Section Notes 6

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Agenda

1. Partial Equilibrium
2. Fundamental Welfare Theorem Failure
3. Practice Problem 9.4
4. Practice Problem 9.5

1 Partial Equilibrium[1]

1.1 The Model (and Notation)

1.1.1 The Consumers

There are a total of I consumers each with the following utility function:

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

where $\phi'(\cdot) > 0$ and $\phi''(\cdot) < 0$. Each individual $i \in I$[2] has an endowment of the numeraire good m equal to w_i and no endowment of good x.[3] The price vector for the two goods $\vec{p} = [p_m, p_x] = [1, p]$.

In an equilibrium, all consumers will maximize their utilities subject to their respective budget constraints.

[1] We call it partial equilibrium because we focus on one good and a composite commodity (all other goods).

[2] Abusing notation, as always.

[3] It's fine to think of the numeraire good as money. However, note that since we're in the partial equilibrium context, we might want to think of the numeraire good as the money value of all other goods in the economy.

1.1.2 The Producers (Firms)

There are a total of J producers each of whom have a cost function, $c_j(q_j)$,[4] which is measured in terms of units of the numeraire good. Producers (or firms) use the numeraire good, which is supplied by the consumers (recall that consumers are endowed with the numeraire good) in exchange for good x, and transform them into the output good x which is denoted in the context of the firm as q.

Note also that the firms themselves are owned by the consumers. The ownership share of consumer i in firm j is equal to θ_{ij}, and so $\sum_{i=1}^{I} \theta_{ij} = 1$. It is also important to remember the budget constraint of the consumer's utility maximization problem must also take into consideration this ownership share of firms. Therefore, the total wealth available to consumer i is equal to:

$$w_i + \sum_{j=1}^{J} \theta_{ij} \Pi_j(p^*),$$

where $\Pi_j(p^*)$ is equal to the profit of firm j at price p^*.

1.1.3 Market Clears at equilibrium price p^*

It must be the case that at the equilibrium price, p^*, the total amount of good x demanded and supplied must be equal:

$$\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} q_j$$

1.2 Partial Equilibrium Example

What is the market equilibrium in the following case: 100 consumers, all with $u_i(m_i, x_i) = m_i + \ln x_i$ and endowment $w_i = \frac{1}{100} w_m$; $\vec{p} = [p_m, p_x] = [1, p]$; and one firm, with cost function $c(q) = 5q^2$ and $\forall i$ the ownership share $\theta_{ij=1} = \frac{1}{100}$.

1.2.1 Producers

Each producer solves the PMP (in our problem, we only have one firm) to derive the net supply function of good x. In short, we are looking for the optimal value of q^* in terms of the parameter p. Therefore,

$$\max_q \Pi(p) = pq - c(q) = pq - 5q^2.$$

The FOCs result in:

$$\frac{\partial \Pi(q^*)}{\partial q} = p - 10q^* = 0$$

[4] Note that MWG may use the more general notation y to be consistent with the production plan vector $\vec{y}$.

[5] and so the supply function can be written as:

$$q^*(p) = \frac{p}{10}, \tag{1}$$

and the optimal profit function is:

$$\Pi^*(p) = \frac{p^2}{10} - \frac{5p^2}{100} = \frac{p^2}{20}$$

1.2.2 Consumers

Each consumer i, of 100 consumers in the market, solves his or her own utility maximization problem subject to a wealth constraint:

$$\max_{m_i, x_i} u_i(m_i, x_i) = m_i + \ln x_i \; s.t. \; m_i + px_i \leq \frac{w_m}{100} + \frac{\Pi^*(p)}{100}$$

where we know that at the optimum, the constraint will bind. This is important since we don't have to resort to the Lagrangian method. We can rewrite the problem as an unconstrained maximization:[6]

$$\max_{x_i} u_i(x_i) = -px_i + \frac{w_m}{100} + \frac{\Pi^*(p)}{100} + \ln x_i.$$

The FOCs result in:

$$\frac{\partial u(x_i^*)}{\partial x_i} = -p + \frac{1}{x_i^*} = 0 \Rightarrow x_i^*(p) = \frac{1}{p}. \tag{2}$$

[7]

1.2.3 Markets Clear at the Equilibrium Price p^*

Since supply must equal demand at the equilibrium price, we have from equations 1 and 2 that the following equation must hold at the equilibrium p^*:

$$q^*(p^*) = \sum_{i=1}^{I} x_i^*(p^*) \Rightarrow \frac{p^*}{10} = \frac{100}{p^*} \Rightarrow p^* = 10\sqrt{10} \approx 31.6 \tag{3}$$

which means that the equilibrium level of good x that is consumed is equal to:

$$q^*(31.6) = \sum_{i=1}^{I} x_i^*(31.6) = \sqrt{10}$$

[5] $\frac{\partial^2 \Pi(q^*)}{\partial q^2} = -10 < 0$, so we're at a maximum.

[6] Do we have to worry about corner solutions? Recall that m_i is allowed to be negative (meaning that the consumer can borrow) and that the Inada condition for the non-numeraire good applies.

[7] Note how the ownership share does not enter into the FOC. Remember that we're using a quasilinear utility function, so there are no wealth effects.

Practice Problem 9.4 Pareto Optimality and Various Utility Functions

Part (a)

FOCs for a Pareto optimal interior equilibrium with Cobb-Douglas preferences is that the indifference curves of consumer 1 and 2 are tangent to each other. Therefore, the allocation $(\vec{x}^*, \vec{y}^*)$ is Pareto optimal if:

$$-\frac{\frac{\partial u(x_1^*,y_1^*)}{\partial x_1}}{\frac{\partial u(x_1^*,y_1^*)}{\partial y_1}} = -\frac{\frac{\partial v(x_2^*,y_2^*)}{\partial x_2}}{\frac{\partial v(x_2^*,y_2^*)}{\partial y_2}} \Rightarrow \frac{a x_1^{*a-1} y_1^{*1-a}}{(1-a) x_1^{*a} y_1^{*-a}} = \frac{a x_2^{*a-1} y_2^{*1-a}}{(1-a) x_2^{*a} y_2^{*-a}} \Rightarrow \frac{x_1^*}{y_1^*} = \frac{x_2^*}{y_2^*}. \tag{4}$$

If we plug in the endowment constraints, equation 4 simplifies to:

$$\frac{x_1^*}{y_1^*} = \frac{x_2^*}{y_2^*} = \frac{e_x}{e_y}. \tag{5}$$

where all $x_i^*, y_i^* > 0$, $\forall i \in \{1, 2\}$. Therefore, any Pareto Optimal division of the endowment will take the form:

$$(x_1^*, y_1^*) = \alpha \left(e_x, e_y\right); \left(x_2^*, y_2^*\right) = (1-\alpha)\left(e_x, e_y\right), \ \alpha \in (0,1).$$

Part (b)

FOCs for Pareto optimal interior equilibrium with quasi-linear preferences is that the indifference curves of consumer 1 and 2 are tangent to each other. Therefore, the allocation $(\vec{x}^*, \vec{y}^*)$ is Pareto Optimal if:

$$-\frac{\frac{\partial u(x_1^*,y_1^*)}{\partial x_1}}{\frac{\partial u(x_1^*,y_1^*)}{\partial y_1}} = -\frac{\frac{\partial v(x_2^*,y_2^*)}{\partial x_2}}{\frac{\partial v(x_2^*,y_2^*)}{\partial y_2}} \Rightarrow \frac{1}{f'(y_1^*)} = \frac{1}{f'(y_2^*)} \Rightarrow f'(y_1^*) = f'(y_2^*), \tag{6}$$

which means that $y_1^* = y_2^*$. If we plug in the endowment constraints, equation 6 simplifies to:

$$y_1^* = y_2^* = \frac{e_y}{2} \tag{7}$$

where all $x_i^*, y_i^* > 0$, $\forall i \in \{1, 2\}$. Therefore, any Pareto Optimal division of the endowment will take the form:

$$(x_1^*, y_1^*) = \left(x_1^*, \frac{e_y}{2}\right); (x_2^*, y_2^*) = \left(e_x - x_1^*, \frac{e_y}{2}\right). \tag{8}$$

Part (c)

Let's consider the case where $x_1^* = 0$, which will generalize to the remaining cases. We know that the marginal rate of substitution for each consumer can be written as follows:

$$MRS_i = \frac{a}{1-a} \cdot \left(\frac{y_i}{x_i}\right).$$

If $x_1^* = 0$ and $y_1^* > 0$ ($\Leftrightarrow x_2^* = e_x$, $y_2^* \geq 0$), then $MRS_1 \to \infty$ and $MRS_2 < \infty$. This means that $MRS_1 > MRS_2$, and that there is a Pareto improving trade to be made such that consumer 1 demands good x and supplies good y and consumer 2 demands good y and supplies good x.

If $x_1^* = 0$ and $y_1^* = 0$ ($\Leftrightarrow x_2^* = e_x$, $y_2^* = e_y$), we know that this is a Pareto optimal equilibrium because you can't make anyone better off without making someone else better off.

Part (d)

Consider the case where $x_1^* = 0$ and $y_1^* > 0$ ($\Leftrightarrow x_2^* = e_x$, $y_2^* \geq 0$). Since consumer 1 doesn't have any good 1, the only Pareto improving trade would have consumer 1 demanding good x and supplying good y, while consumer 2 must demand good y and supply good x. The FOC for this trade to occur is:

$$MRS_1 > MRS_2$$

which means that for there *not* to be a Pareto improving trade, it must be the case that:

$$MRS_1 \leq MRS_2 \Leftrightarrow \frac{1}{f'(y_1^*)} \leq \frac{1}{f'(y_2^*)} \Rightarrow y_1^* \leq y_2^* \Rightarrow y_1^* \leq \frac{e_y}{2}. \tag{9}$$

Therefore, any allocation of the following form is a Pareto optimal equilibrium with $x_1^* = 0$ and $y_1^* > 0$:

$$\left[(0, y_1^*), (e_x, e_y - y_1^*)\right], \, y_1^* \leq \frac{e_y}{2}. \tag{10}$$

The same applies to cases where $x_2^* = 0$ and $y_2^* > 0$ ($\Leftrightarrow x_1^* = e_x$, $y_1^* \geq 0$).

$$\left[(e_x, e_y - y_2^*), (0, y_2^*)\right], \, y_2^* \leq \frac{e_y}{2} \tag{11}$$

Now consider the case where $x_1^* > 0$ and $y_1^* = 0$ ($\Leftrightarrow x_2^* \geq 0$, $y_2^* = e_y$). Using the same argument from above, the only Pareto improving trade would have consumer 1 demanding good y and supplying good x, while consumer 2 must demand good x and supply good y. The FOC for this trade to occur is:

$$MRS_1 < MRS_2$$

which means that for there *not* to be a Pareto improving trade, it must be the case that:

$$MRS_1 \geq MRS_2 \Leftrightarrow \frac{1}{f'(y_1^*)} \geq \frac{1}{f'(y_2^*)} \Rightarrow y_1^* = 0 \geq y_2^*. \tag{12}$$

Inequality (9) is a contradiction because $y_2^* = e_y > 0$. Therefore, we can't have an equilibrium where consumer 1 consumes 0 of good y. The same applies to the case where $x_2^* > 0$ and $y_2^* = 0$ ($\Leftrightarrow x_1^* \geq 0$, $y_1^* = e_y$). This result is a result of the concavity of the non-numeraire good's separable utility function and the quasilinear form of each consumer's utility function.

Therefore, equations 10and 11 define the corner equilibrium for quasi-linear preferences.

Part (e)

This part of the question is asking you to solve for the competitive/exchange equilibrium (not the Pareto equilibrium). Solving each consumer's UMP, we get:

$$x_1^* = a(e_{x1} + pe_{y1});\ y_1^* = \frac{(1-a)(e_{x1} + pe_{y1})}{p};$$

$$x_2^* = a(e_{x2} + pe_{y2});\ y_2^* = \frac{(1-a)(e_{x2} + pe_{y2})}{p}$$

The equilibrium price will clear markets. Recall that we only need to find the equilibrium price that will clear one of the markets.

$$x_1^* + x_2^* = a(e_{x1} + pe_{y1}) + a(e_{x2} + pe_{y2}) = a(e_x + pe_y) = e_x \Rightarrow p^* = \frac{1-a}{a} \cdot \frac{e_x}{e_y} \tag{13}$$

which shows that equilibrium price depends on aggregate endowment of each good.

Now we're asked to confirm that this is a Pareto optimal equilibrium (and so the First Welfare Theorem holds). One way to show this is to plug the equilibrium price into **??** and check to see that equation 5 holds.

An easier way to go about this is to see that at a Pareto optimal equilibrium:

$$\frac{x_1^*}{y_1^*} = \frac{e_x}{e_y} \tag{14}$$

and in the competitive equilibrium:

$$\frac{x_1^*}{y_1^*} = \frac{ap^*}{1-a} = \frac{e_x}{e_y}\ (\because equation\, 13) \tag{15}$$

Therefore, the exchange equi. is a Pareto equi.

Part (f)

The key here is that the problem tells you that each consumer has sufficiently large enough wealth for the FOCs to hold with equality and we have interior solutions. In other words, consumers are not wealth constrained such that they will only consume the non-numeraire good y. Solving the UMP results in the following:

$$x_1^* = (e_{x1} + pe_{y1}) - p\frac{e_y}{2};\ x_2^* = (e_{x2} + pe_{y2}) - p\frac{e_y}{2};\ y_1^* = y_2^* = \frac{e_y}{2}. \tag{16}$$

Further, the UMP FOCs result in the equilibrium price for good y:

$$\frac{f'(y_1^*)}{p} = \frac{f'(y_2^*)}{p} = 1 \Rightarrow p^* = f'\left(\frac{e_y}{2}\right)$$

which doesn't depend on the individual endowments.

Note also that 16 is the same condition as 7, which means that the exchange equilibrium is a Pareto optimal equilibrium.

Part (g)

Now the condition that there is sufficient wealth fails. Let's assume that consumer 1 doesn't have enough wealth to consume $\frac{e_y}{2}$ units of good y at the equilibrium price we solved for in (f) above: $p^* \cdot \frac{e_y}{2} \geq e_{x1} + p^* e_{y1}$. This means that there is an excess supply of good y (since at p^* consumer 2 still consumes $\frac{e_y}{2}$, therefore, the new equilibrium price for good y, $p^{**} < p^*$. At the new equilibrium price p^{**}, consumer 1's consumption bundle is:

$$\left(0, y_1^{**} = \frac{e_{x1}}{p^{**}} + e_{y1} < \frac{e_y}{2}\right)$$

and consumer 2's consumption bundle is:

$$\left(e_x, y_2^{**} = f_y^{-1}(p^{**}) > \frac{e_y}{2}\right).$$

Notice that the following equation implicitly solves for the new equilibrium price, p^{**}:

$$\frac{e_{x1}}{p^{**}} + e_{y1} + f_y^{-1}(p^{**}) = e_y. \tag{17}$$

Finally, does this represent a Pareto optimal equilibrium? Yes. See equation 10. Notice that the Pareto optimal equilibrium first order conditions change once we are at a border.

Practice Problem 9.5 Exchange Equilibrium with Nonconvex Preferences

Part (a)

Each consumer has a quasi-linear utility function but preferences over the non-numeraire good are defined by a convex function (not a concave function), such that $u(x_i, y_i) = x_i + f(y_i)$, where $f'(\cdot) > 0$; $f''(\cdot) > 0$. Therefore, unlike quasi-linear functions with concave utility, the marginal utility of the non-numeraire good is increasing as you increase the consumption of the non-numeraire good, while the marginal utility of the numeraire good is fixed. Therefore, if $\frac{f'(e_{iy})}{p_y} < \frac{1}{p_x}$, then the consumer only consumes good x, the numeraire good, and if $\frac{f'(e_{iy})}{p_y} \geq \frac{1}{p_x}$, then the consumer only consumes good y, the non-numeraire good.

Part (b)

From part (a) above, we know that in any exchange equilibrium consumer 1 will consume all of good x or all of good y (the same holds for consumer 2). Since each consumer has endowment equal to (1,1), total wealth for each consumer is equal to $1 + p$.

Therefore, consumer 1 will consume all of good y if and only if the utility from consuming only good x is less than the utility from consuming only good y:

$$1 + p \leq \frac{1}{3} \cdot \left(\frac{1+p}{p}\right)^2 \Rightarrow 3p^2 - p - 1 \leq 0 \Rightarrow p \in \left[0, \frac{1+\sqrt{13}}{6} \approx 0.76\right] \tag{18}$$

On the other hand, consumer 2 will consume all of good y if and only if:

$$1 + p \leq 3\left(\frac{1+p}{p}\right)^2 \Rightarrow p^2 - 3p - 3 \leq 0 \Rightarrow p \in \left[0, \frac{3+\sqrt{21}}{2} \approx 3.79\right] \tag{19}$$

Part (c)

The *only* exchange equilibrium has consumer 1 consuming good x and consumer 2 consuming good y (which is intuitive, since the marginal utility of good y is greater for consumer 2). Therefore, the equilibrium price at which markets clear, $p^* \in [0.76, 3.79]$, and consumer 1 consumes $1 + p$ which has to equal the total endowment of good x:

$$1 + p = 2 \Rightarrow p^* = 1 \in [0.76, 3.79] \tag{20}$$

Note that we can't use the marginal rate of substitution to calculate the equilibria in this case because we're dealing with a non-concave function for the non-numeraire good.

And the resulting competitive/exchange equilibrium:

$$[(2, 0), (0, 2)] \tag{21}$$

Part (d)

Because there is a numeraire good (both consumers have quasi-linear utility), we can add utility functions. Therefore, Pareto optimal equilibria of the form $[\vec{x}^*, \vec{y}^*]$ maximizes:

$$\begin{aligned} x_1^* + x_2^* + \frac{1}{3}y_1^{*2} + 3y_2^{*2} &= 2 + \frac{1}{3}y_1^{*2} + 3y_2^{*2} = 2 + \frac{1}{3}y_1^{*2} + 3(2 - y_1^*)^2 \\ &\Rightarrow y_1^* = 0,\ y_2^* = 2 \end{aligned}$$

The resulting Pareto optimal equilibrium is the following:

$$[(x_1^*, 0), (2 - x_1^*, 2)] \tag{22}$$

where $x_1^* \in [0, 2]$. The exchange equilibrium identified in 21 above is contained within the Pareto optimal set identified in 22 above, and so the First Welfare Theorem applies.

Part (e)

Recall that in partial equilibrium analysis, the numeraire good could take on negative values so that consumers could consume the optimal amount of the non numeraire good. This is not the case here since good x is limited by its endowment to 2. The result is Pareto optimal equilibria of the form found in 22 above. Note that equilibria of the form following form are Pareto optimal, but not exchange equilibria:

$$[(x_1^*, 0), (2 - x_1^*, 2)],\ x_1^* \in [0, 2)$$

Therefore, the Second Welfare Theorem does not hold because there are Pareto optimal equilibria that cannot be sustained in a competitive equilibrium, which has each consumer only consuming one of the goods.

Part (f)

The initial endowment is the same as before and the same logic applies. The only possible exchange equilibrium has consumer 1 consuming all of good x and consumer 2 consuming all of good y. The equilibrium price will also remain the same at $p^* = 1$.

Recall from 18 and 19 above that the following inequalities must hold:

$$
\begin{aligned}
A\left(\frac{1+p}{p}\right)^2 &\leq 1+p \\
B\left(\frac{1+p}{p}\right)^2 &\geq 1+p
\end{aligned}
$$

which results in the following at $p^* = 1$:

$$
\begin{aligned}
A &\leq \frac{1}{2} \\
B &\geq \frac{1}{2}
\end{aligned}
$$

These conditions are necessary for there to be an exchange equilibrium.

2 Fundamental Welfare Theorem Failure

What assumptions do we need for the Welfare Theorems to hold?[8]

1. Complete markets: no unpriced goods. This assumption fails when we have externalities.

2. No market power: all economic agents are price-takers (competitive markets for all goods). This assumptions fails when we are dealing with monopolies/monopsonies, oligopolies/oligopsonies.

3. Complete information for all agents. This assumption fails if we have uncertainty in the payoffs or when there is asymmetric information or imperfect information.

These are some of the topics we'll discuss after Spring Break.

[8]Recall from the class that convexity of preferences (which leads to what type of utility function?) was necessary for the Second Fundamental Welfare Theorem to hold. See also Practice Problem 9.5 above.