A APPENDIX

For the convenience of the readers, we list definitions, a lemma, some theorems and a corollary that we use from other books and papers but did not explicitly state in the paper.

Theorem A.1. *(Theorem 3.2.2 in [1]) Let $(K, ')$ be a differential field, and t be transcendental over K. Then, for any $w \in K(t)$, there exists a unique derivation δ on $K(t)$ such that $t^\delta = w$ and $(K(t), {}^\delta)$ is a differential extension of $(K, ')$.*

Definition A.2. *(Definition 5.1.1 in [1]) Suppose k is a differential field and K is a differential extension of k. We say that*

(i) $t \in K$ *is a* primitive over k *if* $Dt \in k$,
(ii) $t \in K^*$ *is a* hyperexponential over k *if* $Dt/t \in k$, *and*
(iii) $t \in K$ *is* Liouvillian over k *if t is either algebraic, a primitive, or a hyperexponential over k.*

K *is a* Liouvillian extension *of k if there are $t_1, \ldots, t_n$ in K such that $K = k(t_1, \ldots, t_n)$ and t_i is Liouvillian over $k(t_1, \ldots, t_{i-1})$ for $i \in \{1, \ldots, n\}$.*

Definition A.3. *(Definition 5.1.2 in [1]) Suppose k is a differential field and K is a differential extension of k. We say that $t \in K$ is a* Liouvillian monomial over k *if t is transcendental and Liouvillian over k and $C_{k(t)} = C_k$.*

Definition A.4. *(Definition 5.1.3 in [1]) $t \in K$ is a* logarithm over k *if $Dt = Db/b$ for some $b \in k^*$. $t \in K^*$ is an* exponential over k *if $Dt/t = Db$ for some $b \in k$. $t \in K$ is* elementary over k *if t is either algebraic, or a logarithm or an exponential over k. $t \in K$ is an* elementary monomial over k *if t is transcendental and elementary over k, and $Const(k(t)) = Const(k)$.*

Definition A.5. *(Definition 5.1.4 in [1]) K is an* elementary extension *of k if there are $t_1, \ldots, t_n$ in K such that $K = k(t_1, \ldots, t_n)$ and t_i is elementary over $k(t_1, \ldots, t_{i-1})$ for i in $\{1, \ldots, n\}$. We say that $f \in k$ has an* elementary integral over k *if there exists an elementary extension E of k and $g \in E$ such that $Dg = f$. An* elementary function *is any element of any elementary extension of $(\mathbb{C}(x), d/dx)$.*

Theorem A.6. *(Theorem 5.1.1 in [1]) If t is a primitive over a differential field k and Dt is not the derivative of an element of k, then t is a monomial over k, $C_{k(t)} = C_k$, and $S = k$. Conversely, if t is transcendental and primitive over k and $C_{k(t))} = C_k$, then Dt is not the derivative of an element of k.*

Theorem A.7. *(Theorem 5.3.1 in [1]) Let $f \in k(t)$. Using only the extended Euclidean algorithm in $k[t]$, one can find $g, h, r \in k(t)$ such that h is simple, r is reduced, and $f = Dg + h + r$. Furthermore, the denominators of g, h and r divide the denominator of f, and either $g = 0$ or $\mu(g) < \mu(f)$.*

Lemma A.8. *(Lemma 2.1 in [2]) Let $g \in K[t] + K(t)'$. Then $g = 0$ if it is t-simple.*

Theorem A.9. *(Theorem 3.1.1 (v) in [1], Logarithmic Derivative Identity) Let (R, D) be a differential ring. If R is an integral domain, then*

$$\frac{D(u_1^{e_1} \cdots u_n^{e_n})}{u_1^{e_1} \cdots u_n^{e_n}} = e_1 \frac{Du_1}{u_1} + \cdots + e_n \frac{Du_n}{u_n}$$

for any $u_1, \ldots, u_n \in R^*$ and any integers $e_1, \ldots, e_n$.

Definition A.10. *(Definition 5.1.4 in [1]) K is an elementary extension of k if there are $t_1, \ldots, t_n$ in K such that $K = k(t_1, \ldots, t_n)$ and t_i is elementary over $k(t_1, \ldots, t_{i-1})$ for $i \in [n]$. We say that $f \in k$* has an elementary integral over *k if there exists an elementary extension E of k and $g \in E$ such that $Dg = f$. An* elementary function *is any elementary extension of $(\mathbb{C}(x), d/dx)$.*

Theorem A.11. *(Theorem 5.5.2 in [1], Liouville's Theorem) Let K be a differential field with an algebraically closed constant field and $f \in K$. If there exists an elementary extension E of K and $g \in E$ such that $Dg = f$, then there are $v \in K, u_1, \ldots, u_n \in K^*$ and $c_1, \ldots, c_n \in Const(K)$, such that*

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}.$$

Definition A.12. *(Chapter 2, Definition 3 in [3], Lexicographic Order) Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be in $\mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{lex} \beta$ if the leftmost nonzero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive. We will write $x^\alpha >_{lex} x^\beta$ if $\alpha >_{lex} \beta$.*

Corollary A.13. *(Corollary 1' in [4, Page 124]) Let K be a field and let $F = K(S)$ be a purely transcendental extension of K; here S denotes a set of generators of F/K which are algebraically independent over K. Let $x \to u_x$ be a mapping of S into a field L containing F. If D is any derivation of K with values in L, then there exists one and only one derviation D' of F extending D, such that $D(x) = u_x$ for all x in S.*

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