Section Notes 9

Wonbin Kang

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Agenda

1. Problem Set 5: FOSD and SOSD
2. Pure Exchange Economy
3. Examples

1 Problem Set 5: FOSD and SOSD

Recall the propositions of FOSD and SOSD:

Proposition 1. *The distribution/lottery of monetary payoffs $F(\cdot)$ first-order stochastically dominates the distribution/lottery $G(\cdot)$* ***if and only if*** $F(x) \leq G(x), \forall x$. [1]

Proposition 2. *For two distributions/lotteries with the same mean, $\int xdF(x) = \int xdG(x)$, the distribution/lottery of monetary payoffs $F(\cdot)$ second-order stochastically dominates the distribution/lottery $G(\cdot)$* ***if and only if*** *the distribution $G(x)$ is a mean preserving spread of $F(x)$, which implies that* $\int_{-\infty}^{x} G(t)dt \geq \int_{-\infty}^{x} F(t)dt, \forall x$.[2]

2 Pure Exchange Equilibrium

2.1 The Model

Two consumers: 1 and 2; two goods: x and y; initial total endowments: $\bar{e}_x = 10$ and $\bar{e}_y = 8$; and no production. Note that in this model there appears to be four endogeneous variables which denote the allocation of the two goods: $[x_1, y_1, x_2, y_2]$ and one endogeneous price variable which allows the markets to

[1] See MWG ch 6 for proof.
[2] See MWG ch 6 for proof.

clear: p. However, in reality, because the total endowment is fixed at the outset, it must be the case that: $x_1 + x_2 = \bar{e}_x = 10$ and $y_1 + y_2 = \bar{e}_y = 8$. So in reality, we're only dealing with, at most, three endogeneous variables.

2.2 Edgeworth Box [3]

This model can be depicted using the Edgeworth Box. Draw the line depicting the number of units of good x on the horizontal axis and the number of units of good y on the vertical axis. Note that points within the box are both feasible: $x_1 + x_2 \leq \bar{e}_x = 10$ and $y_1 + y_2 \leq \bar{e}_y = 8$, and non-wasteful: $x_1 + x_2 = \bar{e}_x = 10$ and $y_1 + y_2 = \bar{e}_y = 8$.

In Figure 1, we see an exchange equilibrium where the price vector $\vec{p} = [p_x, p_y]$ clears the markets for both good x and good y. The allocation $[\vec{x}^*, \vec{y}^*]$ is a Pareto Optimal allocation (since the indifference curves of Consumer 1 and 2 are tangent to each other) and the allocation $[\vec{x}^*, \vec{y}^*]$ is on a higher indifference curve for both Consumers than those which pass through the initial endowment, $\vec{e}$.

In Figure 2, we see an instance where the price vector $\vec{p} = [p_x, p_y]$ does not clear the markets for both good x and good y; there is an excess demand for good x and an excess supply for good y.

The difference between the two figures arises from the UMPs that each consumer is solving. In each figure, Consumer 1 and 2 are choosing their optimal bundle such that:

$$\frac{\partial u_i(x_i^*, y_i^*)/\partial x}{\partial u_i(x_i^*, y_i^*)/\partial y} = \frac{p_x}{p_y} \; \forall i \in \{1, 2\}\,.$$

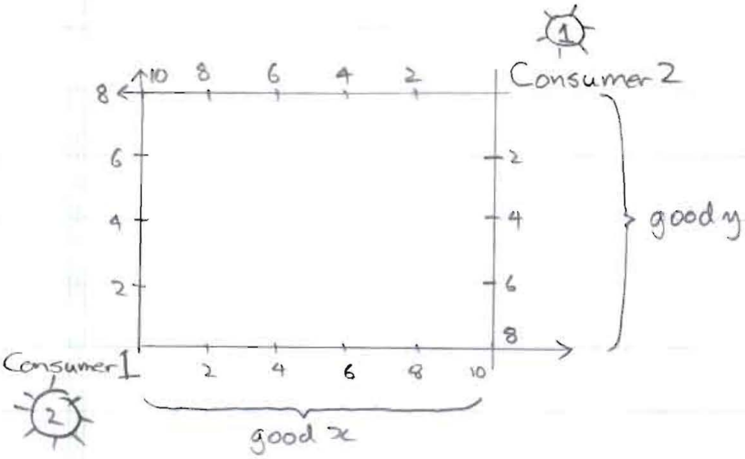
2.3 The Exchange/Walrasian Equilibrium

The exchange equilibrium we are looking for can be thought of as follows:

1. Given an endowment, $\vec{e}$, rotate the budget line through such endowment. This rotation can be thought of as changing the relative prices of good x and good y since the slope of the budget line is equal to $-\frac{p_x}{p_y}$. Note that here we need to assume that prices $p_x,\ p_y \in (0, \infty)$. See MWG ch 15.B, Figure 15.B.5.

[3] This is a very important concept and provides much of the intuition behind general equilibrium, so make sure that you're comfortable with it.

Edgeworth Box



A Pareto Optimal Allocation is an allocation $(x_1^*, x_2^*, y_1^*, y_2^*)$ such that Consumer 1's utility can't be increased without decreasing Consumer 2's utility and vice versa.

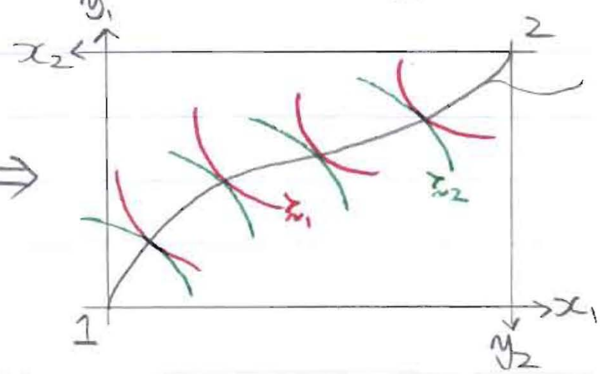
Therefore, an allocation $(\vec{x}^*, \vec{y}^*)$ is Pareto Optimal if $\nexists\, (\vec{x}', \vec{y}')$ s.t. $u_i(x_i', y_i') \geq u_i(x_i^*, y_i^*)\ \forall i$ AND $u_i(x_i', y_i') > u_i(x_i^*, y_i^*)$ for some i.

Note that $\vec{x}^* = (x_1^*, \cdots, x_i^*, \cdots x_I^*)$, $\vec{y}^* = (y_1^*, \cdots, y_i^*, \cdots y_I^*)$.

The implication is that Pareto Optimal allocations $(\vec{x}^*, \vec{y}^*)$ will be at points in the Edgeworth Box where the indifference curves of Consumer 1 and Consumer 2 are tangent to each other:

$$\frac{\dfrac{\partial u_1(x_1^*, y_1^*)}{\partial x_1}}{\dfrac{\partial u_1(x_1^*, y_1^*)}{\partial y_1}} = \frac{\dfrac{\partial u_2(x_2^*, y_2^*)}{\partial x_2}}{\dfrac{\partial u_2(x_2^*, y_2^*)}{\partial y_2}} \iff MRS_{x_1 y_1} = MRS_{x_2 y_2}$$

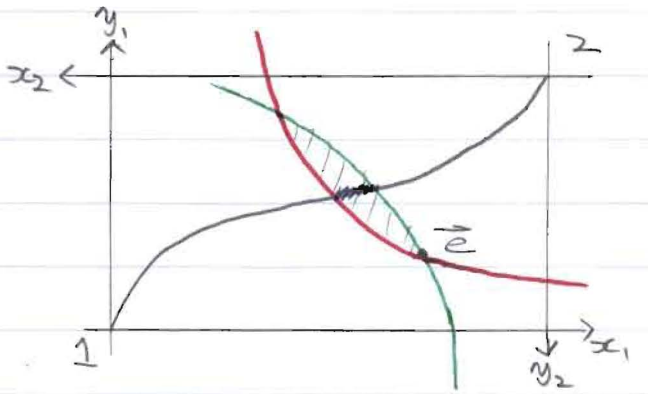
$\Rightarrow$



Pareto Set $\equiv \{(\vec{x}, \vec{y}) \mid MRS_{x_1 y_1} = MRS_{x_2 y_2}\}$

Now we add in an initial endowment

$\vec{e} = (\bar{e}_x, \bar{e}_y) = (e_{x_1} e_{x_2} e_{y_1} e_{y_2})$



From this Box, do you see that the initial allocation $\vec{e}$ is NOT Pareto Optimal? All allocations in the shaded region are prefer by both Consumer 1 and 2.

Further, the set of allocations marked in blue is the contract curve: the set of allocations $(\vec{x}^{**}, \vec{y}^{**})$ s.t. they are Pareto Optimal AND preferred by Consumer 1 and 2 to the endowment:

$$C_{contract} \equiv \{(\vec{x}, \vec{y}) \mid MRS_{x_1 y_1} = MRS_{x_2 y_2} \text{ and } U_i(x_i y_i) \geq U_i(e_{x_i} e_{y_i}), \forall i\}$$

What about price $(\vec{p} = (p_x, p_y))$? Prices play the role of clearing the markets. Note that every $\vec{p}$ does NOT clear markets.

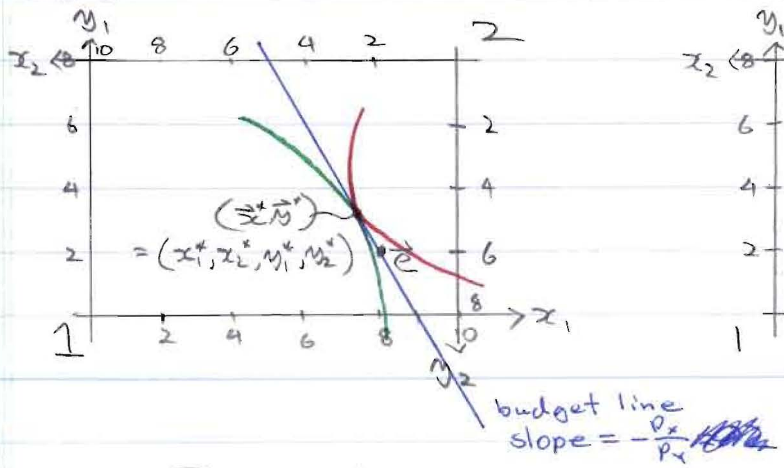


Figure 1

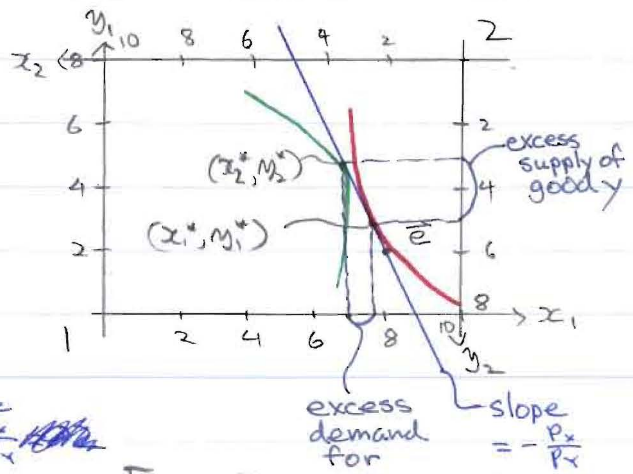


Figure 2

2. Then for each relative price ratio (slope of the budget line), solve the UMP for Consumer 1, which will result in the offer curve. The offer curve can be thought of as the solutions to the Consumer 1's UMP, which are functions (or correspondences) of the relative price ratio, $\frac{p_x}{p_y}$. Do the same for Consumer 2. Equilibrium candidates are at the intersection of the offer curves.

$$\max_{x_i, y_i} u_i(x_i, y_i) \; s.t. \; p_x x_i + p_y y_i \leq p_x e_{xi} + p_y e_{yi} \; \forall i \in \{1, 2\}$$

3. Given these candidates, we need to check to see if there exists a price ratio, given the initial endowment, that will clear the markets for one of the goods (if one of the markets clear, the other will also clear since both Consumer 1 and Consumer 2's demanded bundles lie on the budget line). In short, we need to check whether either:

$$x_1^* + x_2^* = e_{x1} + e_{x2}$$

, or

$$y_1^* + y_2^* = e_{y1} + e_{y2}.$$

Another issue that economists have grappled with is the existence of exchange equilibria. A comprehensive review of this material is well beyond the scope of Ec 2020a, but for those interested see MWG ch 17.C and 17.D. Suffice it to say that a Walrasian equilibrium exists in a pure exchange economy in which $\bar{\omega} = [\omega_1 ... \omega_L] \gg 0$, where $\omega_l = \sum_i \omega_{li}$, and each consumer has continuous, strictly convex, and strongly monotone preferences.

3 Examples

Example 3. Let $u_1(x_1, y_1) = x_1^{\frac{1}{2}} y_1^{\frac{1}{2}}$ and $u_2(x_2, y_2) = x_2^{\frac{1}{2}} y_2^{\frac{1}{2}}$; $e_{x1} = 8$, $e_{x2} = 2$, $e_{y1} = 2$, $e_{y2} = 6$; and $\vec{p} = [1, p]$[4]. Solve for the exchange equilibrium allocation and the equilibrium price.

Consumer 1's UMP is:

$$\max_{x_1, y_1} x_1^{\frac{1}{2}} y_1^{\frac{1}{2}} \; s.t. \; x_1 + p y_1 \leq e_{x1} + p e_{y1} = 8 + 2p$$

, which has the following solutions:

$$x_1^*(p) = \frac{8 + 2p}{2} = 4 + p \tag{1}$$

$$y_1^*(p) = \frac{8 + 2p}{2p} = \frac{4}{p} + 1. \tag{2}$$

[4]Prices have been normalized.

Consumer 2's UMP yields the following solutions:

$$\begin{aligned}
x_2^*(p) &= \frac{e_{x2}+pe_{y2}}{2} = \frac{2+6p}{2} = 1+3p && (3)\\
y_2^*(p) &= \frac{e_{x2}+pe_{y2}}{2p} = \frac{2+6p}{2p} = \frac{1}{p}+3. && (4)
\end{aligned}$$

We now have to determine the p^*such that markets will clear. Note that the price ratio for this problem, as in many problems you will come across in Econ 2020a, is equal to the inverse of the price of good y because we normalized the price of one good to equal 1. If we don't normalize the price of good 1, we would be solving for $\frac{p_x}{p_y}$ and we wouldn't be able to solve for p_x and p_y separately. In short, the exchange equilibrium allows us to pin down the price ratio, but not the exact prices of good x and good y.

Let's use good x to solve for p^* since it has the easier equations:

$$\begin{aligned}
x_1^*(p)+x_2^*(p) &= 5+4p^* = e_{x1}+e_{x2} = 10\\
\therefore p^* &= \frac{5}{4}.
\end{aligned}$$

If we plug the equilibrium price ratio into equations 5, 6, 7, and 8, we can solve for the equilibrium allocation $[\vec{x}^*, \vec{y}^*]$.

Example 4. Let $u_1(x_1,y_1) = x_1$ and $u_2(x_2,y_2) = x_2^{\frac{1}{2}}y_2^{\frac{1}{2}}$; $e_{x1} = 10$, $e_{x2} = 0$, $e_{y1} = 0$, $e_{y2} = 8$; and $\vec{p} = [1,p]$. Solve for the exchange equilibrium allocation and the equilibrium price.

Consumer 1's UMP is:

$$\max_{x_1} x_1 \; s.t. \; x_1 + py_1 \le e_{x1} + pe_{y1} = 10$$

, which has the following border solutions:

$$\begin{aligned}
x_1^*(p) &= 10 && (5)\\
y_1^*(p) &= 0. && (6)
\end{aligned}$$

Consumer 2's UMP yields the following solutions:

$$\begin{aligned}
x_2^*(p) &= \frac{e_{x2}+pe_{y2}}{2} = \frac{8p}{2} = 4p && (7)\\
y_2^*(p) &= \frac{e_{x2}+pe_{y2}}{2p} = \frac{8p}{2p} = 4. && (8)
\end{aligned}$$

Markets clear for good x and so $10 + 4p = 10$, therefore, $p^* = 0$, which is intuitive since Consumer 2 is supplying good y, but Consumer 1 doesn't demand good y. At $p^* = 0$, we have multiple equilibria. See Edgeworth Box at the end of the section notes.

Example 5. Consumers 1 and 2; Goods x and y; Let $u(x_1, y_1) = \min\left\{x_1^2 y_1, 300\right\}$ and $v(x_2, y_2) = x_2 y_2^2$; $\bar{e}_x = \bar{e}_y = 12$; and $\vec{p} = [1, p]$.

Part a) *Explain how you would identify the set of Pareto Optimal allocations for this economy.*

Recall that the set of Pareto Optimal allocations could be written as follows:

$$S \equiv \{(\vec{x}, \vec{y}) \,| MRS_{x_1 y_1} = MRS_{x_2 y_2}\}.$$

For Consumer 1, the marginal utilities are as follows:

$$\frac{\partial u}{\partial x_1} \begin{cases} 2x_1 y_1 & if\, x_1^2 y_1 < 300 \\ 0 & otherwise \end{cases}$$

$$\frac{\partial u}{\partial y_1} \begin{cases} x_1^2 & if\, x_1^2 y_1 < 300 \\ 0 & otherwise \end{cases}$$

, therefore if $x_1^2 y_1 \geq 300$, then the marginal rate of subsitution is undefined and if $x_1^2 y_1 < 300$, the marginal rate of substitution can be written as follows:

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial y_1}} = \frac{2y_1}{x_1} \tag{9}$$

. For Consumer 2, the marginal utilities are as follows:

$$\begin{aligned} \frac{\partial u}{\partial x_2} &= y_2^2 \\ \frac{\partial u}{\partial y_2} &= 2x_2 y_2 \end{aligned}$$

, so that the marginal rate of subsitution is equal to:

$$\frac{\frac{\partial u}{\partial x_2}}{\frac{\partial u}{\partial y_2}} = \frac{y_2}{2x_2} \tag{10}$$

. Therefore, for values of x_1, y_1 s.t. $x_1^2 y_1 < 300$, we set equations 9 and 10 equal to each other noting that $x_1 + x_2 = 12$ and $y_1 + y_2 = 12$ because the total endowment for both goods is equal to 12. Then we get the following for x_1, y_1 s.t. $x_1^2 y_1 < 300$:

$$\frac{2y_1}{x_1} = \frac{y_2}{2x_2} \Rightarrow \frac{2y_1}{x_1} = \frac{12 - y_1}{2(12 - x_1)} \Rightarrow y_1 = \frac{4x_1}{16 - x_1} \tag{11}$$

, which defines the set of Pareto Optimal allocations. See end of the section notes for the graphical representation.

Part b) *Consider the initial endowment of (8,4) for Consumer 1. Normalize* $p_x = 1$. *What price* p_y, *if any, will sustain this allocation as an exchange equilibrium?*

First note that $x_1^2 y_1 = 256 < 300$.

The key to this question is to see that for the initial allocation to be an exchange equilibrium, it must be the case that it is Pareto Optimal (in the sense that you cannot make one consumer better off without making the other worse off, or equivalently, there can be no gains from trade).

From **part a)** above, we have the conditions for Pareto optimality in equations 11. Furthermore, the marginal rates of substitutions for both consumers must equal the price ratio:

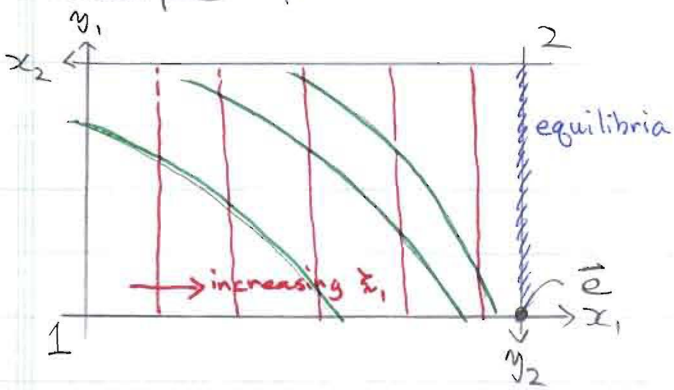
$$\frac{2y_1}{x_1} = \frac{y_2}{2x_2} = \frac{1}{p_y} \Rightarrow p_y^* = 1$$

Part c) *Repeat part b) above, but with the initial endowment of Consumer 1 equal to (10,5) (which means that the Consumer 2's endowment is equal to (2,7)).*

Note that $x_1^2 y_1 = 500 > 300$, which means that the MRS of Consumer 1 is not defined. Intuitively, Consumer 1 is indifferent between any consumption bundle (x_1, y_1) s.t. $x_1^2 y_1 > 300$ and so depending on the assumptions that you make on the behavior of Consumer 1 when indifferent between consumption bundles, a price ratio may or may not exist, or there may be multiple price ratios that allow for gains from trade.

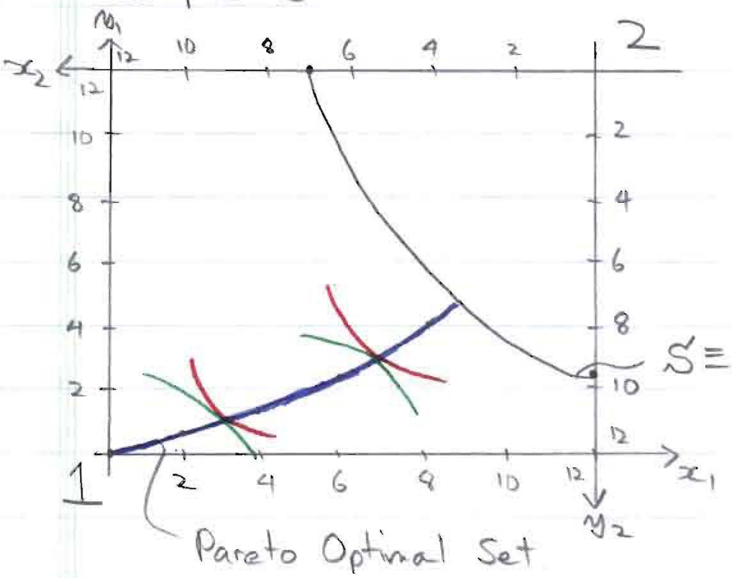
Part d) N/A

Example 4



Note that the preferences for Consumer 1 are not strongly monotonic.

Example 5



Recall that Pareto Optimal Set was defined by $y_1 = \frac{4x_1}{16 - x_1}$

$S \equiv \{(x_1, y_1) \mid x_1^2 y_1 = 300\}$