Methods for packet combining in HARQ systems over bursty channels *

Abdel-Ghani A. Daraiseh [a] and Carl W. Baum [b]

[a] *Wireless Systems Engineering Department, Northern Telecom, Richardson, TX 75082-4399, USA*
[b] *Department of Electrical and Computer Engineering, Clemson University, Clemson, SC 29634-0915, USA*

The performance of ARQ systems can be improved by combining current and prior transmissions at the receiver. Two techniques for combining outputs in a packet-based communication system are presented. In both techniques the fundamental unit of retransmission is a packet, and the fundamental unit of combining is a codeword. The techniques are analyzed for a bursty channel and a system that employs Reed–Solomon coding and bounded-distance errors-and-erasures decoding. Performance results show that the packet-combining schemes provide significant gains in throughput and reductions in error probability when compared with a system that does not employ combining.

1. Introduction

The performance of communication systems subject to bursty noise and interference can be unacceptable without some form of retransmission approach and/or error control. A Type-I hybrid ARQ (HARQ-I) system can provide significant performance improvements [11,13], but more advanced techniques can result in greater gains.

A class of improved techniques employ Type-II HARQ with variable rate coding. In one technique, a packet is first encoded and transmitted with a high rate code for error detection. If the packet fails, a retransmission is requested and a low rate ($1/m$) code for error detection and correction is sent [8]. A second approach based on MDS codes is to first send half of an $(2n, k)$ code for error detection and correction. If decoding fails, the second half of the code is transmitted, and decoding is attempted on the combined set of data [9,10,13].

The variable rate schemes can provide significant improvements in performance over TYPE-I ARQ [13], but the schemes suffer from added complexity at the transmitter, since packets must be reconfigured with retransmissions. In addition, many of these schemes require multiple decoders or, at minimum, require configurability of a single decoder.

In this paper, we present and analyze two HARQ schemes that employ *packet combining*. In particular, the transmitter continues to send the same packet until no more repeat requests are made, as in HARQ-I, but the results of decoding the current packet is combined with the previous packets in such a way so as to make a more accurate decision about the data. We refer to the resulting decision as a *combined packet*.

The transmission requirements for this scheme are nearly as simple as those for HARQ-I, yet the packet combining techniques can reduce the number of required retransmissions and increase the packet reliability. (The only difference is at most a modest additional buffering requirement.) The techniques we consider exploit the power of Reed–Solomon (RS) codes and take advantage of the increased performance that results from erasing unreliable symbols prior to decoding. In wireless channels, several powerful techniques for determining erasures can be used, including Viterbi's ratio threshold test [1,6] and a technique based on Bayesian decision theory [2–4].

This paper is organized as follows. In section 2, system and channel models are presented. In section 3, the two packet combining techniques are described in detail. In section 4, an analysis of the schemes is presented along with several general performance calculations involving RS codes that may be useful in other applications. In section 5, numerical results are presented to illustrate various aspects of the performance of packet combining in HARQ-I systems. Conclusions are presented in section 6.

2. System and channel models

At the transmitter, data is encoded with an (n, k) singly-extended RS code and configured into packets that each contain N codewords. These packets are not reconfigurable; i.e., packets are retransmitted *as is* according to the rules of the retransmission protocol.

The receiver utilizes a bounded-distance errors-and-erasures correction decoder with correction diameter d_e. With this decoder, any pattern of t errors and e erasures is corrected if $2t + e$ does not exceed d_e. If $2t + e > d_e$, a decoding error (decoding into the wrong codeword) occurs if the input vector is within a diameter d_e of some other codeword, and a decoding failure occurs otherwise. The constant d_e can be chosen to be any value between zero and $n - k$, inclusive.

* This research was supported in part by the Army Research Office under grant number DAAH04-95-1-0247.

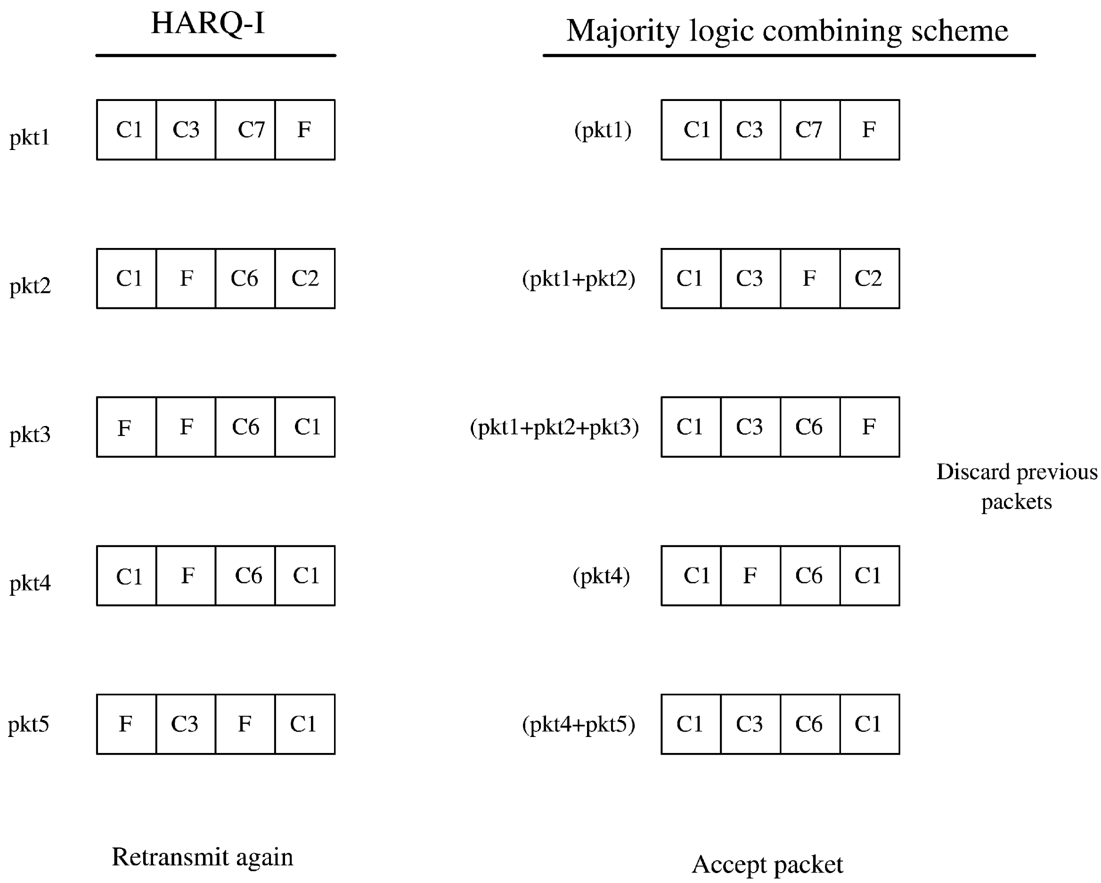

Figure 1. Example of 1-2-3-restart combining.

The channel is assumed to be slowly changing with constant symbol error and erasure probabilities throughout the transmission of a single packet. These probabilities can vary between packet transmissions, however. We model the variation between packets with a Markov chain as follows.

The channel has L states; for the ℓth state, the symbol error and erasure probabilities are $p_t(\ell)$ and $p_e(\ell)$, respectively. Given that the channel is currently in state i, the channel will be in state ℓ for the next packet transmission with probability $P_{i\ell}$. The probability of being in state ℓ during the jth packet transmission is denoted by $\pi_\ell(j)$.

3. Packet combining schemes

In this section we present two techniques for packet combining. With both techniques, decoder outputs of codewords and decoding failures are combined to give aggregate decisions.

3.1. 1-2-3-restart combining

The name for this technique comes from the fact that after three packet retransmissions we throw away previous information and start anew. For example, if a total of five packet transmissions have occurred, only the fourth and fifth packets are combined; if seven have occurred, only the seventh is used.

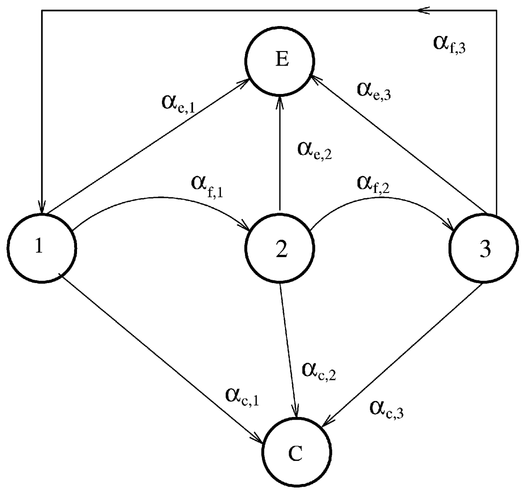

Figure 2. Flow diagram for 1-2-3-restart combining.

Packet combining occurs according to the following rules:

- *Two packets.* Codeword A with codeword A combines into codeword A. Codeword A with codeword B combines into a decoding failure. Codeword A with a decoding failure combines into codeword A. Two decoding failures combine into a decoding failure.

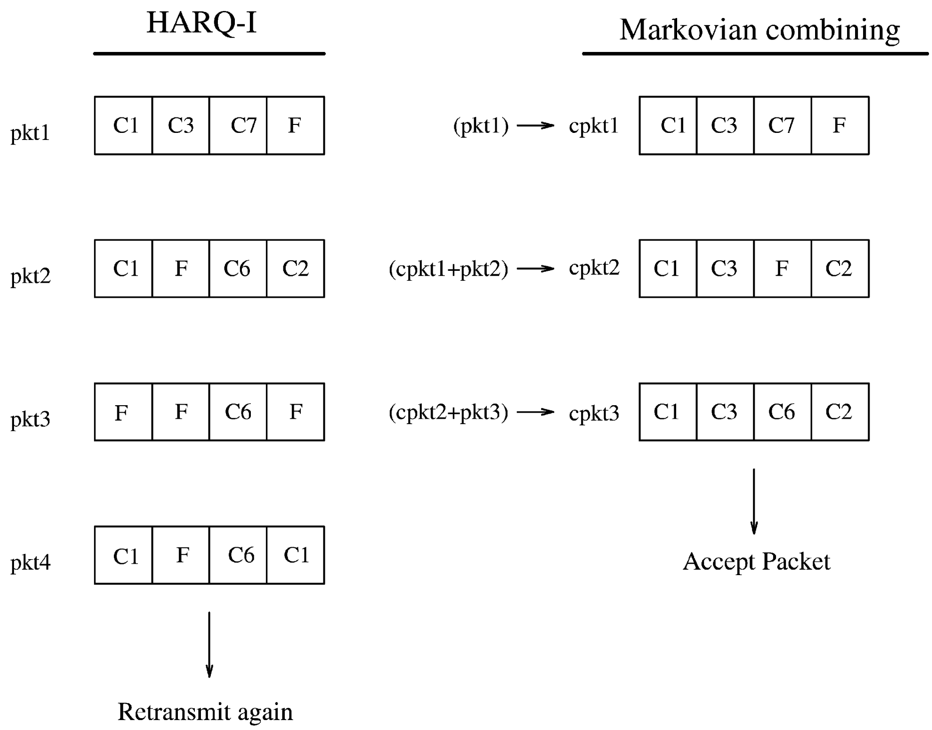

Figure 3. Example of Markovian combining.

- *Three packets*. Two codeword A's with anything else combines into codeword A. Codeword A with codeword B with either codeword C or a failure combines into a failure. Codeword A with two failures combines into codeword A. Three decoding failures combine into a failure.

If any decoding failures exist in the combined packet, then a retransmission is requested. An example of the results of 1-2-3-restart combining is shown in figure 1, and a flow diagram for this scheme is given in figure 2.

3.2. Markovian combining

This method combines the most recently received packet with the previous *combined packet*. Combining occurs according to the rules for two packets in the 1-2-3-restart technique. As with the 1-2-3-restart method, a retransmission occurs if one or more decoding failures exist in the combined packet. An example of the results of Markovian combining is shown in figure 3.

This method of combining can be modeled by a Markov chain (hence the name). A state diagram representing the chain is given in figure 4. The chain has $((N^2+N)/2)+3$ states. State (i,j) represents the state in which the combined packet contains i failures and j correct codewords. State S is the initial state, state C corresponds to the situation in which the combined packet is correct (no codeword errors or failures), and state E is the error state (no codeword failures and at least one codeword error). From state S any other state can be reached, and from any of the numbered states any state except S can be reached. For this reason, the state transition arrows are not included in the diagram.

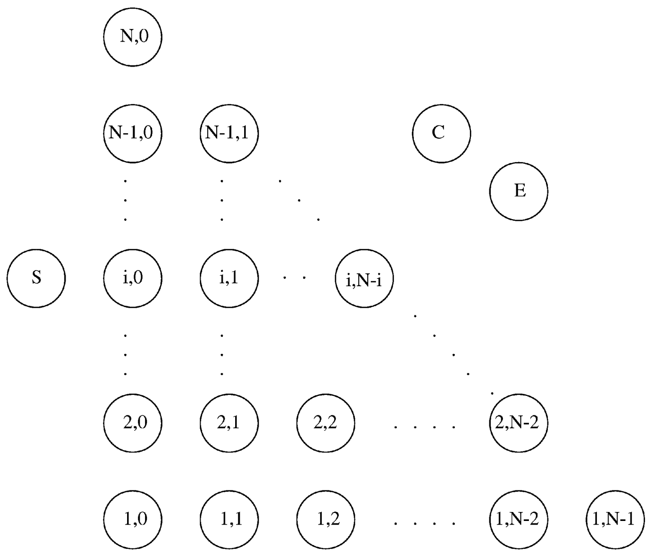

Figure 4. Markov chain representation of Markovian combining.

4. System analysis

4.1. Useful calculations for RS codes

The following results are needed to analyze the combining schemes in this paper. The first result has appeared previously [12]; the others are new. In these results, p_t and p_e denote the probabilities of symbol error and symbol erasure, respectively, and both are assumed to be independent from symbol to symbol.

1. Let A_j denote the number of codewords in an (n,k) RS code with weight j; i.e., $\{A_0, \ldots, A_n\}$ is the weight distribution of the code. For an (n,k) RS code [5],

$$A_j = \binom{n}{j}(2^m - 1)2^{m(j-d_{\min})} \sum_{i=0}^{j-d_{\min}} \binom{j-1}{i} \frac{(-1)^i}{2^{mi}},$$

$j \geqslant d_{\min}$, where $d_{\min} = n-k+1$ and $m = \log_2 n$. Assume that bounded distance decoding is employed with a decoding radius of d_e; i.e., decoding occurs if the received vector lies within a distance d_e of some codeword, and a decoding failure occurs otherwise. Then the probability of decoding correctly, denoted by P_c, is given by

$$P_c = \sum_{i=0}^{d_e} \sum_{j=0}^{\lfloor \frac{d_e-i}{2} \rfloor} \binom{n}{i}\binom{n-i}{j}(p_e)^i(p_t)^j(p_c)^{n-i-j}, \quad (1)$$

where $p_c = 1 - p_t - p_e$. The probabilities of codeword error P_e and codeword failure P_f are given by

$$P_e = \sum_{j=d_{\min}}^{n} A_j P_j^{d_e}, \quad (2)$$

and $P_f = 1 - (P_c + P_e)$, where $P_j^{d_e}$ is the probability of decoding into a sphere of radius d_e surrounding a codeword of weight j. It has been shown that $P_j^{d_e}$ satisfies [12]

$$\begin{aligned} P_j^{d_e} = &\sum_{i_1=0}^{\lfloor \frac{d_e}{2} \rfloor} \sum_{i_2=0}^{d_e-2i_1} \sum_{i_3=0}^{\lfloor \frac{d_e-2i_1-i_2}{2} \rfloor} \sum_{i_4=0}^{d_e-2i_1-i_2-2i_3} \\ &\sum_{i_5=0}^{\lfloor \frac{d_e-2i_1-i_2-2i_3-i_4}{2} \rfloor} \binom{n-j}{i_1}\binom{n-j-i_1}{i_2} \\ &\times \binom{j}{i_3}\binom{j-i_3}{i_4}\binom{j-i_3-i_4}{i_5}(n-2)^{(i_3)} \\ &\times (n-1)^{(i_4+i_5-j)} p_t^{(j+i_1-i_4-i_5)} \\ &\times p_e^{(i_2+i_4)} p_c^{(n-j+i_5-i_1-i_2)}. \end{aligned} \quad (3)$$

2. An alternative simpler approach can be used to obtain P_c, P_e and P_f. Let T_{ji} denote the total number of vectors of length n with j nonzero symbols and i erasures. It is straightforward to show that

$$T_{ji} = \binom{n}{j}\binom{n-j}{i}(q-1)^j.$$

Let $\widehat{P}_{ji}$ denote the probability of obtaining j particular nonzero symbols and i erasures prior to decoding. Then

$$\widehat{P}_{ji} = \left(\frac{p_t}{q-1}\right)^j (p_e)^i (1 - p_t - p_e)^{n-j-i}.$$

Also let a_{ji} denote the total number of decodable vectors with j nonzero symbols and i erasures, and define F_{ji} to be a count of the number of non-decodable vectors. Clearly, $a_{ji} + F_{ji} = T_{ji}$ for all i and j. Using the above definitions for $\widehat{P}_{ji}$, a_{ji} and F_{ji}, determination of the performance of an (n,k) q-ary block code is straightforward. In particular,

$$P_c = \sum_{j=0}^{\lfloor \frac{d_e}{2} \rfloor} \sum_{i=0}^{d_e-2j} a_{ji}\widehat{P}_{ji}, \quad (4)$$

the probability of decoder error is

$$P_e = \sum_{j=0}^{n} \sum_{i=0,\, 2j+i>d_e}^{\min(d_e, n-j)} a_{ji}\widehat{P}_{ji}, \quad (5)$$

and the probability of decoder failure is

$$P_f = \sum_{j=0}^{n} \sum_{i=0}^{n-j} F_{ji}\widehat{P}_{ji}, \quad (6)$$

where [7]

$$\begin{aligned} a_{ji} = &\sum_{m=-\lfloor \frac{d_e-i}{2} \rfloor}^{d_e} A_{j+m} \sum_{i_1=0}^{i} \sum_{i_2=0}^{\lfloor \frac{\lfloor \frac{d_e-i}{2} \rfloor - |i_1-m|}{2} \rfloor} \\ &\sum_{i_3=0}^{\lfloor \frac{d_e-i}{2} \rfloor - |i_1-m| - 2i_2} \binom{n-j-m}{i-i_1}\binom{j+m}{i_1} \\ &\times \binom{n-j-m-(i-i_1)}{i_2+(i_1-m)^+}\binom{j+m-i_1}{i_2+(m-i_1)^+} \\ &\times \binom{j+m-i_1-i_2-(m-i_1)^+}{i_3} \\ &\times (q-1)^{i_2+(i_1-m)^+}(q-2)^{i_3}, \end{aligned} \quad (7)$$

and $(x)^+ = \max(0, x)$. Details of the proof can be found in [7].

3. The probability of decoding twice into any codeword of weight j is $A_j(P_j^{d_e})^2$. Therefore the probability of decoding incorrectly twice into the same codeword, denoted by $P_{E_1E_1}$, is given by

$$P_{E_1E_1} = \sum_{j=d_{\min}}^{n} A_j\left(P_j^{d_e}\right)^2, \quad (8)$$

and similarly, the probability of decoding incorrectly into the same codeword three times is

$$P_{E_1E_1E_1} = \sum_{j=d_{\min}}^{n} A_j\left(P_j^{d_e}\right)^3. \quad (9)$$

4. The probability of decoding into two different codewords, one with weight j and the other with weight i, is $A_j(A_i - \delta(i-j))(P_j^{d_e}P_i^{d_e})$, where $\delta(\ell) = 1$ if $\ell = 0$ and 0 otherwise. It follows that the probability

of decoding incorrectly into two different codewords, denoted by $P_{E_1E_2}$, is equal to

$$\sum_{j=d_{\min}}^{n}\sum_{i=d_{\min}}^{n} A_j\big(A_i-\delta(i-j)\big)\big(P_j^{d_e}P_i^{d_e}\big).$$

A rearrangement of terms gives

$$P_{E_1E_2}=(P_e)^2-P_{E_1E_1}. \tag{10}$$

5. In the same manner, the probability of decoding into three codewords, the first two of which are the same, is denoted by $P_{E_1E_1E_2}$ and determined to be

$$\sum_{j=d_{\min}}^{n}\sum_{i=d_{\min}}^{n} A_j\big(A_i-\delta(i-j)\big)\big(P_j^{d_e}\big)^2\big(P_i^{d_e}\big),$$

and a rearrangement of terms gives

$$P_{E_1E_1E_2}=P_e\ P_{E_1E_1}-P_{E_1E_1E_1}. \tag{11}$$

Note that $P_{E_1E_1E_2}=P_{E_1E_2E_1}$.

6. Let D_1 denote the event that a transmitted codeword is decoded into a particular wrong codeword, and let D_2 denote the event that D_1 occurs, the codeword is resent, and decoding results in the same wrong codeword. The conditional probability of D_2 given D_1 is denoted by P_{E_1/E_1} and is given by

$$P_{E_1/E_1}=\frac{P(D_2\cap D_1)}{P(D_1)}=\frac{P(D_2)}{P(D_1)}=\frac{P_{E_1E_1}}{P_e}. \tag{12}$$

7. Let G_1 be the event of decoding incorrectly into some codeword, resending, and decoding incorrectly into a different codeword. Furthermore, let G_2 be the event that G_1 occurs, the codeword is resent, and decoding results in the first wrong codeword. The conditional probability of G_2 given G_1 is denoted by P_{E_1/E_1E_2} and equals

$$P_{E_1/E_1E_2}=\frac{P(G_2\cap G_1)}{P(G_1)}=\frac{P(G_2)}{P(G_1)}=\frac{P_{E_1E_2E_1}}{P_{E_1E_2}}. \tag{13}$$

Equations (1)–(13) can be generalized to accommodate the channel described in section 2. Let $p_t^{(\ell_i)}$ and $p_e^{(\ell_i)}$ be the conditional symbol error and erasure probabilities given that the channel is in state ℓ_i. By conditioning on the state of the channel for each transmitted codeword, we obtain the following in place of equations (1)–(3) and (8)–(13):

$$P_c(\ell_i)=\sum_{i_1=0}^{d_e}\sum_{j_1=0}^{\lfloor\frac{d_e-i_1}{2}\rfloor}\binom{n}{i_1}\binom{n-i_1}{j_1}\big(p_e^{(\ell_i)}\big)^{i_1}\big(p_t^{(\ell_i)}\big)^{j_1}\times\big(1-p_t^{(\ell_i)}-p_e^{(\ell_i)}\big)^{n-i_1-j_1},$$

$$P_e(\ell_i)=\sum_{j=d_{\min}}^{n}A_jP_{d_e}^{j}(\ell_i),$$

$$P_f(\ell_i)=1-\big(P_c(\ell_i)+P_e(\ell_i)\big),$$

$$P_{E_1E_1}(\ell_i,\ell_m)=\sum_{j=d_{\min}}^{n}A_jP_j^{d_e}(\ell_i)P_j^{d_e}(\ell_m),$$

$$P_{E_1E_1E_1}(\ell_i,\ell_m,\ell_k)=\sum_{j=d_{\min}}^{n}A_jP_j^{d_e}(\ell_i)P_j^{d_e}(\ell_m)P_j^{d_e}(\ell_k),$$

$$P_{E_1E_2}(\ell_i,\ell_m)=P_e(\ell_i)P_e(\ell_m)-P_{E_1E_1}(\ell_i,\ell_m),$$

$$P_{E_1E_1E_2}(\ell_i,\ell_m,\ell_k)=P_e(\ell_i)P_{E_1E_1}(\ell_m,\ell_k)-P_{E_1E_1E_1}(\ell_i,\ell_m,\ell_k),$$

$$P_{E_1/E_1}(\ell_i,\ell_m)=\frac{P_{E_1E_1}(\ell_i,\ell_m)}{P_e(\ell_i)},$$

$$P_{E_1/E_1E_2}(\ell_i,\ell_m,\ell_k)=\frac{P_{E_1E_1E_2}(\ell_i,\ell_m,\ell_k)}{P_{E_1E_2}(\ell_i,\ell_m)},$$

and $P_{d_e}^{j}(\ell_i)$ is obtained by evaluating (3) with $p_t=p_t^{(\ell_i)}$ and $p_e=p_e^{(\ell_i)}$.

4.2. Analysis of HARQ-I

We define a *packet failure* to be the event that one or more codewords in a packet fails to decode and a *packet error* to be the event that all codewords decode but one or more decode incorrectly. Retransmissions occur after packet failures, but packet errors are undetectable.

Let $\alpha_c(\ell_i)$ denote the conditional probability of decoding the entire packet correctly given that the channel is in state ℓ_i, let $\alpha_e(\ell_i)$ denote the conditional probability of a packet error, and let $\alpha_f(\ell_i)$ denote the conditional probability of a packet failure. It is straightforward to show that

$$\alpha_c(\ell_i)=\big(P_c(\ell_i)\big)^N, \tag{14}$$

$$\alpha_e(\ell_i)=\sum_{m=1}^{N}\binom{N}{m}\big[P_e(\ell_i)\big]^m\big[P_c(\ell_i)\big]^{N-m}=\big[P_c(\ell_i)+P_e(\ell_i)\big]^N-\big[P_c(\ell_i)\big]^N, \tag{15}$$

and

$$\alpha_f(\ell_i)=1-\alpha_c(\ell_i)-\alpha_e(\ell_i). \tag{16}$$

An *accepted packet* is defined to be the final decision that results after the receiver determines that no additional retransmissions will be requested. For type-I HARQ, the accepted packet is simply the decoder output of the final transmission. The probability that the accepted packet contains one or more codeword errors is denoted by P_E^* and is given by

$$P_E^*=\sum_{k=1}^{\infty}P(\text{Packet error}\cap k\ \text{transmissions})=\sum_{k=1}^{\infty}\sum_{\ell_1=1}^{L}\sum_{\ell_2=1}^{L}\dots\sum_{\ell_k=1}^{L}\pi_{\ell_1}(0)$$

$$\times \left(\prod_{i=1}^{k-1} \alpha_{\mathrm{f}}(\ell_i) P_{\ell_i \ell_{i+1}} \right) \alpha_{\mathrm{e}}(\ell_k).$$

The average number of transmissions required for a single packet (i.e., the delay), is denoted by D_{av} and is calculated as follows:

$$\begin{aligned} D_{\mathrm{av}} &= \sum_{k=1}^{\infty} k P(\text{Packet accepted} \cap k \text{ transmissions}) \\ &= \sum_{k=1}^{\infty} k \sum_{\ell_1=1}^{L} \sum_{\ell_2=1}^{L} \cdots \sum_{\ell_k=1}^{L} \pi_{\ell_1}(0) \\ &\quad \times \left(\prod_{i=1}^{k-1} \alpha_{\mathrm{f}}(\ell_i) P_{\ell_i \ell_{i+1}} \right) \left[\alpha_{\mathrm{e}}(\ell_k) + \alpha_{\mathrm{c}}(\ell_k) \right]. \end{aligned}$$

The average throughput can be calculated from the relationship

$$\eta_{\mathrm{av}} = k/(nD_{\mathrm{av}}). \tag{17}$$

4.3. Analysis of the 1-2-3-restart scheme

Define $\alpha_{\mathrm{c},1}(\ell_i)$, $\alpha_{\mathrm{c},2}(\ell_i, \ell_m)$, and $\alpha_{\mathrm{c},3}(\ell_i, \ell_m, \ell_k)$ to be the conditional probabilities of accepting a completely correct *combined packet* on the first, second, and third transmissions, respectively, given the states of the channel for each transmission. Similarly, let $\alpha_{\mathrm{e},1}(\ell_i)$, $\alpha_{\mathrm{e},2}(\ell_i, \ell_m)$, $\alpha_{\mathrm{e},3}(\ell_i, \ell_m, \ell_k)$, $\alpha_{\mathrm{f},1}(\ell_i)$, $\alpha_{\mathrm{f},2}(\ell_i, \ell_m)$, and $\alpha_{\mathrm{f},3}(\ell_i, \ell_m, \ell_k)$ denote the conditional packet error and failure probabilities for a combined packet. Also, let C_i denote the event that the packet is accepted in the ith transmission and is correct, and let E_i represent the event that the ith transmission is accepted and the combined packet contains one more errors.

For the first transmission, performance is identical to that of type I HARQ; thus, $\alpha_{\mathrm{c},1}(\ell_i)$, $\alpha_{\mathrm{e},1}(\ell_i)$ and $\alpha_{\mathrm{f},1}(\ell_i)$ are given by (14)–(16). To analyze performance with the second transmission, let Y_{C}, Y_{E}, and Y_{F} denote the number of correct codewords, codeword errors, and decoding failures from the first packet. Then, since at least one failure in the first packet is required for a retransmission, we have

$$\begin{aligned} \alpha_{\mathrm{c},2}(\ell_i, \ell_m) &= \sum_{j_1=1}^{N} P(C_2 \mid Y_{\mathrm{F}} = j_1,\ Y_{\mathrm{C}} = N - j_1) \\ &\quad \times P(Y_{\mathrm{F}} = j_1,\ Y_{\mathrm{C}} = N - j_1). \end{aligned}$$

Note that for the combined packet to be correct, no codeword errors can occur in the first transmission (thus $Y_{\mathrm{E}} = 0$).

It is straightforward to show that

$$P(Y_{\mathrm{F}} = j_1,\ Y_{\mathrm{C}} = N - j_1) = \binom{N}{j_1} \left[P_{\mathrm{f}}(\ell_i) \right]^{j_1} \left[P_{\mathrm{c}}(\ell_i) \right]^{N-j_1}$$

and

$$\begin{aligned} &P(C_2 \mid Y_{\mathrm{F}} = j_1,\ Y_{\mathrm{C}} = N - j_1) \\ &\quad = \left[P_{\mathrm{c}}(\ell_m) \right]^{j_1} \left[1 - P_{\mathrm{e}}(\ell_m) \right]^{N-j_1}. \end{aligned} \tag{18}$$

The rationale for (18) is that if the kth codeword in the first packet failed, then the kth codeword of the second packet must be correct in order to have a correct combined packet. Furthermore, if the kth codeword of the first packet is correct, then the kth codeword of the second packet must be either correct or have failed (not in error) in order to have a correct combined packet. Thus, we obtain

$$\begin{aligned} &\alpha_{\mathrm{c},2}(\ell_i, \ell_m) \\ &\quad = \sum_{j_1=1}^{N} \binom{N}{j_1} \left[P_{\mathrm{f}}(\ell_i) P_{\mathrm{c}}(\ell_m) \right]^{j_1} \left[P_{\mathrm{c}}(\ell_i) \big(1 - P_{\mathrm{e}}(\ell_m)\big) \right]^{N-j_1}. \end{aligned}$$

To determine $\alpha_{\mathrm{e},2}(\ell_i, \ell_m)$, we can similarly write

$$\begin{aligned} &\alpha_{\mathrm{e},2}(\ell_i, \ell_m) \\ &\quad = \sum_{j_1=1}^{N} \sum_{j_2=0}^{N-j_1} P(E_2 \mid Y_{\mathrm{F}} = j_1,\, Y_{\mathrm{E}} = j_2,\, Y_{\mathrm{C}} = N - j_1 - j_2) \\ &\quad \times P(Y_{\mathrm{F}} = j_1,\ Y_{\mathrm{E}} = j_2,\ Y_{\mathrm{C}} = N - j_1 - j_2), \end{aligned}$$

where

$$\begin{aligned} &P(Y_{\mathrm{F}} = j_1,\ Y_{\mathrm{E}} = j_2,\ Y_{\mathrm{C}} = N - j_1 - j_2) \\ &\quad = \binom{N}{j_1} \binom{N - j_1}{j_2} \left[P_{\mathrm{f}}(\ell_i) \right]^{j_1} \left[P_{\mathrm{e}}(\ell_i) \right]^{j_2} \left[P_{\mathrm{c}}(\ell_i) \right]^{N-j_1-j_2} \end{aligned}$$

and

$$\begin{aligned} &P(E_2 \mid Y_{\mathrm{F}} = j_1,\ Y_{\mathrm{E}} = j_2,\ Y_{\mathrm{C}} = N - j_1 - j_2) \\ &\quad = \left[\sum_{j_3=0,\, j_3+j_2>0}^{j_1} \binom{j_1}{j_3} \left[P_{\mathrm{e}}(\ell_m) \right]^{j_3} \left[P_{\mathrm{c}}(\ell_m) \right]^{j_1-j_3} \right] \\ &\qquad \times \left[P_{\mathrm{f}}(\ell_m) + P_{E_1/E_1}(\ell_i, \ell_m) \right]^{j_2} \\ &\qquad \times \left[1 - P_{\mathrm{e}}(\ell_m) \right]^{N-j_1-j_2}. \end{aligned} \tag{19}$$

The expression in (19) is explained by noting that, for E_2 to occur, there must be no failures and at least one error in the combined packet. In particular, if a codeword from the first packet was correct, the second must be correct or a failure; if a codeword from the first packet was a failure, then the second must not be a failure; and if a codeword from the first packet was in error, the second must be a failure or have the *same* error. Thus,

$$\begin{aligned} \alpha_{\mathrm{e},2}(\ell_i, \ell_m) &= \sum_{j_1=1}^{N} \sum_{j_2=0}^{N-j_1} \sum_{j_3=0,\, j_3+j_2>0}^{j_1} \binom{N}{j_1} \binom{N - j_1}{j_2} \\ &\quad \times \binom{j_1}{j_3} \left[P_{\mathrm{f}}(\ell_i) P_{\mathrm{c}}(\ell_m) \right]^{j_1} \\ &\quad \times \left[P_{\mathrm{e}}(\ell_i) \big(P_{\mathrm{f}}(\ell_m) + P_{E_1/E_1}(\ell_i, \ell_m) \big) \right]^{j_2} \\ &\quad \times \left[P_{\mathrm{c}}(\ell_i) \big(1 - P_{\mathrm{e}}(\ell_m) \big) \right]^{N-j_1-j_2} \\ &\quad \times \left[P_{\mathrm{e}}(\ell_m) / P_{\mathrm{c}}(\ell_m) \right]^{j_3}. \end{aligned}$$

To determine $\alpha_{\mathrm{f},2}(\ell_i, \ell_m)$, we note that the sum of the error, failure, and correct probabilities for the combination of two packets is the probability of failure for the first packet (since the first packet must fail for a retransmission to occur). It follows that

$$\alpha_{\mathrm{f},2}(\ell_i,\ell_m)=\alpha_{\mathrm{f},1}(\ell_i)-\left[\alpha_{\mathrm{c},2}(\ell_i,\ell_m)+\alpha_{\mathrm{e},2}(\ell_i,\ell_m)\right].$$

In a similar manner, $\alpha_{\mathrm{c},3}(\ell_i,\ell_m,\ell_k)$, $\alpha_{\mathrm{e},3}(\ell_i,\ell_m,\ell_k)$ and $\alpha_{\mathrm{f},3}(\ell_i,\ell_m,\ell_k)$ can be determined; the results and an outline of the derivation are given in the appendix.

For this combining scheme, P_{E}^* and D_{av} can be determined in a manner similar to the method used for type-I ARQ. The result is that

$$P_{\mathrm{E}}^*=\sum_{k=1}^{\infty}\sum_{\ell_1=1}^{L}\sum_{\ell_2=1}^{L}\cdots\sum_{\ell_k=1}^{L}\pi_{\ell_1}(0)\left(\prod_{i=1}^{k-1}P_{\ell_i\ell_{i+1}}\right)\beta_{\mathrm{e}}(k)\times\prod_{i=1}^{\lfloor(k-1)/3\rfloor}\alpha_{\mathrm{f},3}(\ell_{3(i-1)+1},\ell_{3(i-1)+2},\ell_{3(i-1)+3})$$

and

$$D_{\mathrm{av}}=\sum_{k=1}^{\infty}k\sum_{\ell_1=1}^{L}\sum_{\ell_2=1}^{L}\cdots\sum_{\ell_k=1}^{L}\pi_{\ell_1}(0)\left(\prod_{i=1}^{k-1}P_{\ell_i\ell_{i+1}}\right)\times\left[\beta_{\mathrm{e}}(k)+\beta_{\mathrm{c}}(k)\right]\times\prod_{i=1}^{\lfloor(k-1)/3\rfloor}\alpha_{\mathrm{f},3}(\ell_{3(i-1)+1},\ell_{3(i-1)+2},\ell_{3(i-1)+3}),$$

where

$$\beta_{\mathrm{e}}(k)=\begin{cases}\alpha_{\mathrm{e},1}(\ell_k), & k \bmod 3=1,\\ \alpha_{\mathrm{e},2}(\ell_{k-1},\ell_k), & k \bmod 3=2,\\ \alpha_{\mathrm{e},3}(\ell_{k-2},\ell_{k-1},\ell_k), & k \bmod 3=0,\end{cases}$$

and

$$\beta_{\mathrm{c}}(k)=\begin{cases}\alpha_{\mathrm{c},1}(\ell_k), & k \bmod 3=1,\\ \alpha_{\mathrm{c},2}(\ell_{k-1},\ell_k), & k \bmod 3=2,\\ \alpha_{\mathrm{c},3}(\ell_{k-2},\ell_{k-1},\ell_k), & k \bmod 3=0.\end{cases}$$

The throughput is easily calculated from (17).

4.4. Analysis of the Markovian scheme

The performance of this scheme can be determined by conditioning on a particular state evolution of the channel, determining the conditional performance, and removing the conditioning by averaging appropriately over all state evolutions. The conditional state transition probabilities for the Markovian algorithm (conditioned on the channel states) can be calculated with straightforward counting arguments. The results are given in the following equations, where n and m are nonnegative integers. Note that the explicit dependence on channel states has been suppressed to enhance readability:

$$P_{\mathrm{S},(i,j)}=\binom{N}{i}\binom{N-i}{j}(P_{\mathrm{f}})^i(P_{\mathrm{c}})^j(P_{\mathrm{e}})^{N-i-j},$$

$$P_{\mathrm{S,E}}=\sum_{h=1}^{N}\binom{N}{h}(P_{\mathrm{e}})^h(P_{\mathrm{c}})^{N-h}=(1-P_{\mathrm{f}})^N-(P_{\mathrm{c}})^N,$$

$$P_{\mathrm{S,C}}=(P_{\mathrm{c}})^N,$$

$$\begin{aligned}P_{(i,j),(i+n,j+m)}&=\sum_{h_1=m}^{\min(i,j+m)}\sum_{h_2=0}^{\min(i-h_1,N-i-j-(n+m))}\binom{i}{h_1}\binom{i-h_1}{h_2}\\&\quad\times\binom{j}{h_1-m}\binom{N-i-j}{h_2+n+m}(P_{\mathrm{f}})^{i-(h_1+h_2)}\\&\quad\times(P_{\mathrm{c}})^{h_1}(P_{\mathrm{e}})^{h_2+h_1-m}(P_{\mathrm{f}}+P_{\mathrm{c}})^{j-(h_1-m)}\\&\quad\times(P_{\mathrm{f}}+P_{E_1/E_1})^{N-i-j-(h_2+n+m)}\\&\quad\times(P_{\mathrm{c}}+P_{E_1/E_2})^{h_2+n+m},\end{aligned}$$

$$\begin{aligned}P_{(i,j),(i-n,j-m)}&=\sum_{h_1=0}^{\min(i-n-m,N-i-j)}\sum_{h_2=0}^{\min(i-h_1-n-m,j-m)}\binom{i}{h_1+n+m}\\&\quad\times\binom{i-h_1-n-m}{h_2}\binom{j}{h_2+m}\binom{N-i-j}{h_1}(P_{\mathrm{c}})^{h_2}\\&\quad\times(P_{\mathrm{f}})^{i-(h_2+h_1+n+m)}(P_{\mathrm{e}})^{h_1+h_2+2m+n}\\&\quad\times(P_{\mathrm{f}}+P_{\mathrm{c}})^{j-(h_2+m)}(P_{\mathrm{f}}+P_{E_1/E_1})^{N-i-j-h_1}\\&\quad\times(P_{\mathrm{c}}+P_{E_1/E_2})^{h_1},\end{aligned}$$

$$\begin{aligned}P_{(i,j),(i-n,j+m)}&=\sum_{h_1=0}^{\min(i-m,j)}\sum_{h_2=0}^{\min(i-m-h_1,N-i-j-m+n)}\binom{i}{h_1+m}\\&\quad\times\binom{i-(h_1+m)}{h_2}\binom{j}{h_1}\binom{N-i-j}{h_2-n+m}\\&\quad\times(P_{\mathrm{f}})^{i-(m+h_1+h_2)}(P_{\mathrm{c}})^{m+h_1}(P_{\mathrm{e}})^{h_2+h_1}\\&\quad\times(P_{\mathrm{f}}+P_{\mathrm{c}})^{j-h_1}(P_{\mathrm{f}}+P_{E_1/E_1})^{N-i-j-(h_2-n+m)}\\&\quad\times(P_{\mathrm{c}}+P_{E_1/E_2})^{h_2-n+m},\quad m>n,\end{aligned}$$

$$\begin{aligned}P_{(i,j),(i-n,j+m)}&=\sum_{h_1=0}^{\min(i-m,j)}\sum_{h_2=0}^{\min(i-h_1-n,N-i-j)}\binom{i}{h_1+m}\\&\quad\times\binom{i-h_1-m}{h_2+n-m}\binom{j}{h_1}\binom{N-i-j}{h_2}\\&\quad\times(P_{\mathrm{f}})^{i-(h_1+h_2+n)}(P_{\mathrm{c}})^{m+h_1}(P_{\mathrm{e}})^{h_2+h_1+n-m}\\&\quad\times(P_{\mathrm{f}}+P_{\mathrm{c}})^{j-h_1}(P_{\mathrm{f}}+P_{E_1/E_1})^{N-i-j-h_2}\\&\quad\times(P_{\mathrm{c}}+P_{E_1/E_2})^{h_2},\quad m\leqslant n,\end{aligned}$$

$$\begin{aligned}P_{(i,j),(i+n,j-m)}&=\sum_{h_1=0}^{\min(i,j-m)}\sum_{h_2=0}^{\min(i-h_1,N-i-j-n+m)}\binom{i}{h_1}\\&\quad\times\binom{i-h_1}{h_2}\binom{j}{h_1+m}\binom{N-i-j}{h_2+n-m}(P_{\mathrm{f}})^{i-(h_1+h_2)}\\&\quad\times(P_{\mathrm{c}})^{h_1}(P_{\mathrm{e}})^{h_2+h_1+m}(P_{\mathrm{f}}+P_{\mathrm{c}})^{j-h_1-m}\\&\quad\times(P_{\mathrm{f}}+P_{E_1/E_1})^{N-i-j-(h_2+n-m)}\\&\quad\times(P_{\mathrm{c}}+P_{E_1/E_2})^{h_2+n-m},\quad m\leqslant n,\end{aligned}$$

$$
\begin{aligned}
&P_{(i,j),(i+n,j-m)}\\
&= \sum_{h_1=0}^{\min(i-m+n,N-i-j)} \sum_{h_2=0}^{\min(i-h_1-m+n,j-m)} \binom{i}{h_1+m-n}\\
&\quad\times \binom{i+n-h_1-m}{h_2}\binom{j}{h_2+m}\binom{N-i-j}{h_1}\\
&\quad\times (P_{\mathrm{c}})^{h_2}(P_{\mathrm{f}})^{i-(h_2+h_1)}(P_{\mathrm{e}})^{h_1+h_2+2m-n}\\
&\quad\times (P_{\mathrm{f}}+P_{\mathrm{c}})^{j-h_2-m}(P_{\mathrm{f}}+P_{E_1/E_1})^{N-i-j-h_1}\\
&\quad\times (P_{\mathrm{c}}+P_{E_1/E_2})^{h_1}, \quad m>n,
\end{aligned}
$$

$$
\begin{aligned}
P_{(i,j),\mathrm{E}} = &\sum_{h=0,\, h+N-i-j>0}^{i} \binom{i}{h}(P_{\mathrm{e}})^h(P_{\mathrm{c}})^{i-h}(1-P_{\mathrm{e}})^j\\
&\times (P_{\mathrm{f}}+P_{E_1/E_1})^{N-i-j},
\end{aligned}
$$

$$
P_{(i,j),\mathrm{C}} = \begin{cases} (P_{\mathrm{c}})^i(1-P_{\mathrm{e}})^j, & N-i-j=0,\\ 0, & N-i-j\neq 0, \end{cases}
$$

$$
P_{\mathrm{C,C}} = P_{\mathrm{E,E}} = 1,
$$

$$
P_{(i,j),\mathrm{S}} = 0.
$$

The states can be renumbered as follows, where $M = ((N^2+N)/2)+3$: state S becomes state 1; state C becomes state $M-1$; state E becomes state M; state (i,j) becomes state $w(i,j)$, where

$$
w(i,j) = \begin{cases} j+2, & i=1,\\ j+2+\sum_{k_1=0}^{i-2}(N-k_1), & i\geqslant 2. \end{cases}
$$

Let $\boldsymbol{V}_0$ denote the initial system state probabilities, and let $\boldsymbol{G}(\ell_i,\ell_{i+1})$ be the conditional $M\times M$ one step probability transition matrix. Then $\boldsymbol{V}_0 = [1\ 0\ 0\ 0\ \ldots\ 0\ 0]$ and $\boldsymbol{G}(\ell_i,\ell_{i+1})$ equals

$$
\begin{bmatrix}
0 & P_{1,2} & \ldots & P_{1,M-2} & P_{1,M-1} & P_{1,M}\\
0 & P_{2,2} & \ldots & P_{2,M-2} & P_{2,M-1} & P_{2,M}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & P_{M-3,2} & \ldots & P_{M-3,M-2} & P_{M-3,M-1} & P_{M-3,M}\\
0 & P_{M-2,2} & \ldots & P_{M-2,M-2} & P_{M-2,M-1} & P_{M-2,M}\\
0 & 0 & \ldots & 0 & 1 & 0\\
0 & 0 & \ldots & 0 & 0 & 1
\end{bmatrix}.
$$

Let $\boldsymbol{V}_k(\ell_1,\ell_2,\ldots,\ell_k)$ denote the conditional system state probabilities at time k, and let $P(\ell_1,\ell_2,\ldots,\ell_k)$ represent the probability of the channel state evolution from states ℓ_1 through ℓ_k. Then it follows that

$$
\boldsymbol{V}_k(\ell_1,\ell_2,\ldots,\ell_k) = \boldsymbol{V}_0\boldsymbol{G}(\ell_1,\ell_1)\prod_{i=1}^{k-1}\boldsymbol{G}(\ell_i,\ell_{i+1})
$$

and

$$
P(\ell_1,\ell_2,\ldots,\ell_k) = \pi_{\ell_1}(0)\prod_{i=1}^{k-1}P_{\ell_i\ell_{i+1}}.
$$

If $\boldsymbol{V}_k$ is defined as the unconditional probabilities of the system states at time k, we have

$$
\begin{aligned}
\boldsymbol{V}_k &= \sum_{\ell_1=1}^{L}\sum_{\ell_2=1}^{L}\cdots\sum_{\ell_k=1}^{L}\boldsymbol{V}_k(\ell_1,\ell_2,\ldots,\ell_k)P(\ell_1,\ell_2,\ldots,\ell_k)\\
&= \sum_{\ell_1=1}^{L}\sum_{\ell_2=1}^{L}\cdots\sum_{\ell_k=1}^{L}\boldsymbol{V}_0\pi_{\ell_1}(0)\boldsymbol{G}(\ell_1,\ell_1)\\
&\quad\times\prod_{i=1}^{k-1}\boldsymbol{G}(\ell_i,\ell_{i+1})P_{\ell_i\ell_{i+1}}.
\end{aligned}
$$

We also define the weighted difference vector $\boldsymbol{\gamma}$ as

$$
\begin{aligned}
\boldsymbol{\gamma} = &\sum_{k=1}^{\infty}k\sum_{\ell_1=1}^{L}\sum_{\ell_2=1}^{L}\cdots\sum_{\ell_k=1}^{L}\pi_{\ell_1}(0)\left(\prod_{i=1}^{k-1}P_{\ell_i\ell_{i+1}}\right)\\
&\times\left[\boldsymbol{V}_k(\ell_1,\ell_2,\ldots,\ell_k)-\boldsymbol{V}_{k-1}(\ell_1,\ell_2,\ldots,\ell_{k-1})\right].
\end{aligned}
$$

Now, we can write $\boldsymbol{V}_k = [V_{k1},V_{k2},\ldots,V_{kM}]$ and $\boldsymbol{\gamma} = [\gamma_1,\gamma_2,\ldots,\gamma_M]$. Then it is straightforward to show that

$$
P_{\mathrm{E}}^* = \lim_{k\to\infty}V_{kM}
$$

and

$$
D_{\mathrm{av}} = \gamma_M + \gamma_{M-1}.
$$

The average throughput η_{av} can be calculated from (17).

5. Numerical results

The performance of Type-I hybrid ARQ, the 1-2-3-restart combining scheme, and Markovian combining can be compared through a consideration of the packet error probability P_{E}^* and the average throughput η_{av} for the three schemes. Figures 5–10 present the performance in a variety of channel conditions for a system using a (16, 8) RS code, an effective decoding diameter of $d_{\mathrm{e}} = 6$, and $N = 10$ codewords per packet. This particular value of d_{e} was chosen to provide a good tradeoff between throughput and error

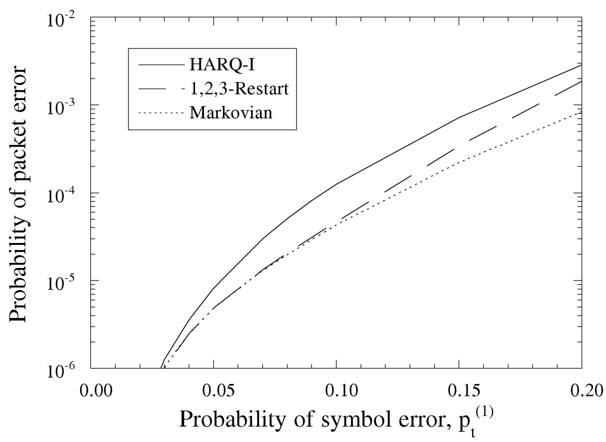

Figure 5. Packet error performance, one channel state, $N = 10$, $p_{\mathrm{e}}^{(1)} = 0.1$.

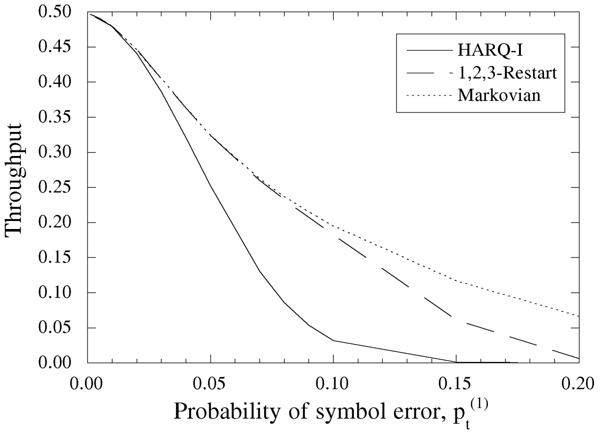

Figure 6. Throughput performance, one channel state, $N = 10$, $p_e^{(1)} = 0.1$.

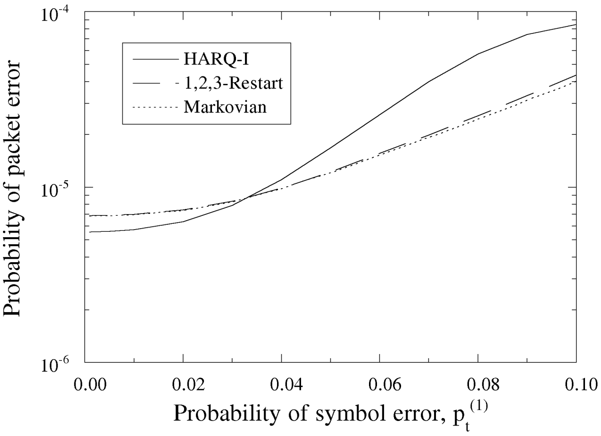

Figure 7. Packet error performance, two channel states, $N = 10$, $p_e^{(1)} = 0.1$, $p_t^{(2)} = 0.05$, $p_e^{(2)} = 0.2$.

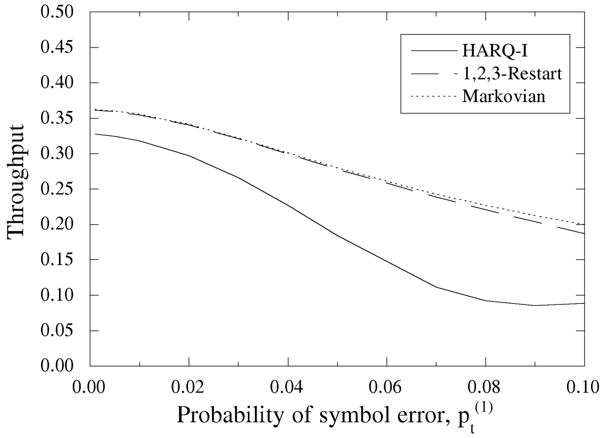

Figure 8. Throughput performance, two channel states, $N = 10$, $p_e^{(1)} = 0.1$, $p_t^{(2)} = 0.05$, $p_e^{(2)} = 0.2$.

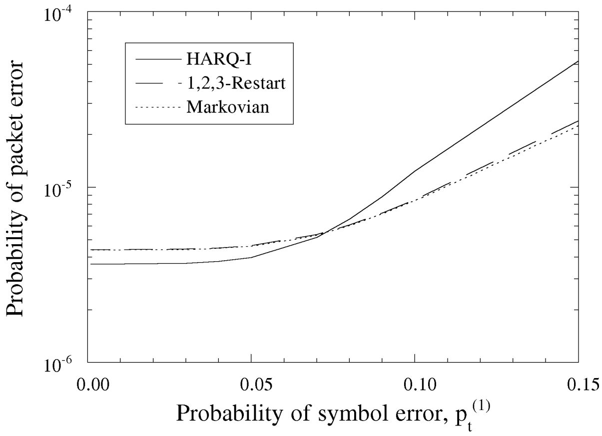

Figure 9. Packet error performance, two channel states, $N = 10$, $p_t^{(2)} = 0.15$, $p_e^{(1)} = p_e^{(2)} = 0$.

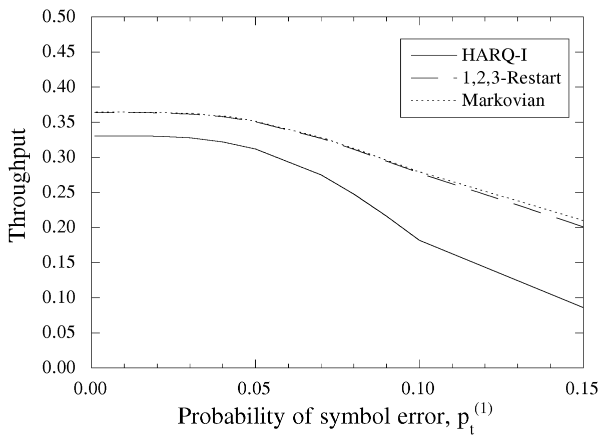

Figure 10. Throughput performance, two channel states, $N = 10$, $p_t^{(2)} = 0.15$, $p_e^{(1)} = p_e^{(2)} = 0$.

detection capability. (A higher value of d_e results in a better probability of undetected error and a worse throughput; a lower value of d_e has the opposite effect.)

Figures 5 and 6 present P_E^* and η_{av} for a channel with a single state with symbol erasure probability $p_e^{(1)} = 0.1$. The figures show that the probability of packet error is lowest and the throughput is highest for the Markovian technique. Both the Markovian and the 1-2-3-restart schemes significantly outperform HARQ-I, especially with regards to throughput. For a symbol error probability of 0.1, for example, the throughput of the two combining schemes is more than triple the throughput of HARQ-I.

Figures 7 and 8 present the performance under a two-state channel model with transition probabilities $P_{11} = 3/4$, $P_{12} = 1/4$, $P_{21} = 1/2$, and $P_{22} = 1/2$. The initial channel state probabilities (which are equal to the stationary state probabilities) are $p_1(0) = 2/3$ and $p_2(0) = 1/3$, and the error and erasure probabilities are $p_e^{(1)} = 0.1$, $p_e^{(2)} = 0.2$, and $p_t^{(2)} = 0.05$. The figures show that the Markovian and 1-2-3-restart techniques give nearly identical performance, and both provide very substantial gains in throughput over HARQ-I. Both schemes also have lower packet error probabilities than HARQ-I except when $p_t^{(1)}$ is very small, in which case the error probabilities are similar.

Figures 9 and 10 show the performance for a two-state channel with the same transition probabilities as the previous two figures but for an errors-only decoding system with

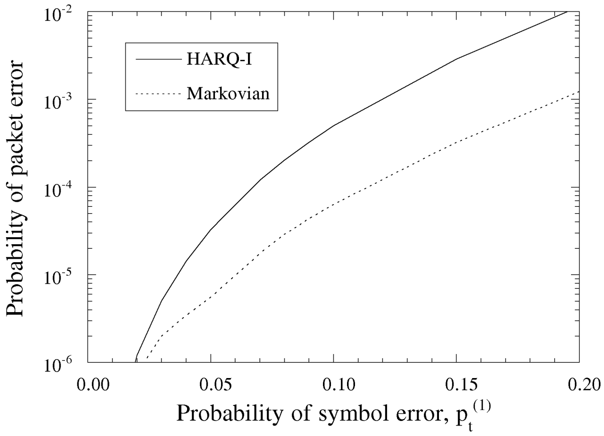

Figure 11. Packet error performance, one channel state, $N = 40$, $p_e^{(1)} = 0.1$.

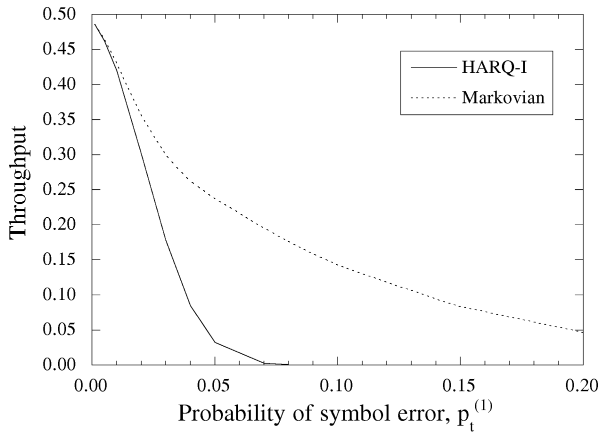

Figure 12. Throughput performance, one channel state, $N = 40$, $p_e^{(1)} = 0.1$.

$p_t^{(2)} = 0.15$. Comparisons of the three schemes reveal similar characteristics to the errors-and-erasures system. We can conclude that the availability of erasure information is not necessary for good performance of the packet combining techniques (although erasure information, if available, should be used to improve performance).

Finally, it is of interest to examine the effect of packet length on the performance of the combining schemes. Figures 11 and 12 present performance for the Markovian and HARQ-I schemes for $N = 40$ codewords per packet on a one-state channel with $p_e^{(1)} = 0.1$. (Calculation of the performance of the 1-2-3-restart scheme becomes intractable with increasing N.) The figures show that the relative performance gain that results from packet combining increases greatly with N.

6. Conclusions

We have presented two techniques for packet combining in block-coded communication systems. The techniques have been analyzed for RS codes used over a bursty errors-and-erasures channel. We note that the analysis techniques presented here can be modified easily to analyze the performance of other block codes.

The results show that significant performance gains can result by using these techniques. The Markovian technique performs slightly better than the 1-2-3-restart scheme in a one-state channel, but both perform nearly identically when Markov dependencies are introduced. We are not willing to conclude that the Markovian scheme is uniformly superior to the 1-2-3-restart scheme; although the performance results shown in this paper are consistent with this fact, it may be that the situation reverses for channels with different state transition probababilities. It is true, however, that the Markovian scheme maintains a slight advantage with regards to buffer requirements.

Appendix

To find $\alpha_{c,3}(\ell_i, \ell_m, \ell_k)$, we follow a similar approach to the one used to calculate $\alpha_{c,2}(\ell_i, \ell_m)$, except that we condition on all possible patterns of the first two transmissions that will give a correct decoding for the three combined packets. Let Y_{CF} be the number of codewords that are correct in the first packet and fail in the second packet, and let Y_{FC}, Y_{CE}, Y_{EC}, Y_{FF}, and Y_{CC} be defined similarly. Then,

$$\alpha_{c,3}(\ell_i, \ell_m, \ell_k) = \sum_{j_1,\dots,j_5 \in S} P(C_3 \mid Z_{(j_1,j_2,j_3,j_4,j_5)}) P(Z_{(j_1,j_2,j_3,j_4,j_5)}),$$

where $Z_{(j_1,j_2,j_3,j_4,j_5)}$ is the event that $Y_{FF} = j_1$, $Y_{FC} = j_2$, $Y_{CF} = j_3$, $Y_{CE} = j_4$, $Y_{EC} = j_5$, and $Y_{CC} = N - \sum_{i=1}^{5} j_i$, and where S is a set that guarantees that a third transmission is necessary. Furthermore,

$$\begin{aligned} P(Z_{(j_1,j_2,j_3,j_4,j_5)}) &= \binom{N}{j_1} \big[P_f(\ell_i)P_f(\ell_m)\big]^{j_1} \\ &\quad \times \binom{N-j_1}{j_2} \big[P_f(\ell_i)P_c(\ell_m)\big]^{j_2} \\ &\quad \times \binom{N-j_1-j_2}{j_3} \big[P_c(\ell_i)P_f(\ell_m)\big]^{j_3} \\ &\quad \times \binom{N-j_1-j_2-j_3}{j_4} \big[P_c(\ell_i)P_e(\ell_m)\big]^{j_4} \\ &\quad \times \binom{N-j_1-j_2-j_3-j_4}{j_5} \big[P_e(\ell_i)P_c(\ell_m)\big]^{j_5} \\ &\quad \times \big[P_c(\ell_i)P_c(\ell_m)\big]^{N-j_1-j_2-j_3-j_4-j_5} \end{aligned}$$

and

$$\begin{aligned} P(C_3 \mid Z_{(j_1,j_2,j_3,j_4,j_5)}) &= \big[P_c(\ell_k)\big]^{j_1} \big[1 - P_e(\ell_k)\big]^{j_2} \big[1 - P_e(\ell_k)\big]^{j_3} \\ &\quad \times \big[P_c(\ell_k)\big]^{j_4} \big[P_c(\ell_k)\big]^{j_5}. \end{aligned}$$

We obtain the following expression for $\alpha_{c,3}(\ell_i, \ell_m, \ell_k)$:

$$\alpha_{c,3}(\ell_i,\ell_m,\ell_k) = \sum_{j_1=0}^{N} \sum_{j_2=0,\, j_1+j_2\geqslant 1}^{N-j_1} \sum_{j_3=0}^{N-j_1-j_2} \sum_{j_4=0}^{N-j_1-j_2-j_3} \sum_{j_5=0,\, j_1+j_4+j_5\geqslant 1}^{N-j_1-j_2-j_3-j_4} \binom{N}{j_1}\binom{N-j_1}{j_2}\binom{N-j_1-j_2}{j_3} \times \binom{N-j_1-j_2-j_3}{j_4}\binom{N-j_1-j_2-j_3-j_4}{j_5} \times \big[P_f(\ell_i)P_f(\ell_m)P_c(\ell_k)\big]^{j_1} \times \big[P_f(\ell_i)P_c(\ell_m)\big(1-P_e(\ell_k)\big)\big]^{j_2} \times \big[P_c(\ell_i)P_f(\ell_m)\big(1-P_e(\ell_k)\big)\big]^{j_3} \times \big[P_c(\ell_i)P_e(\ell_m)P_c(\ell_k)\big]^{j_4} \times \big[P_e(\ell_i)P_c(\ell_m)P_c(\ell_k)\big]^{j_5} \times \big[P_c(\ell_i)P_c(\ell_m)\big]^{N-j_1-j_2-j_3-j_4-j_5}.$$

Note that the condition $j_1+j_2\geqslant 1$ guarantees that the first packet failed and $j_1+j_4+j_5\geqslant 1$ guarantees that the second packet failed.

To determine $\alpha_{e,3}(\ell_i,\ell_m,\ell_k)$, we let $Y_{E_1E_1}$ denote the number of codewords that decode to a wrong codeword for the first packet and decode into the *same* wrong codeword in the second packet. Also, we define $Y_{E_1E_2}$ as the number of codewords that decode into two *different* wrong codewords in the first two packets. Then, if we associate the index i_1 with Y_{FF}, i_2 with Y_{FE}, i_3 with Y_{FC}, i_4 with Y_{EF}, i_5 with $Y_{E_1E_1}$, i_6 with $Y_{E_1E_2}$, i_7 with Y_{EC}, i_8 with Y_{CF}, i_9 with Y_{CE}, and i_{10} with Y_{CC},[1] we can write

$$\alpha_{e,3}(\ell_i,\ell_m,\ell_k) = \sum_{i_1,i_2,\ldots,i_{10}\in S} P(E_3 \mid Z_{(i_1,i_2,\ldots,i_{10})})P(Z_{(i_1,i_2,\ldots,i_{10})}),$$

where

$$P(Z_{(i_1,i_2,\ldots,i_{10})}) = \binom{N}{i_1}\big[P_f(\ell_i)P_f(\ell_m)\big]^{i_1} \times \binom{N-i_1}{i_2}\big[P_f(\ell_i)P_e(\ell_m)\big]^{i_2} \times \binom{N-\sum_{j=1}^{2} i_j}{i_3}\big[P_f(\ell_i)P_c(\ell_m)\big]^{i_3} \times \binom{N-\sum_{j=1}^{3} i_j}{i_4}\big[P_e(\ell_i)P_f(\ell_m)\big]^{i_4} \times \binom{N-\sum_{j=1}^{4} i_j}{i_5}\big[P_{E_1E_1}(\ell_i,\ell_m)\big]^{i_5} \times \binom{N-\sum_{i=1}^{5} i_j}{i_6}\big[P_{E_1E_2}(\ell_i,\ell_m)\big]^{j_6} \times \binom{N-\sum_{j=1}^{6} i_j}{i_7}\big[P_e(\ell_i)P_c(\ell_m)\big]^{i_7} \times \binom{N-\sum_{j=1}^{7} i_j}{i_8}\big[P_c(\ell_i)P_f(\ell_m)\big]^{i_8} \times \binom{N-\sum_{j=1}^{8} i_j}{i_9}\big[P_c(\ell_i)P_e(\ell_m)\big]^{i_9} \times \big[P_c(\ell_i)P_c(\ell_m)\big]^{N-\sum_{j=1}^{9} i_j},$$

[1] Note that $i_{10} = N - \sum_{j=1}^{9} i_j$.

and

$$P(C_3 \mid Z_{(i_1,i_2,\ldots,i_{10})} = \sum_{m_1=0}^{i_1}\binom{i_1}{m_1}\big[P_e(\ell_k)\big]^{m_1}\big[P_c(\ell_k)\big]^{(i_1-m_1)} \times \big[P_f(\ell_k)+P_{E_1/E_1}(\ell_m,\ell_k)\big]^{i_2} \times \big[1-P_e(\ell_k)\big]^{i_3}\big[P_f(\ell_k)+P_{E_1/E_1}(\ell_i,\ell_k)\big]^{i_4} \times \big[2P_{E_1/E_1E_2}(\ell_i,\ell_m,\ell_k)\big]^{i_6} \times \sum_{m_2=0}^{i_7}\binom{i_7}{m_2}\big[P_{E_1/E_1}(\ell_i,\ell_k)\big]^{m_2}\big[P_c(\ell_k)\big]^{(i_7-m_2)} \times \big[1-P_e(\ell_k)\big]^{i_8} \times \sum_{m_3=0}^{i_9}\binom{i_9}{m_3}\big[P_{E_1/E_1}(\ell_m,\ell_k)\big]^{m_3}\big[P_c(\ell_k)\big]^{(i_9-m_3)}.$$

The resulting expression for $\alpha_{e,3}(\ell_i,\ell_m,\ell_k)$ is

$$\alpha_{e,3}(\ell_i,\ell_m,\ell_k) = \sum_{i_1=0}^{N}\sum_{i_2=0}^{N-i_1}\sum_{i_3=0}^{N-i_1-i_2}\sum_{i_4=0}^{N-i_1-i_2-i_3} \sum_{i_5=0}^{N-\sum_{j=1}^{4} i_j}\sum_{i_6=0}^{N-\sum_{j=1}^{5} i_j}\sum_{i_7=0}^{N-\sum_{j=1}^{6} i_j} \sum_{i_8=0}^{N-\sum_{j=1}^{7} i_j}\sum_{i_9=0}^{N-\sum_{j=1}^{8} i_j}\sum_{m_1=0}^{i_1}\sum_{m_2=0}^{i_7}\sum_{m_3=0}^{i_9}\binom{N}{i_1} \times \binom{N-i_1}{i_2}\binom{N-\sum_{j=1}^{2} i_j}{i_3}\binom{N-\sum_{j=1}^{3} i_j}{i_4} \times \binom{N-\sum_{j=1}^{4} i_j}{i_5}\binom{N-\sum_{i=1}^{5} i_j}{i_6} \times \binom{N-\sum_{j=1}^{6} i_j}{i_7}\binom{N-\sum_{j=1}^{7} i_j}{i_8} \times \binom{N-\sum_{j=1}^{8} i_j}{i_9}\binom{i_1}{m_1}\binom{i_7}{m_2}\binom{i_9}{m_3} \times \big[P_f(\ell_i)P_f(\ell_m)P_c(\ell_k)\big]^{i_1} \times \big[P_f(\ell_i)P_e(\ell_m)\big(P_f(\ell_k)+P_{E_1/E_1}(\ell_m,\ell_k)\big)\big]^{i_2} \times \big[P_f(\ell_i)P_c(\ell_m)\big(1-P_e(\ell_k)\big)\big]^{i_3} \times \big[P_e(\ell_i)P_f(\ell_m)\big(P_f(\ell_k)+P_{E_1/E_1}(\ell_i,\ell_k)\big)\big]^{i_4} \times \big[P_{E_1E_1}(\ell_i,\ell_m)\big]^{i_5} \times \big[P_{E_1E_2}(\ell_i,\ell_m)\big(2P_{E_1/E_1E_2}(\ell_i,\ell_m,\ell_k)\big)\big]^{i_6}$$

$$\times \left[P_{\mathrm{e}}(\ell_i)P_{\mathrm{c}}(\ell_m)P_{\mathrm{c}}(\ell_k)\right]^{i_7}$$
$$\times \left[P_{\mathrm{c}}(\ell_i)P_{\mathrm{f}}(\ell_m)\big(1-P_{\mathrm{e}}(\ell_k)\big)\right]^{i_8}$$
$$\times \left[P_{\mathrm{c}}(\ell_i)P_{\mathrm{e}}(\ell_m)P_{\mathrm{c}}(\ell_k)\right]^{i_9}$$
$$\times \left[P_{\mathrm{c}}(\ell_i)P_{\mathrm{c}}(\ell_m)\right]^{N-\sum_{j=1}^{9} i_j}$$
$$\times \left[P_{\mathrm{e}}(\ell_k)/P_{\mathrm{c}}(\ell_k)\right]^{m_1}\left[P_{E_1/E_1}(\ell_i,\ell_k)/P_{\mathrm{c}}(\ell_k)\right]^{m_2}$$
$$\times \left[P_{E_1/E_1}(\ell_m,\ell_k)/P_{\mathrm{c}}(\ell_k)\right]^{m_3}. \tag{20}$$

Although not explicitly stated, (20) excludes the following cases:

1. $i_1 = i_2 = i_3 = 0$, to ensure that the first transmission failed.
2. $i_1 = i_6 = i_7 = i_9 = 0$, to ensure that the second transmission failed.
3. $i_2 = i_4 = i_5 = i_6 = m_1 = m_2 = m_3 = 0$, in order *not* to have a pattern of correct decoding.

Finally, it is straightforward to show that

$$\alpha_{\mathrm{f},3}(\ell_i,\ell_m,\ell_k) = \alpha_{\mathrm{f},2}(\ell_i,\ell_m) - \left[\alpha_{\mathrm{c},3}(\ell_i,\ell_m,\ell_k) + \alpha_{\mathrm{e},3}(\ell_i,\ell_m,\ell_k)\right].$$

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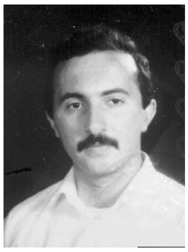

Abdel-Ghani Daraiseh was born on October 19, 1968. He received the B.Sc. degree in electrical engineering from the University of Jordan in June 1991, and M.Sc. and Ph.D. degrees in electrical engineering from Clemson University in May 1993 and December 1995, respectively. From January 1992 until June 1992 he was a Research Assistant at the University of Jordan. From August 1993 until December 1994 he was a Research Assistant at the Center for Computer Communication Systems at Clemson University. Currently he is a member of the scientific staff with the Wireless Systems Engineering Department at Northern Telecom. His research interests lie in the areas of wireless architecture analysis, cellular networks, spread spectrum systems, error control coding, ATM and queueing theory.

Carl W. Baum was born in Pasadena, CA, in 1965. In 1987 he received the B.S. degree in electrical engineering (with the highest distinction) from the University of California, Los Angeles, and in 1989 and 1992 he received the M.S. and Ph.D. degrees in electrical engineering from the University of Illinois, Urbana-Champaign. In 1992, Dr. Baum received the Robert Chien Memorial Award for excellence in research in the field of electrical and computer engineering. In 1996, he was named recipient of the IEEE Browder J. Thompson Prize Award for the outstanding paper by authors under thirty years of age. Dr. Baum was a University of Illinois Fellow from 1987 to 1988. From 1987 to 1992, he held a graduate research assistantship at the Coordinated Science Laboratory, University of Illinois. In August 1992 he joined Clemson University, where he is currently an Assistant Professor in Electrical Engineering. His current research interests are in the general areas of detection theory and communications theory with emphasis on spread-sprectrum communications. Dr. Baum is a member of Tau Beta Pi, Eta Kappa Nu, and Phi Eta Sigma.