Section Notes 10

Abby Friedman, Wonbin Kang

November 12, 2010

Agenda

1. Bayesian Nash Equilibrium
2. Trembling Hand Perfect Nash Equilibrium ("THPNE")

1 Bayes-Nash Equilibrium

Recall that one of the assumptions of the games that we've looked at was *complete information*, the players knew the payoff functions of the other players. Now let's consider the case where the parties are unaware of their opponents' payoffs because the parties are of different types, and the players are only aware of the probability distribution of types. Consider the following examples and its extension.

1.1 High-Low Poker Game

Now let's look at the high-low poker game from lecture. Recall that the initial pot is equal to \$20 and it is divided by the two players, 1 and 2. Nature determines the type of player 1 (or player 1 draws a "High" or "Low" card) such that the probability of a "High" type/card is equal to $\lambda \in (0, 1)$. Player 1 observes the card and then decides to "Bet" or "Pass," where "Pass" ends the game with payoffs equal to (0, 20). If Player 1 chooses "Bet" and player 2 plays "Call," then the payoff is (30, -10) if player 1's card is "High." If player 1's card is "Low," then the payoffs are equal to (-10, 30).

Recall that we can rewrite the extensive form game into a normal form. For simplicity, let's eliminate strategies which has the High type/card player 1 choosing to "Pass." Then the normal form is:

	Call	Fold
Bet, Bet	$30\lambda + (-10)(1-\lambda)$, $-10\lambda + 30(1-\lambda)$	20, 0
Bet, Pass	$30\lambda + 0(1-\lambda)$, $-10\lambda + 20(1-\lambda)$	20λ, $20(1-\lambda)$

which simplifies to the normal form below, and we can underline best responses given that $\lambda \in (0, 1)$:

	Call	Fold
BB	$40\lambda - 10$, $30 - 40\lambda$	20,0
BP	30λ, $20 - 30\lambda$	20λ, $20 - 20\lambda$

Notice that we need more information on λ to figure out the best response of player 2 to player 1's strategy of BB. Recall that in class $\lambda = \frac{1}{2}$, and so there did not exist a PSNE.[1] However, we did find a MSNE in class.

1.2 More Examples

Example 1.1. Player 1 is a graduate student who has strategies work ("W") or shirk ("S"). Player 2 is a professor who must decide whether to guide or ignore the graduate student. Let's assume that the graduate student can be of two types: motivated or lazy. The probability that the graduate student is motivated is equal to $m \in (0, 1)$. The payoffs for each type are:

Motivated:

	G	I
W	8,5	4,-1
S	2,-1	2,0

Lazy:

	G	I
W	3,5	-1,-1
S	2,-1	2,0

.

Draw the extensive form of this game and the Bayesian Normal Form.

1. The extensive form can be found in Appendix A-1. Note that we need the move by "Nature" to draw this game.

2. The Bayesian Normal Form is as follows:[2]

	G	I
WW	$\underline{3+5m}$, $\underline{5}$	-1+5m, -1
WS	2+6m, -1+6m	$\underline{2+2m}$, -m
SW	3-m, 5-6m	-1+3m, -1+m
SS	2,-1	2,0

---

[1]The normal form with underlined best responses would be:

	Call	Fold
BB	10,$\underline{10}$	$\underline{20}$,0
BP	$\underline{15}$,5	10,$\underline{10}$

, and so no PSNE.

[2]To reiterate my notation, [WS] means that the "motivated" type will "Work" and the "lazy" type will "Shirk".

(a) For player 1 (the graduate student) "SS" is strictly dominated by "WS".

(b) For player 1 (the graduate student) "SW" is strictly dominated by "WW".

(c) If we underline best responses, we see that [WW,G] is BNE. Further, if $-1+6m > -m \Rightarrow m > \frac{1}{7}$, then [WW,G] is the unique BNE. Otherwise, we have two BNEs: [WW,G] and [WS, I].

(d) Assuming that $m \leq \frac{1}{7}$, which means that there are two BNEs, what is the mixed strategy BNE? Let the probability of player 1 playing "WW" equal p and the probability of player 2 playing "I" equal q. We won't be putting positive probability on the strictly dominated strategies, and so no need to check against any remaining pure strategies.

   i. For player 1:

$$
\begin{aligned}
U_1(WW) &= (3+5m)\cdot(1-q)+(-1+5m)\cdot q = 3+5m-4q \\
U_1(WS) &= (2+6m)\cdot(1-q)+(2+2m)\cdot q = 2+6m-4mq \\
&\Rightarrow \quad q = \frac{1-m}{4-4m} = \frac{1}{4}
\end{aligned}
$$

   ii. For player 2:

$$
\begin{aligned}
U_2(G) &= 5p+(1-p)\cdot(-1+6m) = 6p-1+6m-6mp \\
U_2(I) &= -p+(1-p)\cdot(-m) = -p-m+mp \\
&\Rightarrow \quad p = \frac{1-7m}{7-7m}
\end{aligned}
$$

Example 1.2. [Free-riding] There are two players $i = 1, 2$; denote player 1's effort level x and player 2's effort level y; and total profit for the two players is $\Pi(x,y) = 4(x+y+kxy)$, where $k \in \left[0, \frac{1}{2}\right]$. The effort cost of player 1 and player 2 are, respectively, $c(x) = x^2$ and $c(y) = y^2$. Each player chooses their level of effort simultaneously. What is the NE of this game?

We set up the maximization problems and solve for $i = 1$:

$$
U_1(x,y) = 2(x+y+kxy) - x^2 \Rightarrow^{FOC} \frac{\partial U_1}{\partial x} = 2(1+ky) - 2x = 0
$$

Since we have a symmetric problem, we have the following equations:

$$
\begin{aligned}
x^* &= 1 + ky^* && (1)\\
y^* &= 1 + kx^* && (2)
\end{aligned}
$$

which results in the following NE:[3]

$$
x^* = y^* = \frac{1}{1-k} \tag{3}
$$

We can do one interesting comparative static on the equilibrium identified in 3

$$
\frac{dx^*}{dk} = \frac{dy^*}{dk} = \frac{1}{(1-k)^2} > 0.
$$

What does this mean? Note that k is the complementary increase in profit if both players increase their effort level.

Example 1.3. Same set up as Example 1.2, except now suppose that player 1 doesn't know (or observe) the payoffs of player 2. All she knows is the distribution of player 2's payoffs: there is 50% chance that player 2's payoff is $U_2^{high}(x,y) = 2(x+y+kxy) - 2y^2$ and a 50% chance that her payoff is $U_2^{low}(x,y) = 2(x+y+kxy) - y^2$, where the "high" and "low" superscripts denote high cost vs low cost type.

The strategy space of each player are: $S_1 \equiv [0,\infty)$ and $S_2 \equiv \left\{S_2^{low}, S_2^{high}\right\} = \{[0.\infty), [0,\infty)\}$.

Solving for each of player 2's best responses, we have:

$$
\begin{aligned}
y^*_{low} &= \arg\max_y 2(x+y+kxy) - y^2 = 1 + kx && (4)\\
y^*_{high} &= \arg\max_y 2(x+y+kxy) - 2y^2 = \frac{1+kx}{2} && (5)
\end{aligned}
$$

Given these payoffs for player 2's two types, player 1 will maximize her expected utility to solve for x^*:

$$
\begin{aligned}
x^* &= \arg\max_x \left[\frac{1}{2}\left\{2(x + y^*_{low} + kxy^*_{low}) - x^2\right\} + \frac{1}{2}\left\{2(x + y^*_{high} + kxy^*_{high}) - x^2\right\}\right]\\
&= \arg\max_x \left[2x + y^*_{low} + y^*_{high} + kxy^*_{low}kxy^*_{high} - x^2\right].
\end{aligned}
$$

FOCs (don't worry about K-T Conditions) result in the following:

$$
\frac{\partial U_1(x, y^*_{low}, y^*_{high})}{\partial x}\Big|_{x=x^*} = 2 + ky^*_{low} + ky^*_{high} - 2x^* = 0
$$

[3] An interesting exercise might be to see what the equilibrium effort levels are when we only have one player or when we have two players but player 1 moves first and player 2 moves after observing player 1's move.

$$\therefore x^* = 1 + k \cdot \left(\frac{y^*_{low} + y^*_{high}}{2} \right). \tag{6}$$

We have three equations and three unknowns (Equations 4, 5, and 6), and if we solve we get the following Bayes-Nash equilibrium:

$$x^* = \frac{4+3k}{4-3k^2};\ y^*_{low} = 1 + k \cdot \frac{4+3k}{4-3k^2};\ y^*_{high} = \frac{1}{2}\left\{1 + k \cdot \frac{4+3k}{4-3k^2}\right\}.$$

We tried some comparative statics in the original game. Try them now and see what results you get.

2 THPNE

By now you should know that in certain games, you will find a number of NEs. SPNE and BE are equilibrium concepts which "refine" the set of NE. THPNE is another refinement we consider. Let's look at the following example:

Example 2.1.

	L	M	R
U	4,-4	2,4	0,-3
D	5,3	2,2	-1,1

$\Rightarrow$

	L	M
U	4,-4	<u>2</u>,<u>4</u>
D	<u>5</u>,<u>3</u>	2,2

Remember, when you first see a game, the first thing you should do is check to see if there is a strictly *dominated* strategy. In the game above, we can simplify the 2×3 matrix into the 2×2 matrix since "R" is strictly dominated by "M". In the reduced form game, there are two NE: (U,M) and (D,L). What we want to think about is which of the two NEs is "better." We're going to use THPNE to solve for the "better" equilibrium.[4]

2.1 Definition of THPNE

Let's start with a couple of definitions:

Definition 2.2. A strategy σ_i is a *completely mixed strategy* if it puts positive probability on all of player i's pure strategies. A strategy profile σ is completely mixed if all players are playing a completely mixed strategy.

To prevent confusion, let's assume that $i \in \{1, ..., I\}$, then:

Definition 2.3. A NE strategy profile σ is trembling hand perfect *if there exists* a sequence of completely mixed strategy profiles $\{\sigma_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \sigma_n = \sigma$ and such that for all players i, σ_i is a best response to σ_{-in} for a large enough n.

[4]Last year in Econ 2020b, Elon called this the equilibrium selection process and the "better" equilibrium as the self-enforcing equilibrium.

2.2 Examples of THPNE

Let's look at a few examples to clarify the definition above:

For Example 2.1 above, consider the NE where player 1 plays "U" and player 2 plays "M". From the definition of THP, we need to consider completely mixed strategy *profiles*, so let's assume that player 1 is playing a completely mixed strategy, denoted σ_1^*, where he places $1-\epsilon_n$ probability on "U" and ϵ_n probability on "D". Also let's assume that player 2 is playing a completely mixed strategy, denoted σ_2^*, where she places $1-\epsilon_n$ probability on "M" and ϵ probability on "L".

Note the following:

1. We can write $\epsilon_n = \frac{1}{n}$ such that $n \to \infty$ is equivalent to $\epsilon_n \to 0$. So we do have a sequence of strategy profiles of the form in the definition.
2. Note that as $\epsilon_n \to 0$, the strategy profiles converge to the NE under consideration, [U,M].

Now let's look at the expected utilities of player 1 given the completely mixed strategy of player 2:

$$\begin{aligned} U_1(Up;\sigma_2^*) &= 4\epsilon_n + 2(1-\epsilon_n) = 2 + 2\epsilon_n \\ U_1(Down;\sigma_2^*) &= 5\epsilon_n + 2(1-\epsilon_n) = 2 + 3\epsilon_n \end{aligned}$$

Because we're looking at values of ϵ_n arbitrarily close to zero (but not equal to zero, because we're considering completely mixed strategy profiles), $U_1(Down;\sigma_2^*) > U_1(Up;\sigma_2^*)$, since $3\epsilon > 2\epsilon$. Therefore, we conclude that the candidate [U,M] is *not* trembling hand perfect.

Note that the definition of THPNE states that there is *no* sequence of completely mixed strategies $\{\sigma_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \sigma_n = \sigma$ and such that for all players i, σ_i is a best response to σ_{-in} for a large enough n. We didn't do that above, but for purposes of this class it is sufficient.

Example 2.4.

	L	C	R
U	3,1	1,0	4,0
M	1,3	2,4	1,2
D	0,1	1,1	2,2

$\Rightarrow$

	L	C	R
U	3,1	1,0	4,0
M	1,3	2,4	1,2

$\Rightarrow$

	L	C
U	$\underline{3}$,$\underline{1}$	1,0
M	1,3	$\underline{2}$,$\underline{4}$

Iterated elimination of strictly dominated strategies gets us to the 2×2 matrix, where we have 2 pure strategy NE: [U,L] and [M,C]; and a mixed strategy NE where $\vec{p}_1^* = [\Pr(U), \Pr(M), \Pr(D)] = \left[\frac{1}{2}, \frac{1}{2}, 0\right]$ and $\vec{p}_2^* = [\Pr(L), \Pr(C), \Pr(R)] = \left[\frac{1}{3}, \frac{2}{3}, 0\right]$.

On your own, check to see whether the two PSNE are THP.

Below, we check to see whether the MSNE is THP.[5]

1. We need to come up with a completely mixed strategy for each player that at its limit is equal to the strategy being played in our equilibrium of interest. Consider the following:

$$\sigma_{1\epsilon} = \left[\frac{1}{2} - \epsilon,\ \frac{1}{2} - \epsilon,\ 2\epsilon\right];\ \sigma_{2\epsilon} = \left[\frac{1}{3} - \epsilon,\ \frac{2}{3} + \frac{\epsilon}{4},\ \frac{3}{4}\epsilon\right]$$

2. Now we calculated the expected utilities for each of the players. Let's start with player 1:

$$\begin{aligned} U_1(U) &= 3\left(\frac{1}{3} - \epsilon\right) + 1\left(\frac{2}{3} + \frac{\epsilon}{4}\right) + 4\left(\frac{3}{4}\epsilon\right) = \frac{5}{3} + \frac{\epsilon}{4} \\ U_1(M) &= 1\left(\frac{1}{3} - \epsilon\right) + 2\left(\frac{2}{3} + \frac{\epsilon}{4}\right) + 1\left(\frac{3}{4} \cdot \epsilon\right) = \frac{5}{3} + \frac{\epsilon}{4}. \end{aligned}$$

   Since $U_1(M) = U_1(U)$ as $\epsilon \to 0$, we've found a sequence of completely mixed strategies which converge to the NE and for which the mixed strategy is a best response (because the payoffs are the same given player 2's mixed strategy $\sigma_{2\epsilon}$.

3. Try this for player 2 and you should see that the same holds.

$$\begin{aligned} U_2(L) &= 1\left(\frac{1}{2} - \epsilon\right) + 3\left(\frac{1}{2} - \epsilon\right) + 1\,(2\epsilon) = 2 - 2\epsilon \\ U_2(C) &= 0\left(\frac{1}{2} - \epsilon\right) + 4\left(\frac{1}{2} - \epsilon\right) + 1\,(2\epsilon) = 2 - 2\epsilon. \end{aligned}$$

Example 2.5. Here's another example from Chris' lecture last year, which we won't cover in section:

	L	M	R
Up	4, 4	1, 4	0, -3
Down	5, 3	2, 2	-999999, -2

Notice that "R" is strictly dominated by "M" and iterated dominance outcome (NE) is [D, L].

1. Assume that the probabilities of player 2's completely mixed strategy is equal to $\vec{q} = [q_L, q_M, q_R]$. Then we can solve for player 1's expected values

[5] Instead of checking to see whether we have convergence to the candidate strategy as $n \to \infty$, I use $\epsilon \to 0$ and drop the subscripts.

of playing "Up" and "Down."

$$\begin{aligned} U_1(Up) &= 4q_L + q_M \\ U_1(Down) &= 5q_L + 2q_M - 999999q_R \\ \therefore U_1(Down) - U_1(Up) &= q_L + q_M - 999,999q_R = 1 - 1000000q_R \end{aligned}$$

2. For player 1, if $q_R \leq \frac{1}{1,000,000}$ then "Down" is a best response for player 1. Any sequence of strategies for player 2, which has as its limit the strategy "L", has the property that eventually $\Pr(R) < \frac{1}{1,000,000}$. Therefore, [D,L] is THPNE.

2.3 Main Results of THPNE

1. All finite games have at least one THPNE.
2. Any completely mixed NE is THPNE. Let the sequence of strategies be equal to the mixed strategy candidate, then the definition of THPNE is satisfied.
3. If $\sigma^*$is a THPNE, then no player is putting positive probability on a weakly dominated strategy.
4. In a two player game, if $\sigma^*$is a NE not involving a weakly dominated strategy, then σ^* is THP (this doesn't hold if you have $I > 2$).