Section Notes 10

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Agenda

1. Practice Problem 17.1
2. Practice Problem 17.3

1 Practice Problem 17.1

Part (a)

FOCs for a Pareto optimal interior equilibrium with Cobb-Douglas preferences is that the indifference curves of consumer 1 and 2 are tangent to each other. Therefore, the allocation $(\vec{x}^*, \vec{y}^*)$ is Pareto Optimal if:

$$\frac{\frac{\partial u(x_1^*,y_1^*)}{\partial x_1}}{\frac{\partial u(x_1^*,y_1^*)}{\partial y_1}} = \frac{\frac{\partial v(x_2^*,y_2^*)}{\partial x_2}}{\frac{\partial v(x_2^*,y_2^*)}{\partial y_2}} \Rightarrow \frac{a x_1^{*a-1} y_1^{*1-a}}{(1-a) x_1^{*a} y_1^{*-a}} = \frac{a x_2^{*a-1} y_2^{*1-a}}{(1-a) x_2^{*a} y_2^{*-a}} \Rightarrow \frac{x_1^*}{y_1^*} = \frac{x_2^*}{y_2^*} \tag{1}$$

. If we plug in the endowment constraints, equation 1 simplifies to:

$$\frac{x_1^*}{y_1^*} = \frac{x_2^*}{y_2^*} = \frac{e_x}{e_y} \tag{2}$$

, where all $x_i^*, y_i^* > 0$, $\forall i \in \{1, 2\}$. Therefore, any Pareto Optimal division of the endowment will take the form:

$$(x_1^*, y_1^*) = \alpha \left(e_x, e_y\right); \left(x_2^*, y_2^*\right) = (1-\alpha)\left(e_x, e_y\right), \, \alpha \in [0, 1].$$

Part (b)

FOCs for Pareto optimal interior equilibrium with quasi-linear preferences is that the indifference curves of consumer 1 and 2 are tangent to each other. Therefore, the allocation $(\vec{x}^*, \vec{y}^*)$ is Pareto Optimal if:

$$\frac{\frac{\partial u(x_1^*,y_1^*)}{\partial x_1}}{\frac{\partial u(x_1^*,y_1^*)}{\partial y_1}} = \frac{\frac{\partial v(x_2^*,y_2^*)}{\partial x_2}}{\frac{\partial v(x_2^*,y_2^*)}{\partial y_2}} \Rightarrow \frac{1}{f'(y_1^*)} = \frac{1}{f'(y_2^*)} \Rightarrow f'(y_1^*) = f'(y_2^*) \tag{3}$$

, which means that $y_1^* = y_2^*$. If we plug in the endowment constraints, equation 3 simplifies to:

$$y_1^* = y_2^* = \frac{e_y}{2} \tag{4}$$

, where all $x_i^*, y_i^* > 0$, $\forall i \in \{1,2\}$. Therefore, any Pareto Optimal division of the endowment will take the form:

$$(x_1^*, y_1^*) = \left(x_1^*, \frac{e_y}{2}\right); (x_2^*, y_2^*) = \left(e_x - x_1^*, \frac{e_y}{2}\right). \tag{5}$$

Part (c)

Let's consider the case where $x_1^* = 0$, which will generalize to the remaining cases. We know that the marginal rate of substitution for each consumer can be written as follows:

$$MRS_i = \frac{a}{1-a} \cdot \left(\frac{y_i}{x_i}\right).$$

If $x_1^* = 0$ and $y_1^* > 0$ ($\Leftrightarrow x_2^* = e_x$, $y_2^* \geq 0$), then $MRS_1 = \infty$ and $MRS_2 < \infty$. This means that $MRS_1 > MRS_2$, and that there is a Pareto improving trade to be made such that consumer 1 demands good x and supplies good y and consumer 2 demands good y and supplies good x.

If $x_1^* = 0$ and $y_1^* = 0$ ($\Leftrightarrow x_2^* = e_x$, $y_2^* = e_y$), we know that this is a Pareto optimal equilibrium because you can't make anyone better off without making someone else better off.

Part (d)

Consider the case where $x_1^* = 0$ and $y_1^* > 0$ ($\Leftrightarrow x_2^* = e_x$, $y_2^* \geq 0$). Since consumer 1 doesn't have any good 1, the only Pareto improving trade would have consumer 1 demanding good x and supplying good y, while consumer 2 must demand good y and supply good x. The FOC for this trade to occur is:

$$MRS_1 > MRS_2$$

, which means that for there *not* to be a Pareto improving trade, it must be the case that:

$$MRS_1 \leq MRS_2 \Leftrightarrow \frac{1}{f'(y_1^*)} \leq \frac{1}{f'(y_2^*)} \Rightarrow y_1^* \leq y_2^* \Rightarrow y_1^* \leq \frac{e_y}{2}. \tag{6}$$

Therefore, any allocation of the following form is a Pareto optimal equilibrium with $x_1^* = 0$ and $y_1^* > 0$:

$$[(0, y_1^*), (e_x, e_y - y_1^*)]\,,\; y_1^* \leq \frac{e_y}{2} \tag{7}$$

. The same applies to cases where $x_2^* = 0$ and $y_2^* > 0$ ($\Leftrightarrow x_1^* = e_x,\, y_1^* \geq 0$).

$$[(e_x, e_y - y_2^*), (0, y_2^*)]\,,\; y_2^* \leq \frac{e_y}{2} \tag{8}$$

Now consider the case where $x_1^* > 0$ and $y_1^* = 0$ ($\Leftrightarrow x_2^* \geq 0,\, y_2^* = e_y$). Using the same argument from above, the only Pareto improving trade would have consumer 1 demanding good y and supplying good x, while consumer 2 must demand good x and supply good y. The FOC for this trade to occur is:

$$MRS_1 < MRS_2$$

, which means that for there *not* to be a Pareto improving trade, it must be the case that:

$$MRS_1 \geq MRS_2 \Leftrightarrow \frac{1}{f'(y_1^*)} \geq \frac{1}{f'(y_2^*)} \Rightarrow y_1^* = 0 \geq y_2^* \tag{9}$$

. Inequality (9) is a contradiction because $y_2^* = e_y > 0$. Therefore, we can't have an equilibrium where consumer 1 consumers 0 of good y. The same applies to the case where $x_2^* > 0$ and $y_2^* = 0$ ($\Leftrightarrow x_1^* \geq 0,\, y_1^* = e_y$).

Therefore, equations 7and 8 define the corner equilibrium for quasi-linear preferences.

Part (e)

This part of the question is asking you to solve for the competitive/exchange equilibrium (not the Pareto equilibrium). Solving each consumer's UMP, we get:

$$x_1^* = a(e_{x1} + pe_{y1});\; y_1^* = \frac{(1-a)(e_{x1} + pe_{y1})}{p}; \tag{10}$$

$$x_2^* = a(e_{x2} + pe_{y2});\; y_2^* = \frac{(1-a)(e_{x2} + pe_{y2})}{p}$$

. The equilibrium price will clear markets. Recall that we only need to find the equilibrium price that will clear only one of the markets.

$$x_1^* + x_2^* = a(e_{x1} + pe_{y1}) + a(e_{x2} + pe_{y2}) = a(e_x + pe_y) = e_x \Rightarrow p^* = \frac{1-a}{a} \cdot \frac{e_x}{e_y}$$

, which shows that equilibrium price depends on aggregate endowment of each good.

Now we're asked to confirm that this is a Pareto optimal equilibrium (and so the First Welfare Theorem holds). One way to show this is to plug the equilibrium price into 10 and check to see that equation 2 holds.

An easier way to go about this is to see that at a Pareto optimal equilibrium:

$$\frac{x_1^*}{y_1^*} = \frac{e_x}{e_y} \tag{11}$$

, and in the competitive equilibrium:

$$\frac{x_1^*}{y_1^*} = \frac{ap^*}{1-a} = \frac{e_x}{e_y} \; (\because \textit{equation} \, \textbf{??}) \tag{12}$$

. Therefore, the exchange equi. is a Pareto equi.

Part (f)

The key here is that the problem tells you that each consumer has sufficiently large enough wealth for the FOCs to hold with equality and we have interior solutions. Solving the UMP results in the following:

$$x_1^* = (e_{x1} + pe_{y1}) - p\frac{e_y}{2};\, x_2^* = (e_{x2} + pe_{y2}) - p\frac{e_y}{2};\, y_1^* = y_2^* = \frac{e_y}{2}. \tag{13}$$

Further, the UMP FOCs result in the equilibrium price for good y:

$$\frac{f'(y_1^*)}{p} = \frac{f'(y_2^*)}{p} = 1 \Rightarrow p^* = f'\left(\frac{e_y}{2}\right)$$

, which doesn't depend on the individual endowments.

Note also that 13 is the same condition as 4, which means that the exchange equilibrium is a Pareto optimal equilibrium.

Part (g)

Now the condition that there is sufficient wealth fails. Let's assume that consumer 1 doesn't have enough wealth to consume $\frac{e_y}{2}$ units of good y at the equilibrium price we solved for in (f) above: $p^* \cdot \frac{e_y}{2} \geq e_{x1} + p^* e_{y1}$. This means that there is an excess supply of good y (since at p^* consumer 2 still consumes $\frac{e_y}{2}$, therefore, the new equilibrium price for good y, $p^{**} < p^*$. At the new equilibrium price p^{**}, consumer 1's consumption bundle is:

$$\left(0, y_1^{**} = \frac{e_{x1}}{p^{**}} + e_{y1} < \frac{e_y}{2}\right)$$

, and consumer 2's consumption bundle is:

$$\left(e_x, y_2^{**} = f_y^{-1}(p^{**}) > \frac{e_y}{2}\right).$$

Notice that the following equation implicitly solves for the new equilibrium price, p^{**}:

$$\frac{e_{x1}}{p^{**}} + e_{y1} + f_y^{-1}(p^{**}) = e_y. \tag{14}$$

Finally, does this represent a Pareto optimal equilibrium? Yes. See equation 7. Notice that the Pareto optimal equilibrium first order conditions change once we are at a border.

2 Practice Problem 17. 3

Part (a)

Each consumer has a quasi-linear utility function but preferences over the non-numeraire good are defined by a convex function (not a concave function), such that $u(x_i, y_i) = x_i + f(y_i)$, where $f'(\cdot) > 0$; $f''(\cdot) > 0$. Therefore, unlike quasi-linear functions with concave utility, the marginal utility of the non-numeraire good is increasing as you increase the consumption of the non-numeraire good, while the marginal utility of the numeraire good is fixed. Therefore, if $\frac{f'(0)}{p_y} < \frac{1}{p_x}$, then the consumer only consumes good x, the numeraire good, and if $\frac{f'(0)}{p_y} \geq \frac{1}{p_x}$, then the consumer only consumes good y, the non-numeraire good.

Part (b)

From part (a) above, we know that in any exchange equilibrium consumer 1 will consume all of good x or all of good y (the same holds for consumer 2). Since each consumer has endowment equal to (1,1), total wealth for each consumer is equal to $1 + p$.

Therefore, consumer 1 will consume all of good y if and only if:

$$1 + p \leq \frac{1}{3} \cdot \left(\frac{1+p}{p} \right)^2 \Rightarrow 3p^2 - p - 1 \leq 0 \Rightarrow p \in \left[0, \frac{1+\sqrt{13}}{6} \approx 0.76 \right] \tag{15}$$

. On the other hand, consumer 2 will consume only good y if and only if:

$$1 + p \leq 3 \left(\frac{1+p}{p} \right)^2 \Rightarrow p^2 - 3p - 3 \leq 0 \Rightarrow p \in \left[0, \frac{3+\sqrt{21}}{2} \approx 3.79 \right] \tag{16}$$

Part (c)

The *only* exchange equilibrium has consumer 1 consuming good x and consumer 2 consuming good y (which is intuitive, since the marginal utility of good y is greater for consumer 2). Therefore, the equilibrium price at which markets clear, $p^* \in [0.76, 3.79]$, and consumer 1 consumes $1 + p$ which has to equal the total endowment of good x:

$$1 + p = 2 \Rightarrow p^* = 1 \in [0.76, 3.79] \tag{17}$$

Note that we can't use the marginal rate of substitution to calculate the equilibria in this case because we're dealing with a non-concave function for the non-numeraire good.

And the resulting competitive/exchange equilibrium:

$$[(2,0),(0,2)] \tag{18}$$

Part (d)

Because there is a numeraire good (both consumers have quasi-linear utility), we can add utility functions. Therefore, one possible Pareto optimal equilibrium $[\vec{x}^*, \vec{y}^*]$ maximizes:

$$\begin{aligned} x_1^* + x_2^* + \frac{1}{3}y_1^{*2} + 3y_2^{*2} &= 2 + \frac{1}{3}y_1^{*2} + 3y_2^{*2} = 2 + \frac{1}{3}y_1^{*2} + 3\left(2 - y_1^*\right)^2 \\ &\Rightarrow y_1^* = 0,\ y_2^* = 2 \end{aligned}$$

. The resulting Pareto optimal equilibrium is the following:

$$[(x_1^*, 0), (2 - x_1^*, 2)] \tag{19}$$

, where $x_1^* \in [0,2]$. The exchange equilibrium identified in 18 above is contained within the Pareto optimal set identified in 19 above, and so the First Welfare Theorem applies.

Part (e)

Recall that in partial equilibrium analysis, the numeraire good could take on negative values so that consumers could consume the optimal amount of the numeraire good. This is not the case here since good x is limited by its endowment to 2. The result is Pareto optimal equilibria of the form found in 19 above. Note that equilibria of the form following form are Pareto optimal, but not exchange equilibria:

$$[(x_1^*, 0), (2 - x_1^*, 2)],\ x_1^* \in [0,2)$$

Therefore, the Second Welfare Theorem does not hold because there are Pareto optimal equilibria that cannot be sustained in a competitive equilibrium, which has each consumer only consuming one of the goods.

Part (f)

The initial endowment is the same as before and the same logic applies. The only possible exchange equilibrium has consumer 1 consuming all of good x and consumer 2 consuming all of good y. The equilibrium price will also remain the same at $p^* = 1$.

Recall from 15 and 16 above that the following inequalities must hold:

$$
\begin{aligned}
A\left(\frac{1+p}{p}\right)^2 &\leq 1+p \\
B\left(\frac{1+p}{p}\right)^2 &\geq 1+p
\end{aligned}
$$

, which results in the following at $p^* = 1$:

$$
\begin{aligned}
A &\leq \frac{1}{2} \\
B &\geq \frac{1}{2}
\end{aligned}
$$

. These conditions are necessary for there to be an exchange equilibrium.