Section Notes 12

Wonbin Kang

April 29, 2010

Agenda

1. Preferences over Two Alternatives
2. Preferences over more than Two Alternatives
3. Strategic Voting

For those of you who are interested in doing any type of work in welfare economics, political economy, or voting models, you should try to understand the proof of Arrow's Impossibility Theorem. MWG's proof in Section 21.C is the most intuitive for me, but there are several versions out there.

1 Preferences over Two Alternatives

Assume that there are two alternatives, x and y, both of which are elements of the set I; agent $i \in \{1, ..., N\}$ has preferences as follows:

$$\alpha_i = \begin{cases} 1, & if\, x \succ_i y \\ 0, & if\, x \sim_i y \\ -1, & if\, x \prec_i y \end{cases}$$

Let A denote the set of all possible preferences or rankings over the set I for each agent. Therefore, the preference ordering $\vec{\alpha} \in A^N$.

Given the representation of individual preferences above, we can aggregate these preferences via majority voting as follows:

$$F(\vec{\alpha}) = \begin{cases} 1, & if\, \sum_{i=1}^N \alpha_i > 0 \\ 0, & if\, \sum_{i=1}^N \alpha_i = 0 \\ -1, & if\, \sum_{i=1}^N \alpha_i < 0 \end{cases}$$

Theorem 1.1. May's Theorem: *Given two alternatives, the majority voting social welfare function is the only social welfare function that satisfies the following conditions:*

1. Unrestricted domain: $F : A^N \to \{-1, 0, 1\}$, $\forall \vec{\alpha} \in A^N$.

2. Anonymity (symmetry across agents): $F(\vec{\alpha})$ only depends on the sums of α_i's equaling -1, 0, or 1. The identity of the individuals does not matter—*i.e.* each voter is identical.

3. Neutrality among alternatives: If we reversed every individual's preference, we would reverse society's preference:

$$F(\vec{\alpha}) = -F(-\vec{\alpha}).$$

   Note that supermajorities are social welfare functions which also satisfy this condition.

4. Positive Responsiveness (monotonicity): Society does not have "thick indifference curves."[1] A single change of preference is sufficient to break a tie. Further, preference shifts in favor of social preference will not change social preference. Formally, if $\vec{\alpha} \in A^N$ s.t. $F(\vec{\alpha}) \geq 0$, then for $\vec{\alpha}' \neq \vec{\alpha}$ where for some i, $\alpha_i' \geq \alpha_i$, $F(\vec{\alpha}') = 1$.

Given a majority voting social welfare function and two alternatives, is there an incentive for agent i to misrepresent her preferences? No. Voting your preference is a weakly dominant strategy.

2 Preferences over more than Two Alternatives[2]

Total of $n \geq 3$ members on committee; all committee members have complete and transitive (weak) preferences over alternatives; complete information on the preferences of each committee member;[3] and the relevant social welfare function is a Borda count.

Part (a)

Let's assume that committee members, denoted i, are elements of the set I. We also use the following notation for social preferences: if x is strictly prefered by society to y, then $x \succ_S y$, and if x is as good as y for society, then $x \succsim_S y$.[4]

A social welfare function is weakly Paretian if $x \succ_i y$, $\forall i$, then $x \succ_S y$. A social welfare function is Paretian if $x \succsim_i y$, $\forall i$ and $x \succ_i y$, for some i, then $x \succ_S y$.[5]

[1] Recall that we had a similar concept in consumer theory. Can you recall which assumption got us to this result?

[2] 2005 Final Exam Problem #5

[3] Note that if we have complete information of preferences, we could have strategic voting. However, this is assumed away in this problem.

[4] MWG uses $xF_p(\succsim_1, ..., \succsim_N)y$ to denote: society strictly prefers x to y; and $xF(\succsim_1, ..., \succsim_N)y$ to denote: x is socialy at least as good as y.

[5] Do you see the relation between this definition and Pareto efficiency?

Part (b)

How are individual preferences aggregated using a Borda count? Let A be the set of all possible rankings over the set I. Then for each agent i, $\vec{\alpha}_i \in A$ is agent i's ranking of each of the committee members in set I. To illustrate, assume that $n = 3$ and agent 1's ranking $\vec{\alpha}_1 = \{3, 1, 2\}$. This is equivalent to $3 \succ_1 1 \succ_1 2$ as the committee chair.

In a Borda count, for each agent i, we define $c_i(x)$ which is equal to the rank of alternative x.[6] In the example above, $c_1(3) = 1$; $c_1(1) = 2$; and $c_1(2) = 3$. We can define this for all $i \in I$. Then the Borda count aggregates individual preferences such that alternatives x with a lower value of $\sum_{i=1}^{n} c_i(x)$ is prefered by society.

Using the definition of weakly Paretian preferences above, given two alternatives x and y, assume that $x \succ_i y$, $\forall i \in I$. This means that $c_i(x) < c_i(y)$, $\forall i \in I$, which results in:

$$\sum_{i=1}^{n} c_i(x) < \sum_{i=1}^{n} c_i(y) \Rightarrow x \succ_S y$$

Therefore, we conclude that a Borda count is a weakly Paretian social welfare function.

The question then asks us to show that this is efficient. By efficient, we mean Pareto efficient. Given any two options, x and y, if all individuals prefer x to y, then x will be chosen by the social welfare function, which is the Pareto efficient result.

Part (c)

To understand the definition of "non-dictatorial"[7] let's first define what it means for agent i to be a dictator.

Definition 2.1. An agent i is a dictator, if $\forall \{x, y\} \subset I$ and $\forall \vec{\alpha} \in A^n$, we have that x is socially preferred to y, that is $x \succ_S y$, whenever $x \succ_i y$.

Therefore, a social welfare function is non-dictatorial iff no individual is a dictator.

The proof that a Borda count is non-dictatorial is by contradiction. Assume that agent i is a dictator with preferences $x \succ_i y \succ_i z$; and all $j \neq i \in I$ have preferences $z \succ_j y \succ_j x$. Consider $\{x, z\} \subset I$, the dictator prefers x to z. However,

$$\sum_{i=1}^{n} c_i(x) > \sum_{i=1}^{n} c_i(z) \Rightarrow z \succ_S x,$$

which contradicts the statement that agent i is a dictator.

[6] If we allow for indifference between two alternatives, then we take the average rank of the two indifferent alternatives.

[7] A weakening of May's symmetry / anonymity.

Definition 2.2. If agent i's preferences are such that $x \succ_i y$ for some $\{x, y\}$ pairing, then there exists some $\vec{\alpha} \in A^n$ such that $y \succ_S x$. Then agent i is not a dictator.

Part (d)

The Independence of Irrelevant Alternatives ("IIA") is defined formally as follows:

Definition 2.3. The social welfare function defined on the domain A^n satisfies IIA (or "pairwise independence" condition) if the social preference between any two alternatives $\{x, y\} \subset I$ *depends only on the ordering of individual preferences over the same alternatives.* For any pair of alternatives $\{x, y\} \subset I$, and for any pair of preference orderings $\vec{\alpha} \in A^n$ and $\vec{\alpha}' \in A^n$ with the property that, for every i,

$$x \succsim_i y \Leftrightarrow x \succsim_i' y$$

and

$$y \succsim_i x \Leftrightarrow y \succsim_i' x,$$

we have that

$$x \succsim_S y \Leftrightarrow x \succsim_S' y$$

and

$$y \succsim_S x \Leftrightarrow y \succsim_S' x.$$

In other words, if two sets of individual preference orderings have the same preference between x and y for all individuals, then the social preference between x and y must be the same.

Consider the following preferences $\vec{\alpha}$ and $\vec{\alpha}'$:[8]

$\vec{\alpha}$			$\vec{\alpha}'$		
$\vec{\alpha}_1$	$\vec{\alpha}_2$	$\vec{\alpha}_3$	$\vec{\alpha}_1'$	$\vec{\alpha}_2'$	$\vec{\alpha}_3'$
a	b	a	a	b	a
b	z	b	z	a	z
z	a	z	b	z	b

Notice that the individual preferences between alternatives "a" and "b" are the same for all agents. For preferences $\vec{\alpha}$, society is indifferent between "a" and "b" since:

[8] I realize that this is a slight abuse of notation, since these aren't really vectors.

$$\sum_{i=1}^{3} c_i(a|\vec{\alpha}) = 5 = \sum_{i=1}^{3} c_i(b|\vec{\alpha}) \Rightarrow a \sim_S b$$

but for preferences $\vec{\alpha}'$, society prefers "a" to "be" since:

$$\sum_{i=1}^{3} c_i(a|\vec{\alpha}') = 4 < \sum_{i=1}^{3} c_i(b|\vec{\alpha}') = 7 \Rightarrow a \succ_S b$$

In short, because the rank of an alternative depends on the placement of every tother alternative, a Borda count does not satisfy the IIA condition. Consider another example that illustrates IIA at work.

Example 2.4. Assume that there are two agents 1 and 2, and three alternatives a, b, and c. Suppose that the preferences of the first agent is given by: $a \succ_1 b \succ_1 c$, which we denote as "abc." There are two preference orderings $\vec{\alpha} = [abc,\, acb]$ and $\vec{\alpha}' = [abc,\, bca]$. Consider a social welfare function $F : A^2 \rightarrow \{a, b, c\}$ which maps each of the preference orderings above as follows: $F(\vec{\alpha} = [abc,\, acb]) = acb$ and $F(\vec{\alpha}' = [abc,\, bca]) = abc$.[9]

If we assume that the social welfare function $F(\cdot)$ satisfies IIA and is weakly Paretian, then by IIA we have that any preference ordering in A which has $b \succ_1 c$, $c \succ_2 b$ results in $c \succ_S b$ and any preference ordering in A^2 which has $a \succ_1 c$, $c \succ_2 a$ results in $a \succ_S c$. Therefore, the social welfare function $F(\cdot)$ must map the preference ordering $\vec{\alpha}'' = [bac, cba]$ such that $c \succ_S b$ and $a \succ_S c$.

Part (e)

The problem has restricted the preferences that each agent i can hold such that each agent has "single-peaked" preferences. With single-peaked preferences we can define a Condorcet winner. Recall in May's Theorem and in Arrow's Impossibility Theorem (as well as Sen's Impossibility Theorem) that we assumed the domain of the social welfare function was unrestricted. However, with single-peaked preferences, we've limited the domain of the social welfare function such that we can get rid of "cycling."

As noted above, with single peaked preferences over a one-dimensional policy space (in this example, it's age), we can apply the Median Voter Theorem.[10] The result is that the Condorcet winner is the most preferred policy of the median voter, which will be chosen by the social welfare function of simple majority rule.

[9] From MIT 14.773 Section Notes, Spring 2010.

[10] The proof of the MVT is very accessible, so for those interested I encourage you to review it.

3 Strategic Voting

An assumption that appears in the context of aggregating individual preferences via a social welfare function based on some form of voting is sincere voting. Of course, in some instances, sincere voting is endogenous to the model because it is a weakly dominant strategy. However, what happens if we allow for strategic voting?

Recall question 2 from the 2010 mid term. A similar example is as follows:

Example 3.1. The relevant voting rule or social welfare function is all vote for one option and drop the lowest performing option. Repeat until one option has been chosen. Assume that there exist 5 agents who have the following preferences:

Agent i:	1	2	3	4	5
1st best	A	A	B	B	C
2nd best	B	B	C	C	A
3rd best	C	C	A	A	B

What is the SPNE?

If all agents vote sincerely, then in the second stage, alternative A faces off against B which results in A winning. However, if agents 3 and 4 vote for C in the first stage, then C vs A in the second stage which results in C winning. However, agents 1 and 2 have an incentive to deviate and vote for B in the first stage, which results in B vs C in the second stage resulting in B winning and no incentive to deviate.

Do you see the intuition behind this SPNE?