

COMS E6232 - Problem Set #3

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Problem 1

a. **Lemma 1a** There is always a directed cut that contains at least $1/4$ of the edges.

Proof: For $i = 1, \dots, |N|$ we assign node i to set S with probability $1/2$ and to set $N - S$ with probability $1/2$. For each directed edge (i, j) , let $X_{(i,j)} = 1$ if $(i, j) \in$ directed cut from S to $N-S$, else 0 otherwise. Thus, we get the equation:

$$E[X_{(i,j)}] = P[(i, j) \in \text{directed cut from } S \text{ to } N-S] = P[i \in S, j \in N - S] = 1/4$$

□

b.

c. The worst-case approximation ratio of the simple local search algorithm is unbounded. Take for example a node i and some number n of nodes where all edges are directed towards i . If we put node i in set S and the rest of the nodes in set $N - S$, then all the edges are directed from $N - S$ to S , giving a cut with 0 edges since only edges from S to $N - S$ count. Using only one local move will not improve the cut, regardless of which node we decide to move. If we move node i to $N - S$, then all nodes will be on that side. If we move any of the n nodes from $N - S$ to S , then the edge will reside within S , still not improving the cut. We can see that the maximum cut would have the n number of nodes in set S and node i in set $N - S$, which would give us a cut with n number of edges. Thus, we can see that the worst-case approximation ratio is unbounded using the simple local search algorithm since we can scale n to any arbitrarily large number.

d. We can improve the local search so that it guarantees ratio $1/4$. The improvement of the local search algorithm is summarized below:

Given some directed graph $D = (N, E)$ and a random assignment of the nodes $i = 1, \dots, |N|$ into either set S or $N - S$, while \exists a node i that has more neighbors on the same side than the opposite side, move i to the opposite side. Based on slides 23 and 24 of the Lecture 5 slides, we will end up with $\geq 1/2|E|$ of edges that go between the sets S and $N - S$. From this, we find the value of the directed cut of these edges. Then, we move all the current nodes that are in S to $N - S$ and the current nodes of $N - S$ to S and find the value of the directed cut of these edges. We take the configuration of S and $N - S$ based on the larger of the two directed cuts that we found, which at least one of them $\geq 1/4|E|$. One of the cuts will be $\geq 1/4|E|$ because there are a total of $\geq 1/2|E|$ edges in the cut, so if $\leq 1/4|E|$ edges go from S to $N - S$ in one configuration, then there must be $\geq 1/4|E|$ edges going from $N - S$ to S , so we would move the nodes to orient the direction of the edges to go from S to $N - S$.

Problem 2

a. The quadratic program for this problem is given below:

$$\begin{aligned} \max & \frac{1}{2} \sum_{i < j} (1 + (y_i \cdot y_j \cdot z_{ij})) \\ \text{s.t. } & y_i \in \{-1, +1\}, \forall i = 1, \dots, n \\ & z_{ij} \in \{-1, +1\}, i < j \end{aligned}$$

The correspondence between the problem and the quadratic program are as follows:

$$\text{object } i \in \text{class } S \Leftrightarrow y_i = +1$$

$$\text{object } i \in \text{class } S' \Leftrightarrow y_i = -1$$

$$\text{similar constraint } i \sim j \Leftrightarrow z_{ij} = +1$$

$$\text{dissimilar constraint } i \nsim j \Leftrightarrow z_{ij} = -1$$

For objects i, j that are in the same class with a similarity constraint:

$$y_i y_j z_{ij} = (-1)(-1)z_{ij} \text{ or } (+1)(+1)z_{ij} = 1 \cdot z_{ij} = (1)(1) = 1 \Leftrightarrow 1 + y_i y_j z_{ij} = 2$$

For objects i, j that are in the same class with a dissimilarity constraint:

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For objects i, j that are in different classes with a similarity constraint:

$$y_i y_j z_{ij} = (-1)(+1)z_{ij} \text{ or } (+1)(-1)z_{ij} = -1 \cdot z_{ij} = (-1)(1) = -1 \Leftrightarrow 1 + y_i y_j z_{ij} = 0$$

For objects i, j that are in different classes with a dissimilarity constraint:

$$y_i y_j z_{ij} = (-1)(+1)z_{ij} \text{ or } (+1)(-1)z_{ij} = -1 \cdot z_{ij} = (-1)(-1) = 1 \Leftrightarrow 1 + y_i y_j z_{ij} = 2$$

where the objective function is maximizing the number of satisfied constraints.

Problem 3

a. **Lemma 3a** If I is an independent set of G then I^k is an independent set of $G^{(k)}$.

Proof: For I to be an independent set of G , there must be no adjacent edges between any of the nodes of I . Given the definition of $E^{(k)}$ of $G^{(k)}$,

$$E^{(k)} = \{(u, v) \in N^{(k)} \times N^{(k)} \mid \exists i, j \in [k], (u_i, u_j) \in E \text{ or } (v_i, v_j) \in E \text{ or } (u_i, v_j) \in E\}$$

we can see that there are no edges between any pairs (u, v) of k -tuples of nodes that only contain nodes of I . When taking the Cartesian product of I, I^k , we create k -tuples using only those nodes from I so $I^k \subseteq N^{(k)}$ since $I \subseteq N$. Also, as explained earlier, the k -tuples of I^k do not have adjacent edges between them since they only contain nodes of I , thus showing that I^k is an independent set of $G^{(k)}$. \square

b. **Lemma 3b** If J is an independent set of $G^{(k)}$ then we can construct in polynomial time an independent set I of G of size at least $|J|^{1/k}$ and conclude that $\alpha(G^{(k)}) = (\alpha(G))^k$

Proof: From Lemma 3a, if I is an independent set of G , then I^k is an independent set of $G^{(k)}$. Since we have defined J to be an independent set of $G^{(k)}$, we can use Lemma 3a to say that $J = I^k$. The size of J is then equal to the size of I^k , which is $|I|^k$, thus $|J| = |I|^k \implies |I| = |J|^{1/k}$, showing that we can construct in polynomial time an independent set I of G of size at least $|J|^{1/k}$. Also, if given the maximum independent set of G , $\alpha(G)$, we can use our earlier conclusion and see that $(\alpha(G^{(k)}))^{1/k} = \alpha(G) \implies \alpha(G^{(k)}) = (\alpha(G))^k$. \square

c. **Lemma 3c** If the Maximum Independent Set problem can be approximated in polynomial time within some constant factor $c > 1$, then it has a PTAS.

Proof: We first define $\alpha(G)$ as the maximum independent set of G . From the PCP theorem, we can trivially say that the Maximum Independent Set problem has a 2-approximation algorithm. Let us define I as an independent set of G that agrees with the 2-approximation algorithm. This means that the size of I will be $1/2$ the size of the maximum independent set of G , thus $\frac{|\alpha(G)|}{|I|} = 2$. Now, using Lemma 3b, if we take the k -th power of the graph G , we know that $\alpha(G^{(k)}) = (\alpha(G))^k$ and by Lemma 3a, I^k is an independent set of $G^{(k)}$. Thus, for some k -th power graph $G^{(k)}$, the approximation ratio is $\frac{|\alpha(G^{(k)})|}{|I^k|} = \left(\frac{|\alpha(G)|}{|I|}\right)^k = 2^k$ which shows that the approximation ratio grows as we apply the approximation algorithm to larger and larger k -th powers of graph G . Thus, the Maximum Independent Set problem can not be approximated in polynomial time within some constant factor, which means it does not have a PTAS unless $\mathbf{P} = \mathbf{NP}$. \square

Problem 4

a. **Lemma 4a** MDAS can be trivially approximated within a factor of 2.

Proof: Suppose we have an arbitrary ordering of nodes v_1, \dots, v_n and we have two subsets of edges A_1 and A_2 where

$$A_1 = \{(v_i, v_j) \mid (v_i, v_j) \in A, i < j\}$$

$$A_2 = \{(v_i, v_j) \mid (v_i, v_j) \in A, i > j\}$$

By separating the edges into these two subsets, the only way for A_1 to contain a cycle is if there is an edge where $i > j$, which would be contained in the A_2 subset, and the only way for A_2 to contain a cycle is if there is an edge where $i < j$, which would be contained in the A_1 subset. Thus, neither subset will contain a cycle and at least one of the two subsets will contain at least $|A|/2$ edges since $A = A_1 + A_2$. Thus, MDAS can be trivially approximated within a factor of 2 by taking the larger of the two subsets. \square

b. **Theorem 4.2** The Maximum Directed Acyclic Subgraph (MDAS) problem does not have a PTAS unless $\mathbf{P} = \mathbf{NP}$.

Proof: A known problem that does not have a PTAS is the Maximum Independent Set (MIS) problem. MIS does not have a PTAS for any graph with a maximum degree ≥ 3 . We can do a linear reduction $MIS(3) \leq_L MDAS$: Given an undirected graph $G = (N, E)$ with maximum degree 3, we construct a directed graph $D = (V, A)$ where $V = \{(u_1, u_2) \mid u \in N\}$ and $A = \{(u_1, u_2) \mid u \in N\} \cup \{(u_2, v_1), (v_2, u_1) \mid (u, v) \in E\}$. We let $\alpha(G)$ denote the size of the maximum independent set of G and $\gamma(D)$ denote the number of edges of the maximum acyclic subgraph of D . From this, we have the following lemma:

Lemma 4.2.1 For all $\epsilon > 0$, if we are given an acyclic subgraph of D that has at least $(1 - (\epsilon/13))\gamma(D)$ edges, then we can compute in polynomial time an independent set G that has at least $(1 - \epsilon)\alpha(G)$ nodes.

Proof:

(4.2a) If I is an independent set of G , then $D' = (V, A')$ where $A' = \{(u_1, u_2) \mid u \in I\} \cup \{(u_2, v_1), (v_2, u_1) \mid (u, v) \in E\}$ is an acyclic subgraph of D . We can see that all edges $\{(u_2, v_1), (v_2, u_1) \mid (u, v) \in E\}$ do not create a cycle because as we have shown in Lemma 4a, a set of edges $\{(v_i, v_j) \mid (v_i, v_j) \in A, i > j\}$ do not create a cycle. We also know that the independent set I contains nodes that do not have any adjacent edges with each other. Thus, since A' contains all the linear reduction of edges of E , the only way to keep the edge set acyclic would be to only add the edges $\{(u_1, u_2) \mid u \in N\}$ where the nodes do not have any adjacent edges, which would be the independent set

I and gives us $\{(u_1, u_2) \mid u \in I\}$. Thus, if I is an independent set of G , then A' is an acyclic subgraph of D .

(4.2b) Now we show that if H is an acyclic subgraph of D with h edges, we can then derive efficiently from H an independent set of G with at least $h - 2|E|$ nodes. As we have shown in **4.2a**, acyclic subgraphs of D includes all edges $\{(u_2, v_1), (v_2, u_1) \mid (u, v) \in E\}$. We can see that each edge of E corresponds to two edges in A . If we took away all those edges from H , we would be left with edges $\{(u_1, u_2) \mid u \in N\}$ which we know are nodes that could create an independent set, as we have shown in **4.2a**. Thus, we can derive efficiently from H an independent set of G with at least $h - 2|E|$ nodes.

(4.2c) Lastly, since G has maximum degree 3, we can show that $\alpha(G) \geq |E|/6$. Without loss of generality, let's take a node $u \in N$ that has degree 3, which would imply 4 connected nodes. If we were to make all these nodes connected together, we would have a complete graph where each of the 4 nodes would have a maximum degree of 3. This complete graph has a 6 total edges. From this complete graph, we can only choose any one node as the maximum independent set $\alpha(G)$ as all nodes are adjacent to all other nodes. Thus, we can see that $\alpha(G) \geq |E|/6$.

Let us do the linear reduction from MIS to MDAS. From **4.2b** we know we can get an acyclic subgraph of D with $|I| + 2|E|$ edges. If $I = \alpha(G)$, then getting an acyclic subgraph D with $\alpha(G) + 2|E|$ would imply a maximum acyclic subgraph $\gamma(D)$. We can manipulate our conclusion from **4.2c** where $\alpha(G) \geq |E|/6 \implies |E| \leq 6 \cdot \alpha(G)$. Thus, we can say

$$\gamma(D) = \alpha(G) + 2|E| \leq \alpha(G) + 12 \cdot \alpha(G) = 13 \cdot \alpha(G)$$

Based on this L-reduction, it satisfies the first property that $OPT_{MDAS} \leq \alpha \cdot OPT_{MIS}$ where $\alpha = 13$. We can also see it satisfies the second property $|C_1 - OPT_{MIS}| = \beta \cdot |C_2 - OPT_{MDAS}|$ where $\beta = 1$. Combining both properties, we get the relative error:

$$\frac{|C_1 - OPT_{MIS}|}{OPT_{MIS}} \leq \frac{\alpha \cdot \beta \cdot |C_2 - OPT_{MDAS}|}{OPT_{MDAS}} \leq \alpha\beta\epsilon$$

Showing that the relative error of MIS is $13 \cdot \epsilon$ in relation to the relative error of MDAS being ϵ . Thus, if we wanted to express the relative error of MIS as just ϵ , that would mean that the relative error of MDAS would be $\epsilon/13$. Thus for all $\epsilon > 0$, if we are given an acyclic subgraph of D that has at least $(1 - (\epsilon/13))\gamma(D)$ edges, then we can compute in polynomial time an independent set G that has at least $(1 - \epsilon)\alpha(G)$ nodes, which proves the lemma. \square

But because MIS does not have a PTAS as we have proven in problem 3, and since $MIS \leq_L MDAS$, that tells us that MDAS also does not have a PTAS unless $\mathbf{P} = \mathbf{NP}$, thus the theorem is proved. \square