Section Notes 7

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Agenda

1. Stochastic Dominance
2. Profit Maximization Problem (PMP)
3. Cost Minimization Problem (CMP)
4. Example of CMP (2003 Final Exam Problem 3)

1 Stochastic Dominance[1]

1.1 First-Order Stochastic Dominance (FOSD)

By way of background, for those of you who have taken statistics, you know that there are multiple ways to represent the distribution of a random variable. For example, you can use the method of moments (the first, second, third, fourth,...moment), maximum likelihood estimators, etc. The most frequently used is the expected value and the variance of a distribution. Then for any two distributions of money outcomes, $F(\cdot)$ and $G(\cdot)$,[2]we can define the preferences of an economic agent over these two money outcome distributions based on their respective expected values and variances. The former is first-order stochastic dominance and the latter is second-order stochastic dominance.

For any two distributions over money outcomes, $F(\cdot)$ and $G(\cdot)$, if $F(\cdot)$ first-order stochastically dominates $G(\cdot)$, then all agents with weakly increasing utility functions will weakly prefer $F(\cdot)$ to $G(\cdot)$. Note that first-order stochastic dominance is a very strict condition and typically we will not have either distribution first-order stochastically dominating another.

More formally, we can define FOSD as follows:

[1] The handout on stochastic dominance on the course website is a great source to review the key concepts.

[2] The random variable X is the money outcome and $F(\cdot)$ and $G(\cdot)$ are the cdfs which represent the lotteries over which we defined an economic agent's preferences in the previous section. We can think of these cdfs as lotteries because the lotteries we considered in the previous section was merely a vector of probabilities over a discrete number of events.

Definition 1. The distribution over money outcomes $F(\cdot)$ first-order stochastically dominates $G(\cdot)$ if, for $\forall u : \mathfrak{R} \rightarrow \mathfrak{R}$, s.t. $u(\cdot)$ is a non-decreasing function, we have that:

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

, or

$$\int u(x)f(x)dx \geq \int u(x)g(x)dx$$

, where $f(x)$ and $g(x)$ are the respective pdfs of $F(x)$ and $G(x)$. This implies that the distribution of monetary payoffs $F(\cdot)$ first-order stochastically dominates the distribution $G(\cdot)$ if and only if $F(x) \leq G(x)$, $\forall x$.

Try drawing a picture. Draw two cdfs, s.t. $F(x) \leq G(x)$, $\forall x$, and reverse the axis to see that for a given probability you want to have a greater value of the random variable X.

1.2 Second-Order Stochastic Dominance (SOSD)

SOSD uses the variance of a distribution to determine whether an economic agent prefers one money lottery, represented again by a cdf, over another.

If $F(\cdot)$ second-order stochastically dominates $G(\cdot)$, then the 2 distributions over money (or money lotteries) have the same expected value, but all risk-averse economic agents prefer $F(\cdot)$ to $G(\cdot)$, which implies that $F(\cdot)$ is less risky than $G(\cdot)$. In short, because the two distributions over money outcomes have the same expected value, $G(\cdot)$ can be represented as a mean-preserving spread of $F(\cdot)$.

More formally, we can define SOSD as follows:

Definition 2. For any two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ second-order stochastically dominates (or is less risky than) $G(\cdot)$ if for every non-decreasing *concave* function $u : \mathfrak{R} \rightarrow \mathfrak{R}$, we have that:

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

. This implies that the distribution of monetary payoffs $F(\cdot)$ second-order stochastically dominates the distribution $G(\cdot)$ if and only if:

$$\int_{-\infty}^{x} G(t)dt \geq \int_{-\infty}^{x} F(t)dt, \forall x.$$

Again, drawing a picture may provide you with some intuition.

1.3 Combining FOSD and SOSD

If L_1 FOSDs L_2, and L_2 SOSDs L_3, then all risk-averse economic agents will prefer L_1 to L_3 due to transitivity. However, we don't say that L_1 SOSDs L_3.

2 PMP

If you've been reading MWG, you'll notice that it uses two different notations: the general and the one output case. Therefore, the two different types of product sets can be defined as follows: $Y \equiv \left\{\vec{y} \in \Re^L : F(\vec{y})\right\}$, where $\vec{y}$ is a vector of net outputs and inputs and $F(\cdot)$ is the transformation function, and $Y \equiv \left\{(-\vec{z}, q) \in \Re_-^{L-1} \times \Re_+ : \vec{z} \geq 0,\, f(\vec{z}) \geq q\right\}$, where $\vec{z}$ is a vector of inputs, you have one output q, and $f(\cdot)$ is the production function. For the most part in this class, we'll be looking at the latter version, but you should familiarize yourself with the former.

Before setting up the PMP, let's consider the parameters: output is equal to q and has prices p, inputs are represented by $\vec{z}$ with prices $\vec{\omega}$,[3] technology of producer is represented by $f(\cdot)$ s.t. $f(\vec{z}) \geq q$, where we normalize the production function so that $f(\vec{0}) = 0$, which MWG terms as the possibility of inaction is within the production set, and $f(\cdot)$ is a monotonically increasing strictly concave function, which means that the firm has a technology that exhibits decreasing returns to scale.

2.1 Problem

The PMP can be written as follows:

$$\max_{q,\vec{z}} \Pi(q, \vec{z}) = pq - \vec{\omega} \cdot \vec{z}\, s.t.\, f(\vec{z}) \geq q$$

, which can be rewritten as:

$$\max_{\vec{z} \geq \geq 0} \Pi(q, \vec{z}) = pf(\vec{z}) - \vec{\omega} \cdot \vec{z}$$

, since at the optimum the firm will choose $\vec{z}$ such that $f(\vec{z}) = q$ given $\vec{\omega} \gg 0$ and a continuous and monotonic production function.

Because we have no information on the exact functional form of $f(\cdot)$, we still have to worry about the non-negativity constraints with regards to the choice variables $\vec{z}$. Therefore, we use the K-T FOCs and we get:

$$\frac{\partial \Pi}{\partial z_i} = p\frac{\partial f(\vec{z}^*)}{\partial z_i} - \omega_i \leq 0\, \forall i$$

, and the following complementary slackness constraints:

$$z_i \left(p\frac{\partial f(\vec{z}^*)}{\partial z_i} - \omega_i\right) = 0.$$

If we assume that $z_i^* > 0\, \forall i$, then:

$$\frac{\partial f(\vec{z}^*)}{\partial z_i} = \frac{\omega_i}{p} \Rightarrow p\frac{\partial f(\vec{z}^*)}{\partial z_i} = \omega_i \tag{1}$$

[3] We'll usually be dealing with two inputs and one output for this class.

, which can be interpreted as the marginal return of increasing production by increasing the input i is equal to the marginal cost of that additional input i at the optimum. Draw a picture to see what this looks like.

2.2 Solution

If we solve the K-T first order conditions and the complementary slackness conditions, you get the following solutions:

$$\begin{aligned}
\vec{z}^* &= \vec{z}(\vec{\omega}, p) \Rightarrow z_i^* = z_i(\vec{\omega}, p) \Leftrightarrow factor\, demand \\
q^* &= q(\vec{\omega}, p) = f(\vec{z}(\vec{\omega}, p)) \Leftrightarrow supply\, function \\
\Pi^* &= \Pi(\vec{\omega}, p) = pf(\vec{z}(\vec{\omega}, p)) - \vec{\omega} \cdot \vec{z}(\vec{\omega}, p) \Leftrightarrow profit\, function
\end{aligned}$$

From the equations above, we can derive the following Lemma:

Lemma 3. *Hotelling's Lemma: If there is a unique solution to the PMP, then the following holds:*

$$\begin{aligned}
\nabla_{\vec{\omega}} \Pi(\vec{\omega}, p) &= -\vec{z}(\vec{\omega}, p) \Leftrightarrow \frac{\partial \Pi(\vec{\omega}, p)}{\partial \omega_i} = -z_i(\vec{\omega}, p) \\
\frac{\partial \Pi(\vec{\omega}, p)}{\partial p} &= q(\vec{\omega}, p)
\end{aligned}$$

2.3 Marginal Rate of Technical Substitution

In consumer theory, we had the marginal rate of substitution between two goods. We have something analogous in production theory as well, the marginal rate of technical substitution (note that if we're dealing with the more general case with $\vec{y} \in Y$ as a vector of inputs and outputs, we call it the marginal rate of transformation). We define the marginal rate of technical substitution at the point $\vec{z}^*$ on the production boundary as follows:

$$\frac{\frac{\partial f(\vec{z}^*)}{\partial z_i}}{\frac{\partial f(\vec{z}^*)}{\partial z_j}} = MRTS_{ij}$$

, which can be interpreted as how much should the firm decrease the amount of resource i in the production of q units of the output if the firm increases the input of resource j. Note that if we have an interior solution, we can use equation 1 above to show that:

$$MRTS_{ij} = \frac{\frac{\partial f(\vec{z}^*)}{\partial z_i}}{\frac{\partial f(\vec{z}^*)}{\partial z_j}} = \frac{\frac{\omega_i}{p}}{\frac{\omega_j}{p}} = \frac{\omega_i}{\omega_j}$$

. Try to interpret this. What happens if $MRTS_{ij} > \frac{\omega_i}{\omega_j}$?

3 CMP

As in consumer theory, where we had the EMP to the UMP, we have the cost minimization problem (CMP) for the producer's profit maximization problem. The CMP is especially useful for the following reasons:

1. We can derive the optimal values of a firm even if the production function $f(\cdot)$ is not concave (has decreasing returns to scale technology).

2. We can relax the assumption that all firms act as price takers (we'll see a lot of this when we look at market failures).

3.1 Problem

We can write it up as follows:

$$\min_{\vec{z} \geq \geq 0} \vec{\omega} \cdot \vec{z}\, s.t.\, f(\vec{z}) \geq \bar{q}$$

. We know that at the optimum the constraint will bind, however, it's too difficult to plug the constraint into the objective function. Therefore, we turn to the Lagrangian.

$$\mathfrak{L}(\vec{z}, \lambda) \equiv \vec{\omega} \cdot \vec{z} - \lambda[f(\vec{z}) - \bar{q}].$$

[4] Since we still have the non-negativity constraints on our choice variables and we don't know the exact functional form of the constraint, we use the K-T FOCs and the complementary slackness condition:

$$\begin{aligned} \frac{\partial \mathfrak{L}(\vec{z}^*, \lambda^*)}{\partial z_i} &= \omega_i - \lambda^* \cdot \frac{\partial f(\vec{z}^*)}{\partial z_i} \geq 0 \\ z_i^* \left(\omega_i - \lambda^* \frac{\partial f(\vec{z}^*)}{\partial z_i} \right) &= 0 \end{aligned}$$

[5] At an interior, we have the following equality:

$$\omega_i = \lambda^* \frac{\partial f(\vec{z}^*)}{\partial z_i} \Rightarrow \lambda^* = \frac{\omega_i}{\frac{\partial f(\vec{z}^*)}{\partial z_i}} \Rightarrow \frac{\frac{\partial f(\vec{z}^*)}{\partial z_i}}{\frac{\partial f(\vec{z}^*)}{\partial z_j}} = \frac{\omega_i}{\omega_j}$$

. Let's draw a picture to see what we're dealing with.

[4] Be very careful of the sign of the multiplier and the form of the constraint.

[5] Notice that the direction of the inequality changes since we have a minization problem.

3.2 Solution

If we solve the K-T first order conditions and the complementary slackness conditions, you get the following solutions:

$$\begin{aligned}\vec{z}^* &= \vec{z}(\vec{\omega},\bar{q}) \Rightarrow z_i^* = z_i(\vec{\omega},\bar{q}) \Leftrightarrow conditional\,factor\,demand\\ c^* &= c(\vec{\omega},\bar{q}) = \vec{\omega}\cdot\vec{z}(\vec{\omega},\bar{q}) \Leftrightarrow cost\,function\end{aligned}$$

From the equations above, we can derive the following Lemma:

Lemma 4. *Shepard's Lemma: If there is a unique solution to the CMP, then the following holds:*

$$\nabla_{\vec{\omega}} c(\vec{\omega},\bar{q}) = \vec{z}(\vec{\omega},\bar{q}) \Leftrightarrow \frac{\partial c(\vec{\omega},\bar{q})}{\partial \omega_i} = z_i(\vec{\omega},\bar{q})$$

4 CMP Example (2003 Final Problem 3)

Firm produces q using z_1 and z_2 according to the following production function: $f(\vec{z}) = 4\left(z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}}\right)^{\frac{1}{2}}$. The price vector is equal to $[p,\omega_1,\omega_2] \gg 0$.

Problem 5. State the firm's CMP.

$$\min_{\vec{z}\geq\geq 0} \omega_1 z_1 + \omega_2 z_2 \; s.t.\; 4\left(z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}}\right)^{\frac{1}{2}} \geq \bar{q}$$

Problem 6. Derive the FOCs while assuming that the solution to the above minimization problem are interior.

If the problem didn't ask you to assume interior solutions, we'd have to use the K-T FOCs and the complementary slackness conditions, or do we?

Notice the following (which applies when we take the first derivative with regards to both $i = 1$ and $i = 2$:

$$\frac{\partial f(\vec{z})}{\partial z_1} = 2\left(z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}}\right)^{-\frac{1}{2}} \times \frac{z_1^{-\frac{1}{2}}}{2} = \frac{1}{\sqrt{\left(z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}}\right)\cdot z_1}}$$

, which means that

$$\lim_{z_1\to 0} \frac{\partial f(\vec{z})}{\partial z_1} = \infty$$

, and so you would never utilize 0 of either resource 1 or resource 2.

Now going back to the problem, we should set up a Lagrangian and take FOCs.

$$\mathfrak{L}(\vec{z},\lambda) = \omega_1 z_1 + \omega_2 z_2 - \lambda\left[\bar{q} - 4\left(z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]$$

, and the FOCs are:

$$\begin{aligned}
\frac{\partial \mathfrak{L}(\vec{z}^*,\lambda^*)}{\partial z_1} &= \omega_1 - \frac{\lambda^*}{\sqrt{\left(z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}}\right)\cdot z_1}} = 0 \\
\frac{\partial \mathfrak{L}(\vec{z}^*,\lambda^*)}{\partial z_2} &= \omega_2 - \frac{\lambda^*}{\sqrt{\left(z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}}\right)\cdot z_2}} = 0 \\
\frac{\partial \mathfrak{L}(\vec{z}^*,\lambda^*)}{\partial \lambda} &= 4\left(z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}}\right)^{\frac{1}{2}} - \bar{q} = 0
\end{aligned}$$

[6]

Problem 7. Find the conditional factor demand and cost functions.

Solving the three equations above, you can solve for the conditional factor demand (notice that $\bar{q}$ is a parameter), and if you plug your solutions back into the objective function you get the cost function. The results is:

$$z_1^* = z_1(\vec{\omega},\bar{q}) = \left(\frac{\bar{q}^4}{256}\right)\left(\frac{\omega_2}{\omega_1+\omega_2}\right)^2$$

$$z_2^* = z_2(\vec{\omega},\bar{q}) = \left(\frac{\bar{q}^4}{256}\right)\left(\frac{\omega_1}{\omega_1+\omega_2}\right)^2$$

$$c(\vec{\omega},\bar{q}) = \omega_1 z_1^* + \omega_2 z_2^* = \frac{\bar{q}^4\omega_1\omega_2}{256(\omega_1+\omega_2)}$$

Problem 8. Show that the cost function is convex in output and interpret.

We need to show either that $\frac{\partial^2 c(\vec{\omega},q)}{\partial q^2} > 0$ or that for $\alpha \in [0,1]$ $c(\vec{\omega},\alpha q_0 + (1-\alpha)q_1) < \alpha\cdot c(\vec{\omega},q_0) + (1-\alpha)\cdot c(\vec{\omega},q_1)$. Let's use try the former:

$$\frac{\partial c(\vec{\omega},q)}{\partial q} = \frac{q^3\omega_1\omega_2}{64(\omega_1+\omega_2)}$$

$$\frac{\partial^2 c(\vec{\omega},q)}{\partial q^2} = \frac{3q^2\omega_1\omega_2}{64(\omega_1+\omega_2)} > 0.$$

Convex cost functions imply that the marginal cost of producing more units of output is increasing, which means that there are decreasing returns to scale and that the production function is concave.

[6] Note that all of the z_1 and z_2 should be starred.

Problem 9. Show that the cost function is concave in ω_1.

You could derive the second derivative of the cost function with regards to ω_1: $\frac{\partial^2 c(\vec{\omega},q)}{\partial \omega_1^2}$, and check the sign. When doing so it is helpful to note that $c \propto \frac{\omega_1 \omega_2}{\omega_1+\omega_2}$ and derive the second derivative with regards to ω_1 of the RHS. You will find that:

$$\frac{\partial^2 c(\vec{\omega},q)}{\partial \omega_1^2} \propto \frac{-2w_2^2}{(\omega_1+\omega_2)^3} < 0.$$

The intuition is that if the price of inputs goes up, the firm is able to reoptimize the bundle used to produce the output and so costs will go up less than linearly. Recall that the same applied for the expenditure function (which was also concave).

However, a more general method of solving this problem is as follows:

Let $\vec{\omega} = \alpha\vec{\omega}' + (1-\alpha)\vec{\omega}''$, where $\alpha \in [0,1]$. Then:

$$\begin{aligned}
c(\vec{\omega},q) &= c(\alpha\vec{\omega}' + (1-\alpha)\vec{\omega}'', q) \\
&= (\alpha\vec{\omega}' + (1-\alpha)\vec{\omega}'') \cdot \vec{z}(\vec{\omega},q) \\
&= \alpha\vec{\omega}' \cdot \vec{z}(\vec{\omega},q) + (1-\alpha)\vec{\omega}'' \cdot \vec{z}(\vec{\omega},q) \\
&> \alpha\vec{\omega}' \cdot \vec{z}(\vec{\omega}',q) + (1-\alpha)\vec{\omega}'' \cdot \vec{z}(\vec{\omega}'',q) \\
&= \alpha c(\vec{\omega}',q) + (1-\alpha)c(\vec{\omega}'',q)
\end{aligned}$$

, which means that the cost function is concave in the prices of the inputs.