Section Notes 8

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Agenda

1. Dominance and Iterated Dominance
2. Mixed Strategies and Dominance
3. Nash Equilibrium (hereinafter "NE")
4. Mixed Strategy Nash Equilibrium (hereinafter "MSNE")
5. Nash Equilibrium Revisited

1 Dominance and Iterated Dominance

1.1 Definitions

Define the player set $I \equiv \{1, ..., n\}$ and consider $i, k \in I$.

- Player i's strategy s_i is *strictly dominant* iff $u_i(s_i, \vec{s}_{-i}) > u_i(\hat{s}_i, \vec{s}_{-i})$, $\forall \hat{s}_i \in S_i$, $\hat{s}_i \neq s_i$ and $\forall \vec{s}_{-i} \in \times_{k \in I, k \neq -i} S_k$.
- Player i's strategy s_i is *weakly dominant* iff $u_i(s_i, \vec{s}_{-i}) \geq u_i(\hat{s}_i, \vec{s}_{-i})$, $\forall \hat{s}_i \in S_i$, $\hat{s}_i \neq s_i$ and $\forall \vec{s}_{-i} \in \times_{k \in I, k \neq -i} S_k$; and $u_i(s_i, \vec{s}_{-i}) > u_i(\hat{s}_i, \vec{s}_{-i})$, $\forall \hat{s}_i \in S_i$, $\hat{s}_i \neq s_i$ and for some $\vec{s}_{-i} \in \times_{-i} S_k$
- Player i's strategy s_i is *strictly dominated* iff $\exists \hat{s}_i \in S_i$, $\hat{s}_i \neq s_i$ s. t. $u_i(s_i, \vec{s}_{-i}) < u_i(\hat{s}_i, \vec{s}_{-i})$, $\forall \vec{s}_{-i} \in \times_{k \in I, k \neq -i} S_k$.
- Player i's strategy s_i is *weakly dominated* iff $\exists \hat{s}_i \in S_i$, $\hat{s}_i \neq s_i$ s. t. $u_i(s_i, \vec{s}_{-i}) \leq u_i(\hat{s}_i, \vec{s}_{-i})$, $\forall \vec{s}_{-i} \in \times_{k \in I, k \neq -i} S_k$; and $u_i(s_i, \vec{s}_{-i}) < u_i(\hat{s}_i, \vec{s}_{-i})$, for some $\vec{s}_{-i} \in \times_{-i} S_k$.

Note that I used $\vec{s}_{-i} \in \times_{k \in I, k \neq -i} S_k$ to represent the *pure* strategy profiles of the "other" players. However, keep in mind that the definitions above apply to *mixed* strategies ($\vec{\sigma}_{-i} \in \times_{-i} \Sigma_k$) as well, or combinations of both.[1] Also note that although I used Bernoulli utility functions in the definitions, we could switch these out with expected utility functions, $U(\cdot)$.

[1] More on this below when we consider mixed strategies.

1.2 Examples

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

	L	M	R
U	5,3	2,2	2,-2
C	0,0	3,5	0,-2
D	0,0	0,2	3,1

Note that the order of eliminating strictly dominated strategies does not matter. However, the same does not apply when we're dealing with *weakly* dominated strategies. Check Problem 12.3 in the Practice Problems to see why.

2 Mixed Strategies and Dominance

Definition. In the normal form game (simultaneous move), suppose that $S_i \equiv \{s_{i1}, ..., s_{ik}, ..., s_{iK}\}$, then a mixed strategy for player i is a probability distribution over the possible strategies, $\vec{p}_i = [p_{i1}, ..., p_{ik}, ..., p_{iK}]$, where $p_{ik} \in [0,1]$ and $\sum_{k=1}^{K} p_{ik} = 1$. Consider the following games and check to see which pure strategies are strictly dominated.

	L	R
U	-1,3	5,-1
C	5,-1	-1,3
D	0,3	0,2

	L	M	R
U	3,1	1,0	4,0
C	1,3	2,4	1,2
D	0,1	1,1	2,2

In the first example the mixed strategy of the form $\vec{p}^* = [p, 1-p, 0]$, over the pure strategies $[U, C, D]$, where $p \in \left(\frac{1}{6}, \frac{5}{6}\right)$, strictly dominates strategy D for player 1; and in the second example, a mixed strategy of the form $\vec{q}^* = [q, 1-q, 0]$, over the pure strategies $[U, C, D]$, where $q \in \left(\frac{1}{3}, 1\right)$, strictly dominates strategy D for player 1. Therefore, in both examples, strategy D is strictly dominated.

3 Nash Equilibrium[2]

Definition 3.1. In the normal form game where the n player set is I, the strategy profile $(s_1^*, ..., s_I^*)$ is a NE if, for each player i, her strategy s_i^* is a best response to the strategies specified for the other $n-1$ players, or for $\vec{s}_{-i} = (s_1^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_I^*)$. Formally,

$$u_i(s_i^*, \vec{s}_{-i}) \quad \geq \quad u_i(s_i', \vec{s}_{-i}),\ \forall i,\ \forall s_i' \in S_i,$$

or

[2] I'm not going to go through solving for NE in the normal form of the game.

$$s_i^* = \arg\max_{s_i} u_i(s_i, \vec{s}_{-i}) \, \forall i \in I$$

where

$$\vec{s}_{-i} = (s_1^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_I^*)$$

This has a number of less formal intepretations:

1. *Given what others are doing*, there is no profitable deviation for any player.
2. The point where the best response correspondences of all of the players meet, or where each player is best responding to each other players' best response.

Personally, I think the easiest way to understand NE is as follows: Given (or *guess*) a specific strategy profile, *checking* to see whether such strategy profile is NE, is the same as *checking* to see whether there are any deviations that would be profitable for any of the players *given that you do not change the strategies of the other players specified by such strategy profile.*

4 MSNE

If a player "mixes" strategies, or puts *positive probability* on pure strategies, she must be *indifferent* between the strategies. Let's look at the following example to see how we get mixed strategies.

	L	C	R
U	3,3	0,0	1,1
M	0,0	3,3	2,1
D	2,2	2,2	3,1

$\Rightarrow$

	L	C
U	3,3	0,0
M	0,0	3,3
D	2,2	2,2

1. First, iteratively eliminate *strictly dominated* strategies. In other words, mixed strategies will not put positive probability on a *strictly dominated* strategy. Does this extend to *weakly dominated* strategies? We'll discuss this when we cover Trembling Hand Perfect Nash Equilibrium (hereinafter "THPNE"). We can eliminate "R" because it is strictly dominated by player 2's mixed strategy: $\vec{p} = \left[\frac{1}{2}, \frac{1}{2}, 0\right]$, and infinitely many others.
2. Second, assuming that the probability of player 2 playing "L" is $\pi \in (0, 1)$, let's draw the expected utilities of player 1 for varying values of π.[3] Since a player mixes when she is indifferent between strategies, we can see that player 1 mixes "U" and "D" when $\pi = \frac{2}{3}$, and mixes "M" and "D" when $\pi = \frac{1}{3}$. Why don't we consider "U" and "M"?
   Further, player 1 is mixing only two out of three strategies. Could we have

[3] Graph omitted.

an instance where player 1 mixes all three strategies? Normally the answer is no. And to see why, note that the three lines representing the expected utilities of player 1 do not meet at one point. Of course, if the expected utility of player 1 playing "D" was equal to $\frac{3}{2}$, then we could have player 1 mixing all three strategies. There is a question on this week's problem set which covers this.

3. To continue with the example, assume that player 1 is mixing "U" and "D" and the probability of player 2 playing "L", or $\pi = \frac{2}{3}$. Since, player 2 is playing a mixed strategy of "L" and "C", it must also be the case that *player 2* is indifferent between "L" and "C". Given that the probability of player 1 playing "U" is equal to μ, then the expected utility (denoted $U(\cdot;\cdot)$) of player 2 for playing "L" and "C" can be written as:

$$\begin{aligned} U_2(L;\mu) &= 3\mu + 2\cdot(1-\mu) = 2+\mu \\ U_2(C;\mu) &= 0\mu + 2\cdot(1-\mu) = 2-2\mu, \end{aligned}$$

   resulting in $\mu = 0$. This means that player 1 is playing a pure strategy "D" which is a contradiction. This can't be a mixed strategy.

4. Now assume that player 1 is mixing "M" and "D" and the probability of player 2 playing "L" is now $\pi = \frac{1}{3}$. Since, player 2 is playing a mixed strategy of "L" and "C", it must also be the case that *player 2* is indifferent between "L" and "C". Given that the probability of player 1 playing "M" is equal to η, then the expected utility of player 2 for playing "L" and "C" can be written as:

$$\begin{aligned} U_2(L;\eta) &= 0\eta + 2\cdot(1-\eta) = 2-2\eta \\ U_2(C;\eta) &= 3\eta + 2\cdot(1-\eta) = 2+\eta, \end{aligned}$$

   resulting in $\eta = 0$. This means that player 1 is again playing a pure strategy "D" which is a contradiction. This can't be a mixed strategy.

5. Then what is the MSNE? We know that there should be one (at least in the games that we will be dealing with).[4] The above analysis leads to the conclusion that given the mixed strategy of player 2, player 1 doesn't have a mixed strategy best response. Therefore, it must be the case that player 1 is playing a pure strategy while player 2 is playing a mixed strategy at the MSNE.
   From the graph above, we see that if $\pi \in \left[\frac{1}{3}, \frac{2}{3}\right]$, then the best response by player 1 is to play "D" and player 2 is indifferent between playing any of the series of mixed strategies where $\pi \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and the probability of playing "C" is $1-\pi$.

[4] See the question on number and oddness of NE in this week's problem set.

5 Nash Equilibrium Revisited[5]

5.1 Existence of NE

5.1.1 Fixed Point Theorems[6]

For those interested, here are the Fixed Point Theorems used in Nash's famous paper:

Theorem 5.1. *Brouwer's Fixed Point Theorem: Suppose that $\mathbb{A} \subset \mathbb{R}^n$ is a nonempty compact, convex set, and that $f : \mathbb{A} \to \mathbb{A}$ is a continuous function from $\mathbb{A}$ unto itself. Then $f(\cdot)$ has a fixed point, or there exists an $\vec{x}^* \in \mathbb{A}$ such that $f(\vec{x}^*) = \vec{x}^*$.*

Note that the key here is that the domain and the target of the function f are the identical, compact, and convex; just like the strategy space of the aggregate best response correspondence we saw in class.

The version for correspondences, which applies to the best response correspondence graphs we will see later on, is:

Theorem 5.2. *Kakutani's Fixed Point Theorem: Suppose that $\mathbb{A} \subset \mathbb{R}^n$ is a nonempty compact, convex set, and that $f : \mathbb{A} \to \mathbb{A}$ is an upper hemi-continuous correspondence from $\mathbb{A}$ unto itself with the property that the set $f(\vec{x}) \subset \mathbb{A}$ is nonempty and convex for every $x \in \mathbb{A}$. Then $f(\cdot)$ has a fixed point, or there exists an $\vec{x}^* \in \mathbb{A}$ such that $f(\vec{x}^*) = \vec{x}^*$.*

5.1.2 Best Response Correspondences/Functions

We define the aggregate best response correspondence: $\vec{b}(\Theta_1, ..., \Theta_I) = b_1(\Theta_1, ..., \Theta_I) \times b_2(\Theta_1, ..., \Theta_I) \times ... \times b_I(\Theta_1, ..., \Theta_I)$, or

$$\vec{b} : \Theta_1 \times ... \times \Theta_I \to \Theta_1 \times ... \times \Theta_I.$$

Further, for all player i, who has a total of n finite strategies available to her, we can think of $\Theta_i \subseteq [0,1]^n$, which is compact and convex. This means that $\Theta \equiv \Theta_1 \times ... \times \Theta_I$ is also compact and convex. Recall that a pure strategy can be thought of as a degenerate mixed strategy—*e.g.* if player i has four possible strategies: $S_i \equiv \{(In, Fight); (In, Acquiece); (Out, Fight); (Out, Acquiesce)\}$, the pure strategy $(In, Fight)$ can be written as $p = (1,0,0,0) \in [0,1]^4$.

If we assume that the best response functions/correspondences are continous/upper hemi-continuous, we have all of the requirements to apply one of the Fixed Point Theorems Chris went over in class.

5.2 Best Response vs Dominance

Example 5.3. Consider the following game:

[5] Only for those who are interested in some of the technical aspects of NE.

[6] >From Section M.I of MWG.

	L	C	R
U	3,3	0,0	1,1
M	0,0	3,3	2,1
D	2,2	2,2	3,1

First, notice that the strategy "R" is strictly *dominated*. Can you provide an example of a strategy that strictly *dominates* "R"? Second, in the reduced game, what is the NE? Do you see that it is (U,L) and (M,C)? Finally, since strategy "D" is not a best response to any pure strategy of player 2 (either "L" or "C"), does this mean that the strategy "D" is strictly *dominated* for player 1? For what strategies of player 2 would "D" be a best response? If you assume that the probability of player 2 playing "L" is equal to π, and player 2 is playing a mixed strategy equal to $\vec{p} = [\pi, 1-\pi, 0]$, then for values of $\pi \in \left[\frac{1}{3}, \frac{2}{3}\right]$, note that "D" is a best response for player 1.