- Cartesian form: $z = x + iy$
  Polar form: $z = r(\cos\theta + i\sin\theta)$
  Exponential form: $z = re^{i\theta}$
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
- de Moivre's formula: $\cos(n\theta)+i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$
- $|z|^2 = z\bar{z} \quad \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2} = \dfrac{x}{|z|^2} - i\dfrac{y}{|z|^2}$
  $Re(z) = \dfrac{z+\bar{z}}{2}, Im(z) = \dfrac{z-\bar{z}}{2i}$
- Triangle equality: $|z + w| \leq |z| + |w|$
  Reverse triangle equality: $||z| - |w|| \leq |z - w|$
- $\arg(z) = \{\theta : z = |z|e^{i\theta}\}$
  $= \{\mathrm{Arg}(z) + 2\pi k : k \in \mathbb{Z}\}$
  $-\pi < \mathrm{Arg(z)} \leq \pi$ satisfies $z = |z|e^{i\mathrm{Arg}(z)}$
- Principal value: $\mathrm{Arg}(z)$, has discontinuity at all points $z$ on the negative real axis
- $\arg(zw) = \arg(z) + \arg(w)$
  $\arg(\bar{z}) = -\arg(z)$
  $\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w) + 2k\pi$
  $\mathrm{Arg}(\bar{z}) = -\mathrm{Arg}(z) + 2k\pi$
- open $\epsilon$-disc: $D_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$
  closed $\epsilon$-disc: $\bar{D}_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$
  punctured $\epsilon$-disc: $D'_\epsilon(z_0) = \{z \in \mathbb{C} : 0 < |z-z_0| < \epsilon\} = D_\epsilon(z_0)\backslash\{z_0\}$

# 1 Holomorphicity

- $f(z) = f(x + iy) = u(x, y) + iv(x, y)$
- $f$ continuous at $z_0$: $\forall\epsilon > 0, \exists\delta > 0$ s.t. $|f(z_0) - f(z)| < \epsilon$ whenever $z \in S$ satisfies $|z_0 - z| < \delta$
  e.g. $f(z) = z, \bar{z}, |z|$ is continuous
- $f$ continuous $\Leftrightarrow$ the preimage $f^{-1}(U) = \{z \in \mathbb{C} : f(z) \in U\}$ is open for all open $U \subseteq \mathbb{C}$
- $S \subseteq \mathbb{C}$ is closed & bounded + $f$ continuous $\Rightarrow f(S)$ is closed & bounded
- $f$ differentiable at $z_0$: $U \subseteq \mathbb{C}$ is a neighbourhood of $z_0$, $f : U \to \mathbb{C}$, the limit $f'(z_0) = \dfrac{df}{dz}(z_0) = \lim\limits_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists
- $f$ holomorphic on $U$: $f$ is differentiable at $z_0$ for every $z_0 \in U$
- $f$ differentiable at $z_0 \Rightarrow f$ continuous at $z_0$
- Chain rule: $(f \circ g)'(z_0) = f'(g(z_0))g'(z_0)$
- C-R equations: $z_0 = x_0 + iy_0$, $f = u + iv$ differentiable at $z_0 \Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

- $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$
- $f = z$ hol
  $f = \bar{z}$ nowhere hol, nowhere differentiable
  $f = |z|^2$ only diff at the origin, nowhere hol
- hol $\Rightarrow$ C-R
  C-R + partial derivatives continuous $\Rightarrow$ hol
- $f, g$ hol $\Rightarrow f + g, fg, f/g(g \neq 0)$ hol
- complex polynomial $P(z) = \sum_{n=0}^{N} a_n z^n$ is hol on $\mathbb{C}$
- rational function $R = P/Q$ is hol on $\{z \in \mathbb{C} : Q(z) \neq 0\}$
- $U$ open + $g$ hol on $U$ + $f$ hol on $g(U) \Rightarrow f \circ g$ is hol on $U$
- Harmonic: $\dfrac{\partial^2 h}{\partial x^2}(x, y) + \dfrac{\partial^2 h}{\partial y^2}(x, y) = 0$
- Harmonic conjugate: $U$ open, $u$ harmonic, then $v$ is the harmonic conjugate of $u$ if $f = u+iv$ is hol on $U$
- $u, v$ twice cont differentiable (i.e. all the 2nd partial derivatives of $u, v$ are exist and continuous) + $f(z) = u + iv$ is hol $\Rightarrow u, v$ are harmonic
- $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + \frac{i}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = 0$ iff C-R
- $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$

# 2 Some Complex functions

**Exponential**

- $\exp(z) = \exp(x + iy) = e^x(\cos y + i\sin y)$
- $\exp(z + 2\pi i) = \exp(z)$
  $|\exp(a + ib)| = e^a$
  $\arg(\exp(a + ib)) = \{b + 2k\pi, k \in \mathbb{Z}\}$
  $\exp(nz) = (\exp(z))^n$
- $z = x + iy, w = a + ib$, $\exp(z) = \exp(w) \Rightarrow x = a, y - b = 2k\pi$
- Given $z \in \mathbb{C}\backslash\{0\}$, $\exp(a + ib) = z \Leftrightarrow e^a = |z| \Leftrightarrow a = \ln|z|, b \in \arg(z)$

**Cosine & sine & hyperbolic**

- $\cos(z) := \dfrac{e^{iz} + e^{-iz}}{2}, \sin(z) := \dfrac{e^{iz} - e^{-iz}}{2i}$
- $\cos(z)$ and $\sin(z)$ are hol on $\mathbb{C}$
- $\tan(z) = \frac{\sin(z)}{\cos(z)}, \cot(z) = \frac{1}{\tan(z)}, \sec(z) = \frac{1}{\cos(z)}, \csc(z) = \frac{1}{\sin(z)}$
- $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
  $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
- $\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$
  $\cos(x + iy) = \cos(x)\cosh(y) + i\sin(x)\sinh(y)$
- $\cosh(z) := \dfrac{e^z + e^{-z}}{2}, \sinh(z) = \dfrac{e^z - e^{-z}}{2}$

- $\sinh(iz) = i\sin z, \cosh(iz) = \cos z$

**Logarithm**

- $\log(z) := \{w \in \mathbb{C} : \exp(w) = z\}$
- $w = \log(z) = \log(re^{i\theta}) = \ln|z| + i\arg(z)$
  $= \{\ln|z| + i\theta : \theta \in \arg z\}$
  $= \{\ln|z| + i\text{Arg}(z) + 2\pi ik : k \in \mathbb{Z}\ \}$
  $= \{\ln r + i\theta + i2\pi k : k \in \mathbb{Z}\}$
- $\log(zw) = \log(z) + \log(w)$ i.e. $u \in \log(zw)$ iff $\exists v_i \in \log(z), v_2 \in \log(w)$ s.t. $u = v_1 + v_2$
- $\log(1/z) = -\log(z)$ i.e. $u \in \log(1/z)$ iff $-u \in \log(z)$
- Principal branch: $\text{Log}(z) := \ln|z| + i\text{Arg}(z), z$ is non-zero

  the non-positive real axis is a branch cut of this function, the origin is a branch point
- Branch cut: $L_{z_0,\phi} = \{z \in \mathbb{C} : z = z_0 + re^{i\theta}, r \geq 0\}$

  Cut plane: $D_{z_0,\phi} = \mathbb{C}\backslash L_{z_0,\phi}$

  ( $L_{0,\phi} = L_\phi, D_{0,\phi} = D_\phi$, the cut plane associated with the principal branch of the arg and log functions will be denoted by $D$, i.e. $D = D_{-\pi}$
- $\phi < \text{Arg}_\phi(z) \leq \phi + 2\pi$, so $\text{Arg} = \text{Arg}_{-\pi}$
- $\text{Log}_\phi(z) = \ln|z| + i\text{Arg}_\phi(z) : \mathbb{C}\backslash\{0\} \to \{a + bi : a \in \mathbb{R}, \phi < b \leq \phi + 2\pi\}$
- $z = \exp(\text{Log}_\phi(z))$, but $\text{Log}_\phi(\exp(z)) = z$ is not true
- $\text{Log}_\phi$ is hol on $D_{0,\phi} = \mathbb{C}\backslash L_{0,\phi}$, $\text{Log}'_\phi(z) = 1/z$
- $g : U \to \mathbb{C}$ hol on $U \Rightarrow \text{Log}_\phi(g(z))$ is hol on $U \cap g^{-1}(D_\phi)$

  in particular, if $g$ is hol on $\mathbb{C} \Rightarrow \text{Log}_\phi(g(z))$ is hol on $g^{-1}(D_\phi)$ i.e. at points $z$ s.t. $g(z) \in D_\phi)$

**Complex powers**

- $z^\alpha = \{\exp(\alpha w) : w \in \log(z)\}$
  $= \{\exp(\alpha \ln|z| + i\alpha\text{Arg}(z) + i\alpha 2\pi k) : k \in \mathbb{Z}\}$
  $= \{\exp(\alpha\text{Log}(z))\exp(i\alpha 2\pi k) : k \in \mathbb{Z}\}$

  $\exp(i\alpha 2\pi k) = 1, \alpha = n \in \mathbb{Z}$
- if $\alpha \in \mathbb{Z} \Rightarrow$ exactly one value of $z^\alpha$

  if $\alpha = p/q$, where $p, q$ coprime, $q \neq 0 \Rightarrow$ exactly $q$ values of $z^\alpha$

  if $\alpha$ irrational or non-real $\Rightarrow$ infinitely many values of $z^\alpha$
- $1^{1/q} = \{\exp(i2\pi k/q) : k \in \mathbb{Z}\}$
  $= \{1, w, w^2, ..., w^{q-1}\}$

  where $w := \exp(i2\pi/q)$ are the $q$ roots of unity

  $z^{1/q} = \{\exp(\text{Log(z/}q)\exp(i2\pi k/q) : k \in \mathbb{Z}\}$
  $= \{|z|^{1/q}\exp(i\text{Arg}(z)/q)w^k : k = 0, ..., q-1\}$

  $z^{p/q} = \{|z|^{p/q}\exp(ip\text{Arg}(z)/q)w^{pk} : k = 0, ..., q-1\}$
- for $\alpha = a + ib$ where $b \neq 0, \exp(i\alpha 2\pi k) = \exp(i(a + ib)2\pi k) = \exp(i2\pi ka)\exp(-2\pi kb)$
- Principal branch: $z^\alpha = \exp(\alpha\text{Log}(z))$
- a branch of $z^\alpha$ is hol on $D_\phi$ on which the associated branch $\text{Log}_\phi$ is hol, and for all $z \in D_\phi$, $\frac{d}{dz}z^\alpha = \alpha z^{\alpha-1}$

  e.g. $z^z = \exp(z\text{Log}z)$ is hol on set $D$ on which $\text{Log}(z)$ is hol, $\frac{d}{dz}z^z = \frac{d}{dz}\exp(z\text{Log}(z)) = \exp(z\text{Log}(z))(\text{Log}(z) + 1)$
- $z^\alpha z^\beta = z^{\alpha+\beta}$ is true if principal branch is chosen for each power function

  $(zw)^\alpha = z^\alpha w^\alpha$ is not true in general for the principal branch in each case

# 3 Conformal maps & MT

- MT: $f(z) = \dfrac{az + b}{cz + d}, ad \neq bc$
- Extended complex plane: $\tilde{\mathbb{C}} \cup \{\infty\}$

  $a + \infty = \infty, b \cdot \infty = \infty, b/0 = \infty, b/\infty = 0$
- $f$ fixes the point at infinity: $f(\infty) = \infty$
- MT $f$ = composition of finite translations + rotations + dilations + (one inversion, if do not fix the point at infinity)
- MT maps circlines to circlines
- two triplets of distinct points $(z_1, z_2, z_3), (w_1, w_2, w_3)$, there exists a unique MT $f$ s.t. $f(z_i) = w_i$
- Cross ratio $[z_1, z_2, z_3, z_4] = \dfrac{z_1 - z_3}{z_1 - z_4}\dfrac{z_2 - z_4}{z_2 - z_3}$, sends $(z_2, z_3, z_4)$ to $(1, 0, \infty)$
- MT $f \Rightarrow [f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]$
- if $[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4] \Rightarrow \exists$ MT s.t. $h(z_i) = w_i$

# 4 Integral

- $f$ is integrable $\Leftrightarrow u, v$ are integrable in the real sense
- continuous function are integrable
- $\int_a^b f(t)dt = F(b) - F(a)$, $F$ is antiderivative of $f$
- $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$
- Parametrized curve $\Gamma$ connecting $z_0$ and $z_1$: a continuous function $\gamma : [t_0, t_1] \to \mathbb{C}$ s.t. $\gamma(t_0) = z_0, \gamma(t_1) = z_1$

  writing $z_0 = z_0 + iy_0 \to \gamma(t) = x(t) + iy(t) \to x(t_0) = x_0, y(t_0) = y_0$

  $\Gamma$ is regular: if $\gamma$ is continuously differentiable and $\gamma'(t) \neq 0$ for all $t \in (t_0, t_1)$
- $\int_\Gamma f(z)dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)ft$
- arclength $\ell(\Gamma) := \int_{t_0}^{t_1} |\gamma'(t)|dt$
  $= \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}dt$

- M-L lemma: $\Gamma$ is a regular curve, $f : \Gamma \to \mathbb{C}$ is continuous, then
$$\left|\int_\Gamma f(z)dz\right| \le \int_\Gamma |f(z)||dz| \le \max_{z\in\Gamma}|f(z)|\ell(\Gamma)$$
since $f$ continuous on $\Gamma, \Gamma$ is a closed and bounded subset of $\mathbb{C}$, $f$ is indeed bounded on $\Gamma$, and the values $\max|f|, min|f|$ are attained on $\Gamma$
- $\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz$
- $\int_\Gamma f(z)dz = \sum_{i=1}^n \int_{\Gamma_i} f(z)dz$
- Path-independence lemma: $D$ domain, $f$ continuous, then
  $f$ has antiderivative $F$ in $D$
  $\Leftrightarrow \int_\Gamma f(z)dz = 0$ for all closed contous $\Gamma$ in $D$
  $\Leftrightarrow \int_\Gamma f(z)dz$ are independent of the path $\Gamma$, and depend only on the endpoints
- Jordan curve theorem: a loop $\Gamma, \mathbb{C} = \mathrm{Int}(\Gamma) \cup \Gamma \cup \mathrm{Ext}(\Gamma)$
- CIT: $\Gamma$ loop, $f$ hol inside & on $\Gamma \Rightarrow \int_\Gamma f(z)dz = 0$
- $D$ simply-connected domain, $f$ hol on $D \Rightarrow f$ has antiderivative on $D$
- $\displaystyle\int_\Gamma \frac{1}{z-z_0} = \begin{cases} 2\pi i & \text{if } z_0 \in \mathrm{Int}(\Gamma) \\ 0 & \text{otherwise} \end{cases}$
- CIF: $\Gamma$ loop, $f$ hol inside & on $\Gamma, z_0 \in \mathrm{Int}(\Gamma) \Rightarrow$ $f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f(z)}{z-z_0}dz$
- GCIF: $\Gamma$ loop, $f$ hol inside & on $\Gamma, z_0 \in \mathrm{Int}(\Gamma) \Rightarrow$ $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\displaystyle\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dw$
- Morera's theorem: $D$ domain, $f$ continuous, $\int_\Gamma f(z)dz = 0$ for all loops $\Gamma \subseteq D \Rightarrow f$ hol
- Cauchy estimate: $f$ hol on $D, \bar{D}_R(z_0) \subseteq D, |f(z)| \le M \Rightarrow |f^{(n)}(z_0)| \le \dfrac{n!M}{R^n}$,
  where $R = |z - z_0|$
- Liouville's theorem: $f$ hol & bounded on $\mathbb{C} \Rightarrow f$ is constant
- Maximum modulus principle: $f$ hol & bounded on $D$, if $|f(z)|$ achieves its max at $z_0 \in D \Rightarrow f$ is constant on $D$
- Max/Min principle: $\phi$ harmonic & bounded above or below $M$, $\phi(z_0) = M$ for some $z_0 \in D \Rightarrow \phi$ is constant on $D$

# 5 Series

- $\sum_{j=0}^\infty z_j$ is convergent if partial sums $S_n = \sum_{j=0}^n z_j$ is a convergent sequence, with limit $S$, we say $\sum_{j=0}^\infty z_j = S$
- $\sum_{j=0}^\infty z_j$ convergent $\Rightarrow z_n \to 0$ as $n \to 0$
  $z_n \not\to 0$ as $n \to 0 \Rightarrow$ divergent
  $z_n \to 0$ as $n \to 0 \not\Rightarrow$ convergent
  eg. $\sum_{j=1}^\infty \frac{1}{j}(\sum_{j=1}^n \frac{1}{j} = \ln(n+1) = \infty$ as $n \to \infty)$
- $\sum_{n=1}^\infty \frac{1}{n^p}$ convergent if $p > 1$
- Comparison test: $|z_n| \le M_n$ for all sufficiently large $n$, where $\sum_{j=0}^\infty M_j$ is convergent, $M_n \ge 0 \Rightarrow \sum_{j=0}^\infty z_j$ is convergent
- $\sum_{j=0}^\infty c^j$ is convergent iff $|c| < 1, \sum_{j=0}^\infty c^j = \frac{1}{1-c}$
- Ratio test: $L = \lim\limits_{n\to\infty}\left|\dfrac{z_{n+1}}{z_n}\right|$
  $L < 1$ convergent, $L > 1$ divergent, $L = 1$ you know nothing

**When the terms become functions**

- Converge pointwise: for each $z \in S, \forall\epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|f_n(z) - f(z)| < \epsilon$ whenever $n \ge N$
  Converge uniformly: $\forall\epsilon > 0, \exists N \in \mathbb{N}$ s.t. for all $z \in S, |f_n(z) - f(z)| < \epsilon$ whenever $n \ge N$
  uniform $\Rightarrow$ pointwise
- converge pointwise / uniformly $\Leftrightarrow$ partial sum converge pointwise / uniformly
- $f_n : S \to \mathbb{C}$ a sequence of continuous functions, $f_n$ converges uniformly to $f : S \to \mathbb{C} \Rightarrow f$ is continuous
- Weierstrass M-test: $|f_n(z)| \le M_n$ for all sufficiently large $n$, where $\sum_{j=0}^\infty M_j$ is convergent, $M_n \ge 0 \Rightarrow \sum_{j=0}^\infty f_j(z)$ converges uniformly
- $\sum_{j=0}^\infty M_j$ converges $\Leftrightarrow$ $\exists n_1 \in \mathbb{N}$ s.t. $\left|\sum_{j=0}^\infty M_j - \sum_{j=0}^n M_j\right| < \epsilon$ whenever $n \ge n_1$
- $f_n : S \to \mathbb{C}$ a sequence of continuous functions, $f_n$ converges uniformly to $f, \Gamma$ a contour inside $S \Rightarrow \int_\Gamma f_n(z)dz$ converges to $\int_\Gamma f(z)dz$
- Integral and Sum: $f_n : S \to \mathbb{C}$ a sequence of continuous functions, $\sum_{j=0}^\infty f_j(z)$ converges uniformly on $S, \Gamma$ a contour inside $S \Rightarrow$
$$\int_\Gamma \sum_{j=0}^\infty f_j(z)dz = \sum_{j=0}^\infty \int_\Gamma f_j(z)dz$$
- $D$ simply-connected domain, $f_n$ hol on $D$, $f_n$ converges uniformly to $f : D \to \mathbb{C} \Rightarrow f$ hol on $D$

**Power series**

- $\sum_{j=0}^\infty a_j(z-z_0)^j$ (Given $f$ hol at $z_0$)
- For power series, $\exists R \in [0,\infty] \cup \infty$ s.t. the series
  converges on $D_R(z_0)$
  converges uniformly on $\bar{D}_r(z_0)$ for any $r \in [0, R)$
  diverges on $\mathbb{C}\backslash\bar{D}_R(z_0)$
  $R$ is the radius of convergence

- $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

  (NOTE: doesn's assert that $R$ can always be evaluated by taking the limit, since this limit does not in general exist)

  e.g. $a_j = \begin{cases} 1 & j \text{ is oven} \\ 2 & j \text{ is odd} \end{cases}$

- $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$ is hol on $D_R(z_0)$

**Taylor series**

- $\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j$

  Maclaurin series: $z_0 = 0$

- Taylor series theorem: $f$ hol on $D_R(z_0) \Rightarrow$ the Taylor for $f$ centred at $z_0$ converges to $f(z)$ for all $z \in D_R(z_0)$ & converges uniformly on $\bar{D}_r(z_0)$ for all $0 \le r < R$

  i.e. Taylor of $f$ centred at $z_0$ will converge to $f(z)$ everywhere inside the largest open disc centred at $z_0$, on which $f$ is hol

- $U$ is open + $f : U \to \mathbb{C}$ is analytic: if at every point $z \in U, f$ can be expressed as a convergent power series

- $U$ is open, $f : U \to \mathbb{C}$ hol, $\Rightarrow f$ is analytic

- $\exp(z) = \displaystyle\sum_{j=0}^{\infty} \frac{z^j}{j!} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

  $\cos(z) = \displaystyle\sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!}$

  $\sin(z) = \displaystyle\sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$

- $f'(z) = \displaystyle\sum_{j=0}^{\infty} \frac{f^{(j+1)}(z_0)}{j!}(z-z_0)^j$ for $z \in D_R(z_0)$, where $f$ hol on $D_R(z_0)$

  i.e. Taylor for $f'$ is found by differentiating Taylor for $f$ term-by-term

**Laurent series**

- $\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$

- $A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z-z_0| < R\}$

  $\bar{A}_{r,R}(z_0) = \{z \in \mathbb{C} : r \le |z-z_0| \le R\}$

- Laurent series theorem: $f$ hol on $A_{r,R}(z_0) \Rightarrow f$ can be expressed as a Laurent series centred at $z_0$ which converges on $A_{r,R}(z_0)$ & converges uniformly on $\bar{A}_{r_1,R_1}(z_0)$ where $r < r_1 \le R_1 < R \le \infty$, the coefficients are given by

  $$a_j = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{(z-z_0)^{j+1}} dz$$

  for any loop $\Gamma$ lying inside $A_{r,R}(z_0)$ and containing $z_0$ in its interior

- trick: $\dfrac{1}{1-z} = \dfrac{1}{-z(1-\frac{1}{z})}$

**Singularities**

- **singularity**: if $f$ is not hol at $z_0$

  **isolated sigularity**: if $\exists R > 0$ s.t. $f$ is hol on $D'_R(z_0)$

- **zero**: $f$ is hol on the neighbourhood of $z_0$, if $f(z_0) = 0$

  **zero of order** $m$: if $f(z_0) = f'(z_0) = ... = f^{(m-1)}(z_0) = 0, f^{(m)}(z_0) \ne 0$

  **isolated zero**: if $\exists R > 0$ s.t. $f(z) \ne 0$ for $z \in D'_R(z_0)$

- $f$ is hol on the neighbourhood of $z_0$, with a zero of finite order at $z_0 \Rightarrow z_0$ is isolated

- $f$ is hol on the neighbourhood of $z_0$, $f(z_n) = 0$ for a sequence of distinct points $z_n \in U$ which converge to $z_0 \Rightarrow f$ is identically zero on some disc centred at $z_0$

- $z_0 \in \mathbb{C}$ is s singularity of a rational function $f = P/Q \Rightarrow z_0$ is isolated

- **isolated singularity**:
  - **removable singularity**: if $a_j = 0$ for all $j < 0 \Leftrightarrow$ no negative powers, $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$
  - **pole of order** $m$: if $a_j = 0$ for $j < -m$ and $a_{-m} \ne 0 \Leftrightarrow f(z) = \sum_{j=-m}^{\infty} a_j(z-z_0)^j$
  - **essential singularity**: infinite numbers of non-zero terms with nagative powers

- for removable singularity $z_0$ of $f$ which is hol on $D'_R(z_0)$, $f(z_0)$ can be re-defined so taht $f$ is hol at $z_0$

  $$f(z) = \begin{cases} f(z) & z \ne z_0 \\ \lim_{\zeta\to z_0} f(\zeta) & z = z_0 \end{cases}$$

- $f, g$ hol at $z_0$, where $z_0$ is a zero of $g$ of order $m$,

  if $z_0$ not a zero of $f \Rightarrow$

  $f/g$ has **pole of order** $m$ at $z_0$

  if $z_0$ zero of order $k$ of $f \Rightarrow$

  $f/g$ has **pole of order** $m-k$ at $z_0$ if $m > k$,

  has **removable singularity** at $z_0$ if $m \le k$

# 6 Residue calculus

- $f$ hol on $D'_R(z_0)$, isolated singularity at $z_0$, $\mathrm{Res}(f, z_0) = a_{-1}$ the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion of $f$ centred at $z_0$ valid on $D'_R(z_0)$

- $f$ hol on $D'_R(z_0)$, **isolated singularity** at $z_0$, $\Gamma$ a loop inside $D'_R(z_0), z_0 \in \mathrm{Int}(\Gamma) \Rightarrow$

  $\int_\Gamma f(z)dz = 2\pi i a_{-1} = 2\pi i \mathrm{Res}(f, z_0)$

- $f$ hol on $D'_R(z_0)$, **removable singularity** at $z_0 \Rightarrow \mathrm{Res}(f, z_0) = 0$

- $f$ hol on $D'_R(z_0)$, with **a pole of order** $m$ at $z_0 \Rightarrow$

  $\mathrm{Res}(f, z_0) = \displaystyle\lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z))$

  1st order: $\mathrm{Res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z)$

  2nd order: $\mathrm{Res}(f, z_0) = \lim_{z\to z_0}\left[(z-z_0)^2 f(z)\right]'$

- $g, h$ hol on $D'_R(z_0)$, $h$ has **a simple zero** at $z_0$, $g(z_0) \neq 0$, define $f = g/h \Rightarrow \text{Res}(f, z_0) = \dfrac{g(z_0)}{h'(z_0)}$
- Cauchy residue theorem: $\Gamma$ loop, $f$ hol inside & on $\Gamma$ except for finitely many isolated singularities $z_1, ..., z_k \in \text{Int}(\Gamma) \Rightarrow$

$$\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{k} \text{Res}(f, z_j)$$

- $f$ is meromorphic on $D$: if for all $z \in D$
  either $f$ has a pole of some finite order at $z$
  or $f$ is hol at $z$
- The argument principle: $\Gamma$ loop, $f$ meromorphic on $\text{Int}(\Gamma)$, $f$ hol & non-zero on $\Gamma$ $\Rightarrow$ $\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = N_0(f) - N_\infty(f)$
  \# **zeros** of $f$ inside $\Gamma$, counted with multiplicity: $N_0(f) = \sum_{j=1}^{l}$ order of $w_j$
  \# **poles** of $f$ inside $\Gamma$, counted with multiplicity: $N_\infty(f) = \sum_{j=1}^{k}$ order of $z_j$
- Rouche's theorem: $\Gamma$ loop, $f, g$ hol inside & on $\Gamma$ s.t. for all $z \in \Gamma, |f(z) - g(z)| < |f(z)| \Rightarrow N_0(f) = N_0(g)$
- Open mapping theorem: $f$ non-constant and hol on $D \Rightarrow$ the image of $D$ under $f, f(D) = \{f(z) : z \in D\}$, is an open subset of $\mathbb{C}$
- Maximum modulus theorem: $f$ non-constant and hol on $D \Rightarrow |f(z)|$ does not attain a maximum on $D$

**Trigonometric integrals**

- $\cos\theta = \frac{1}{2}(z + \frac{1}{z}), \sin\theta = \frac{1}{2i}(z - \frac{1}{z}), d\theta = \frac{dz}{iz}$
  $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta = \int_{C_1(0)} \frac{1}{iz} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) dz$

**Improper integrals**

- Cauchy principal value:
  p.v. $\int_{-\infty}^{\infty} f(x)dx := \lim_{\rho\to\infty} \int_{-\rho}^{\rho} f(x)dx$
- Jordan lemma: $R = P/Q$ rational function, $Q \neq 0$, $\deg(Q) \geq \deg(P) + 1$, $a \in \mathbb{R}$ non-zero
  $\lim_{\rho\to\infty} \int_{C_\rho^+} \exp(iaz)\frac{P(z)}{Q(z)}dz = 0, \text{ if } a > 0$
  $\lim_{\rho\to\infty} \int_{C_\rho^-} \exp(iaz)\frac{P(z)}{Q(z)}dz = 0, \text{ if } a < 0$
- Convert $R(x)\cos(ax), R(x)\sin(ax)$ into real or imaginary part of $R(x)\exp(iax)$
  e.g. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}dx$ is the imaginary part of $\displaystyle\int_{-\infty}^{\infty} \frac{x\exp(ix)}{1+x^2}dx$, consider contour $C_\rho^+$
- e.g. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx$, consider contour rectangular loop $\Gamma_\rho$ (contour $C_\rho^+$ contains infinity poles in $\rho \to \infty$, hard to calculate)

**Improper integrals with pole $z = c$**

- $\int_a^c f(x)dx = \lim_{r\downarrow 0} \int_a^{c-r} f(x)dx$
  $\int_c^b f(x)dx = \lim_{s\downarrow 0} \int_{c+s}^{b} f(x)dx$
  $\int_a^b f(x)dx = \lim_{r\downarrow 0} \int_a^{c-r} + \lim_{s\downarrow 0} \int_{c+s}^{b}$
  $r \downarrow 0$: $r \to 0$ through positive values only
- p.v. $\int_{-\infty}^{\infty} f(x)dx = \lim_{\rho\to\infty, r\downarrow 0} \left(\int_{-\rho}^{c-r} + \int_{c+r}^{\rho}\right)$
- Consider 2 types of contour:
  - the contour around a singularity, from $c - r$ to $c + r$, and $-C_r^+(c)$
  - a small circular arc $S_r$, parametrized by $\gamma(\theta) = c + r\exp(i\theta)$ for $\theta \in [\theta_0, \theta_1]$ for some $0 \leq \theta_0 < \theta_1 \leq 2\pi$
- $\lim_{r\downarrow 0} \int_{S_r} f(z)dz = i(\theta_1 - \theta_0)\text{Res}(f, c)$

**Evaluate infinite series, calc $\int_{\Gamma_N} f(z)dz$**

- $R = P/Q, Q \neq 0, \deg Q - \deg P \geq 2$
  $\displaystyle\int_0^\infty R(x)dx = -\sum_{\text{poles } z_k} \text{Res}(f, z_k)$
  $f(z) = \log(z)R(z)$
- For $\int_0^\infty R(x)dx, f(z) = \log(z-a)R(z)$
- $R = P/Q, \deg Q - \deg P \geq 2$
  $\displaystyle\sum_{n=-\infty, n\neq z_k}^{\infty} R(n) = -\sum_{\text{poles } z_k \text{ of } R} \text{Res}(f, z_k)$
  $f(z) = \pi\cot(\pi z)R(z)$
- $R = P/Q, \deg Q - \deg P \geq 2$
  $\displaystyle\sum_{n=-\infty, n\neq z_k}^{\infty} (-1)^n R(n) = -\sum_{\text{poles } z_k \text{ of } R} \text{Res}(f, z_k)$
  $f(z) = \pi\csc(\pi z)R(z)$
- eg. $\sum_{n=-\infty}^{\infty} \frac{1}{n^2} \to f(z) = \frac{\cot(\pi z)}{z^2}$
  eg. $\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{n^2} \to f(z) = \frac{\csc(\pi z)}{z^2}$
  eg. $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} \to f(z) = \frac{\pi\cot(\pi z)}{z^2+1}$
  eg $\sum_{n=-\infty}^{\infty} \frac{1}{(n-1/2)^2} \to f(z) = \frac{\pi\cot(\pi z)}{(z-1/2)^2}$
- $\Gamma$ loop with 0 in its interior
  $\displaystyle\binom{n}{k} = \frac{1}{2\pi i}\int_\Gamma \frac{(1+z)^n}{z^{k+1}}dz$