Preliminaries

1. $P(X > Y) = \iint_{\{(x,y):x>y\}} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{\infty}^{\infty}\int_{y}^{\infty} f(x,y)\mathrm{d}x\mathrm{d}y$
2. $E[E(X|Y)] = E(X)$, $Var(X) = E(X^2) - E(X)^2$

Discrete Time Markov Chain (DTMC)

3. If present state given, future events are independent of the past.
4. Matrix P row sum up to 1
5. $p_{ij}^{(n)} = P(X_n = j|X_0 = i) = [P^n]_{ij}$
6. Chapman-Kolmogorov equations: $p_{ij}^{(n+m)} = \sum_{k\in S} p_{ik}^{(n)} p_{kj}^{(m)}$, $P^{(n+m)} = P^{(n)}P^{(m)}$, $P^{(n)} = P^n$
7. $a^{(n)} = a^{(0)}P^n$, $a_k^{(0)} = P(X_0 = k)$
8. $P(X_{n_1} = i_1, X_{n_2} = i_2, \cdots, X_{n_k} = i_k) = [P^{n_k - n_{k-1}}]_{i_{k-1},i_k} \cdots [P^{n_2-n_1}]_{i_2,i_1}[a_0 P^{n_1}]_{i_1}$
9. Accessibility of states: $p_{ii}^{(0)} = 1$
10. Communicating class:
    - $i \in C$, $j \in C \Rightarrow i \leftrightarrow j$
    - $i \in C$, $i \leftrightarrow j \Rightarrow j \in C$
11. Cannot LEAVE $C \Rightarrow$ CLOSED communicating class
12. State space S is a SINGLE class $\Rightarrow$ IRREDUCIBLE
13. First arrival time: $T_j = \min\{n \geq 1 : X_n = j\}$
14. Start at i, prob of ever reaching j: $\varrho_{ij} = P(T_j < \infty|X_0 = i) = \sum_{n=1}^{\infty} P(T_j = n|X_0 = i)$
15. Mean time to return to i: $m_{ii} = E(T_i|X_0 = i)$
16.
    - TRANSIENT: $\varrho_{ii} < 1$
    - POSITIVE RECURRENT: $\varrho_{ii} = 1$, $m_{ii} < \infty$
    - NULL RECURRENT: $\varrho_{ii} = 1$, $m_{ii} = \infty$

    (Class properties)
17. $i \to j$, $\varrho_{ji} < 1 \Rightarrow i$ is TRANSIENT
18. Mean number of visits: $E(N_i|X_0 = i) = \begin{cases} \infty & \text{if } i \text{ is RECURRENT} \\ \frac{1}{1-\varrho_{ii}} & \text{if } i \text{ is TRANSIENT} \end{cases}$
19. RECURRENT $\Leftrightarrow \sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$
20.
    - FINITE CLOSED communicating class $\Rightarrow$ POSITIVE RECURRENT
    - OPEN communicating class $\Rightarrow$ TRANSIENT
21. $p_{ij}^{*} = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} p_{ij}^{(n)}$

22. POSITIVE RECURRENT $\Leftrightarrow p^*_{ii} = \frac{1}{m_i i} > 0$
TRANSIENT or NULL RECURRENT $\Leftrightarrow p^*_{ii} = \frac{1}{m_i i} = 0$

23. State i has period d, $d = \gcd\{n | P^{(n)}_{ii} > 0\}$, class properties

24. Stationary probability: $\pi = \pi P$, $\sum_{j \in S} \pi_j = 1$ (global balance)

25. Aperiodic (irreducible): $\pi_j = \lim_{n \to \infty} p^{(n)}_{jj} = \lim_{n \to \infty} p^{(n)}_{ij}$
Periodic: DO NOT EXIST

26. $\pi_j = p^*_{jj}$

27. Positive recurrent $\Rightarrow \pi_j = \lim_{n \to \infty} p^{(n)}_{ij}$ is unique

28. Detailed balance equations: $\pi_i p_{ij} = \pi_j p_{ji}$, $\sum_{j \in S} \pi_j = 1$
Detailed balance equations satisfied $\Rightarrow$ global balance equations satisfied

29. First passage time: $\hat{T}_A = \min\{n \geq 0 : X_n \in A\}$

30. Reaching set A before set B starting from i:

$$h_i = P(\hat{T}_A < \hat{T}_B | X_0 = i) = \begin{cases} 0 & \text{if } i \in B \\ 1 & \text{if } i \in A \\ \sum_{j \in S} & \text{if } i \in S - (A \cup B) \end{cases}$$

31. Mean time to reach set A starting from i:

$$g_i = E(\hat{T}_A | X_0 = i) = \begin{cases} 0 & \text{if } i \in A \\ 1 + \sum_{j \in S} p_{ij} g_j & \text{if } i \in S - A \end{cases}$$

32. Long-run average cost: $\psi_i = \sum_{j \in S} c(j) \pi_j$ (independent of initial state)

33. Total expected cost (transient): $\phi_i = c(i) + \sum_{j \in S} p_{ij} \phi_j$

Poisson Processes

34. Exponential distribution:
    - c.d.f.: $P(X \leq x) = F(x) = 1 - e^{-\lambda x}$
    - p.d.f.: $f(x) = \lambda e^{-\lambda x}$

35. Memoryless property: $P(X > s + t | X > s) = P(X > t)$
In continuous distributions, only Exponential is memoryless

36. $P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$, $P(\min\{X_1, X_2\} > x) = e^{-(\lambda_1 + \lambda_2)x}$

37. Strong memoryless property: $P(X_2 > X_1 + x | X_2 > X_1) = P(X_2 > x) = e^{-\lambda x}$,
where $X_2 \sim Exp(\lambda)$, X_1 is continuous r.v. ≥ 0.

38. Sum of i.i.d. exponentials: $f_n(z) = \lambda e^{-\lambda z} \frac{(\lambda z)^{n-1}}{(n-1)!}$,
where $Z = X_1 + X_2 + \cdots + X_n$, $X_i \sim Exp(\lambda)$,
$P(Z \leq t) = 1 - \sum_{r=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^r}{r!}$

39.
    - N_t: the number of events that has occurred

- τ_n: the time between $(n-1)$-st and n-th event
- $S_n = \tau_1 + \tau_2 + \cdots + \tau_n$: the time when n-th event occurs

40. Def1: $\tau_n \overset{i.i.d.}{\sim} Exp(\lambda)$. $N_t \sim Pois(\lambda t)$, $P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$
Def2: $N_t^{(s)} = N_{s+t} - N_s$, it is independent of $\{N_u, 0 \leq u \leq s\}$ (previous time)
Def3:
    - has stationary and independent increments
    - $P(N_h = 0) = 1 - \lambda h + o(h)$
    - $P(N_h = 1) = \lambda h + o(h)$
    - $P(N_h \geq 2) = o(h)$
41.
    - Stationary increments: $N_{s+t} - N_s$ is identical, does not dependent on s
    - Independent increments: $N_t - N_s$ and $N_v - N_u$ are independent for $s < t \leq u < v$
42. Superpositioning: $N_t = N_t^{(1)} + \cdots + N_t^{(k)}$ is a $PP(\lambda_1 + \cdots \lambda_k)$
43. Splitting: $\{N_t^{(1)}, t \geq 0\}$, …, $\{N_t^{(k)}, t \geq 0\}$ are PP with rates λp_1, …, λp_k
44. Campbell's Theorem: if $N_t = n$ is given, then these n events occur uniformly over t hours
45. Nonhomogeneous Poisson process:
    - has independent increments
    - $P(N_{t+h} - N_t = 0) = 1 - \lambda(t)h + o(h)$
    - $P(N_{t+h} - N_t = 1) = \lambda(t)h + o(h)$
    - $P(N_h \geq 2) = o(h)$
46. $\Lambda(t) = \displaystyle\int_0^t \lambda(u)\mathrm{d}u$, $N_t \sim Pois\left(\Lambda(t)\right)$, $P(N_t = k) = \dfrac{e^{-\Lambda(t)}(\Lambda(t))^k}{k!}$
47. $E(N_t) = \Lambda(t)$, $Var(N_t) = \Lambda(t)$
48. Unordered arrival are uniformly distributed with $(s < t)$
    - c.d.f.: $\Lambda(s)/\Lambda(t)$
    - p.d.f.: $\lambda(s)/\Lambda(t)$
49. Compound Poisson process: when event n occurs, we incur a random cost Y_n,
$Z_t = \displaystyle\sum_{n=1}^{N_t} Y_n$, $E(Z_t) = \lambda t E(Y_1)$, $Var(Z_t) = \lambda t E(Y_1^2) = \lambda t[Var(Y_1) + E(Y_1)^2]$

Continuous Time Markov Chain (CTMC)

50. CTMC: $\{X_t : t \geq 0\}$, DTMC embedded chain: $\{\tilde{X}_n : n \in \mathbb{N}\}$
51. $\tau_n \sim Exp(q_i)$, is the time spent in state $\tilde{X}_{n-1}$
52. $\tilde{p}_{ii} = 0$ for all i
53. $q_{ij} = q_i\tilde{p}_{ij}$ if $i \neq j$; $q_{ii} = -q_i = -\sum_{j\in S, j\neq i} q_i j$; row sum up to 0
54. $p_{ij}(t) = P(X_t = j | x_0 = i)$

55. Chapman-Kolmogorov equations: $p_{ij}(t+s) = \sum_{k \in S} p_{ik}(t) p_{kj}(s)$, $P(t+s) = P(t)P(s)$

56. $a^{(n)} = a^{(0)}P(t)$, $a_i^{(0)} = P(X_0 = i)$

57. $P(X_{t_1} = j_1, \cdots, X_{t_k} = j_k) = [P(t_k - t_{k-1}]_{j_{k-1}, j_k} \cdots [a^{(0)}P(t_1)]_{j_1}$

58. $P'(0) = Q$

59. Forward Kolmogorov equation: $P'(t) = P(t)Q$
Backward Kolmogorov equation: $P'(t) = QP(t)$
Initial conditions: $P(0) = I$

60. TRANSIENT and RECURRENT: same as DTMC (14, 15, 16)

61. CTMC is POSITIVE RECURRENT $\Leftrightarrow \sum_{i \in S} \frac{\tilde{\pi}_i}{q_i} < \infty$, where $\tilde{\pi} = \tilde{\pi}\tilde{P}$

62. Stationary probability: $\pi Q = 0$, $\sum_{i \in S} \pi_i = 1$ (global balance)

63. Detailed balance equations: same as DTMC (28)

64. Reaching set A before set B staring from i:

$$\begin{cases} h_i = 0 & \text{if } i \in B \\ h_i = 1 & \text{if } i \in A \\ \sum_{j \in S} q_{ij} h_j = 0 & \text{if } i \in S - (A \cup B) \end{cases}$$

65. Mean time to reach set A starting from i:

$$\begin{cases} g_i = 0 & \text{if } i \in A \\ \sum_{j \in S} q_{ij} g_j = -1 & \text{if } i \in S - A \end{cases}$$

66. Long-run average cost: same as DTMC (32)