- Check if any of the players has an incentive to deviate (Because you calculated the BR of the "other" players, you only need to check the incentives of the first player).
- If possible, calculate beliefs consistent with the behavioral strategy profile off the equilibrium path.

[2]Bullets 1-3 are required for "weak" PBE and bullet 4 is required for PBE. Usually there won't be a difference, but see the example below.

2 PBE Examples[3]

Example 2.1. See the game outlined in Appendix A-1.

Some preliminary findings:

1. Only one proper subgame starting at player 2's decision node. The normal form representation of this subgame between player 2 and 3 is:

	L'	R'
L	$\underline{2}$,1	$\underline{3}$,$\underline{3}$
R	1,$\underline{2}$	1,1

2. Therefore, the SPNE of the entire game is [D,L,R']. Further, if we couple this strategy profile with $p = 1$, we have that [D,L,R',$p = 1$] is a PBE. Note that bullets 1-4 in section 1.2 are satisfied.

Now consider the behavioral strategy profile, [A,L,L',$p = 0$], which leads to payoffs (2,0,0).

1. Is this a NE? No player has an incentive to deviate, so YES. We only need to check whether player 1 has an incentive to deviate. Do you see why?

2. Let's see if this is weakly PBE.

   (a) Player 3 is the only player with a non-singleton information set. We've specified his beliefs as $p = 0$. Given these beliefs and the other player's strategies, he is best responding by playing L' (sequentially rational).

   (b) Are players 1 and 2 best responding? If player 1 deviates to D, her payoff is 1 (< 2). Player 2 is indifferent between L and R since either way he's getting a payoff of 0.

   (c) Therefore, [A,L,L',$p = 0$] is weakly PBE based on bullets 1-3 (the definition of weak PBE) in section 1.2 above. However, this isn't SPNE.

3. But we said that PBE could be considered a refinement of SPNE. What's the problem? The problem is that the definition of weak PBE doesn't consider bullet 4: constraints on beliefs off the equilibrium path. [A,L,L',$p =$ 0] has beliefs, $p = 0$, which are inconsistent with the player 2's strategy of playing L. Adding bullet 4, and thereby strengthening the definition of PBE so that beliefs correspond to the play of other players, results in player 3's belief being $p = 1$. Then player 3's strategy of L' does not satisfy sequential rationality, and so this equilibrium is not a PBE.

[3]From Gibbons (1992)

Example 2.2. Now what if player 2 had an additional strategy "A"' which would end the game.[4] Consider the behavioral strategy profiles [A,A',L',$p = 0$] and [A; μ_1L + $(1-\mu_1)$R + 0A'; L' ; $p = 0$]. We're not going to consider whether these strategies are PBE or not, but concentrate on the beliefs.

1. For [A,A',L',$p = 0$], player 3's information set is again off the equilibrium path. However, unlike in the example above, player 2's strategy is to end the game by playing A'. Therefore, player 2's strategy does not constrain player 3's beliefs and so player 3 may have any beliefs that he wants. Hence the language in bullet 4: "if possible."

2. For [A; μ_1L + $(1 - \mu_1)$R + 0A'; L' ; $p = 0$], player 3's information set is again off the equilibrium path, but player 2's mixed strategy constrains player 3's beliefs. Therefore, $p \neq 0$ and it must be the case that $p = \mu_1$ for bullet 4 to be satisfied.

3 Signaling Examples

Example 3.1. Let's look at an abstract version of the signaling model.[5] There are two types of Senders: $t \in \{t_1, t_2\}$, and the *ex ante* probability distribution is $\Pr(t = t_1) = \frac{1}{2}$; there are two signals that any type of Sender can send: $m \in \{L, R\}$; the beliefs of the Receiver are denoted $[p, 1-p]$ and $[q, 1-q]$, respectively for Receiver's two information sets; the Receiver has two actions at each information set: $a \in \{u, d\}$. The payoffs are as depicted in the extensive form of the game in Appendix A-2.

1. What are the strategy spaces of each player?

$$\begin{aligned} S_{Sender} &\equiv \{LL,\ LR,\ RL,\ RR\} \\ S_{Receiver} &\equiv \{uu,\ ud,\ du,\ dd\} \end{aligned}$$

2. Per the guess and check method outlined above in section 1.2, find all *pure strategy* PBE. Let's consider each of the Senders strategies.

   (a) "Pooling" on L (or LL): Receiver's beliefs for the information set on the equilibrium path are $p = \frac{1}{2}$, $1 - p = \frac{1}{2}$. Sequential rationality dictates that the Receiver best responds to LL by playing "u" since:[6]

   $$U_{Receiver}(u|p = \frac{1}{2}, LL) = \frac{1}{2}(3+4) = \frac{7}{2} > U_{Receiver}(d|p = \frac{1}{2}, LL) = 0.$$

   Sender receives either 1 or 2 (depending on type). Does either type of Sender have an incentive to deviate? Clearly, type t_2 does not

[4] Extensive form not in the appendix. Just draw a branch from player 2's decision node.
[5] Gibbons (1992).
[6] Note that "u" is strictly dominant.

want to deviate and send signal R, since she'll get a lower payoff. But type t_1 can get a higher payoff of 2 if she sends the signal R and the Receiver plays "u". Therefore, off the equilibrium path, we need to have the Receiver playing "d", which is consistent with all beliefs where $q < \frac{2}{3}$:

$$\because U_{Receiver}(d|q) = q \cdot 0 + (1-q) \cdot 2 > U_{Receiver}(u|a) = q \cdot 1 + (1-q) \cdot 0$$

Therefore, we have that [LL, ud, $p = \frac{1}{2}$, $q < \frac{2}{3}$] is a PBE.

(b) "Pooling" on R (or RR): Receiver's beliefs for the information set on the equilibrium path are $q = \frac{1}{2}$, $1 - q = \frac{1}{2}$. Sequential rationality dictates that the Receiver best responds to LL by playing "d" since:

$$U_{Receiver}(d|q = \frac{1}{2}, RR) = \frac{1}{2}(0+2) > U_{Receiver}(u|q = \frac{1}{2}, RR) = \frac{1}{2}(1+0).$$

We could go through the same steps we did in (a) above, or we can note that Sender type t_1 receives 0 with the behavioral strategy profile. However, Sender type t_1 is guaranteed at least a payoff of 1 if she deviates. Of course, we know from (a) above that Receiver's best response to the signal "L" is to play "u" so Sender type t_1 will get 1. Therefore, there is no PBE (or equilibrium) when the Sender pools on R.

(c) "Separating" with type t_1 playing L and types t_2 playing R: Since both of Receiver's information sets are now on the equilibrium path, we need to figure out the beliefs for both information sets using Bayes' Rule and players' strategies. This is trivial: $p = 1$ and $q = 0$. The Receiver's best response to her beliefs are "ud" so that the payoff for the Sender is equal to 1 for both types. Now let's check for Sender's incentive to deviate. Notice that Sender type t_2 could get 2 if she deviates and sends signal L (since the Receiver best responds to the signal L with "u"). Therefore, this cannot be an equilibrium.

(d) "Separating" with type t_1 playing R and types t_2 playing L: Since both of Receiver's information sets are now on the equilibrium path, we need to figure out the beliefs for both information sets using Bayes' Rule and players' strategies. This is again trivial: $p = 0$ and $q = 1$. Receiver's best response given her beliefs are "uu" so that the payoff for the Sender is equal to 2 for both types. Check that each type of Sender does not have an incentive to deviate. Therefore, [RL, uu, $p = 0$, $q = 1$] is a PBE.

Example 3.2. Mark Buehrle's throwing arm is either in good or bad shape, with $\Pr(t = \textit{Good}) = \frac{1}{2}$. Value to the ChiSox is \$20 million if Buehrle's arm is in good shape and worth \$0 if it's in bad shape. Buehrle knows what his type is, but the ChiSox do not. The ChiSox can offer Buehrle an \$8 million dollar contract or offer him nothing. Buehrle can send a signal to the ChiSox organization by choosing to take a physical which would reveal the state of his arm for \$1 million, or send a signal by doing nothing at all. ChiSox's beliefs as to Buehrle's arm is denoted by p. Assume that Buehrle's utility is monotonically increasing in the offer he receives from the ChiSox.

The extensive form of the game is Appendix A-3. Note that if testing was not a possible action for Buehrle, the ChiSox would offer him an \$8 million dollar contract.

What is the strategy space of Buehrle and the ChiSox?

$$\begin{aligned} S_{M.Buerhle} &\equiv \{TT;\, TN;\, NT;\, NN\} \\ S_{ChiSox} &\equiv \{OOO;\, OONo;\, ONoO;\, NoOO;\, ...;\, NoNoNo\} \end{aligned}$$

Let's do the same analysis as above:

1. Pooling on "no test": Therefore, ChiSox's beliefs are such that $p = \frac{1}{2}$; ChiSox best respond given these beliefs and Buehrle's strategy by playing "offer" (since \$2 million > 0). Also, note that both of Buehrle's types, good or bad, cannot do better regardless of what the ChiSox do by deviating and sending the signal "no test." Therefore, the behavioral strategy profile [(no test, no test); (offer, offer, no offer); $p = \frac{1}{2}$] is PBE.

2. Pooling on "test': Again, we could go through all of the messy steps as we did above, but note that Buehrle's "bad" type will get a payoff of -\$1 million (since the ChiSox know exactly where they are on the game tree). If the "bad" type were to deviate and play "no test" then he's assured of at least "0" regardless of the ChiSox's beliefs. Therefore, this can't be an equilibrium.

3. Separating with type "bad" playing "no test" and type "good" playing "test": ChiSox's beliefs are such that $p = 0$. ChiSox best respond by playing "offer" if the signal is "test" and "no offer" if the signal is "no test." Can either the "good" or the "bad" type do better by deviating given these beliefs? No because the "good" type would get \$0 (since ChiSox have beliefs such that $p = 0$ and so will play "no offer") and because the "bad" type would get -\$1 million if he deviates (since ChiSox know what type Buehrle is if he tests). Therefore, [(test, no test); (no offer, offer, no offer); $p = 0$] is a PBE.

Between the two PBEs above, one seems a bit odd. Is there an equilibrium refinement that might be able to get rid of the less desirable equilibrium?