Closed-Form Approximation of LoRa Modulation BER Performance

Tallal Elshabrawy and Joerg Robert

***Abstract*— LoRa has been demonstrated as a front runner when it comes to evolving low-power wide area networks. At its core is LoRa modulation which depicts a patented chirp spread-spectrum modulation technique that supports energy-efficient and reliable long-range communication. The underlying bit error rate (BER) performance of LoRa modulation has not been yet rigorously analyzed in the literature. In this letter, closed-form approximations for LoRa BER performance are derived for both additive white Gaussian noise and Rayleigh fading channels. Accuracy of the derived approximations is confirmed by comparisons against numerical results.**

***Index Terms*— LP-WAN, LoRa, chirp modulation, spread spectrum, BER performance analysis.**

I. Introduction

COMMON CONSENSUS for the realization of a multitude of Internet-of-Things (IoT) scenarios particularly within smart cities envisions deployment over Low Power Wide Area Networks (LP-WAN) [1]. Recently, LoRa has been exhibiting tremendous commercial growth to establish itself among the front runners of emerging LP-WAN. LoRa has been primarily pillared on the patented LoRa modulation. LoRa modulation features a chirp spread spectrum modulation that supports energy-efficient/reliable long-range communication. Given its patented protection, thorough theoretical description for LoRa modulation may still be considered sparse in the scientific research community state-of-the-art. In fact, the first research to shed some light on the mathematical representation and signal processing-based implementation for LoRa modulation has been recently presented in [2]. Vangelista [2] also refers to some high level description presented in [3] as well as some theoretical analog-based description in [4] and [5]. While the theoretical description of LoRa modulation is becoming evident, to the best of the authors' knowledge, the theoretical BER analysis of LoRa modulation is still completely lacking. The only formula for LoRa BER performance reported in the literature has been estimated through curve fitting of simulated data in [5]. Nevertheless, LoRa modulation may still be categorized under the classical problem of error probability of orthogonal signaling with non-coherent detection. The underlying BER performance has been well-established to be evaluated numerically rather than in closed form [6, Sec. 4.5-3, pp. 216]. Accordingly, the main contribution of this letter is the derivation of closed form approximations for LoRa BER performance in both AWGN and Rayleigh fading channel environments. The accuracy of the derived approximations is then confirmed by comparisons against numerical results.

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II. Basics of LoRa Chirp Modulation

LoRa adopts a frequency shift chirp modulation scheme [2]. Consider a LoRa system in the baseband where the chirp signals have bandwidth B. Consequently, one LoRa sample is transmitted every $T = 1/B$. LoRa modulation is achieved by spreading the frequency change of the chirp signal across 2^{SF} samples within a symbol duration of $T_S = 2^{SF} \cdot T$, where $SF \in \{7, 8, ..., 12\}$ depicts the spreading factor. The LoRa encoder maps every SF bits to a symbol $s_k = k$, $k \in \{0, 1, 2, \ldots, 2^{SF} - 1\}$. Each LoRa symbol s_k is characterized by a frequency offset $f_k = B \cdot k/2^{SF}$ that sets the starting frequency of the corresponding chirp signal. During the symbol duration T_S, the signal frequency corresponding to s_k increases linearly with a slope of $B^2/2^{SF}$ from f_k until it reaches B. The chirp signal frequency then wraps around to 0 before it continues to increase linearly again until the end of T_S. With a sampling rate of $1/T$, the LoRa waveforms $w_k(nT)$, $k \in \{0, 1, 2.., 2^{SF} - 1\}$, corresponding to the LoRa symbols s_k are comprised of 2^{SF} samples and are expressed as

$$
\begin{aligned}
w_k(nT) &= \sqrt{E_S} \times \omega_k(nT) \\
&= \sqrt{\frac{E_S}{2^{SF}}} \exp\left[j2\pi \cdot (k+n) \mod 2^{SF} \cdot \frac{n}{2^{SF}}\right],
\end{aligned}
\tag{1}
$$

where $\omega_k(nT)$, $k \in \{0, 1, 2..., 2^{SF} - 1\}$, are the 2^{SF} orthonormal basis functions that characterize the multi-dimensional space of LoRa signaling, E_S depicts the symbol energy, and $n = 0, 1, 2, \ldots, 2^{SF} - 1$ depicts the sample index at time nT.

LoRa demodulation is based on the orthogonality of chirp signals with different offsets [2], where the correlation properties of the 2^{SF} possible LoRa waveforms are governed by

$$
\sum_{n=0}^{2^{SF}-1} w_k(nT) \cdot \omega_i^*(nT) = \begin{cases} \sqrt{E_S} & i = k \\ 0 & i \neq k \end{cases}, \tag{2}
$$

where $\omega_i^*(nT)$ is the complex conjugate of the basis function $\omega_i(nT)$ that corresponds to the LoRa symbol $s_i = i$.

Accordingly, for a received signal $r_k(nT)$ that corresponds to a transmitted waveform $w_k(nT)$ over an AWGN channel, the correlator outputs of the LoRa demodulator are given as

$$
\sum_{n=0}^{2^{SF}-1} r_k(nT) \cdot \omega_i^*(nT) = \begin{cases} \sqrt{E_S} + \phi_i & i = k \\ \phi_i & i \neq k \end{cases}, \tag{3}
$$

where ϕ_i depicts a complex Gaussian noise process.

The detection of LoRa symbols reverts to selection of the index of the LoRa waveform that has the highest correlation magnitude with the received signal. Therefore, for a received waveform $r_k(nT)$ that corresponds to a transmitted symbol $s_k = k$, the detected symbol $\widetilde{s}_k$ is computed as

$$
\widetilde{s}_k = \arg_i \max\left(\left|\delta_{k,i}\sqrt{E_S} + \phi_i\right|\right), \tag{4}
$$

where $\delta_{k,i} = 1$ for $i = k$ and $\delta_{k,i} = 0$ otherwise.

Let us define $\rho_i = |\phi_i|$ as the magnitude of the complex noise envelope. Given that ϕ_i is a complex zero-mean Gaussian noise process, then ρ_i depicts a Rayleigh distributed random variable with the cumulative distribution function

$$F_{\rho_i}(\rho) = 1 - \exp\left[-\frac{\rho^2}{2\sigma^2}\right], \tag{5}$$

where $\sigma^2 = N_0/2$ and N_0 is the single-sided noise power spectral density.

Let us define $P_{e|k}$ as the probability of symbol error given that symbol $s_k = k$ is transmitted. Given the outputs of the correlation process defined in (3), $P_{e|k}$ can be expressed as

$$P_{e|k} = \Pr\left[\max_{i,i\neq k}(\rho_i) > \beta_k\right], \tag{6}$$

where $\beta_k = \left|\sqrt{E_S} + \phi_k\right|$ and β_k accordingly follows a Rician distribution with the shape parameter $\kappa_\beta = E_S/2\sigma^2 = E_S/N_0$.

Let $\hat{\rho} = \max_{i,i\neq k}(\rho_i)$ depict a random variable for the maximum of $2^{SF} - 1$ i.i.d. Rayleigh random variables. The cumulative distribution function for $\hat{\rho}$ can be given as

$$F_{\hat{\rho}}(\rho) = \left[1 - \exp\left[-\frac{\rho^2}{2\sigma^2}\right]\right]^{2^{SF}-1}. \tag{7}$$

Given (6), (7) and assuming equally probable symbols, the average bit error probability P_b can be expressed as

$$P_b = 0.5 \times \int_0^\infty \left[1 - \left[1 - \exp\left[-\frac{\beta^2}{2\sigma^2}\right]\right]^{2^{SF}-1}\right] \cdot f_\beta(\beta)d\beta, \tag{8}$$

where $f_\beta(\beta)$ is the probability density function for the Rician distributed β. Given that LoRa falls under the category of orthogonal signaling, the multiplier of 0.5 before the integral reflects the fact that for each symbol error only half of the underlying symbol bits are expected to be in error [6, Sec. 4.4-1, pp. 205]. The integral in (8) could be shown to be evaluated as [6, Sec. 4.5-3, pp. 218]

$$P_b = 0.5 \times \sum_{q=1}^{2^{SF}-1} \frac{(-1)^{q+1}}{q+1} \cdot \binom{2^{SF}-1}{q} \cdot \exp\left[-\frac{q}{q+1} \cdot \frac{E_S}{N_0}\right]. \tag{9}$$

Given that LoRa depicts a relatively high order of M-ary modulation that reaches $M = 4096$ when $SF = 12$, the combination term in (9) would suffer from precision problems during the evaluation of P_b. The contribution of this letter is that it uses the high order feature of LoRa modulation to derive an approximation for the distribution of $\hat{\rho}$. This in turn leads to the derivation of concise closed form and highly accurate approximation expressions for (8).

III. Closed Form Approximation for LoRa Modulation BER Performance

A. LoRa BER Performance in AWGN Channels

As in (8), the BER performance of LoRa modulation is attributed to the probabilistic comparison of a Rician distributed variable against the maximum of $2^{SF} - 1$ Rayleigh distributed variables. Let us first focus on the Rician distribution with shape parameter $\kappa_\beta = E_S/N_0$ that reflects the correlator output when correlating with the basis $\omega_k(nT)$ that corresponds to the actual transmitted symbol $s_k = k$. Let us define Γ as the SNR for the LoRa communication. Given that

$$\Gamma = \frac{E_s/T_S}{N_0 \cdot B} = \frac{E_S}{N_0 \cdot 2^{SF}}, \tag{10}$$

then the shape parameter for the Rician distributed β can be expressed as $\kappa_\beta = \Gamma \cdot 2^{SF}$. In other words, due to spreading in LoRa, the target symbol enjoys an effective SNR of $\Gamma_{eff} = \Gamma \cdot 2^{SF}$. The Rician distribution for high SNR can be approximated with a Gaussian distribution [7]. Since LoRa modulation deploys $SF \geq 7$, the scale of attainable Γ_{eff} may justify a Gaussian approximation for $f_\beta(\beta)$. Hence,

$$f_\beta(\beta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\left(\beta - \sqrt{E_S}\right)^2}{2\sigma^2}\right], \tag{11}$$

where $\sqrt{E_S}$ and $\sigma^2 = N_0/2$ are the mean and variance of β, respectively.

Let us also assume that the distribution of $\hat{\rho}$ for the maximum of $2^{SF} - 1$ i.i.d. Rayleigh random variables follows a Gaussian distribution. Then, we get

$$f_{\hat{\rho}}(\rho) = \frac{1}{\sqrt{2\pi\sigma_{\hat{\rho}}^2}} \exp\left[-\frac{(\rho - \mu_{\hat{\rho}})^2}{2\sigma_{\hat{\rho}}^2}\right], \tag{12}$$

where $\mu_{\hat{\rho}}$ and $\sigma_{\hat{\rho}}^2$ are the mean and variance for $\hat{\rho}$, respectively.

From (6) and assuming that every symbol error entails an underlying 50% of bits errors, the bit error probability P_b is expressed as

$$P_b = 0.5 \times \Pr\left[\hat{\rho} - \beta > 0\right]. \tag{13}$$

It follows from the Gaussian approximations for $\hat{\rho}$ and β that the bit error probability then becomes

$$P_b \approx 0.5 \times Q\left(\frac{\sqrt{E_S} - \mu_{\hat{\rho}}}{\sqrt{\sigma_{\hat{\rho}}^2 + N_0/2}}\right), \tag{14}$$

with $Q(z) = \frac{1}{\sqrt{2\pi}} \cdot \int_z^\infty \exp\left[-\frac{u^2}{2}\right] du$ being the Q-function, i.e., the tail function of the standard normal distribution. It is evident that a closed form approximation for P_b in (14) is attainable if $\mu_{\hat{\rho}}$ and $\sigma_{\hat{\rho}}^2$ corresponding to the mean and variance of $\hat{\rho} = \max_{i,i\neq k}(\rho_i)$, respectively, are derived. Let us define $\gamma_i = \rho_i^2$, where ρ_i is Rayleigh distributed and governed by (5). The distribution of γ_i is Chi-square with one degree of freedom, where γ_i has the cumulative distribution function

$$F_{\gamma_i}(\gamma) = 1 - \exp\left[-\frac{\gamma}{2\sigma^2}\right]. \tag{15}$$

Let $\hat{\gamma} = \max_i(\gamma_i)$ depict the maximum of $2^{SF} - 1$ variables γ_i. Accordingly, $\hat{\rho} = \sqrt{\hat{\gamma}}$ and the following relations hold

$$\begin{aligned} E\left[\hat{\rho}^2\right] &= E\left[\hat{\gamma}\right] = \mu_{\hat{\gamma}} \\ E\left[\hat{\rho}^4\right] &= E\left[\hat{\gamma}^2\right] = \sigma_{\hat{\gamma}}^2 + \mu_{\hat{\gamma}}^2, \end{aligned} \tag{16}$$

where $E\left[X^m\right]$ is the m^{th} moment for the distribution of some variable X and $\mu_{\hat{\gamma}}$, $\sigma_{\hat{\gamma}}^2$ are the mean, variance of $\hat{\gamma}$, respectively. Assuming a Gaussian distribution for $\hat{\rho}$, $E\left[\hat{\rho}^m\right]$ could be derived from the m^{th} differential of the moment generating function $M_{\hat{\rho}}[t] = e^{\left(\mu_{\hat{\rho}} t + \sigma_{\hat{\rho}}^2 t^2/2\right)}$ evaluated at $t = 0$. It is then straightforward to derive the following relations

$$\begin{aligned} E\left[\hat{\rho}^2\right] &= \sigma_{\hat{\rho}}^2 + \mu_{\hat{\rho}}^2 \\ E\left[\hat{\rho}^4\right] &= 3\sigma_{\hat{\rho}}^4 + 6\sigma_{\hat{\rho}}^2\mu_{\hat{\rho}}^2 + \mu_{\hat{\rho}}^4. \end{aligned} \tag{17}$$

By combining (16) and (17) it is possible to show that

$$\begin{aligned} \mu_{\hat{\rho}} &= \left(\mu_{\hat{\gamma}}^2 - \sigma_{\hat{\gamma}}^2/2\right)^{1/4} \\ \sigma_{\hat{\rho}}^2 &= \mu_{\hat{\gamma}} - \sqrt{\mu_{\hat{\gamma}}^2 - \sigma_{\hat{\gamma}}^2/2}. \end{aligned} \tag{18}$$

The Chi-square distributed γ_i with one degree of freedom as per (15) essentially depicts an exponentially distributed random variable. Therefore, $\hat{\gamma}$ corresponds to the maximum of $2^{SF}-1$ i.i.d. exponentially distributed variables each with mean $2\sigma^2$. Due to the memoryless property of the exponential distribution, it is possible to alternatively express $\hat{\gamma}$ as the summation of $2^{SF}-1$ non-identical independent exponential distributions with mean values of $\frac{2\sigma^2}{2^{SF}-1}, \frac{2\sigma^2}{2^{SF}-2}, \ldots, 2\sigma^2$. Therefore, the mean of $\hat{\gamma}$ can be expressed as

$$\mu_{\hat{\gamma}} = 2\sigma^2 \cdot \sum_{i=1}^{2^{SF}-1} \frac{1}{i} = 2\sigma^2 \cdot H_{2^{SF}-1}, \tag{19}$$

where $H_m = \sum_{i=1}^{m} \frac{1}{i}$ depicts the m^{th} harmonic number. Additionally, the variance of $\hat{\gamma}$ can be expressed as

$$\sigma^2_{\hat{\gamma}} = 4\sigma^4 \cdot \sum_{i=1}^{2^{SF}-1} \frac{1}{i^2} \approx 4\sigma^4 \cdot \frac{\pi^2}{6}, \tag{20}$$

with $\sum_{i=1}^{m} \frac{1}{i^2}$ being the so-called Basel problem that converges to $\pi^2/6$ for large m [8]. One key remark in (19) and (20) is that while the mean $\mu_{\hat{\gamma}}$ is derived from a divergent function H_m, the variance $\sigma^2_{\hat{\gamma}}$ follows a convergent series of the Basel problem. Accordingly, with $SF \geqslant 7$ in LoRa, $\hat{\gamma}$ exhibits a very low coefficient of variation. Given that $\hat{\rho} = \sqrt{\hat{\gamma}}$, then it could be anticipated that $\hat{\rho}$ features a very low coefficient of variation as well. This could support the applicability of the Gaussian approximation that has been assumed for $\hat{\rho}$ in (12).

Substituting for (18), (19) and (20) in (14) and by utilizing the relation for SNR against E_S/N_0 in (10) results in the LoRa bit error probability approximation expression

$$P_b \approx 0.5 \times Q\left(\frac{\sqrt{\Gamma \cdot 2^{SF}} - \left(\left(H_{2^{SF}-1}\right)^2 - \frac{\pi^2}{12}\right)^{1/4}}{\sqrt{H_{2^{SF}-1} - \left(\left(H_{2^{SF}-1}\right)^2 - \frac{\pi^2}{12}\right)^{1/2} + 0.5}}\right). \tag{21}$$

For large m, the harmonic number H_m approaches

$$H_m \approx \ln(m) + \frac{1}{2m} + 0.57722, \tag{22}$$

where 0.57722 is the Gamma or the Euler-Mascheroni constant [9]. Given that $SF \geq 7$, we can assume that $\left(H_{2^{SF}-1}\right)^2 \gg \frac{\pi^2}{12}$. Moreover, if we consider that $\ln\left(2^{SF}-1\right) \approx \ln\left(2^{SF}\right)$, a more concise approximation for P_b becomes

$$P_b \approx 0.5 \times Q\left(\sqrt{\Gamma \cdot 2^{SF+1}} - \sqrt{1.386 \cdot SF + 1.154}\right). \tag{23}$$

It worth noting that the approximation in (23) is consistent with the form provided in [5] that has been attained using curve fitting of simulation results and has been expressed as

$$P_b = 0.5 \times Q\left(1.28 \cdot \sqrt{\Gamma \cdot 2^{SF}} - 1.28 \cdot \sqrt{SF} + 0.4\right). \tag{24}$$

B. LoRa BER Performance in Rayleigh Channels

In the case of Rayleigh channels, the correlation outputs at the LoRa demodulator from (3) are alternatively given as

$$\sum_{n=0}^{2^{SF}-1} r_k(nT) \cdot \omega_i^*(nT) = \begin{cases} \sqrt{\alpha E_S} + \phi_i & i = k \\ \phi_i & i \neq k \end{cases}, \tag{25}$$

where $\sqrt{\alpha}$ is the Rayleigh distributed complex envelope amplitude and α is the corresponding instantaneous normalized channel power. Accordingly, for the Rayleigh fading scenario, the bit error probability in (13) could be expressed as

$$P_{b|\alpha} = 0.5 \times \Pr\left[\hat{\rho} - \left|\sqrt{\alpha E_S} + \phi_k\right| > 0\right], \tag{26}$$

where $P_{b|\alpha}$ is the conditional bit error probability corresponding to a Rayleigh coefficient $\sqrt{\alpha}$, $\left|\sqrt{\alpha E_S} + \phi_k\right|$ represents the resultant amplitude of correlation between the received signal and the basis function of a LoRa symbol s_k, while $\hat{\rho}$ remains to depict maximum of $2^{SF}-1$ Rayleigh variables.

Since $\hat{\rho}$ exhibits a very low coefficient of variation, let us approximate $\hat{\rho} \approx \sqrt{N_0 \cdot H_{2^{SF}-1}}$ to a constant value. Accordingly, (26) could be reformulated as

$$P_{b|\alpha} \approx 0.5 \times \Pr\left[\left|\sqrt{N_0 \cdot H_{2^{SF}-1}} - \phi_k\right| - \sqrt{\alpha E_S} > 0\right]. \tag{27}$$

Given that ϕ_k follows a Gaussian distribution with variance of $N_0/2$, the term $\left|\sqrt{N_0 \cdot H_{2^{SF}-1}} - \phi_k\right|$ in (27) would then follow a Rician distribution with a shape parameter $\kappa = H_{2^{SF}-1}$. Given that typically in LoRa $SF \geqslant 7$, then $\left|\sqrt{N_0 \cdot H_{2^{SF}-1}} - \phi_k\right|$ would feature a relatively high SNR and could be approximated to a Gaussian distribution. Accordingly, the expression for bit error probability after integrating over the distribution of the channel power α becomes

$$P_b \approx 0.5 \times \int_0^\infty Q\left(\sqrt{2\alpha\Gamma_{eff}} - \sqrt{2H_{2^{SF}-1}}\right) \cdot e^{-\alpha} d\alpha, \tag{28}$$

where $\Gamma_{eff} = E_S/N_0$, α in (28) follows a Chi-square distribution with one degree of freedom with an expected normalized channel power of 1.

The integration in (28) could be solved by parts as

$$P_b \approx -0.5 \times \left[e^{-\alpha} Q\left(\sqrt{2\alpha\Gamma_{eff}} - \sqrt{2H_{2^{SF}-1}}\right)\Big|_0^\infty + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[Q\left(\sqrt{2\alpha\Gamma_{eff}} - \sqrt{2H_{2^{SF}-1}}\right)\right] \cdot e^{-\alpha} d\alpha \right]. \tag{29}$$

Using Leibniz's integral rule, the differentiation term in (29) could be given as

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[Q\left(\sqrt{2\alpha\Gamma_{eff}} - \sqrt{2H_{2^{SF}-1}}\right)\right] - \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\sqrt{2\alpha\Gamma_{eff}} - \sqrt{2H_{2^{SF}-1}}\right)^2}{2}} \cdot \sqrt{2\Gamma_{eff}} \frac{\alpha^{-\frac{1}{2}}}{2}. \tag{30}$$

Using a change of variables and substituting for (30) in (29), the approximation for probability of bit error becomes

$$P_b \approx 0.5 \times \left[Q\left(-\sqrt{2H_{2^{SF}-1}}\right) - \frac{1}{\sqrt{2\pi}} \times \int_{-\sqrt{2H_{2^{SF}-1}}}^{\infty} e^{-\frac{\left[\left(\Gamma_{eff}+1\right)y^2 + 2\sqrt{2H_{2^{SF}-1}}y + 2H_{2^{SF}-1}\right]}{2\Gamma_{eff}}} dy \right]. \tag{31}$$

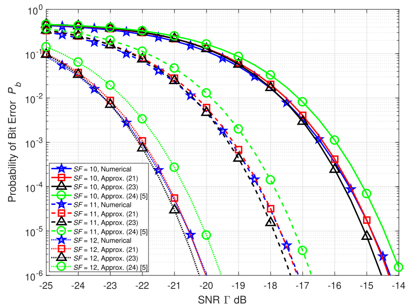

Fig. 1. LoRa BER Performance in AWGN Channel: $SF = 10, 11, 12$.

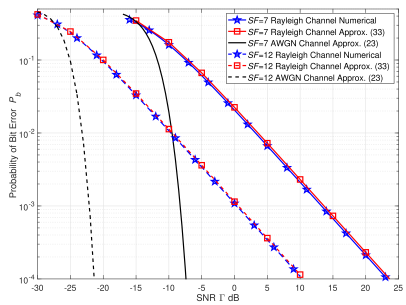

Fig. 2. LoRa BER Performance in Rayleigh Channel: $SF = 7$, $SF = 12$.

By completing the square of the quadratic equation in the exponent and another substitution of variables, (31) becomes

$$P_b \approx 0.5 \times \left[Q\left(-\sqrt{2H_{2^{SF}-1}}\right) - \sqrt{\frac{\Gamma_{eff}}{\Gamma_{eff}+1}}.e^{-\frac{2H_{2^{SF}-1}}{2(\Gamma_{eff}+1)}} \times \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \right], \quad (32)$$

where $x = \sqrt{\frac{\Gamma_{eff}+1}{\Gamma_{eff}}}\left(-\sqrt{2H_{2^{SF}-1}} + \frac{\sqrt{2H_{2^{SF}-1}}}{\Gamma_{eff}+1}\right)$. Given that the integral in (32) depicts the Q-function and $\Gamma_{eff} = 2^{SF}\Gamma$, the approximation for LoRa probability of bit error within a Rayleigh fading environment finally reduces to

$$P_b \approx 0.5 \times \left[Q\left(-\sqrt{2H_{2^{SF}-1}}\right) - \sqrt{\frac{2^{SF}\Gamma}{2^{SF}\Gamma+1}}.e^{-\frac{2H_{2^{SF}-1}}{2.(2^{SF}\Gamma+1)}} \times Q\left(\sqrt{\frac{2^{SF}\Gamma+1}{2^{SF}\Gamma}}.\left[-\sqrt{2H_{2^{SF}-1}} + \frac{\sqrt{2H_{2^{SF}-1}}}{2^{SF}\Gamma+1}\right]\right)\right]. \quad (33)$$

IV. Analytical Results

Figure 1 presents comparisons of the derived LoRa BER approximations in (21) and (23) against the theoretical BER performance computed by numerical solving of the integral in (8) at $SF = 10, 11, 12$. The figures also evaluate the curve fitting solution (24) from [5]. As shown in the figure, the expression in (21) exhibits an extremely accurate approximation while the concise formula in (23) provides a small approximation gap in the order of $0.2\,\text{dB}$. On the other hand, the curve-fitted equation in (24) from [5] exhibits a wider performance gap of around $0.5\,\text{dB}$. It is to be noted that similar tendencies have been witnessed for $SF = 7, 8, 9$. It is important to point out that the LoRa specifications typically report their receiver sensitivities and the corresponding expected coverage ranges within an AWGN type of environment. This could be evident if we consider as an example that the SX1272 data sheet reports a sensitivity of $P_{Rx} = -137\,\text{dBm}$ for LoRa modulation within a bandwidth of $B = 125\,\text{KHz}$ with $SF = 12$ [10]. Assuming a Noise Figure of $NF = 6\,\text{dB}$, such sensitivity would correspond to an SNR $\Gamma = P_{Rx} + 174 - 10.\log_{10}(B) - NF = -20\,\text{dB}$. By referring back to BER performance of $SF = 12$ in Figure 1, it becomes clear that in fact such sensitivity assumes an AWGN channel.

In Figure 2, BER performance curves for LoRa modulation with $SF = 7$, $SF = 12$ in a Rayleigh fading environment using the analytical approximation in (33) are compared to those attained with numerical solving of the integral

$$P_b = 0.5 \times \int_0^{\infty}\int_0^{\infty} \left[1 - \left[1 - \exp\left[-\frac{\beta^2}{2\sigma^2}\right]\right]^{2^{SF}-1}\right] \times f_{\beta|\alpha}(\beta).e^{-\alpha} d\beta\, d\alpha, \quad (34)$$

where similar to (8), $f_{\beta|\alpha}(\beta)$ follows the Rician distribution but with the shape parameter $\kappa_\beta = \alpha E_S/N_0$. The figure confirms the extreme accuracy of the presented analysis. One significant remark in the Rayleigh fading channel is its severe impact on the link budget for LoRa communication. Figure 2 indicates that LoRa would suffer more than a $30\,\text{dB}$ hit to the maximum allowable pathloss for a target BER of $P_b = 10^{-4}$. This could have a severe impact on the ability of LoRa to support its desired long range communication within urban environments where Rayleigh fading characteristics might prevail.

V. Conclusions

In this letter, closed form approximations for the LoRa BER performance in both AWGN and Rayleigh channel environments have been presented. Comparison against numerical results confirm a quite accurate approximation of the presented analysis. The results for the Rayleigh fading environment suggest that LoRa might not be able to sustain long-range communication within urban environments with Rayleigh fading characteristics. Accordingly, research on suitable FEC strategies to help LoRa deal with the underlying link budget loss is of the ultimate essence.

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