Homework assignment 1: Foundations of quantum mechanics

Note: This is the first of three homework assignments. In each homework assignment, you can collect up to four points. After all three homework assignments are completed, you need at least 6 points to participate in the final exam, and at least 9 points to qualify for passing the exam with grades 4 or 5 (Chalmers) or VG (GU), respectively.

Task 1: Sequential Stern-Gerlach type measurement. Consider a beam of spin $\frac{1}{2}$ atoms going through a series of Stern-Gerlach type measurements.

1. The first accepts $s_{z,-} = -\frac{\hbar}{2}$ and rejects $s_{z,+} = +\frac{\hbar}{2}$
2. The second measurement accepts $s_{n,-} = -\frac{\hbar}{2}$ and rejects $s_{n,+} = +\frac{\hbar}{2}$, where $s_{n,\pm}$ are the eigenvalues of the operator $\vec{S} \cdot \hat{\vec{n}}$, with $\hat{\vec{n}} = (\sin(\gamma), 0, \cos(\gamma))^T$.

   ($\vec{S}$ is the spin operator $\vec{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)^T$ and σ_i are the Pauli matrices.)
3. The third measurement accepts $s_{z,+} = +\frac{\hbar}{2}$ and rejects $s_{z,-} = -\frac{\hbar}{2}$

Please answer the following questions:

a) Briefly explain the term "wave-function collapse" on the example of the first Stern-Gerlach measurement, concerning S_z.

b) Compute the intensity of the final $s_{z,+}$ beam, when the intensity of the first beam with $s_{z,-}$ is normalized to unity.

c) For which angle γ do you obtain the maximal intensity?

1 point

Task 2: Linear algebra.

a) Use the definition of the commutator and anticommutator to show:

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$$

b) Use the bra-ket formalism and show:

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

c) Assume $|n\rangle$ and $|p\rangle$ form complete orthonormal bases of a Hilbert space and evaluate:

$$\sum_n \phi_n^*(p)\phi_n(p'), \qquad \phi_n(p) = \langle p|n\rangle$$

Hint: scalars commute!

1 point

Task 3: Exactly solvable models.

a) Show that the Pauli matrices anticommute: $\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$.

b) Use the above property to compute the eigenvalues ($\mathrm{H}\,|\psi\rangle = E\,|\psi\rangle$) of the following Hamiltonian:

$$\mathrm{H} = a_0\sigma_0 + \sum_{i=1}^{3} a_i\sigma_i,$$

with σ_0 being the identity matrix.

Hint: calculate $(E - a_0)^2$.

c) Let $\gamma_{ij} = \sigma_i \otimes \sigma_j$ denote the Kronecker product of the two Pauli matrices σ_i and σ_j. Show that one can construct four matrices γ_{ij} which mutually anticommute. Find one such example.

Hint: $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$.

d) Assume you found four anticommuting matrices α_i ($i = 1, \ldots, 4$), $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$. What are the eigenvalues of the so-called Dirac Hamiltonian

$$\mathrm{H} = \sum_{i=1}^{3} c p_i \alpha_i + \alpha_4 m c^2$$

Remark: the matrix α_4 *is usually denoted with* β*. However, the algebraic structure is the same.*

Historical remark: In a successful attempt of finding a Lorentz-invariant theory for the electron, Paul Dirac formulated the above Hamiltonian in 1928, i.e., about 3 years after the Schrödinger equation. However, the negative energy solutions were puzzling at the time and first miss-interpreted as protons. 1931 Dirac published a follow-up paper identifying the negative energy solutions as "anti-electrons", which were experimentally confirmed by Anderson in 1933. The journal editor convinced Anderson to call the anti-electrons positrons.

1 point

Task 4: Density matrix. Verify the following properties of the density matrix, $\rho = \sum_i p_i \,|\psi_i\rangle\,\langle\psi_i|$, with $0 \le p_i \le 1$, and $\sum_i p_i = 1$.

a) $\mathrm{Tr}\,\rho = 1$ (note that $|\psi_i\rangle$ are NOT necessarily orthogonal).

b) ρ is positive semi-definite, i.e., $\langle\phi|\,\rho\,|\phi\rangle \ge 0$ for all $\phi \in \mathcal{H}$.

c) $\mathrm{Tr}\,\rho^2 \le 1$ (hint: show that $\mathrm{Tr}\,\rho^2 < (\mathrm{Tr}\,\rho)^2$)

d) For a pure state, $\mathrm{Tr}\,\rho^2 = 1$

1 point

Appendix

The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$