Basic algebra and geometry of complex numbers

1. $\overline{zw} = \bar{z}\bar{w}$, $|z| = z\bar{z}$, $|zw| = |z||w|$
2. For $z = x + iy$,
   $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$
3. For $f(x,y) = u(x,y) + iv(x,y)$,
   $u(x,y) = \frac{1}{2}\left(f(x,y) + \overline{f(x,y)}\right)$,
   $v(x,y) = \frac{1}{2i}\left(f(x,y) - \overline{f(x,y)}\right)$
4. Triangle inequality: $|z_1 + z_2| \le |z_1| + |z_2|$
5. Reverse triangle inequality: $|z+w| \ge |z| - |w|$
6. $\arg(z) = \{\mathrm{Arg}(z) + 2\pi k | k \in \mathbb{Z}\}$,
   $-\pi < \mathrm{Arg}(z) \le \pi$
7. De Moivre's formula:
   $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$
8. Roots of unity:
   $1^{1/q} = \{\exp(i2\pi(k/q))\, | k \in \mathbb{Z}\}$,
   $z^{1/q} = |z|^{1/q}\exp(i\theta/q)\omega^k$,
   $z^{p/q} = |z|^{p/q}\exp(ip\theta/q)\omega^{pk}$
9. $\arg(zw) = \arg(z) + \arg(w)$,
   $\arg(1/z) = \arg(\bar{z}) = -\arg(z)$
10. Möbius transformation:
    rational function $f(z) = \frac{az+b}{cz+d}$, $ad \ne bc$,
    because $f'(z) \ne 0 \Rightarrow f(z) \ne \mathrm{Const}$,
    $f(z) = \frac{az+b}{cz+d} = \frac{a+\frac{b}{z}}{c+\frac{d}{z}} \Rightarrow f(\infty) = \frac{a}{c}$
11. Extended complex plane: $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,
    $a + \infty = \infty$, $b\infty = \infty$, $(a \in \mathbb{C},\, b \in \mathbb{C}^\times)$,
    $\frac{a}{0} = \infty$, $\frac{a}{\infty} = 0$, $(a \in \mathbb{C}^\times)$
12. Riemann sphere:
    $\varphi : \tilde{\mathbb{C}} \to S^2$, $\varphi(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$,
    $\varphi(\infty) = \lim\limits_{|z|\to\infty} \varphi(z) = (0,0,1)$,
    $\psi : S^2 \to \tilde{\mathbb{C}}$,
    $$\psi(X,Y,Z) = \begin{cases} \frac{X+iY}{1-Z} & (X,Y,Z) \ne (0,0,1) \\ \infty & (X,Y,Z) = (0,0,1) \end{cases}$$

Holomorphicity

13. f is holomorphic at $z_0 \Leftrightarrow$
    f' exists in some neighbourhood U of z_0,
    $\oint_\Gamma f(z)\mathrm{d}z = 0$ for all $\Gamma \subset D$,
    f is analytic in D,
    power series expansion of f converges in D,
    obey Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ in D,
    $\bar{\partial} f = 0$ in D, where $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$,
    the dependence on $\bar{z}$ cancels
14. f is entire if it is holomorphic in the whole complex plane.
15. Harmonic: $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$
16. Real & imaginary part of holo func f are harmonic. (showed by C-R equations)
17. u is harmonic, $f = u + iv$ is holomorphic $\Rightarrow$
    v is its harmonic conjugate
18. $f(z)$ and $g(z)$ are holomorphic $\Rightarrow$
    $f(z) + g(z)$ and $f(z)g(z)$ are holomorphic.
19. $P(z) = \sum_{n=0}^N a_n z^n$ is entire because $f(z) = z$ is entire,
    $P'(z_0) = \sum_{n=1}^N n a_n z_0^{n-1}$
20. $R(z) = \frac{P(z)}{Q(z)}$ is a rational function and holomorphic away from the zeros of $Q(z)$
21. Composition $(f \circ g)(z) = f(g(z))$ is holomorphic. $(f \circ g)'(z_0) = f'(g(z_0))g'(z_0)$.
22. $\exp(z) = \exp(x + iy) = e^x(\cos y + i\sin y)$,
    entire, addition property:
    $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$,
    periodic: $\exp(z + 2\pi i) = \exp(z)$
23. $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$,
    entire, not bounded,
    $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$,
    $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$
24. $\sinh(z) = \frac{e^z - e^{-z}}{2}$, $\cosh(z) = \frac{e^z + e^{-z}}{2}$,
    $\sinh(iz) = i\sin(z)$, $\cosh(iz) = \cos(z)$
25. $G(z) = \int_\Gamma \frac{g(\zeta)}{\zeta - z}\mathrm{d}\zeta$ is holomorphic,
    g is continuous in Γ, Γ need not be closed
26. f is holomorphic around $z_0 \Rightarrow$
    so are f', f'', ...
27. Log def: $\log(z) = \{w | \exp(w) = z\}$,
    $\log(z) = \{\ln|z| + i\theta | \theta \in \arg(z)\}$,
    $\log(z) = \{\ln|z| + i\mathrm{Arg}(z) + 2\pi i k | k \in \mathbb{Z}\}$,
    $\log\left(re^{i\theta}\right) = \{\ln r + i\theta + i2\pi k | k \in \mathbb{Z}\}$,
    $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ (equality of sets)
    $\log(1/z) = -\log(z)$
28. $\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$,
    $\mathrm{Log}_0(z) = \ln|z| + i\mathrm{Arg}_0(z)$

29.
$$\begin{aligned} z^\alpha &= \{\exp(\alpha w) | w \in \log(z)\} \\ &= \{\exp(\alpha \ln|z| + i\alpha \text{Arg}(z) + i\alpha 2\pi k) \,|k \in \mathbb{Z}\} \\ &= \{\exp(\alpha \text{Log}(z)) \exp(i\alpha 2\pi k) | k \in \mathbb{Z}\} \end{aligned}$$

30.
$$z^n = \begin{cases} 1 & \text{for } n = 0 \\ \underbrace{zz\cdots z}_{n \text{ times}} & \text{for } n > 0 \\ \frac{1}{z^{-n}} & \text{fot } n < 0 \end{cases}$$

31. Principal branch: $z^\alpha = \exp(\alpha \text{Log}(z))$,
$z^\alpha z^\beta = z^{\alpha+\beta}$, $\frac{\mathrm{d}}{\mathrm{d}z}(z^\alpha)\,|_{z=z_0} = \alpha z_0^{\alpha-1}$,
$z_1^\alpha z_2^\alpha = (z_1 z_2)^\alpha$ NOT TRUE IN GENERAL

Complex integration

32. $\left|\int_a^b f(t)\mathrm{d}t\right| \le \int_a^b |f(t)|\mathrm{d}t$
33. $\int_\Gamma f(z)\mathrm{d}z = \int_{z_1}^{z_2} f(z(t))\dot{z}(t)\mathrm{d}t$
34. Arclength $\ell(\Gamma) = \int_\Gamma |\mathrm{d}z|$
35. ML Lemma:
$$\left|\int_\Gamma f(z)\mathrm{d}z\right| \le \int_\Gamma |f(z)||\mathrm{d}z| \le \max_{z\in\Gamma} |f(z)|\ell(\Gamma) = ML$$
36. Complex linear
37. $\int_{-\Gamma} f(z)\mathrm{d}z = -\int_\Gamma f(z)\mathrm{d}z$
38. $$\int_\Gamma f(z)\mathrm{d}z = \sum_{j=1}^n \int_{\Gamma_j} f(z)\mathrm{d}z,$$
where Γ_j is a regular component of Γ.
39. Contour integral only depends on the end-points.
40. FT$\mathbb{C}$: $\int_\Gamma f(z)\mathrm{d}z = F(z_1) - F(z_0)$
41. Closed contour: $\oint_\Gamma f(z)\mathrm{d}z = 0$
42. Path-independence Lemma:
F exists in $D \Leftrightarrow$
$\oint_\Gamma f(z)\mathrm{d}z$ vanishes for all closed contours Γ in $D \Leftrightarrow$
$\int_\Gamma f(z)\mathrm{d}z$ are independent of the path
43. Jordan Curve Theorem: loop separates the plane into 2 domains (interior and exterior), with it as common boundary
44. C$\int$T: D is a simply-connected domain, f is a holomorphic function, $\oint_\Gamma f(z)\mathrm{d}z = 0$
45.
$$\oint_\Gamma \frac{1}{z - z_0}\mathrm{d}z = \begin{cases} 2\pi i & \text{for } z_0 \text{ in the interior of } \Gamma \\ 0 & \text{otherwise} \end{cases}$$
46. C$\int$F: $f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$,
$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta$
47. Morera's theorem: f is continuous in D, all the loop integrals vanish $\Rightarrow$
f is holomorphic
48. Cauchy estimate: $\left|f^{(n)}(z_0)\right| \le \frac{n!M}{R^n}$,
where $R = |z - z_0|$, $|f(z)| \le M$.
49. Liouville's theorem:
Bounded, Entire $\Rightarrow$ CONSTANT func
50. Maximum modulus principle:
f is holomorphic, $|f(z)| \le M$,
If $|f(z)|$ achieves max at $z_0 \in D \Rightarrow$
f is Constant in D
51. Max-min principle:
ϕ is harmonic, bounded above or below M,
If $\exists z_0 \in D$, $\phi(z_0) = M \Rightarrow$
ϕ is Constant in D

Series expansions

52. Sequence: $\{z_n\}$ or $\{z_0, z_1, z_2, \ldots\}$,
converge: $\lim_{n\to\infty} z_n = z$
53. Cauchy criterion (be a Cauchy sequence):
$\forall \varepsilon > 0$, $\exists N(\varepsilon)$, $\forall n, m \ge N$, s.th. $|z_n - z_m| < \varepsilon$
54. Sequence converges $\Rightarrow$ Cauchy
55. $$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=0}^n c_j = S \Rightarrow$$
series converges to S, $S = \sum_{j=0}^\infty c_j$
56. $$\sum_{j=0}^\infty c_j \text{ converges} \Rightarrow \lim_{j\to\infty} c_j = 0$$
57. $$\sum_{j=1}^\infty \frac{1}{j^p} \text{ converges for any real } p > 1$$
58. Comparison test: If $\sum_{j=0}^\infty M_j$ is convergent, $M_j \ge 0$, $|c_j| \le M_j$ for all sufficiently large j, then $\sum_{j=0}^\infty c_j$ converges.
Comparing with $\frac{1}{j_p}$ $(p > 1)$ or c^j $(|c| < 1)$.

59. $\sum_{j=0}^{\infty} c^j = \dfrac{1}{1-c}$, if $|c| < 1$.

60. Ratio test: $L = \lim_{j\to\infty} \left|\dfrac{c_{j+1}}{c_j}\right|$,
$L < 1$ converges; $L > 1$ diverges; $L = 1$ dk

61. Sequence $\{f_n\}$ converges POINTWISE to f (for each z, $\{f_n(z)\}$ converges)

62. UNIFORMLY converge: $\forall \varepsilon > 0$, $\exists N$, $\forall n \geq N$, s.th. $|f_n(z) - f(z)| < \varepsilon$

63. Uniform limit of continuous function
$\Rightarrow$ Continuous

64. Weierstrass M-test: If $\sum_{j=0}^{\infty} M_j$ is convergent, $M_j \geq 0$, $|f_j(z)| \leq M_j$ for all sufficiently large j, then $\sum_{j=0}^{\infty} f_j(z)$ converges UNIFORMLY

65. Taylor series:
$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j$$

66. Maclaurin series: $z_0 = 0$

67. Taylor series theorem:
$f(z)$ is holomorphic in $|z - z_0| < R \Rightarrow$
Taylor series of f converges UNIFORMLY on $|z - z_0| \leq r < R$

68. Taylor series for $\alpha f(z)$:
$$\sum_{j=0}^{\infty} \frac{\alpha f^{(j)}(z_0)}{j!}(z-z_0)^j$$

69. Taylor series of $f(z) + g(z)$:
$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0) + g^{(j)}(z_0)}{j!}(z-z_0)^j$$

70. Cauchy-Hadamard theorem:
Power series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$,
radius of convergence $0 \leq R \leq \infty$,
converges absolutely in $|z - z_0| < R$,
converges uniformly on any closed disk,
diverges in $|z - z_0| > R$

71. $R = \dfrac{1}{\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right|}$

72. $R > 0$ convergent,
otherwise divergent.
If admits a convergent power series, analytic

73. Laurent series theorem:
$f(z)$ is holomorphic in $r < |z - z_0| < R$,
has a Laurent series on $r_1 \leq |z - z_0| \leq R_1$,
for all $r_1 > r$ and $R_1 < R$.
$a_j = \dfrac{1}{2\pi i}\oint_\Gamma \dfrac{f(z)}{(z-z_0)^{j+1}}\mathrm{d}z$

74. ZERO of order m:
$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$,
$f^{(m)}(z_0) \neq 0$,
SIMPLE ZERO: $m = 1$,
zeros of finite order of Holo func are isolated

75. Singularities of a rational function are isolated

76. REMOVABLE SINGULARITY:
no negative powers,
$f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$

77. POLE of order m:
$a_j = 0$ for all $j < -m$, $a_{-m} \neq 0$,
$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + a_0 + a_1(z-z_0) + \cdots$,
SIMPLE POLE: $m = 1$

78. ESSENTIAL SINGULARITY:
∞ number of nonzero terms with negative powers of $(z - z_0)$

79. Riemann's removable singularities theorem:
f is holomorphic,
bounded in $D^\times$ centred at z_0,
$$\hat{f}(z) = \begin{cases} f(z) & z \neq z_0 \\ \lim_{\zeta\to z_0} f(\zeta) & z = z_0 \end{cases}$$

Residue theory

80. Residue $\mathrm{Res}(f; z_0)$ is a_{-1} of the Laurent series at z_0

81. Simple pole at z_0:
$\mathrm{Res}(f; z_0) = \lim_{z\to z_0}(z-z_0)f(z)$, or
$\mathrm{Res}(f; z_0) = \lim_{z\to z_0}(z-z_0)\frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$,
because $h(z_0) = 0$

82. Pole of order m at z_0:
$\mathrm{Res}(f; z_0) = \lim_{z\to z_0} \dfrac{1}{(m-1)!}\dfrac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left((z-z_0)^m f(z)\right)$

83. Cauchy residue theorem (CRT):
$$\oint_\Gamma f(z)\mathrm{d}z = 2\pi i \sum_{\substack{\text{singularities} \\ z_k \in \mathrm{Int}\Gamma}} \mathrm{Res}(f; z_k)$$

84. Meromorphic: for all $z \in D$,
either f has a pole at z,
or f is holomorphic around z

85. Argument principle:
$$\frac{1}{2\pi i}\oint_\Gamma \frac{f'(z)}{f(z)}\mathrm{d}z = N_0(f) - N_\infty(f),$$
$N_0^\Gamma(f) = \sum_{\text{zeros inside }\Gamma}$ order of each zero,
$N_\infty^\Gamma(f) = \sum_{\text{poles inside }\Gamma}$ order of each pole

86. Rouché's theorem:
f: holomorphic, inside a loop Γ,
g: holomorphic, on a loop Γ,
$|f(z) - g(z)| < |f(z)|$,
Then, $N_0(f) = N_0(g)$

87. Open mapping theorem:
f is nonconstant, holomorphic func on D,
$f(D) = \{w \in \mathbb{C} | w = f(z) \quad \exists z \in D\}$ is open

88. Max modulus principle:
f is nonconstant, holo func in $D \subset \mathbb{C}$,
$|f(z)|$ cannot attain its max at $\forall z \in D$

89. By substituting $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$,
$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, $\mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$,
$$\begin{aligned} I_T &= \int_0^{2\pi} R(\cos\theta, \sin\theta)\mathrm{d}\theta \\ &= \oint_\Gamma \frac{1}{iz} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\mathrm{d}z \\ &= 2\pi \sum_{\substack{\text{singularities} \\ |z_k|<1}} \mathrm{Res}(f; z_k) \end{aligned}$$
where $f(z) = \frac{1}{z}R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)$

90. Cauchy principal value:
p.v. $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\mathrm{d}x$

91. $R(x) = P(x)/Q(x)$:
$Q(x) \neq 0$, $\deg Q - \deg P > 1$,
$$\text{p.v.} \int_{-\infty}^{\infty} R(x)\mathrm{d}x = 2\pi i \sum_{\substack{\text{poles } z_k \\ \mathrm{Im}(z_k)>0}} \mathrm{Res}(R; z_k)$$

92. $R(x)\cos(ax)$ and/or $R(x)\sin(ax)$,
$\Rightarrow R(x)\exp(iax)$,
$R(x) = P(x)/Q(x)$:
$Q(x) \neq 0$, $\deg Q - \deg P \geq 1$,
$$\text{p.v.} \int_{-\infty}^{\infty} R(x)\exp(iax)\mathrm{d}x$$
$$= \begin{cases} 2\pi i \sum_{\substack{\text{poles } z_k \\ \mathrm{Im}(z_k)>0}} \mathrm{Res}(f; z_k) & \text{if } a > 0 \\ -2\pi i \sum_{\substack{\text{poles } z_k \\ \mathrm{Im}(z_k)<0}} \mathrm{Res}(f; z_k) & \text{if } a < 0 \end{cases}$$

93. Jordan lemma:
$\lim_{\rho\to\infty}\int_{C_\rho^+}\exp(iax)\frac{P(z)}{Q(z)}\mathrm{d}z = 0$,
$a > 0$, $\deg Q > \deg P$

94. $R(x) = P(x)/Q(x)$,
$Q(x) \neq 0$, $\deg Q - \deg P \geq 2$,
$$\int_0^\infty R(x)\mathrm{d}x = -\sum_{\text{poles } z_k} \mathrm{Res}(f; z_k)$$
$f(z) = \log(z)R(z)$

95. For $\int_a^\infty R(x)\mathrm{d}x$, $f(z) = \log(z-a)R(z)$

96. IMPROPER INTEGRALS WITH POLES