Section Notes 3

Wonbin Kang

Feb 11, 2010

Agenda

1. Problem Set 2
2. Game Trees
3. Backward Induction
4. Subgame Perfect Nash Equilibrium

1 Problem Set 2

Problem 1:

	L	C	R
U	0,0	2,4	1,1
M	4,2	1,1	0,0
D	1,1	0,0	3,3

1. The question asked you to consider mixing two strategies. Many people turned the 3×3 matrix into a 2×2 matrix. This is *not* how you should approach solving these problems. Part (e) of the problem shows you the problem with changing the games: Notice that in the 3×3 matrix, if player 1 is restricted to playing either M or D, or some mix of the two, then player 2's strategy "C" is strictly dominated by "L." Since, "C" is strictly dominated, you cannot mix.

2. In part (c), many of you failed to check whether the mixed strategy was an equilibrium by comparing it to the payoff of the remaining pure strategy.

   (a) For player 1, her expected utility of playing a mix of "U" and "M" is equal to $U_1(U) = 0 \cdot \frac{1}{5} + 2 \cdot \frac{4}{5} = U_1(M) = 4 \cdot \frac{1}{5} + 1 \cdot \frac{4}{5} = \frac{8}{5}$. This is greater than the expected utility of playing "D" which is equal to $\frac{1}{5}$.

   (b) For player 2, his expected utility of playing a mix of "L" and "C" is equal to $U_2(L) = 0 \cdot \frac{1}{5} + 2 \cdot \frac{4}{5} = U_2(C) = 4 \cdot \frac{1}{5} + 1 \cdot \frac{4}{5} = \frac{8}{5}$. This is greater than the expected utility of playing "R" which is equal to $\frac{1}{5}$.

   (c) Now we can conclude based on (a) and (b) that the mixed strategy profile is in fact an equilibrium.

2 Game Trees

Consider the following normal form game:

	L	C
U	0,0	<u>2</u>,<u>4</u>
M	<u>4</u>,<u>2</u>	1,1

We can rewrite the game above in game tree form.[1]

1. Note that player 1 is at a *singleton information set*, while player 2's *information set* contains two *decision nodes*. The game ends at *terminal nodes*.

2. In the game above, how many proper subgames and how many subgames? There are zero proper subgames and one subgame.

What is wrong with the game in Appendix A-2?[2] Note the actions available at each decision node within the same information set must be the same.

3 Backward Induction

Change the rules of the game to a sequential game of perfect information: player 1 moves first and player 2 observes player 1's move. If we assume perfect information, each player knows exactly where he or she is on the game tree. In short, all players observe the actions of the players who came before him or her, and information sets for all players have one decision node, or are singletons. See Appendix A-3.

How many subgames exist in the above extensive form game? Note that in a game of perfect information, each singleton decision node is the start of a subgame. Therefore, there are a total of 3 subgames and 2 proper subgames.

Let's backward induct the equilibrium for this game:

1. At decision node 2.1, player 2 will play "C" ($\because$ $4 > 0$), and at decision node 2.2, player 2 will play "L" ($\because 2 > 1$).

2. The reduced form from the perspective of player 1 is shown in Appendix A-4. Therefore, the unique backward induction NE, σ^* =[M, CL]. Remember that a strategy for player 2 must identify player 2's action at *each of player 2's decision nodes*: 2.1 and 2.2. Therefore, the strategy set for player 2 is $S \equiv \{LL, LC, CL, CC\}$.

3. Note that the game above has a first-mover advantage. Matching pennies is a game where there is a second-mover advantage. What about the Cournot model?

[1]See Appendix A-1. Note that we could've flipped the order of player 1 and 2, but we retain the usual convention.

[2]Game tree omitted.

The next question to consider is what does all of the above buy us? Rewrite the extensive form game above in normal form and underline best-responses:

Case 1.

	LL	LC	CL	CC
U	0,0	0,0	2,$\underline{4}$	$\underline{2}$,$\underline{4}$
M	$\underline{4}$,$\underline{2}$	$\underline{1}$,1	$\underline{4}$,$\underline{2}$	1,1

Therefore, there are three NEs: [M, LL], [M, CL], and [U, CC], of which backward induction tells us the only *credible* NE is [M, CL]. Keep this idea of credibility in mind. Can you see why [M, LL] and [U, CC] are not credible? Later on, we're going to consider NE when one of the parties can credibly commit to one of these "non-credible" strategies.

Theorem 3.1. *Zermelo's Theorem: Every finite sequential game of perfect information has a* <u>*unique*</u> *backward induction solution if there are no ties in payoffs (in other words, $\forall i \in I$, player i cannot get the same payoff for two different actions). If there are ties in payoffs, a backward induction solution exists, but need not be unique.*

In win-lose-draw type zero sum games of perfect information, there is a unique backward induction *payoff*, but not a unique *equilibrium* (different strategies lead to the same payoffs).

- Tic-tac-toe: draw
- Connect four: first-mover wins (solved in 1988)
- Checkers: draw (solved in 2007)
- Chess: almost certainly draw or first-mover win (as Elon noted, this is a very difficult backward induction problem to solve).

4 Subgame Nash Equilibrium ("SPNE")

Definition 4.1. A strategy profile $\sigma^*$is a SPNE if it induces a NE in every subgame.

Example 4.2. Consider the following game based on the **Cold War**. Let player 1 be the US and player 2 be the USSR. Each player moves simultaneously and $S_1 = S_2 = \{E, D, N\}$, where "E" is to escalate the confrontation, "D" is to de-escalate, and "N" is to "nuke" the other player. The payoffs are as follows:

	E	D	N
E	$\underline{1}$,$\underline{1}$	$\underline{3}$,0	$\underline{-100}$,0
D	0,$\underline{3}$	2,2	$\underline{-100}$,0
N	0,$\underline{-100}$	0,$\underline{-100}$	-200,-200

Consider a single-period game. Note that the "E" is a weakly *dominant* strategy for both the US and the USSR, and [E,E] is the only NE.[3]

Now let's consider an extensive form game where the above game is played twice. Assume that the payoffs of this extensive game is equal to the sum of each *stage game*. What important assumption have we done away with in this game? *PERFECT INFORMATION* is no longer applicable. Therefore, the simplistic backward induction we used above is no longer feasible. We have to use the definition of SPNE and find NE in each and every subgame.

The game tree for this game is shown in Appendix A-5. Note that I've filled in some of the payoffs.

The strategy space for the US can be written as follows:

$$S_1 \equiv \{\vec{s}_1 = [s_{1.1}, s_{1.2}, ..., s_{1.10}] : s_{1.n} \in \{E, D, N\}, n = 1, ..., 10\},$$

and the strategy space for the USSR can be written as:

$$S_2 \equiv \{\vec{s}_2 = [s_{2.1}, s_{2.2}, ..., s_{2.10}] : s_{2.n} \in \{E, D, N\}, n = 1, ..., 10\}.$$

In short, a strategy for each player must specify what the player will do at each of its information sets. Therefore, there are 3^{10} possible pure strategies for each player. Then the number of strategy profiles is equal to 3^{20}. In this type of game, we'd be hard pressed to solve analytically for a NE. Further, there is the possibility that there are multiple NEs. So, what do we do? We consider certain strategies which intuitively make sense as equilibria and check to see whether they are NE (or even SPNE).

Let's consider the following questions for this game:

1. How many subgames are there? Total of ten (10).

2. Are there any NE involving playing anything other than "E" in any of the info sets? Consider the following strategy for both players: Play "D" in the first period; then "E" if the first period outcome is [D,D], or "N" otherwise.[4]

   (a) What is the payoff to both players in this strategy profile? Players 1 and 2 each gets 2 in the first period and 1 in the second period for a total of 3.

   (b) Deviation by a player in the first period:

      i. Consider one possible unilateral deviation by player 1 (since this is a symmetric game, we can do this for one player and generalize

[3] Note that the game is very similar to a prisoner's dilemma.

[4] In the repeated prisoner's dilemma context, players cooperate in the first period and punish in the next period if opponent does not cooperate. Also note that in my representation of strategies, the candidate strategy profile is: [(DNNNNENNNN), (DNNNNENNNN)].

to the second player) where player 1 plays "E" in the first period,[5] and plays "E" in the second period. Player 1's total payoff is $3 + (-100) = -97 < 3$. Remember that when you're looking for NE, the strategy of the other players do not change. Since the first period outcome was [E,D], player 2 will following the strategy profile and play "N" in the second period.

ii. Consider another possible unilateral deviation by player 1. Player 1 again plays "E" in the first period, and then following the prescription of the strategy profile above, player 1 plays "N" in the second round. Player 1's total payoff is $3 + (-200) = -197 < 3$.

iii. In short, if player 1 deviates in the first round, the strategy profile requires player 2 to play "N" in the second period where the highest payoff for player 1 is -100.

iv. Therefore, a player will not deviate in the first period.

(c) Deviation by a player in the second period:

i. [E,E] is the NE of the second period game. Therefore, "E" is a best response in the second period, and so no player will deviate.

ii. To check point i. above:

A. what happens if player 1 unilaterally deviates in the second period from "E" to "D": Total payoff is $= 2 + 0 < 3$.

B. what happens if player 1 unilaterally deviates in the second period from "E" to "N": Total payoff is $= 2 + 0 < 3$.

iii. Therefore, a player will not deviate in the second period.

(d) Deviation by a player in the first and the second period: This is covered by deviation in the first period.

(e) Therefore, (b), (c), and (d) above prove that this the strategy profile above is a NE.

(f) One final comment,[6] what about the following strategy for each player: Play "D" in the first period; then "E" if the first period outcome is [D,D], or "E" otherwise.[7]

i. What is the payoff from this strategy? It is $2 + 1 = 3$, which is equal to the payoff of the original strategy profile [(DNNNNENNNN), (DNNNNENNNN)].

ii. To use some of the terminology Chris mentioned in class, whether a strategy profile is NE or not doesn't depend on what the strategy tells the players to do "off the equilibrium path."

[5] Note that the only possible deviation which increases player 1's payoff is to move to "E". Deviating to "N" would not increase player 1's payoff.

[6] Thanks to Anil for making me think through this.

[7] Using my notation, the strategy profile under consideration is: [(DEEEEEEEEE), (DEEEEEEEEE)].

iii. If you don't understand point ii., then consider the payoff matrix in Case 1 above. Recall that the NE [U, CC] was the result of a non-credible threat where player 2 threatens to play "C" if player 1 plays "M." So player 1 plays "U" and player 2 then plays "C" to get her maximum payoff for the game. The resulting payoff is (2,4). Notice what happens to the payoffs if we change player 2's strategy from "CC" to "CL":[8] *NOTHING*! The payoff is still (2,4). This is exactly the same thing as changing the "threat" from "N" to "E" in the Cold War game above.

3. Is the NE above a SPNE? No. Recall that in each subgame, the players must be playing the NE strategy: "E". Therefore, there is only one SPNE: where the players play "E" in all subgames.

The bottom line for identifying SPNE is that backward induction by subgame. This can become difficult when subgames have multiple NE because SPNE requires the players to play a NE in each subgame.

Consider what would happen to the game above if we change the payoffs for the [nuke, nuke] outcome to [-100,-100]. Then the strategy considered above is SPNE.

Now let's consider a case where we have continous action sets for each player.

Example 4.3. Empty Threat Game with continous strategies:[9] Firm 1 is a potential entrant and firm 2 is the incumbent. Entry costs 400, and post-entry competition is Cournot (which means that the players simultaneously move) with inverse demand function given by $P(Q) = 120 - Q$ and marginal costs equal to zero (0).

The game tree can be found in Appendix A-6. How many subgames?

To find the PSNE, recall that there were two types of NE in the entry game: one where the entrant firm stayed out and one where the entrant entered. Just because we're in the world of continuous variables, we don't abandon the backward induction approach. Therefore, we're going to have to start with the Cournot model.

The solution to the Cournot competition is:

$$q_1^* = q_2^* = 40,$$

and we know that these quantities (or actions) define the NE of a Cournot game. At these quantities firm 1 and firm 2's profits are:

$$\begin{aligned} \Pi_1^* &= 1600 - 400 = 1200 \\ \Pi_2^* &= 1600 \end{aligned}$$

[8] In short, change the off equilibrium "action" from "C" to "L". "Off equilibrium" means that the players don't reach that node in equilibrium.

[9] Therefore, this game is not finite.

What happens, given the parameters above, if there is only one firm in the market? Since the firm is a monopolist, that firm should produce monopoly quantity and get monopoly profits:

$$q_M^* = 60, \;\; \Pi_M^* = 3600.$$

Notice the payoffs if player 1 plays "Out" at 1.1: It is (0, 3600). In short, if player 1 stays out of the market, player 2 is monopolist and so gets monopolist profits.

Player 1 plays "In"

Consider a NE candidate strategy profile where player 1 enters the market: $[(In, \, q_1^* = 40), \, q_2^* = 40]$. Is this strategy profile in fact NE? Given player 2's action, player 1 cannot do better by deviating from producing 40 units (which is in fact the definition of "best-responding"). Can player 1 do better by playing "Out"? No. At the strategy profile above, player 1 gets $1600 - 400 = 1200$, vs 0 if player 1 plays "Out." Therefore, this is a NE.

However, is this NE a SPNE? There are two subgames, the whole game and the Cournot subgame. The NE above requires players to play NE strategies at each of the subgames and so it is a SPNE.

Player 1 plays "Out"

Now, let's first look at a NE candidate strategy profile where player 1 stays out of the market.

The objective functions for player 1 if she decides to enter was:

$$U_1(q_1, q_2) \;\; = \;\; (120 - q_1 - q_2) \cdot q_1 - 400 \Rightarrow q_1^* = \frac{120 - q_2}{2}$$

When would player 1 be indifferent from entering and not entering? We want to know this because this will limit the set of strategies that produces a NE.

$$\begin{aligned} U_1(q_1^*, q_2) &= \left(\frac{120 - q_2}{2}\right) \cdot \left(120 - \frac{120 - q_2}{2} - q_2\right) = 400 \\ &\Rightarrow \frac{q_2^2}{2} - 120q_2 + 6400 = 0 \\ &\therefore \;\; q_2 = \frac{120 \pm \sqrt{1600}}{1} = 80, \, 160 \end{aligned}$$

Therefore, if player 2 produces more than 80 units (160 units doesn't make sense for this model), then it would be better for player 1 to stay out. The related

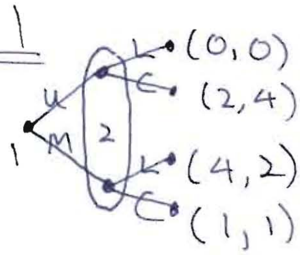
NE is where player 1 plays (Out,$q_1^* \in [0, \infty]$) and player 2 plays $q_2^* \in [80, \infty]$.[10]

However, is it a SPNE? Clearly it isn't because the players are not playing NE strategies in the Cournot subgame. Therefore, this NE, is based on a non-credible threat that player 2 is going to produce more than 80 units. But that would be player 2 shooting itself in the foot.

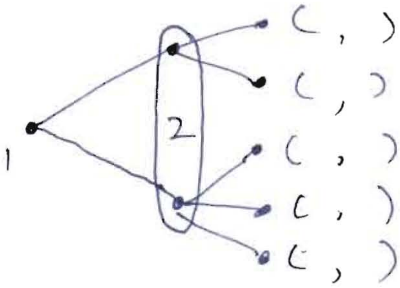
[10]Understanding why all of these infinite strategies is a NE is very important. The reason is that NE does NOT care about the actions off the equilibrium path. *In short, in the Cold War game above, we could change "otherwise play N" to "otherwise play E" and this wouldn't affect the payoffs. The same principle applies here. The payoffs are equal to [0, 3600] if player 1 plays "Out" regardless of what player 1 and player 2 do in the Cournot subgame, subject to the important condition that player 2 produces more than 80 units.*

Appendix Section Note 3

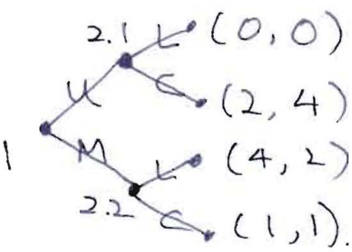
A-1



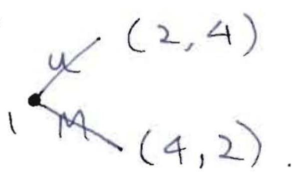
A-2



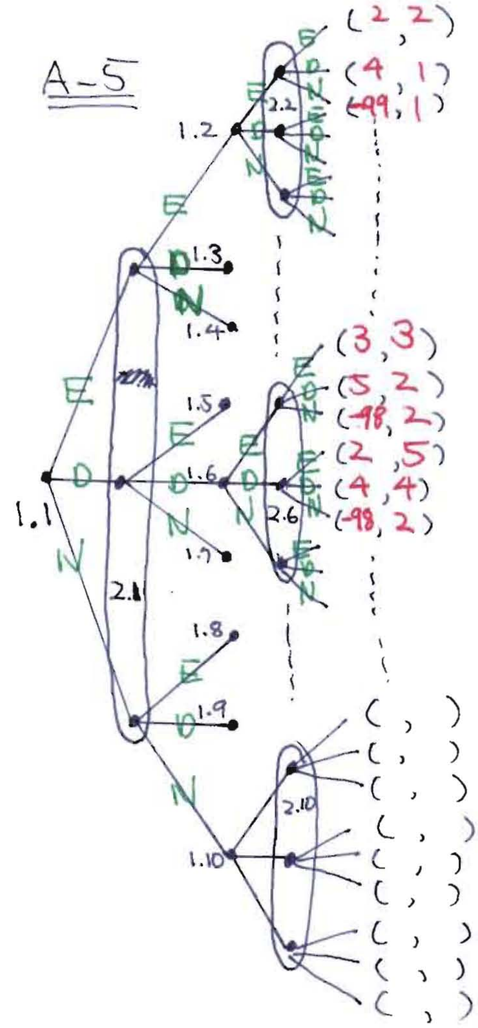
A-3



A-4



A-5



A-6