Section Notes 2

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Feb 4, 2010

Agenda

1. Clarifications
2. Nash Equilibrium (hereinafter "NE")
3. Mixed Strategy Nash Equilibrium (hereinafter "MSNE")
4. Best Response Correspondences
5. Practice Problem 3.9 (a) and (b) (if we have time)

1 Clarifications[1]

1.1 Existence of NE[2]

1.1.1 Fixed Point Theorems[3]

For those interested, here are the Fixed Point Theorems Chris mentioned in class:

Theorem 1.1. *Brouwer's Fixed Point Theorem: Suppose that* $\mathbb{A} \subset \mathbb{R}^n$ *is a nonempty compact, convex set, and that* $f : \mathbb{A} \rightarrow \mathbb{A}$ *is a continuous function from* $\mathbb{A}$ *unto itself. Then* $f(\cdot)$ *has a fixed point, or there exists an* $\vec{x}^* \in \mathbb{A}$ *such that* $f(\vec{x}^*) = \vec{x}^*$.

Note that the key here is that the domain and the target of the function f are the identical, compact, and convex; just like the strategy space of the aggregate best response correspondence we saw in class.

The version for correspondences, which applies to the best response graphs we see in this class, is:

[1] Section 1.1 is only for those interested. You should review section 1.2 and try to answer the questions.

[2] Like Chris said in class, these are *not* proofs!

[3] From Section M.I of MWG.

Theorem 1.2. *Kakutani's Fixed Point Theorem: Suppose that $\mathbb{A} \subset \mathbb{R}^n$ is a nonempty compact, convex set, and that $f : \mathbb{A} \to \mathbb{A}$ is an upper hemi-continuous correspondence from $\mathbb{A}$ into itself with the property that the set $f(x) \subset \mathbb{A}$ is nonempty and convex for every $x \in \mathbb{A}$. Then $f(\cdot)$ has a fixed point, or there exists an $\vec{x}^* \in \mathbb{A}$ such that $f(\vec{x}^*) = \vec{x}^*$.*

1.1.2 Best Response Correspondences

In class, there were a lot of questions on how the Fixed Point Theorems were applied to the best response correspondences we saw in Monday's class. We defined the aggregate best response correspondence: $\vec{b}(\Theta_1, ..., \Theta_I) = b_1(\Theta_1, ..., \Theta_I) \times b_2(\Theta_1, ..., \Theta_I) \times ... \times b_I(\Theta_1, ..., \Theta_I)$, or

$$\vec{b} : \Theta_1 \times ... \times \Theta_I \to \Theta_1 \times ... \times \Theta_I.$$

Further, for all player i, who has a total of n finite strategies available to her, we can think of $\Theta_i \subseteq [0, 1]^n$, which is compact and convex. This means that $\Theta \equiv \Theta_1 \times ... \times \Theta_I$ is also compact and convex. Recall that a pure strategy can be thought of as a degenerate mixed strategy—*e.g.* if player i has four possible strategies: $S_i \equiv \{(In, Fight); (In, Acquiece); (Out, Fight); (Out, Acquiesce)\}$, the pure strategy $(In, Fight)$ can be written as $p = (1, 0, 0, 0) \in [0, 1]^4$.

If we assume that the best response functions/correspondences are continous/upper hemi-continuous, we have all of the requirements to apply one of the Fixed Point Theorems Chris went over in class.

1.2 Best Response vs Dominance

Example 1.3. Consider the following game:

	L	C	R
U	3,3	0,0	1,1
M	0,0	3,3	2,1
D	2,2	2,2	3,1

First, notice that the strategy "R" is strictly *dominated.* Can you provide an example of a strategy that strictly *dominates* "R"? Second, in the reduced game, what is the NE? Do you see that it is (U,L) and (M,C)? Finally, since strategy "D" is not a best response to any pure strategy of player 2 (either "L" or "C"), does this mean that the strategy "D" is strictly *dominated* for player 1? For what strategies of player 2 would "D" be a best response? If you assume that the probability of player 2 playing "L" is equal to π, and player 2 is playing a mixed strategy equal to $\vec{p} = [\pi, 1 - \pi, 0]$, then for values of $\pi \in \left[\frac{1}{3}, \frac{2}{3}\right]$, note that "D" is a best response for player 1.

2 NE

Definition 2.1. In the I player normal form game, the strategies $(s_1^*, ..., s_I^*)$ are a NE if, for each player i, her strategy s_i^* is a best response to the strategies specified for the other $I-1$ players, or for $\vec{s}_{-i} = (s_1^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_I^*)$. Formally,

$$u_i(s_i^*, \vec{s}_{-i}) \quad \geq \quad u_i(s_i', \vec{s}_{-i}), \forall i, \forall s_i' \in S_i$$

, or

$$s_i^* = \arg\max_{s_i} u_i(s_i, \vec{s}_{-i})$$

This has a number of less formal intepretations:

1. *Given what others are doing*, there is no profitable deviation for any player.
2. The point where the best response correspondences of all of the players meet, or where each player is best responding to each other players' best response.

Personally, I think the easiest way to understand NE is as follows: Given (or *guess*) a specific strategy profile, *checking* to see whether such strategy profile is NE, is the same as *checking* to see whether there are any deviations that would be profitable for any of the players *given that you do not change the strategies of the other players specified by such strategy profile.*

3 MSNE[4]

If a player "mixes" strategies, or puts *positive probability* on a pure strategy, she must be *indifferent* between the strategies. Let's look at Example 1.3 above to see how we get mixed strategies.

	L	C	R
U	3,3	0,0	1,1
M	0,0	3,3	2,1
D	2,2	2,2	3,1

$\Rightarrow$

	L	C
U	3,3	0,0
M	0,0	3,3
D	2,2	2,2

1. First, iteratively eliminate *strictly dominated* strategies. In other words, mixed strategies will not put positive probability on a *strictly dominated* strategy. Does this extend to *weakly dominated* strategies? We'll discuss this when we cover Trembling Hand Perfect Nash Equilibrium (hereinafter "THPNE"). We can eliminate "R" because it is strictly dominated by player 2's mixed strategy: $\vec{p} = \left[\frac{1}{2}, \frac{1}{2}, 0\right]$, and infinitely many others.

[4]Practice Problem 3.10 and its solutions provide a great step-by-step explanation of the guess and verify method.

2. Second, assuming that the probability of player 2 playing "L" is π, let's draw the expected utilities of player 1 for varying values of π.[5] Since a player mixes when she is indifferent between strategies, we can see that player 1 mixes "U" and "D" when $\pi = \frac{2}{3}$, and mixes "M" and "D" when $\pi = \frac{1}{3}$. Why don't we consider "M" and "U"?
   Further, player 1 is mixing only two out of three strategies. Could we have an instance where player 1 mixes all three strategies? Normally the answer is no. And to see why, note that the three lines representing the expected utilities of player 1 do not meet at one point. Of course, if the expected utility of player 1 playing "D" was equal to $\frac{3}{2}$, then we could have player 1 mixing all three strategies. For another example where we have an instance where one player mixes two strategies while the other mixes three strategies, see Practice Problem 3.3.

3. To continue with the example, assume that player 1 is mixing "U" and "D" and the probability of player 2 playing "L", or $\pi = \frac{2}{3}$. Since, player 2 is playing a mixed strategy of "L" and "C", it must also be the case that *player 2* is indifferent between "L" and "C". Given that the probability of player 1 playing "U" is equal to μ, then the expected utility (denoted $U(\cdot;\cdot)$) of player 2 for playing "L" and "C" can be written as:

$$\begin{aligned} U(L;\mu) &= 3\mu + 2\cdot(1-\mu) = 2+\mu \\ U(C;\mu) &= 0\mu + 2\cdot(1-\mu) = 2-2\mu \end{aligned}$$

   , resulting in $\mu = 0$. This means that player 1 is playing a pure strategy "D" which is a contradiction. This can't be a mixed strategy.

4. Now assume that player 1 is mixing "M" and "D" and the probability of player 2 playing "L" is now $\pi = \frac{1}{3}$. Since, player 2 is playing a mixed strategy of "L" and "C", it must also be the case that *player 2* is indifferent between "L" and "C". Given that the probability of player 1 playing "M" is equal to η, then the expected utility of player 2 for playing "L" and "C" can be written as:

$$\begin{aligned} U(L;\eta) &= 0\eta + 2\cdot(1-\eta) = 2-2\eta \\ U(C;\eta) &= 3\eta + 2\cdot(1-\eta) = 2+\eta \end{aligned}$$

   , resulting in $\eta = 0$. This means that player 1 is again playing a pure strategy "D" which is a contradiction. This can't be a mixed strategy.

5. Then what is the mixed strategy? We know that there should be one (at least in the games that we will be dealing with).[6] From the graph above, we see that if $\pi \in \left[\frac{1}{3}, \frac{2}{3}\right]$, then the best response by player 1 is to play

[5] Graph omitted.
[6] See Practice Problem 3.5 for an example.

"D" and player 2 is indifferent between playing any of the series of mixed strategies where $\pi \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and the probability of playing "C" is $1 - \pi$.

4 Best Response Correspondences

Note that with best response correspondences, the argument that the correspondence takes is the strategy of the other players. Let's review best response correspondences using the following example, which is slightly different from Example 1.3 above. Also don't forget to try Practice Problem 3.5.

Example 4.1.

	L	C	R
U	3,3	0,0	1,1
M	0,0	3,3	2,1
D	1,2	1,2	3,1

$\Rightarrow$

	L	C
U	3,3	0,0
M	0,0	3,3

Note that "R" is strictly dominated per the argument above, and a similar argument shows us that certain mixed strategies strictly dominate "D".[7] Therefore, we can iteratively eliminate "R" and "D".

Let's try to draw the best response correspondence graph.[8] First, we need to set up some notation. Assume that the probability of player 1 playing "U" in the reduced game is x, which can be written as $x(y)$ and the probability of player 2 playing "C" in the reduced game is y, which can be written as $y(x)$. Being very careful to note exactly what the axes are, we can draw the following best response correspondence graph using the following equations:[9]

$$\begin{aligned} U_1(U;y) &= U_1(M;y) \Rightarrow 3 \cdot (1-y) + 0 \cdot y = 0 \cdot (1-y) + 3 \cdot y \Rightarrow y = \frac{1}{2} \\ U_2(L;x) &= U_2(C;x) \Rightarrow 3 \cdot x + 0 \cdot (1-x) = 0 \cdot x + 3 \cdot (1-x) \Rightarrow x = \frac{1}{2} \end{aligned}$$

, which results in the following best response correspondences:

[7] Can you provide an example?

[8] Best response correspondence graphs are like Edgeworth boxes:

1. They provide insight to a very simple model (normally two players and two strategies per player) which can be generalized...at least for some people.
2. For others, it completely confuses them. If you are in this group, don't worry because you don't need them to find MSNEs. In fact, you can solve for MSNEs as above and then reverse engineer the best response correspondence graphs.

[9] Graph omitted.

$$
\begin{aligned}
y(x) &= \begin{cases} 1 & if\, x < \frac{1}{2} \\ 0 & if\, x > \frac{1}{2} \\ [0,1] & if\, x = \frac{1}{2} \end{cases} \\
x(y) &= \begin{cases} 1 & if\, y < \frac{1}{2} \\ 0 & if\, y > \frac{1}{2} \\ [0,1] & if\, y = \frac{1}{2} \end{cases}
\end{aligned}
$$

Example 4.2. Recall the Cournot model where the inverse demand function was given by $P(Q) = 20 - Q$; $q_1$and $q_2$were the choice variables of each firm 1 and 2; and $MC = 8$.

Since this is a symmetric problem, we know that for $i \neq j$:

$$
\begin{aligned}
q_i(q_j) &= q_i^* = \arg\max_{q_i} (20 - q_i - q_j)q_i - 8q_i \\
q_i(q_j) &= 6 - \frac{q_j}{2}\; i \neq j
\end{aligned}
$$

Try drawing the best response correspondence graph.