Section Notes 9

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Agenda

1. Spence Education Model and PBE[1]
2. Separating Equilibrium
3. Pooling Equilibrium and PBE refinements
4. Imperfect Signals and Partial Pooling

1 Spence Education (Signaling) Model and PBE

What's the motivation for signaling? Adverse selection: high types[2] want to overcome adverse selection and low types want to act like they are high types. In Contract Theory, you can learn about what happens when there are a continuum of infinite types.

1.1 Model Set-Up with Two Types

- Potential employees (who are the agents and the Senders in this model) are of productivity type $\theta \in \{\theta_L, \theta_H\}$, where $\theta_L < \theta_H$. The probability of an employee being a high productivity type given a level of education is equal to $\mu(\theta_H|e) = \mu_e$.
- Potential employees can send a signal $e \in [0, \bar{e}]$; and the cost of education for employees of type $j \in \{L, H\}$ is equal to $c_j(\cdot)$, where $c_j'(\cdot) > 0$; $c_j''(\cdot) > 0$; $c_j(0) = 0$.
- *Single-crossing assumption*: assume that the marginal cost of eduction is greater for the low types than the high types for all levels of education.[3]

$$c_H'(e) < c_L'(e)$$

[1] A more involved version of the Spence model can be found in Gibbons (1992).
[2] The high types in the Spence model are like the "peaches" in the Akerlof model.
[3] See Appendix A-1.

- The utility for employee $j \in \{L, H\}$ is:

$$\begin{aligned} u_j(w,e) &= w(e) - c_j(e) \\ u(w,e|\theta) &= w(e) - c(e|\theta) \end{aligned}$$

- We further assume a competitive market for employees and so wage is equal to marginal productivity, which in turn is equal to θ for an employee of type θ. In short,

$$\begin{aligned} w_L &= \theta_L \\ w_H &= \theta_H. \end{aligned}$$

  However, if the employer does not know the type of employee, then the employer sets wage equal to the expected value of θ given the observable value of e:

$$w(e) = \mathbb{E}\left[\theta|e\right] = \mu_e\theta_H + (1-\mu_e)\theta_L.$$

1.2 Game Theoretic Application

Given the set up above, what equilibrium concept would work? BNE and PBE both are acceptable. However, PBE is more applicable since wage is bounded by the productivity of the employee and because it makes sense to explicitly state the beliefs of the employer (or Receiver) on and off the equilibrium path. Therefore, we're going to focus on PBE.

Recall the requirements of (weak) PBE:

1. Beliefs μ_e assigned to each node in an information set on the equilibrium path.
2. Given these beliefs, players best respond in each information set (sequential rationality).
3. Beliefs consistent with Bayes' Rule on the equilibrium path.

Recall also that in our signaling models (including the Beer-Quiche game), we had pooling, separating, and partial pooling equilibrium. The same should apply to this model as well.

Now let's consider the following example:

Example 1.1. A professor (the principal, receiver, or employer) wants to hire students (the agent, sender, or employee) as RAs for the summer. The relevant signal is the average score on econometrics problem sets: $e \in [0, 100]$.

This is unrealistic in that the signal (or the problem sets) does not increase marginal productivity.[4]

[4] In the Gibbons (1992) version of the Spence education model, education e increases the productivity—*i.e.* total production y is a function of the employee's type and education: $y_e > 0$ and $y_\theta > 0$.

Let's assume that $\theta_H = \$10,000$ and $\theta_L = \$5,000$; $c_H(e) = e^2$ and $c_L(e) = 2e^2$; and $\mu(H) = \frac{4}{5}$ and $\mu(L) = \frac{1}{5}$. Then student's optimization problem is as follows:

$$\max_e u_j(e) = w(e) - c_j(e), \forall j \in \{L, H\}.$$

If the employer (professor) was able to distinguish between the two types of students, what is the optimal level of education for each type? Do you see that it is:

$$e_L^{FB} = e_H^{FB} = 0.$$

In Contract Theory, we denote this no information asymmetry level of education with the superscript "FB", or first-best. It is the first best in the sense that it is the Pareto optimal outcome. Do you see why? If not, review the asymmetric information model below and then return to this question.

2 Separating Equilibrium

In a separating equilibrium, the high types choose $e_H > 0$ and the low types choose $e_L = 0$. Why would the low type of employee choose a level of education greater than 0? Because education is costly, it would be profitable for the low type to increase education greater than 0 if and only if it would increase the wage she would receive, and hence make her better off. Since we're in a separating equilibrium, it must be the case that the low type invest 0 in education (and has no incentive to deviate to a higher level of education).

Since we assume a competitive labor market, $w(e_H) = \theta_H$ and $w(e_L = 0) = \theta_L$. Now we can solve for the PBE using each employee types' Incentive Compatibility Constraints (ICC):[5]

1. ICC_H:[6] $u_H(e_H) \geq u_H(e_L = 0) \Rightarrow w(e_H) - c_H(e_H) \geq w(e_L) - c_H(e_L) \Rightarrow 10,000 - e_H^2 \geq 5000 - 0 \Rightarrow e_H \leq 70.7$.

2. ICC_L: $u_L(e_L = 0) \geq u_L(e_H) \Rightarrow w(0) - c_L(0) \geq w(e_H) - c_L(e_H) \Rightarrow 5,000 - 0 \geq 10,000 - 2e_H^2 \Rightarrow e_H \geq 50$.

3. Therefore, $e_H \in [50, 70.7]$. Note that of these equilibrium levels of e_H, dominance refinement[7] eliminates all of equilibria except for $e_H = 50$.

To provide some intuition for the results above: for the high type of student, she wouldn't invest more than 70.7 in education because she'd be worse off than if she was considered a low type and received \$5,000. Also for the high type, she'd want to invest more than 50 in education or else she would not be able to separate herself from the low type.

[5]If ICCs are satisfied, no type wants to deviate and pretend to be another type.
[6]Note that the ICC is normally written as: $e_H = \arg\max_e u_H(e_H)$.
[7]More on this below.

What beliefs of the Receiver (employer) on and off the equilibrium path get us the separating equilibrium? Consider the following:

$$\mu(H|e) = \begin{cases} 1, & if\, e \geq e_H \\ 0, & otherwise \end{cases}$$

See Appendix A-2 for a representation of the separating equilibrium.

3 Pooling Equilibrium and PBE Refinements

3.1 Pooling PBE

Let's assume that the initial (prior) distribution of high type and low type students is $\mu(H) = \frac{4}{5}$ and $\mu(L) = \frac{1}{5}$. If both the high and low types pooled at the same level of education e_p, then:

$$\mathbb{E}\left[\theta|e_p\right] = \lambda\theta_H + (1-\lambda)\theta_L = 9000.$$

At the pooling equilibrium, the low type must be as well off as if the low type received the low type payoff (ICC_L must hold):

$$9000 - c_L(e_p) \geq 5000 - c_L(e_L = 0) \Rightarrow 9000 - 2e_p^2 \geq 5000 \Rightarrow e_p \leq 44.7.$$

Therefore, it is possible to sustain pooling for any $e_p \leq 44.7$.

What beliefs of the Receiver (employer) on and off the equilibrium path get us the pooling equilibrium at education level e_p, which has to be less than 44.7? One possibility is:

$$\mu(H|e_p) = \begin{cases} \frac{4}{5}, & if\, e = e_p \in [0,\, 44.7] \\ 0, & otherwise \end{cases}$$

Now you should be able to figure out the sequentially rational strategies given these beliefs.

See Appendix A-3 for a representation of the pooling equilibrium.

3.2 Refinement of PBE

3.2.1 Dominance Refinement

Definition 3.1. If possible, each player's reasonable beliefs off the equilibrium path should not assign positive probability to nodes that are reached only if another player plays *a strategy that is strictly dominated beginning at some information set.*[8]

[8]Note that this is different from not assigning positive probabilities to strictly dominated strategies. See Gibbons (1992, pp. 232-35).

In the context of the model above, given that $\forall e \in [0, 100]$, $w(e) \in (5000, 10000)$, low types should never choose a level of education $e > 50$ ($\because$even if a low type is able to deceive the employer into thinking she is a high type and get wage equal to 10000, if she increases her education to a level greater than 50, then she's worse off than if she just got the low type wage with zero education). This means that the probability of a high type for values of $e > 50$: $\mu(H|e > 50) = 1$, which means that $w(e > 50) = 10,000$.

Now consider pooling at $e_p = 44.7$: high types strictly prefer $e_H = 50 + \epsilon$,[9] with $w = 10,000$ to a pooling equilibrium at $e_p = 44.7$ and wage equal to 9000. Therefore, there can be no pooling at $e_p = 44.7$ once we apply the dominance refinement.

Now consider pooling at $e_p = 0$ and $w(0) = 9000$. Here the high types will not be better off by deviating from the pooling equilibrium—*i.e.* payoff for the high types of pooling is equal to $9000 - 0^2 = 9000$, which is greater than deviating to $e = 50 + \epsilon \Rightarrow 10000 - (50 + \epsilon)^2 \approx 7500$. We can set $w(e)$ after defining the beliefs such that they satisfy the dominance refinement. For example:

$$\mu(H|e) = \begin{cases} \frac{4}{5}, & if\, e = e_p = 0 \\ 0, & if\, e \in (0, 50] \\ 1, & if\, e > 50 \end{cases}$$

which leads to the following sequentially rational strategies by the employer/Receiver/principal:

$$w(e) = \begin{cases} 9000, & if\, e = 0 \\ 5000, & if\, e \in (0, 50] \\ 10000, & if\, e > 50 \end{cases}$$

In conclusion, we've found a pooling PBE which satisfies the dominance refine-

[9] I'm trying to be careful with the strict and weak inequalities, but we could just have said $e_H = 50$.

ment.[10] However, this pooling PBE will not survive the next PBE refinement.

Since you've seen the dominance refinement in action once, go back to the separating equilibrium above and show that only the separating PBE with $e_H = 50$ survives the dominance refinement.

3.2.2 Equilibrium Dominance Refinement (Cho-Kreps Intuitive Criterion)

Definition 3.2. Given a PBE in a signaling game, the signal s_j is equilibrium dominated for type $t_i \in T$, where T is a set of all types, if t_i's equilibrium payoff is greater than t_i's highest possible payoff from sending the signal s_j.

Given the definition of equilibrium dominance, we can define the following PBE refinement:

Definition 3.3. (Cho-Kreps Intuitive Criterion): If the information set following $s_j$is off the equilibrium path and it is equilibrium dominated for type t_i then (if possible) the Receiver's belief $\mu(t|s_j)$ should place zero probability on type t_i.

Let's consider pooling at $e_p = 0$ and $w(e_p) = 9000$. Therefore, the payoff to the low type at $[e_p, w(e_p)]$ is equal to $u_L = 9000$, and the low types should never choose a level of e such that:

$$\begin{aligned} u_L(e, w &= 10000) < 9000 \Rightarrow \\ 2e^2 &> 1000 \Rightarrow e > 22.4, \end{aligned}$$

which ends up breaking all pooling equilibrium. Do you see why? Given the pooling equilibrium, the low type would never set her level of education greater than 22.4, which means that $\mu(H|e > 22.4) = 1$. Given these beliefs of the

[10]We can refine the pooling equilibrium even more by noting the following: the levels education that will sustain a pooling equilibrium after dominance refinement requires that:

$$\begin{aligned} 9000 - e_p^2 &> 10,000 - (50 + \epsilon^2) \approx 7500 \\ e_p &< 38.73. \end{aligned}$$

This means that we can rewrite the on and off equilibrium beliefs and sequentially rational strategies above as follows:

$$\mu(H|e) = \begin{cases} \frac{4}{5}, & if\, e \in [0,\ 38.73) \\ 0, & if\, e \in [38.73,\ 50] \\ 1, & if\, e > 50 \end{cases}$$

and

$$w(e) = \begin{cases} 9000, & if\, e \in [0,\ 38.73) \\ 5000, & if\, e \in [38.73,\ 50] \\ 10000, & if\, e > 50 \end{cases}$$

Receiver/employer, the high type employee/Sender would be better off by deviating to $e = 22.4 + \epsilon$ for a payoff greater than what the high type was getting at the pooling equilibrium.

Now that you've seen Cho-Kreps in action for $e_p = 0$, try it for a general $e_p \in [0,\ 38.73]$ that we discussed in footnote 10.

4 Imperfect Signals and Partial Pooling[11]

4.1 Imperfect Signals

Now assume that some proportion $\alpha = \frac{1}{2}$ of low types work on problem sets with high types in a group:

$$c_{L,group}(e) = c_H(e) = e^2 < c_{L,no\,group}(e) = 2e^2,$$

but $\theta_{L,group} = \theta_{L,no\,group} = 5000$. We're asked to solve for a separating equilibrium. We know that $e_{L,ng} = 0$ and wage equal to 5000 in a separating equilibrium, while the other students solve the following maximization problem:

$$\max_e w(e) - e^2 \Rightarrow e_{L,g} = e_H$$

in equilibrium. The intuition here is that since the low types who work with the high types have the same costs as the high types, they are able to act like the high types and thereby get the high type wages. However, the professors know that there is a mix of high and low types in the high equilibrium level of education. Therefore, they will fix wages such that they are equal to the expected productivity given a high level of education.

$$\begin{aligned}
\mathbb{E}\left[\theta | e_H = e_{L,g}\right] &= \Pr(\theta_H|e_H)\cdot\theta_H + \Pr(\theta_L|e_H)\cdot\theta_L \\
&= \frac{\Pr(\theta_H \cap e_H)\cdot\theta_H}{\Pr(e_H)} + \frac{\Pr(\theta_L \cap e_H)\cdot\theta_L}{\Pr(e_H)} \\
&= \frac{\lambda\cdot\theta_H + (1-\lambda)\alpha\cdot\theta_L}{\lambda + (1-\lambda)\alpha} = \frac{\frac{4}{5}\cdot 10000 + \frac{1}{5}\cdot\frac{1}{2}\cdot 5000}{\frac{4}{5} + \frac{1}{5}\cdot\frac{1}{2}} \\
&\approx 9444.
\end{aligned}$$

Therefore, $w(e_H = e_{L,g}) = \mathbb{E}\left[\theta | e_H = e_{L,g}\right] = 9444$, with associated level of education for the low type who does not work as a group equal to $e_{L,ng} \approx 47.1$.

The result is that the high types are worse off because some of the low types are able to pool with them.

[11] I doubt we'll have time for this in section, but please review to check that you've understood the concepts above.

4.2 Partial Pooling Equilibrium[12]

Partial pooling (as we saw in the Beer-Quiche game) is a mixed strategy where usually one of the Sender types plays a mixed strategy and randomizes the signals she sends to the Receiver. This implies that this type is indifferent between the two signals—*i.e.* gets the same payoff from both signals. Consider the following beliefs:

$$\mu(H|e) = \begin{cases} \frac{8}{9}, & if\, e = e_H = e_{L,g} = 47.1 \\ 0, & otherwise \end{cases}$$

which leads to the following (sequentially rational) wage schedule:

$$w(e) = \begin{cases} 9444, & if\, e = e_H = e_{L,g} = 47.1 \\ 5000, & otherwise \end{cases}$$

In the solutions to part (d) above we found that the equilibrium [47.1, 9444] by finding where the low types were indifferent, and we know that the average productivity is 9444 when $\frac{1}{2}$ of the low types pool with the high types.

This partial pooling equilibrium is not consistent with either of the refinements discussed above because the high types will want to choose a level of education greater than 50 and get 10,000.

[12] Again, Gibbons (1992) provides a great overview of the partial pooling equilibrium.