Section Notes 9

Abby Friedman, Wonbin Kang

November 5, 2010

Agenda

1. Best Response Correspondences[1]
2. Game Trees
3. Backward Induction
4. Subgame Perfect Nash Equilibrium

1 Best Response Correspondences

Note that with best response correspondences, the argument that the correspondence takes is the strategy of the other players. Let's review best response correspondences using the following example, which is slightly different from Example ?? above. Also don't forget to try Practice Problem 3.5.

Example 1.1.

	L	C	R
U	3,3	0,0	1,1
M	0,0	3,3	2,1
D	1,2	1,2	3,1

$\Rightarrow$

	L	C
U	3,3	0,0
M	0,0	3,3

Note that "R" is strictly dominated per the argument above, and a similar argument shows us that certain mixed strategies strictly dominate "D".[2] Therefore, we can iteratively eliminate "R" and "D".

Let's try to draw the best response correspondence graph.[3] First, we need to set up some notation. Assume that the probability of player 1 playing "U"

[1] Will not cover in class.

[2] Can you provide an example?

[3] Best response correspondence graphs are like Edgeworth boxes:

1. They provide insight to a very simple model (normally two players and two strategies per player) which can be generalized...at least for some people.
2. For others, it completely confuses them. If you are in this group, don't worry because you don't need them to find MSNEs. In fact, you can solve for MSNEs as above and then reverse engineer the best response correspondence graphs.

in the reduced game is x, which can be written as $x(y)$ and the probability of player 2 playing "C" in the reduced game is y, which can be written as $y(x)$. Being very careful to note exactly what the axes are, we can draw the following best response correspondence graph using the following equations:[4]

$$\begin{aligned} U_1(U;y) &= U_1(M;y) \Rightarrow 3\cdot(1-y)+0\cdot y = 0\cdot(1-y)+3\cdot y \Rightarrow y=\frac{1}{2} \\ U_2(L;x) &= U_2(C;x) \Rightarrow 3\cdot x+0\cdot(1-x) = 0\cdot x+3\cdot(1-x) \Rightarrow x=\frac{1}{2} \end{aligned}$$

, which results in the following best response correspondences:

$$\begin{aligned} y(x) &= \begin{cases} 1 & if\, x<\frac{1}{2} \\ 0 & if\, x>\frac{1}{2} \\ [0,1] & if\, x=\frac{1}{2} \end{cases} \\ x(y) &= \begin{cases} 1 & if\, y<\frac{1}{2} \\ 0 & if\, y>\frac{1}{2} \\ [0,1] & if\, y=\frac{1}{2} \end{cases} \end{aligned}$$

Example 1.2. Recall the Cournot model where the inverse demand function was given by $P(Q)=20-Q$; $q_1$and $q_2$were the choice variables of each firm 1 and 2; and $MC=8$.

Since this is a symmetric problem, we know that for $i\neq j$:

$$\begin{aligned} q_i(q_j) &= q_i^* = \arg\max_{q_i}(20-q_i-q_j)q_i-8q_i \\ q_i(q_j) &= 6-\frac{q_j}{2}\; i\neq j \end{aligned}$$

Try drawing the best response correspondence graph.

2 Game Trees

Consider the following normal form game:

	L	C
U	0,0	<u>2</u>,<u>4</u>
M	<u>4</u>,<u>2</u>	1,1

We can rewrite the game above in game tree form.[5]

[4] Graph omitted.

[5] See Appendix A-1. Note that we could've flipped the order of player 1 and 2, but we retain the usual convention.

1. Note that player 1 is at a *singleton information set*, while player 2's *information set* contains two *decision nodes*. The game ends at *terminal nodes*.

2. In the game above, how many proper subgames and how many subgames? There are zero proper subgames and one subgame.

What is wrong with the game in Appendix A-2?[6] Note the actions available at each decision node within the same information set must be the same.

3 Backward Induction

Change the rules of the simultaneous move game to a sequential game of perfect information: player 1 moves first and player 2 observes player 1's move. If we assume perfect information, each player knows exactly where he or she is on the game tree. In short, all players observe the actions of the players who came before him or her, and information sets for all players have one decision node, or are singletons. See Appendix A-3.

How many subgames exist in the above extensive form game? Note that in a game of perfect information, each singleton decision node is the start of a subgame. Therefore, there are a total of 3 subgames and 2 proper subgames.

Let's backward induct the equilibrium for this game:

1. At decision node 2.1, player 2 will play "C" ($\because 4 > 0$), and at decision node 2.2, player 2 will play "L" ($\because 2 > 1$).

2. The reduced form from the perspective of player 1 is shown in Appendix A-4. Therefore, the unique backward induction NE, $\sigma^* =$[M, CL]. Remember that a strategy for player 2 must identify player 2's action at *each of player 2's decision nodes*: 2.1 and 2.2. Therefore, the strategy set for player 2 is $S \equiv \{LL, LC, CL, CC\}$.

3. Note that the game above has a first-mover advantage. Matching pennies is a game where there is a second-mover advantage.

The next question to consider is what does all of the above buy us? Rewrite the extensive form game above in normal form and underline best-responses:

Case 1.

	LL	LC	CL	CC
U	0,0	0,0	2,$\underline{4}$	$\underline{2}$,$\underline{4}$
M	$\underline{4}$,$\underline{2}$	$\underline{1}$,1	$\underline{4}$,$\underline{2}$	1,1

[6]Game tree omitted.

Therefore, there are three NEs: [M, LL], [M, CL], and [U, CC], of which backward induction tells us the only *credible* NE is [M, CL]. Keep this idea of credibility in mind. Can you see why [M, LL] and [U, CC] are not credible? Later on, we're going to consider NE when one of the parties can credibly commit to one of these "non-credible" strategies.

Example 3.1. Ultimatum Game / Divide the Dollar Game

Assume that the minimum unit in the divide the dollar game is 1 cent. Do you see that $S_1 = \{0, 1, ..., 100\}$ where $s \in S_1$ is the amount offered to Player 2 (the offeree). Player 2 can either "accept" or "reject" the offer.[7]

Remarks:

1. To test your understanding of strategies, do you see that Player 2 (the offeree) has 2^{101} pure strategies. Further, do you see that, using the notation and terminology from Section Note 7, that Player 2's strategy is a 101-dimensional vector?
2. Do you see that offering the minimum to Player 2 ($x^* = 1$) by the offeror and "accept" by the offeree is the SPNE? This is what will happen on the equilibrium path. What must be the offeree's actions off the equilibrium path in this unique SPNE? Use backward induction to get this result.
3. More importantly, do you see that any non zero distribution of the dollar can be sustained as a NE? Check the following strategies:
   (a) Player 2's strategy: "accept" if $x \geq x^*$ and "reject" otherwise. The important thing to see is that this strategy doesn't constrain the behavior of Player 2 off the equilibrium path. However, on the equilibrium path (after the offer of Player 1 s.t. $x^* = 1$), Player 2's strategy results in "accept."
   (b) Player 1's strategy: offer Player 2 $x = x^*$
   Note that $x^* \in (0, 100]$
4. The SPNE results/predictions do not match with the results that you see in the laboratory.

Example 3.2. Centipede Game: Before doing this week's problem set, read "The Centipede Game" starting on page 350 of the Avery Notes ch. 12.

Let's look at the game depicted in Figure 12.7 of the Avery Notes.

1. The Avery Notes state that a critique of SPNE is that "it relies too heavily on the assumption of common knowledge in games...."[8] What does this mean?

[7] This doesn't mean that the strategies available to Player 2 are "accept" and "reject." Her action set A at each node is equivalent to $\{accept, reject\}$.

[8] Avery Notes ch. 12. p.350.

2. What is the unique SPNE? With an abuse of notation and where S represents "Stop" and C represents "Continue," we find that the SPNE is

$$[\sigma_1^*, \sigma_2^*] = [SS,\, SS]$$

which results in a NE in each of the 4 subgames. The final payout is $[3, 1]$.

3. Non-credibility of "Continue": At each node Player 1 moves, Player 2 would find it optimal for her to commit to playing "Continue." She can promise this, but that promise is not credible and so Player 1 will play "Stop."[9]

4. If we extend the game to 200 stages with increasing payoffs, a unique SPNE with each player playing "Stop" at each of his or her nodes exists. But is this an equilibrium that we find realistic?

Theorem 3.3. *Zermelo's Theorem: Every finite sequential game of perfect information has a unique backward induction solution if there are no ties in payoffs (in other words, $\forall i \in I$, player i cannot get the same payoff for two different actions). If there are ties in payoffs, a backward induction solution exists, but need not be unique.*

In win-lose-draw type zero sum games of perfect information, there is a unique backward induction *payoff*, but not a unique *equilibrium* (different strategies lead to the same payoffs).

- Tic-tac-toe: draw
- Connect four: first-mover wins (solved in 1988)
- Checkers: draw (solved in 2007)
- Chess: almost certainly draw or first-mover win (as Elon noted, this is a very difficult backward induction problem to solve).

4 Subgame Perfect Nash Equilibrium ("SPNE")

Definition 4.1. A strategy profile $\sigma^*$is a SPNE if it induces a NE in every subgame.

Example 4.2. Consider the following game based on the **Cold War**. Let player 1 be the US and player 2 be the USSR. Each player moves simultaneously and $S_1 = S_2 = \{E,\, D,\, N\}$, where "E" is to escalate the confrontation, "D" is to de-escalate, and "N" is to "nuke" the other player. The payoffs are as follows:

[9]The inability of players to commit to Pareto improving actions within the extensive form game is something you will run into frequently in the future. A good example is in Acemoglu and Robinson's Economic Origins of Dictatorship and Democracy.

	E	D	N
E	1,1	3,0	-100,0
D	0,3	2,2	-100,0
N	0,-100	0,-100	-200,-200

Consider a single-period game. Note that the "E" is a weakly *dominant* strategy for both the US and the USSR, and [E,E] is the only NE.[10]

Now let's consider an extensive form game where the above game is played twice. Assume that the payoffs of this extensive game are equal to the sum of each *stage game*. What important assumption have we done away with in this game? *PERFECT INFORMATION* is no longer applicable. Therefore, the simplistic backward induction we used above is no longer feasible. We have to use the definition of SPNE and find NE in each and every subgame.

The game tree for this game is shown in Appendix A-5. Note that I've filled in some of the payoffs.

The strategy space for the US can be written as follows:

$$S_1 \equiv \{\vec{s}_1 = [s_{1.1}, s_{1.2}, ..., s_{1.10}] : s_{1.n} \in \{E, D, N\}, \, n = 1, ..., 10\},$$

and the strategy space for the USSR can be written as:

$$S_2 \equiv \{\vec{s}_2 = [s_{2.1}, s_{2.2}, ..., s_{2.10}] : s_{2.n} \in \{E, D, N\}, \, n = 1, ..., 10\}.$$

In short, a strategy for each player must specify what the player will do at each of its information sets. Therefore, there are 3^{10} possible pure strategies for each player. Then the number of strategy profiles is equal to 3^{20}. In this type of game, we'd be hard-pressed to solve analytically for a NE. Further, there is the possibility that there are multiple NEs. So, what do we do? We consider certain strategies which intuitively make sense as equilibria and check to see whether they are NE (or even SPNE).

Let's consider the following questions for this game:

1. How many subgames are there? Total of ten (10).
2. Are there any NE involving playing anything other than "E" in any of the info sets? Consider the following strategy for both players: Play "D" in the first period; then "E" if the first period outcome is [D,D], or "N" otherwise.[11]
   (a) What is the payoff to both players in this strategy profile? Players 1 and 2 each gets 2 in the first period and 1 in the second period for a total of 3.

[10] Note that the game is very similar to a prisoner's dilemma.

[11] In the repeated prisoner's dilemma context, players cooperate in the first period and punish in the next period if opponent does not cooperate. Also note that in my representation of strategies, the candidate strategy profile is: [(DNNNNENNNN), (DNNNNENNNN)].

(b) Deviation by a player in the first period:

   i. Consider one possible unilateral deviation by player 1 (since this is a symmetric game, we can do this for one player and generalize to the second player) where player 1 plays "E" in the first period,[12] and plays "E" in the second period. Player 1's total payoff is $3 + (-100) = -97 < 3$. Remember that when you're looking for NE, the strategy of the other players do not change. Since the first period outcome was [E,D], player 2 will following the strategy profile and play "N" in the second period.

   ii. Consider another possible unilateral deviation by player 1. Player 1 again plays "E" in the first period, and then following the prescription of the strategy profile above, player 1 plays "N" in the second round. Player 1's total payoff is $3 + (-200) = -197 < 3$.

   iii. In short, if player 1 deviates in the first round, the strategy profile requires player 2 to play "N" in the second period where the highest payoff for player 1 is -100.

   iv. Therefore, a player will not deviate in the first period.

(c) Deviation by a player in the second period:

   i. [E,E] is the NE of the second period game. Therefore, "E" is a best response in the second period, and so no player will deviate.

   ii. To check point i. above:

      A. what happens if player 1 unilaterally deviates in the second period from "E" to "D": Total payoff is $= 2 + 0 < 3$.

      B. what happens if player 1 unilaterally deviates in the second period from "E" to "N": Total payoff is $= 2 + 0 < 3$.

   iii. Therefore, a player will not deviate in the second period.

(d) Deviation by a player in the first and the second period: This is covered by deviation in the first period.

(e) Therefore, (b), (c), and (d) above prove that this the strategy profile above is a NE.

(f) One final comment, what about the following strategy for each player: Play "D" in the first period; then "E" if the first period outcome is [D,D], or "E" otherwise.[13]

   i. What is the payoff from this strategy? It is $2 + 1 = 3$, which is equal to the payoff of the original strategy profile [(DNNNNENNNN), (DNNNNENNNN)].

   ii. To use some of the terminology Chris mentioned in class, whether a strategy profile is NE or not doesn't depend on what the strategy tells the players to do "off the equilibrium path."

[12] Note that the only possible deviation which increases player 1's payoff is to move to "E". Deviating to "N" would not increase player 1's payoff.

[13] Using my notation, the strategy profile under consideration is: [(DEEEEEEEEE), (DEEEEEEEEE)].

iii. If you don't understand point ii., then consider the payoff matrix in Case 1 above. Recall that the NE [U, CC] was the result of a non-credible threat where player 2 threatens to play "C" if player 1 plays "M." So player 1 plays "U" and player 2 then plays "C" to get her maximum payoff for the game. The resulting payoff is (2,4). Notice what happens to the payoffs if we change player 2's strategy from "CC" to "CL":[14] *NOTHING*! The payoff is still (2,4). This is exactly the same thing as changing the "threat" from "N" to "E" in the Cold War game above.

iv. Is the strategy profile [(DEEEEEEEEE), (DEEEEEEEEE)] a NE?

3. Is the NE above a SPNE? No. Recall that in each subgame, the players must be playing the NE strategy: "E". Therefore, there is only one SPNE: where the players play "E" in all subgames.

The bottom line for identifying SPNE is that backward induction by subgame. This can become difficult when subgames have multiple NE because SPNE requires the players to play a NE in each subgame.

Consider what would happen to the game above if we change the payoffs for the [nuke, nuke] outcome to [-100,-100]. Then the strategy considered above is SPNE.

Now let's consider a case where we have continous action sets for each player.

Example 4.3. [15]Empty Threat Game with continous strategies:[16] Firm 1 is a potential entrant and firm 2 is the incumbent. Entry costs 400, and post-entry competition is Cournot (which means that the players simultaneously move) with inverse demand function given by $P(Q) = 120 - Q$ and marginal costs equal to zero (0).

The game tree can be found in Appendix A-6. How many subgames?

To find the PSNE, recall that there were two types of NE in the entry game: one where the entrant firm stayed out and one where the entrant entered. Just because we're in the world of continuous variables, we don't abandon the backward induction approach. Therefore, we're going to have to start with the Cournot model.

The solution to the Cournot competition is:

$$q_1^* = q_2^* = 40,$$

and we know that these quantities (or actions) define the NE of a Cournot game. At these quantities firm 1 and firm 2's profits are:

[14]In short, change the off equilibrium "action" from "C" to "L". "Off equilibrium" means that the players don't reach that node in equilibrium.

[15]Will not cover in class.

[16]Therefore, this game is not finite.

$$
\begin{aligned}
\Pi_1^* &= 1600 - 400 = 1200 \\
\Pi_2^* &= 1600
\end{aligned}
$$

What happens, given the parameters above, if there is only one firm in the market? Since the firm is a monopolist, that firm should produce monopoly quantity and get monopoly profits:

$$q_M^* = 60,\ \ \Pi_M^* = 3600.$$

Notice the payoffs if player 1 plays "Out" at 1.1: It is (0, 3600). In short, if player 1 stays out of the market, player 2 is monopolist and so gets monopolist profits.

Player 1 plays "In"

Consider a NE candidate strategy profile where player 1 enters the market: $[(In,\ q_1^* = 40),\ q_2^* = 40]$. Is this strategy profile in fact NE? Given player 2's action, player 1 cannot do better by deviating from producing 40 units (which is in fact the definition of "best-responding"). Can player 1 do better by playing "Out"? No. At the strategy profile above, player 1 gets $1600 - 400 = 1200$, vs 0 if player 1 plays "Out." Therefore, this is a NE.

However, is this NE a SPNE? There are two subgames, the whole game and the Cournot subgame. The NE above requires players to play NE strategies at each of the subgames and so it is a SPNE.

Player 1 plays "Out"

Now, let's first look at a NE candidate strategy profile where player 1 stays out of the market.

The objective functions for player 1 if she decides to enter was:

$$U_1(q_1, q_2) \quad = \quad (120 - q_1 - q_2) \cdot q_1 - 400 \Rightarrow q_1^* = \frac{120 - q_2}{2}$$

When would player 1 be indifferent from entering and not entering? We want to know this because this will limit the set of strategies that produces a NE.

$$
\begin{aligned}
U_1(q_1^*, q_2) &= \left(\frac{120 - q_2}{2}\right) \cdot \left(120 - \frac{120 - q_2}{2} - q_2\right) = 400 \\
&\Rightarrow \frac{q_2^2}{2} - 120q_2 + 6400 = 0 \\
&\therefore \quad q_2 = \frac{120 \pm \sqrt{1600}}{1} = 80,\ 160
\end{aligned}
$$

Therefore, if player 2 produces more than 80 units (160 units doesn't make sense for this model), then it would be better for player 1 to stay out. The related NE is where player 1 plays (Out,$q_1^* \in [0, \infty]$) and player 2 plays $q_2^* \in [80, \infty]$.[17]

However, is it a SPNE? Clearly it isn't because the players are not playing NE strategies in the Cournot subgame. Therefore, this NE, is based on a non-credible threat that player 2 is going to produce more than 80 units. But that would be player 2 shooting itself in the foot.

[17] Understanding why all of these infinite strategies is a NE is very important. The reason is that NE does NOT care about the actions off the equilibrium path. *In short, in the Cold War game above, we could change "otherwise play N" to "otherwise play E" and this wouldn't affect the payoffs. The same principle applies here. The payoffs are equal to [0, 3600] if player 1 plays "Out" regardless of what player 1 and player 2 do in the Cournot subgame, subject to the important condition that player 2 produces more than 80 units.*