

A APPENDIX

For the convenience of the readers, we list definitions, a lemma, some theorems and a corollary that we use from other books and papers but did not explicitly state in the paper.

THEOREM A.1. (Theorem 3.2.2 in [1]) Let $(K, ')$ be a differential field, and t be transcendental over K . Then, for any $w \in K(t)$, there exists a unique derivation δ on $K(t)$ such that $t^\delta = w$ and $(K(t), \delta)$ is a differential extension of $(K, ')$.

DEFINITION A.2. (Definition 5.1.1 in [1]) Suppose k is a differential field and K is a differential extension of k . We say that

- (i) $t \in K$ is a primitive over k if $Dt \in k$,
- (ii) $t \in K^*$ is a hyperexponential over k if $Dt/t \in k$, and
- (iii) $t \in K$ is Liouvillian over k if t is either algebraic, a primitive, or a hyperexponential over k .

K is a Liouvillian extension of k if there are t_1, \dots, t_n in K such that $K = k(t_1, \dots, t_n)$ and t_i is Liouvillian over $k(t_1, \dots, t_{i-1})$ for $i \in \{1, \dots, n\}$.

DEFINITION A.3. (Definition 5.1.2 in [1]) Suppose k is a differential field and K is a differential extension of k . We say that $t \in K$ is a Liouvillian monomial over k if t is transcendental and Liouvillian over k and $C_{k(t)} = C_k$.

DEFINITION A.4. (Definition 5.1.3 in [1]) $t \in K$ is a logarithm over k if $Dt = Db/b$ for some $b \in k^*$. $t \in K^*$ is an exponential over k if $Dt/t = Db$ for some $b \in k$. $t \in K$ is elementary over k if t is either algebraic, or a logarithm or an exponential over k . $t \in K$ is an elementary monomial over k if t is transcendental and elementary over k , and $\text{Const}(k(t)) = \text{Const}(k)$.

DEFINITION A.5. (Definition 5.1.4 in [1]) K is an elementary extension of k if there are t_1, \dots, t_n in K such that $K = k(t_1, \dots, t_n)$ and t_i is elementary over $k(t_1, \dots, t_{i-1})$ for i in $\{1, \dots, n\}$. We say that $f \in k$ has an elementary integral over k if there exists an elementary extension E of k and $g \in E$ such that $Dg = f$. An elementary function is any element of any elementary extension of $(\mathbb{C}(x), d/dx)$.

THEOREM A.6. (Theorem 5.1.1 in [1]) If t is a primitive over a differential field k and Dt is not the derivative of an element of k , then t is a monomial over k , $C_{k(t)} = C_k$, and $S = k$. Conversely, if t is transcendental and primitive over k and $C_{k(t)} = C_k$, then Dt is not the derivative of an element of k .

THEOREM A.7. (Theorem 5.3.1 in [1]) Let $f \in k(t)$. Using only the extended Euclidean algorithm in $k[t]$, one can find $g, h, r \in k(t)$ such that h is simple, r is reduced, and $f = Dg + h + r$. Furthermore, the denominators of g, h and r divide the denominator of f , and either $g = 0$ or $\mu(g) < \mu(f)$.

LEMMA A.8. (Lemma 2.1 in [2]) Let $g \in K[t] + K(t)'$. Then $g = 0$ if it is t -simple.

THEOREM A.9. (Theorem 3.1.1 (v) in [1], Logarithmic Derivative Identity) Let (R, D) be a differential ring. If R is an integral domain, then

$$\frac{D(u_1^{e_1} \cdots u_n^{e_n})}{u_1^{e_1} \cdots u_n^{e_n}} = e_1 \frac{Du_1}{u_1} + \cdots + e_n \frac{Du_n}{u_n}$$

for any $u_1, \dots, u_n \in R^*$ and any integers e_1, \dots, e_n .

DEFINITION A.10. (Definition 5.1.4 in [1]) K is an elementary extension of k if there are t_1, \dots, t_n in K such that $K = k(t_1, \dots, t_n)$ and t_i is elementary over $k(t_1, \dots, t_{i-1})$ for $i \in [n]$. We say that $f \in k$ has an elementary integral over k if there exists an elementary extension E of k and $g \in E$ such that $Dg = f$. An elementary function is any elementary extension of $(\mathbb{C}(x), d/dx)$.

THEOREM A.11. (Theorem 5.5.2 in [1], Liouville's Theorem) Let K be a differential field with an algebraically closed constant field and $f \in K$. If there exists an elementary extension E of K and $g \in E$ such that $Dg = f$, then there are $v \in K, u_1, \dots, u_n \in K^*$ and $c_1, \dots, c_n \in \text{Const}(K)$, such that

$$f = Dv + \sum_{i=1}^n c_i \frac{Du_i}{u_i}.$$

COROLLARY A.12. (Corollary 1' in [?, Page 124]) Let K be a field and let $F = K(S)$ be a purely transcendental extension of K ; here S denotes a set of generators of F/K which are algebraically independent over K . Let $x \rightarrow u_x$ be a mapping of S into a field L containing F . If D is any derivation of K with values in L , then there exists one and only one derivation D' of F extending D , such that $D'(x) = u_x$ for all x in S .

REFERENCES

- [1] M. Bronstein. *Symbolic Integration I: transcendental functions*. Berlin: Springer-Verlag, 2005.
- [2] S. Chen, H. Du and Z. Li. Additive decompositions in primitive extensions. *Proceedings of the 2018 International Symposium on Symbolic and Algebraic Computation*. New York, USA: ACM, 135-142.