Binomial Determinants for Tiling Problems Yield to the Holonomic Ansatz

Elaine Wong

This is joint work with Hao Du, Christoph Koutschan and Thotsaporn Thanatipanonda, and was mostly supported by:

ÖAW RICAM

and

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Our Problem

Consider the following matrices:

$$\mathcal{D}^{\mu}_{s,t}(n) := \left(\binom{\mu+i+j+s+t-4}{j+t-1} + \delta_{i+s,j+t} \right)_{1\leqslant i,j\leqslant n},$$
$$\mathcal{E}^{\mu}_{s,t}(n) := \left(\binom{\mu+i+j+s+t-4}{j+t-1} - \delta_{i+s,j+t} \right)_{1\leqslant i,j\leqslant n},$$

where μ is an indeterminate, $n \in \mathbb{N}$ and $s, t \in \mathbb{Z}$.

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where μ is an indeterminate, $n \in \mathbb{N}$ and $s, t \in \mathbb{Z}$.

We denote their determinants to be $D^{\mu}_{s,t}(n)$ and $E^{\mu}_{s,t}(n)$, respectively, and we want to find nice formulas for them.

Our Problem for $n = 5$

$$\left(\binom{\mu+i+j+s+t-4}{j+t-1}\right)_{1\leqslant i,j\leqslant n}$$

$$\begin{pmatrix}
\binom{\mu+s+t-2}{t} & \binom{\mu+s+t-1}{t+1} & \binom{\mu+s+t}{t+2} & \binom{\mu+s+t+1}{t+3} & \binom{\mu+s+t+2}{t+4} \\
\binom{\mu+s+t-1}{t} & \binom{\mu+s+t}{t+1} & \binom{\mu+s+t+1}{t+2} & \binom{\mu+s+t+2}{t+3} & \binom{\mu+s+t+3}{t+4} \\
\binom{\mu+s+t}{t} & \binom{\mu+s+t+1}{t+1} & \binom{\mu+s+t+2}{t+2} & \binom{\mu+s+t+3}{t+3} & \binom{\mu+s+t+4}{t+4} \\
\binom{\mu+s+t+1}{t} & \binom{\mu+s+t+2}{t+1} & \binom{\mu+s+t+3}{t+2} & \binom{\mu+s+t+4}{t+3} & \binom{\mu+s+t+5}{t+4} \\
\binom{\mu+s+t+2}{t} & \binom{\mu+s+t+3}{t+1} & \binom{\mu+s+t+4}{t+2} & \binom{\mu+s+t+5}{t+3} & \binom{\mu+s+t+6}{t+4}
\end{pmatrix}$$

Our Problem for $n = 5$

$$(\mathcal{D}|\mathcal{E})^{\mu}_{s,t}(n) = \left(\binom{\mu+i+j+s+t-4}{j+t-1} \pm \delta_{i+s,j+t} \right)_{1 \leqslant i,j \leqslant n}$$

- $s = 0, t = 0 : (\mathcal{D}|\mathcal{E})^{\mu}_{0,0}(5)$

$$\begin{pmatrix} \binom{\mu-2}{0} \pm 1 & \binom{\mu-1}{1} & \binom{\mu}{2} & \binom{\mu+1}{3} & \binom{\mu+2}{4} \\ \binom{\mu-1}{0} & \binom{\mu}{1} \pm 1 & \binom{\mu+1}{2} & \binom{\mu+2}{3} & \binom{\mu+3}{4} \\ \binom{\mu}{0} & \binom{\mu+1}{1} & \binom{\mu+2}{2} \pm 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} \pm 1 & \binom{\mu+5}{4} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1} & \binom{\mu+4}{2} & \binom{\mu+5}{3} & \binom{\mu+6}{4} \pm 1 \end{pmatrix}$$

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$$(\mathcal{D}|\mathcal{E})^{\mu}_{s,t}(n) = \left(\binom{\mu+i+j+s+t-4}{j+t-1} \pm \delta_{i+s,j+t}\right)_{1\leqslant i,j\leqslant n}$$

- $s = 2, t = 2 : (\mathcal{D}|\mathcal{E})^{\mu}_{2,2}(5)$

$$\begin{pmatrix}
\binom{\mu+2}{2} \pm 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} & \binom{\mu+5}{5} & \binom{\mu+6}{6} \\
\binom{\mu+3}{2} & \binom{\mu+4}{3} \pm 1 & \binom{\mu+5}{4} & \binom{\mu+6}{5} & \binom{\mu+7}{6} \\
\binom{\mu+4}{2} & \binom{\mu+5}{3} & \binom{\mu+6}{4} \pm 1 & \binom{\mu+7}{5} & \binom{\mu+8}{6} \\
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\end{pmatrix}$$

$$(\mathcal{D}|\mathcal{E})^{\mu}_{s,t}(n) = \left(\binom{\mu+i+j+s+t-4}{j+t-1} \pm \delta_{i+s,j+t}\right)_{1\leqslant i,j\leqslant n}$$

- $s = 3, t = 2 : (\mathcal{D}|\mathcal{E})^{\mu}_{3,2}(5)$

$$\begin{pmatrix} \binom{\mu+3}{2} & \binom{\mu+4}{3} \pm 1 & \binom{\mu+5}{4} & \binom{\mu+6}{5} & \binom{\mu+7}{6} \\ \binom{\mu+4}{2} & \binom{\mu+5}{3} & \binom{\mu+6}{4} \pm 1 & \binom{\mu+7}{5} & \binom{\mu+8}{6} \\ \binom{\mu+5}{2} & \binom{\mu+6}{3} & \binom{\mu+7}{4} & \binom{\mu+8}{5} \pm 1 & \binom{\mu+9}{6} \\ \binom{\mu+6}{2} & \binom{\mu+7}{3} & \binom{\mu+8}{4} & \binom{\mu+9}{5} & \binom{\mu+10}{6} \pm 1 \\ \binom{\mu+7}{2} & \binom{\mu+8}{3} & \binom{\mu+9}{4} & \binom{\mu+10}{5} & \binom{\mu+11}{6} \end{pmatrix}$$

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- $s = 2, t = 3 : (\mathcal{D}|\mathcal{E})^{\mu}_{2,3}(5)$

$$\begin{pmatrix}
\binom{\mu+3}{3} & \binom{\mu+4}{4} & \binom{\mu+5}{5} & \binom{\mu+6}{6} & \binom{\mu+7}{7} \\
\binom{\mu+4}{3} \pm 1 & \binom{\mu+5}{4} & \binom{\mu+6}{5} & \binom{\mu+7}{6} & \binom{\mu+8}{7} \\
\binom{\mu+5}{3} & \binom{\mu+6}{4} \pm 1 & \binom{\mu+7}{5} & \binom{\mu+8}{6} & \binom{\mu+9}{7} \\
\binom{\mu+6}{3} & \binom{\mu+7}{4} & \binom{\mu+8}{5} \pm 1 & \binom{\mu+9}{6} & \binom{\mu+10}{7} \\
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- $s = -2, t = -1 : (\mathcal{D}|\mathcal{E})^{\mu}_{-2,-1}(5)$

$$\begin{pmatrix} \binom{\mu-5}{-1} & \binom{\mu-4}{0} & \binom{\mu-3}{1} & \binom{\mu-2}{2} & \binom{\mu-1}{3} \\ \binom{\mu-4}{-1} \pm 1 & \binom{\mu-3}{0} & \binom{\mu-2}{1} & \binom{\mu-1}{2} & \binom{\mu}{3} \\ \binom{\mu-3}{-1} & \binom{\mu-2}{0} \pm 1 & \binom{\mu-1}{1} & \binom{\mu}{2} & \binom{\mu+1}{3} \\ \binom{\mu-2}{-1} & \binom{\mu-1}{0} & \binom{\mu}{1} \pm 1 & \binom{\mu+1}{2} & \binom{\mu+2}{3} \\ \binom{\mu-1}{-1} & \binom{\mu}{0} & \binom{\mu+1}{1} & \binom{\mu+2}{2} \pm 1 & \binom{\mu+3}{3} \end{pmatrix}$$

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$$(\mathcal{D}|\mathcal{E})^{\mu}_{s,t}(n) = \left(\binom{\mu+i+j+s+t-4}{j+t-1} \pm \delta_{i+s,j+t} \right)_{1\leqslant i,j\leqslant n}$$

- $s = 3, t = -1 : (\mathcal{D}|\mathcal{E})^{\mu}_{3,-1}(5)$

$$\begin{pmatrix} \binom{\mu}{-1} & \binom{\mu+1}{0} & \binom{\mu+2}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} \pm 1 \\ \binom{\mu+1}{-1} & \binom{\mu+2}{0} & \binom{\mu+3}{1} & \binom{\mu+4}{2} & \binom{\mu+5}{3} \\ \binom{\mu+2}{-1} & \binom{\mu+3}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3} \\ \binom{\mu+3}{-1} & \binom{\mu+4}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+7}{3} \\ \binom{\mu+4}{-1} & \binom{\mu+5}{0} & \binom{\mu+6}{1} & \binom{\mu+7}{2} & \binom{\mu+8}{3} \end{pmatrix}$$

Consider the matrix

$$\mathcal{E}^{\mu}_{2,1}(2) = \begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} - 1 \\ \binom{\mu+2}{1} & \binom{\mu+3}{2} \end{pmatrix}.$$

Evaluating Determinants: Laplace Expansion

Consider the matrix

$$\mathcal{E}^{\mu}_{2,1}(2) = \begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} - 1 \\ \binom{\mu+2}{1} & \binom{\mu+3}{2} \end{pmatrix}.$$

We can compute its determinant using a combinatorial interpretation. To see how this is possible, we specialize to $\mu = 2$ and compute the determinant by expanding along the first row:

$$\begin{aligned} E^{2}_{2,1}(2) = \det \begin{pmatrix} \binom{3}{1} & \binom{4}{2} - 1 \\ \binom{4}{1} & \binom{5}{2} \end{pmatrix} &= \binom{3}{1}\binom{5}{2} - \left(\binom{4}{2} - 1\right)\binom{4}{1} \\ &= \underbrace{\left(\binom{3}{1}\binom{5}{2} - \binom{4}{2}\binom{4}{1}\right)}_{=6} + \underbrace{1\binom{4}{1}}_{=4} \\ &= 10. \end{aligned}$$

Evaluating Determinants: Rhombus Tilings

$$\binom{\mu+s+t+i+j-4}{j+t-1} \to \begin{cases} \text{start:} & (\mu+s+i-3,0) \\ \text{end:} & (0,j+t-1) \end{cases} \to \begin{cases} \text{start:} & (i+1,0) \\ \text{end:} & (0,j) \end{cases}$$

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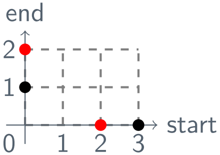

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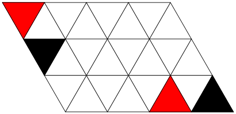

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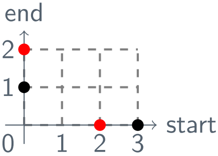

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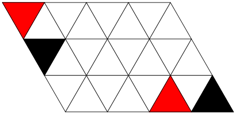

There are six ways to realize these 2-tuples and tile this region:

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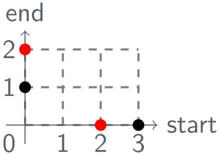

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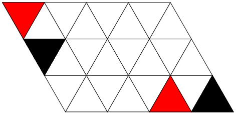

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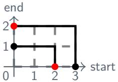

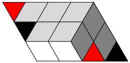

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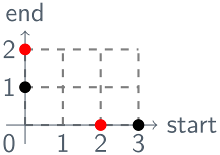

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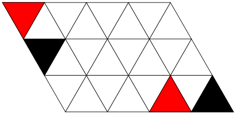

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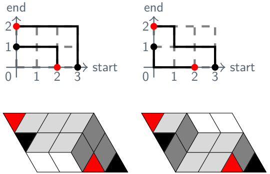

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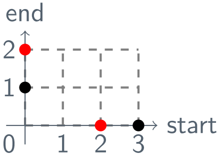

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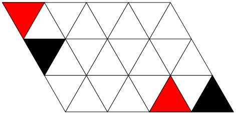

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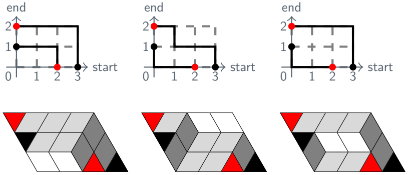

Evaluating Determinants: Rhombus Tilings

$$\binom{\mu+s+t+i+j-4}{j+t-1} \rightarrow \begin{cases} \text{start:} & (\mu+s+i-3,0) \\ \text{end:} & (0,j+t-1) \end{cases} \rightarrow \begin{cases} \text{start:} & (i+1,0) \\ \text{end:} & (0,j) \end{cases}$$

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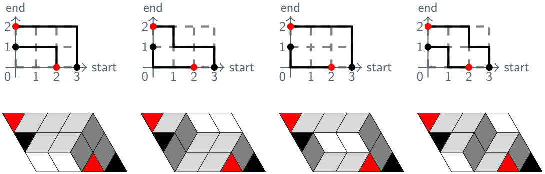

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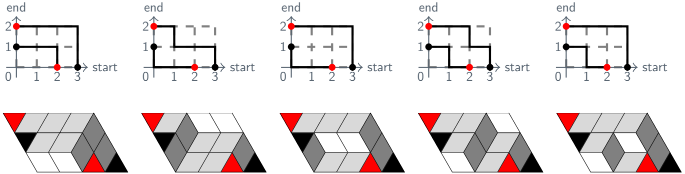

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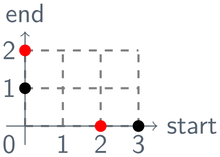

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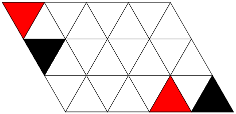

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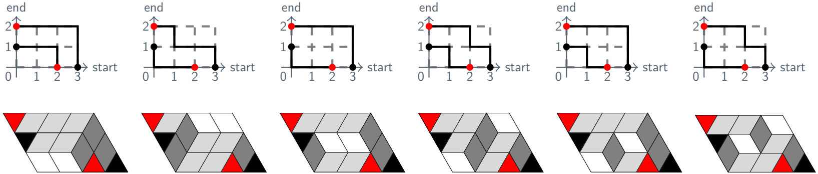
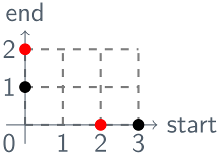


The minor(s) associated to the Kronecker delta(s) corresponds to the *removal* of starting and ending points:



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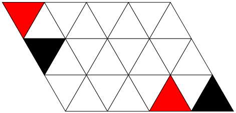

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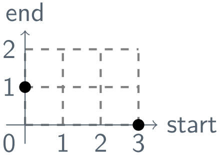

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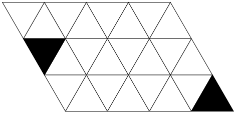

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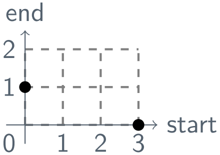

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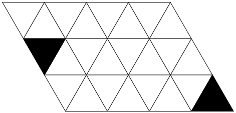
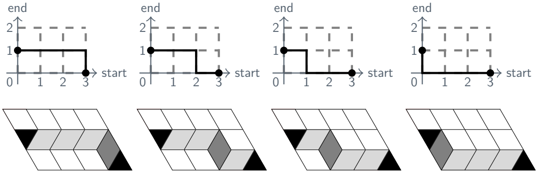
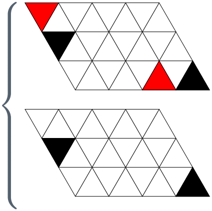


There are four ways to realize the 1-tuples and tile this region:



We can reframe this into one tiling problem, namely, to count the number of cyclically symmetric tilings of one holey hexagonal region:



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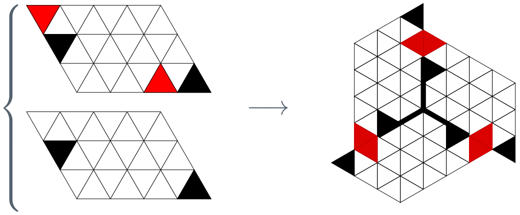

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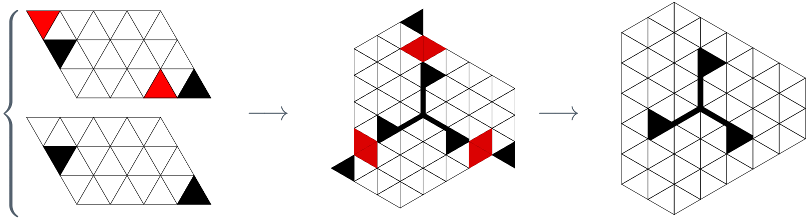

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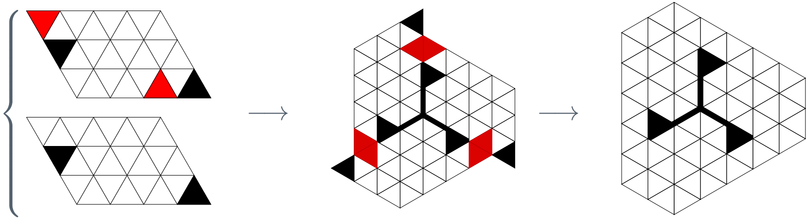
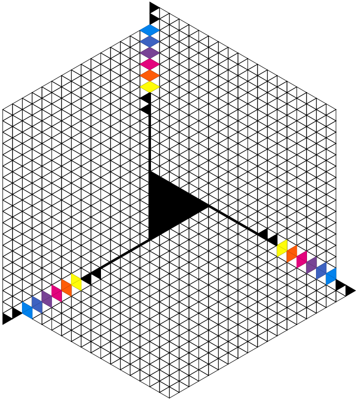


$$E_{2,1}^{2}(2) = 10$$

Evaluating Determinants: Rhombus Tilings



Region (left) associated to the determinant $D^{8}_{5,7}(8)$ and illustration (right) of one cyclically symmetric tiling of this region.

Sum of Minors Formula

$$E^{\mu}_{s,t}(n) = (-1)^{s-t+1} \cdot M^{i}_{i+s-t} + \sum_{j=1}^{n} (-1)^{i+j} \cdot b_{i,j} \cdot M^{i}_{j}$$

where $s \geqslant t$,

$$b_{i,j} := \binom{\mu + i + j + s + t - 4}{j + t - 1},$$

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We can apply the removal of the Kronecker delta recursively so that what remains are minors of $\mathcal{B}^{\mu}_{s,t}(n) := (b_{i,j})_{1 \leqslant i,j \leqslant n}$:

$$E^{\mu}_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n-(s-t)\}} (-1)^{(s-t+1)\cdot|I|} \cdot B^{I}_{I+s-t}.$$

The formulation for $s \leqslant t$ is analogous.

Sum of Minors Formula

$$E^{\mu}_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n-(s-t)\}} (-1)^{(s-t+1)\cdot|I|} \cdot B^{I}_{I+s-t}$$

$$D^{\mu}_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n-(s-t)\}} (-1)^{(s-t)\cdot|I|} \cdot B^{I}_{I+s-t}$$

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Example: Consider the associated regions to the following determinants.

$D^{3}_{2,0}(4)$ $\qquad$ $E^{6}_{1,0}(3)$

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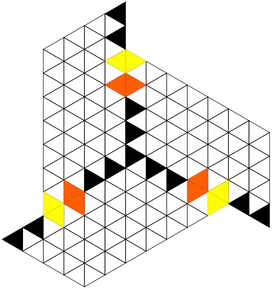

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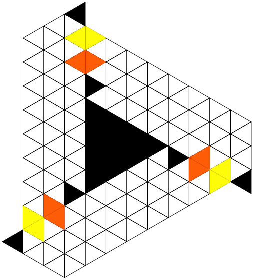

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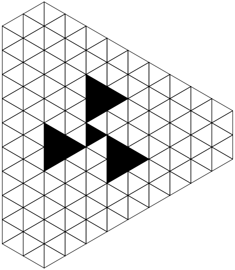

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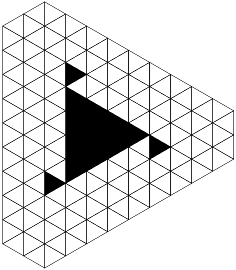

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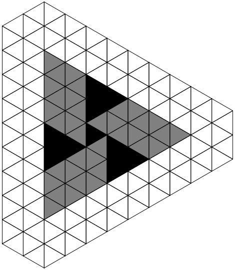

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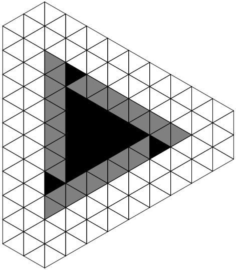

Relationships

$$E^{\mu}_{s,0}(n) = D^{\mu+3}_{s-1,0}(n-1)$$
$$D^{\mu}_{s,0}(n) = E^{\mu+3}_{s-1,0}(n-1)$$

Determinant Closed Forms

Prove conjectured closed forms.

Determinant	Formula Conjectured By	Reference
$E^{\mu}_{1,2r-1}(2m-1)$	Krattenthaler, Lascoux	[1, Conj37]
$D^{\mu}_{2r,1}(2m-1)$	Koutschan, Thanatipanonda	[2, Conj20]
$D^{\mu}_{-1,2r}(2m)$	Koutschan, Thanatipanonda	[2, Conj21]

[1] C. Krattenthaler. Advanced determinant calculus: A complement. *Linear Algebra and its Applications*, 411:68–166, 2005.

[2] C. Koutschan and T. Thanatipanonda. A curious family of binomial determinants that count rhombus tilings of a holey hexagon. *Journal of Combinatorial Theory, Series A*, 166:352–381, 2019. arXiv: 1709.02616.

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Find (and prove) more closed forms for and relationships between the D and E determinant families.

[1] C. Krattenthaler. Advanced determinant calculus: A complement. *Linear Algebra and its Applications*, 411:68–166, 2005.

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The Pochhammer Symbol

For an indeterminate x, and $y \in \mathbb{Z}$:

$$(x)_y := \begin{cases} x(x+1)\cdots(x+y-1), & y > 0, \\ 1, & y = 0, \\ \frac{1}{(x+y)_{-y}}, & y < 0. \end{cases}$$

Determinant Closed Forms

Conjecture 37 (Krattenthaler and Lascoux, 2005)

Let μ be an indeterminate and $m, r \in \mathbb{Z}$. If $m \geqslant r \geqslant 1$, then

$$E^{\mu}_{1,2r-1}(2m-1) \;=\; (-1)^{m-r} \cdot 2^{m^2-2mr+3m+r^2-2r} \cdot \prod_{i=0}^{m-1} \frac{i!\,(i+1)!}{(2i)!\,(2i+2)!}$$

$$\times \prod_{i=0}^{2r-3} i! \cdot \prod_{i=0}^{r-2} \frac{((2m-2i-3)!)^2}{((m-i-2)!)^2\,(2m+2i-1)!\,(2m+2i+1)!}$$

$$\times\ (\mu-1) \cdot \left(\frac{\mu}{2}+r-\frac{1}{2}\right)_{m-r} \cdot \prod_{i=1}^{2r-2} (\mu+i-1)_{2m+2r-2i-1}$$

$$\times \prod_{i=0}^{\left\lfloor \frac{m-r-1}{2} \right\rfloor} \left(\frac{\mu}{2}+3i+3r-\frac{1}{2}\right)^2_{m-r-2i-1} \cdot \left(-\frac{\mu}{2}-3m+3i+3\right)^2_{m-r-2i}.$$

Determinant Closed Forms

Our Version of Conjecture 37

Let μ be an indeterminate and $m, r \in \mathbb{Z}$. If $m \geqslant r \geqslant 1$, then

$$E^{\mu}_{2r-1,1}(2m-1) \; = \; \frac{(-1)^{m-r}\,(\mu-1)\,(\mu+2r-1)_{2m-2}}{(2r-2)!\,(m+r-1)_{m-r+1}\left(\frac{\mu}{2}+r\right)_{m-r}}$$

$$\times \prod_{i=1}^{m-r} \frac{(\mu+2i+6r-5)_{i-1}^{2}\,\left(\frac{\mu}{2}+2i+3r-2\right)_{i}^{2}}{(i)_{i}^{2}\,\left(\frac{\mu}{2}+i+3r-2\right)_{i-1}^{2}}.$$

Determinant Closed Forms

Conjecture 20 (Koutschan and Thanatipanonda, 2019)

Let μ be an indeterminate and $m, r \in \mathbb{Z}$. If $m \geqslant r \geqslant 1$, then

$$D^{\mu}_{2r,1}(2m) = \frac{(-1)^{m-r}\,(\mu-1)\,(\mu+2r)_{2m-1}}{(2r-1)!\,(m+r)_{m-r+1}\,\left(\frac{\mu}{2}+r+\frac{1}{2}\right)_{m-r}}$$

$$\times \prod_{i=1}^{m-r} \frac{(\mu+2i+6r-2)^2_{i-1}\,\left(\frac{\mu}{2}+2i+3r-\frac{1}{2}\right)^2_i}{(i)^2_i\,\left(\frac{\mu}{2}+i+3r-\frac{1}{2}\right)^2_{i-1}}.$$

Determinant Closed Forms

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Conjecture 21 (Koutschan and Thanatipanonda, 2019)

Let μ be an indeterminate and $m, r \in \mathbb{Z}$. If $m > r \geqslant 0$, then

$$D^{\mu}_{-1,2r}(2m) = \frac{(-1)^{m-r}\,(\mu-3)\,\left(\frac{\mu}{2}+r-\frac{1}{2}\right)_{m-r-1}}{(2r+1)_{m-r}} \cdot \prod_{i=1}^{2m} \frac{(\mu+i-3)_{2r}}{(i)_{2r}}$$

$$\times \prod_{i=1}^{m-r-1} \frac{(\mu+2i+6r)_{i}^{2}\,\left(\frac{\mu}{2}+2i+3r+\frac{1}{2}\right)_{i-1}^{2}}{(i)_{i}^{2}\,\left(\frac{\mu}{2}+i+3r+\frac{1}{2}\right)_{i-1}^{2}}.$$

The Holonomic Ansatz

Define $A(n) := \det(a_{i,j})_{1 \leqslant i,j \leqslant n}$ for $n \geqslant 1$ and the $a_{i,j}$ form a bivariate holonomic sequence not depending on n.

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Suppose $A(n) \neq 0$ for all n. Using the Laplace expansion,

$$A(n) = \sum_{k=1}^{n} a_{n,k} \cdot \mathrm{Cof}_{n,k}(n-1),$$

where $a_{n,k}$ is the k-th term in the expansion row and $\mathrm{Cof}_{n,k}(n-1)$ is the corresponding cofactor.

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For each fixed n, the quantities $(c_{n,1}, \ldots, c_{n,n})$ satisfy the following system of equations:

$$\begin{cases} c_{n,n} = 1, & n \geqslant 1, \\ \sum_{k=1}^{n} a_{\ell,k} \cdot c_{n,k} = 0, & 1 \leqslant \ell \leqslant n-1. \end{cases}$$

The Holonomic Ansatz

Now, if we have a conjectured formula $F(n)$ for the determinant $A(n)$, then it suffices to prove

$$\sum_{k=1}^{n} a_{n,k} \cdot c_{n,k} = \frac{F(n)}{F(n-1)}$$

for all $n \geqslant 2$ to conclude that $A(n) = F(n)$.

$$E^{\mu}_{s,0}(n) = D^{\mu+3}_{s-1,0}(n-1)$$
$$D^{\mu}_{s,0}(n) = E^{\mu+3}_{s-1,0}(n-1)$$

$$\frac{D^{\mu}_{2r,1}(2m)}{E^{\mu+3}_{2r-1,1}(2m-1)} = \text{nice}$$
$$\frac{E^{\mu}_{2r+1,1}(2m+1)}{D^{\mu+3}_{2r,1}(2m)} = \text{nice}$$

Applying the Holonomic Ansatz to our Problem

Consider the following two matrices in $\mathbb{R}^{n\times n}$:

$$\mathcal{L}_n := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{R}_n := \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Suppose $t = 1$. Multiplying,

$$\mathcal{L}_n \cdot \left(\binom{\mu+i+j+s-3}{j} \pm \delta_{i+s,j+1} \right)_{1\leqslant i,j\leqslant n} \cdot \mathcal{R}_n$$

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$$\mathcal{L}_n \cdot \left(\tbinom{\mu+i+j+s-3}{j} \pm \delta_{i+s,j+1}\right)_{1\leqslant i,j\leqslant n} \cdot \mathcal{R}_n$$

and using

$$\binom{x+1}{y} - \binom{x}{y} = \binom{x}{y-1},$$

$$\sum_{\ell=0}^{j-1} \binom{x+\ell}{y+\ell} = \binom{x+j}{y+j-1} - \binom{x}{y-1},$$

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$$\mathcal{L}_n \cdot \left(\binom{\mu+i+j+s-3}{j} \pm \delta_{i+s,j+1} \right)_{1\leqslant i,j \leqslant n} \cdot \mathcal{R}_n$$

yields

$$\left(\begin{array}{c:c} \binom{\mu+s-1}{1} & \binom{\mu+j+s-1}{j} - 1 \pm \sum_{k=1}^{j} \delta_{s,k} \\ & (2 \leqslant j \leqslant n) \\ \hdashline 1 & \binom{\mu+i+j+s-3}{j-1} \mp \delta_{i+s,j+2} \\ (2 \leqslant i \leqslant n) & (2 \leqslant i,j \leqslant n) \end{array} \right).$$

Applying the Holonomic Ansatz to our Problem

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We expand about the **first row** (rather than the last row) to get

$$\mathcal{A}^{\mu}_{s,1}(n) = a_{1,1} \cdot \mathrm{Cof}_{1,1}(n-1) + \cdots + a_{1,n} \cdot \mathrm{Cof}_{1,n}(n-1).$$

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Then our goal is to prove that for all $n \geqslant s$:

$$\sum_{j=1}^{n} a_{1,j} \cdot c_{n,j} = \text{nice},$$

where "nice" is a conjectured nice rational function where the numerator and denominators factors into linear factors of μ.

The following identities *uniquely* characterize the $c_{n,j}$'s:

$$\begin{cases} c_{n,1} = 1, & n \geqslant 1, \\ \displaystyle\sum_{j=1}^{n} a_{i,j} \cdot c_{n,j} = 0, & 2 \leqslant i \leqslant n, \end{cases}$$

because $\mathcal{B}^{\mu+3}_{s-1,1}(n-1) = (a_{i,j})_{2 \leqslant i,j \leqslant n}$ has full rank.

Applying the Holonomic Ansatz to our Problem

$$\frac{D^{\mu}_{2r,1}(2m)}{E^{\mu+3}_{2r-1,1}(2m-1)} = \text{nice}$$

$$\frac{E^{\mu}_{2r+1,1}(2m+1)}{D^{\mu+3}_{2r,1}(2m)} = \text{nice}$$

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$$\left(\begin{array}{c|c} \binom{\mu+s-1}{1} & \binom{\mu+j+s-1}{j} - 1 \pm \sum_{k=1}^{j} \delta_{s,k} \\ & (2 \leqslant j \leqslant n) \\ \hline 1 & \binom{\mu+i+j+s-3}{j-1} \mp \delta_{i+s,j+2} \\ (2 \leqslant i \leqslant n) & (2 \leqslant i,j \leqslant n) \end{array}\right)$$

$$\text{nice} \;\to\; \frac{(n+s-2)(\mu-1)(\mu+n+1)(\mu+s)}{2n(s-1)(\mu+2)(\mu+n+s-1)} =: R^{\mu}_{s,1}(n).$$

Applying the Holonomic Ansatz to our Problem

$$\frac{D^{\mu}_{2r,1}(2m)}{E^{\mu+3}_{2r-1,1}(2m-1)} = \text{nice}$$

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For the first equation, we prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m$:

$$c_{2m,1} = 1,$$

$$\sum_{j=1}^{2m} \binom{\mu+i+j+2r-3}{j-1} \cdot c_{2m,j} - c_{2m,i+2r-2} = 0,$$

$$\sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

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For the second one, we prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m+1$:

$$c_{2m+1,1} = 1,$$

$$\sum_{j=1}^{2m+1} \binom{\mu+i+j+2r-2}{j-1} \cdot c_{2m+1,j} + c_{2m+1,i+2r-1} = 0,$$

$$\sum_{j=1}^{2m+1} \binom{\mu+j+2r}{j} \cdot c_{2m+1,j} - \sum_{j=1}^{2r} c_{2m+1,j} - \sum_{j=2r+1}^{2m+1} 2 \cdot c_{2m+1,j} = R^{\mu}_{2r+1,1}(2m+1).$$

Computational Problems

Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m$:

$$\begin{aligned} c_{2m,1} &= 1, \\ \sum_{j=1}^{2m} \binom{\mu+i+j+2r-3}{j-1} \cdot c_{2m,j} - c_{2m,i+2r-2} &= 0, \\ \sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} &= R^{\mu}_{2r,1}(2m). \end{aligned}$$

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We now use our computer algebra machinery and make very good use of Christoph's HolonomicFunctions.m and Manuel's Guess.m to prove these three identities.

Computational Problems

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$$\begin{aligned} c_{2m,1} &= 1, \\ \sum_{j=1}^{2m} \binom{\mu+i+j+2r-3}{j-1} \cdot c_{2m,j} - c_{2m,i+2r-2} &= 0, \\ \sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} &= R^{\mu}_{2r,1}(2m). \end{aligned}$$

We now use our computer algebra machinery and make very good use of Christoph's HolonomicFunctions.m and Manuel's Guess.m to prove these three identities.

- We "pull out of a hat" an implicit description of $c_{2m,j}$.

Computational Problems

Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m$:

$$
\begin{aligned}
c_{2m,1} &= 1, \\
\sum_{j=1}^{2m} \binom{\mu+i+j+2r-3}{j-1} \cdot c_{2m,j} - c_{2m,i+2r-2} &= 0, \\
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\end{aligned}
$$

We now use our computer algebra machinery and make very good use of Christoph's HolonomicFunctions.m and Manuel's Guess.m to prove these three identities.

- We can use creative telescoping to deduce and certify recurrences for each summation and closure properties to combine them.

Computational Problems

Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m$:

$$\begin{aligned}
c_{2m,1} &= 1,\\
\sum_{j=1}^{2m} \binom{\mu+i+j+2r-3}{j-1} \cdot c_{2m,j} - c_{2m,i+2r-2} &= 0,\\
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\end{aligned}$$

We now use our computer algebra machinery and make very good use of Christoph's HolonomicFunctions.m and Manuel's Guess.m to prove these three identities.

- Once we have the set of recurrences (i.e., generators of the annihilating ideals) for both sides, we confirm that the ideals are either the same, or one is a subideal of the other.

Computational Problems

Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m$:

$$\begin{aligned} c_{2m,1} &= 1, \\ \sum_{j=1}^{2m} \binom{\mu+i+j+2r-3}{j-1} \cdot c_{2m,j} - c_{2m,i+2r-2} &= 0, \\ \sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} &= R^{\mu}_{2r,1}(2m). \end{aligned}$$

We now use our computer algebra machinery and make very good use of Christoph's HolonomicFunctions.m and Manuel's Guess.m to prove these three identities.

- We check for annihilator singularities and initial values.

$$\sum_{j} \text{summand}$$

Computational Problems: Creative Telescoping

$$\sum_{j} \text{summand}$$

Creative telescoping outputs operators P and Q such that:

$$\sum_{j} (P - (S_j - 1) \cdot Q) \cdot \text{summand} = 0,$$

$$\sum_j \text{summand}$$

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Creative telescoping outputs operators P and Q such that:

$$\sum_j (P - (S_j - 1) \cdot Q) \cdot \text{ summand} = 0,$$

$$\sum_{j=m}^{n} P \cdot \text{summand} - \sum_{j=m}^{n} (S_j - 1) \cdot Q \cdot \text{summand} = 0,$$

$$\sum_j \text{summand}$$

Creative telescoping outputs operators P and Q such that:

$$\sum_j (P - (S_j - 1) \cdot Q) \cdot \text{ summand} = 0,$$

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$$P \cdot \sum_j \text{summand} = 0.$$

This occurs under ideal circumstances.

Computational Problems: Back to our Problem

Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m$:

$$\begin{aligned} c_{2m,1} &= 1, \\ \sum_{j=1}^{2m} \binom{\mu+i+j+2r-3}{j-1} \cdot c_{2m,j} &= c_{2m,i+2r-2}, \\ \sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} &= R^{\mu}_{2r,1}(2m). \end{aligned}$$

Applying creative telescoping resulted in the following challenges:

Computational Problems: Back to our Problem

Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m$:

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\begin{aligned}
c_{2m,1} &= 1, \\
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\sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} &= R^{\mu}_{2r,1}(2m).
\end{aligned}
$$

Applying creative telescoping resulted in the following challenges:

- Creative telescoping for the summation in the second identity did not finish.

Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m$:

$$
\begin{aligned}
c_{2m,1} &= 1, \\
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\sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} &= R^{\mu}_{2r,1}(2m).
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Applying creative telescoping resulted in the following challenges:

- Creative telescoping for the summation in the second identity did not finish.
- In the third identity, a singularity appeared in the certificate Q at $j = 1$ (for both summations) and we were not able to automatically certify our telescoper.

Computational Problems: Back to our Problem

Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m+1$:

$$\begin{aligned} c_{2m+1,1} &= 1, \\ \sum_{j=1}^{2m+1} \binom{\mu+i+j+2r-2}{j-1} \cdot c_{2m+1,j} + c_{2m+1,i+2r-1} &= 0, \\ \sum_{j=1}^{2m+1} \binom{\mu+j+2r}{j} \cdot c_{2m+1,j} - \sum_{j=1}^{2r} c_{2m+1,j} - \sum_{j=2r+1}^{2m+1} 2 \cdot c_{2m+1,j} &= R^{\mu}_{2r+1,1}(2m+1). \end{aligned}$$

Applying creative telescoping resulted in the following challenges:

- Creative telescoping for the summation in the second identity did not finish.
- In the third identity, a singularity appeared in the certificate Q at $j = 1$ (for both summations) and we were not able to automatically certify our telescoper.
- The other "relationship" took even more computational resources due to the additional sum in the third identity.

Getting Closed Forms from the "Relationships"

$$\frac{D^{\mu}_{2r,1}(2m)}{E^{\mu+3}_{2r-1,1}(2m-1)} = \text{nice}$$

$$\frac{E^{\mu}_{2r+1,1}(2m+1)}{D^{\mu+3}_{2r,1}(2m)} = \text{nice}$$

$$E^{\mu}_{1,1}(2m-1) = \text{nice}$$

$$E^{\mu}_{2r-1,1}(2m-1)$$

K and T Conjecture 20: $D^{\mu}_{2r,1}(2m)$

K and L Conjecture 37: $E^{\mu}_{1,2r-1}(2m-1)$

Relationships

$$E^{\mu}_{s,0}(n) = D^{\mu+3}_{s-1,0}(n-1)$$
$$D^{\mu}_{s,0}(n) = E^{\mu+3}_{s-1,0}(n-1)$$

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$$\frac{D^{\mu}_{2r,1}(2m)}{E^{\mu+3}_{2r-1,1}(2m-1)} = \text{nice}$$
$$\frac{E^{\mu}_{2r+1,1}(2m+1)}{D^{\mu+3}_{2r,1}(2m)} = \text{nice}$$

$$\lim_{\varepsilon\to 0}\left(\frac{D^{\mu}_{2r+\varepsilon,-1+\varepsilon}(2m)}{E^{\mu+3}_{2r-1+\varepsilon,-1+\varepsilon}(2m-1)}\right) = \text{nice}$$
$$\lim_{\varepsilon\to 0}\left(\frac{E^{\mu}_{2r+1+\varepsilon,-1+\varepsilon}(2m+1)}{D^{\mu+3}_{2r+\varepsilon,-1+\varepsilon}(2m)}\right) = \text{nice}$$

Redefining the Binomial Coefficient

We make great use of the gamma function, which is defined for all $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ such that $\Gamma(z+1) = z\Gamma(z)$.

Definition

For an indeterminate x and $y \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, we define

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(x-y+1)\,\Gamma(y+1)}.$$

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The properties from before easily follow for $j \in \mathbb{N}$:

$$\binom{x+1}{y} - \binom{x}{y} = \binom{x}{y-1},$$

$$\sum_{\ell=0}^{j-1} \binom{x+\ell}{y+\ell} = \binom{x+j}{y+j-1} - \binom{x}{y-1}.$$

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Definition

For an indeterminate x and $y \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, we define

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(x-y+1)\,\Gamma(y+1)}.$$

Furthermore, we can write $\binom{x+2\varepsilon}{k+\varepsilon}$ as a Taylor series in ε around $\varepsilon = 0$ for integers $k < 0$ to get

$$\binom{x+2\varepsilon}{k+\varepsilon} = (-1)^{k+1} \cdot \frac{(-k-1)!}{(x+1)_{-k}} \cdot \varepsilon + O(\varepsilon^2),$$

where the first (constant) term is zero and the coefficient of the ε-term is computed using the properties of the logarithmic derivative of $\Gamma(z)$.

Applying the Holonomic Ansatz to our Problem

We use the same two matrices in $\mathbb{R}^{n\times n}$:

$$\mathcal{L}_n := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{R}_n := \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Suppose $s = s + \varepsilon$ and $t = -1 + \varepsilon$. Multiplying

$$\mathcal{L}_n \cdot \left(\binom{\mu+i+j+s-5+2\varepsilon}{j-2+\varepsilon} \pm \delta_{i+s,j-1} \right)_{1\leqslant i,j\leqslant n} \cdot \mathcal{R}_n,$$

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Suppose $s = s + \varepsilon$ and $t = -1 + \varepsilon$. Multiplying

$$\mathcal{L}_n \cdot \left(\binom{\mu+i+j+s-5+2\varepsilon}{j-2+\varepsilon} \pm \delta_{i+s,j-1} \right)_{1\leqslant i,j\leqslant n} \cdot \mathcal{R}_n,$$

yields

$$\left(\begin{array}{c:c} \binom{\mu+s-3+2\varepsilon}{-1+\varepsilon} & \begin{array}{c} \binom{\mu+j+s-3+2\varepsilon}{j-2+\varepsilon} - \binom{\mu+s-3+2\varepsilon}{-2+\varepsilon} \pm \displaystyle\sum_{k=1}^{j} \delta_{s,k-2} \\ (2 \leqslant j \leqslant n) \end{array} \\ \hdashline \begin{array}{c} \binom{\mu+i+s-5+2\varepsilon}{-2+\varepsilon} \\ (2 \leqslant i \leqslant n) \end{array} & \begin{array}{c} \binom{\mu+i+j+s-5+2\varepsilon}{j-3+\varepsilon} - \binom{\mu+i+s-5+2\varepsilon}{-3+\varepsilon} \mp \delta_{s,j-i} \\ (2 \leqslant i,j \leqslant n) \end{array} \end{array}\right).$$

Applying the Holonomic Ansatz to our Problem

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yields

$$\left(\begin{array}{c:c} \binom{\mu+s-3+2\varepsilon}{-1+\varepsilon} & \binom{\mu+j+s-3+2\varepsilon}{j-2+\varepsilon} - \binom{\mu+s-3+2\varepsilon}{-2+\varepsilon} \pm \sum_{k=1}^{j} \delta_{s,k-2} \\ & (2\leqslant j\leqslant n) \\ \hdashline \binom{\mu+i+s-5+2\varepsilon}{-2+\varepsilon} & \binom{\mu+i+j+s-5+2\varepsilon}{j-3+\varepsilon} - \binom{\mu+i+s-5+2\varepsilon}{-3+\varepsilon} \mp \delta_{s,j-i} \\ (2\leqslant i\leqslant n) & (2\leqslant i,j\leqslant n) \end{array}\right).$$

Applying the Holonomic Ansatz to our Problem

We use the same two matrices in $\mathbb{R}^{n\times n}$:

$$\mathcal{L}_n := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{R}_n := \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Suppose $s = s + \varepsilon$ and $t = -1 + \varepsilon$. Multiplying

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$$\mathcal{L}_n := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{R}_n := \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Suppose $s = s + \varepsilon$ and $t = -1 + \varepsilon$. Multiplying

$$\mathcal{L}_n \cdot \left(\binom{\mu+i+j+s-5+2\varepsilon}{j-2+\varepsilon} \pm \delta_{i+s,j-1} \right)_{1\leqslant i,j\leqslant n} \cdot \mathcal{R}_n,$$

and taking the first non-constant term from each entry yields

$$\left(\begin{array}{c:c:c} \frac{1}{\mu+s-2}\cdot\varepsilon & 1 & \binom{\mu+j+s-3}{j-2} \pm \sum_{k=1}^{j} \delta_{s,k-2} \\ & & (3 \leqslant j \leqslant n) \\ \hdashline \frac{-1}{(\mu+i+s-4)_2}\cdot\varepsilon & \frac{1}{\mu+i+s-2}\cdot\varepsilon & \binom{\mu+i+j+s-5}{j-3} \mp \delta_{s,j-i} \\ (2 \leqslant i \leqslant n) & (2 \leqslant i \leqslant n) & (2 \leqslant i \leqslant n,\ 3 \leqslant j \leqslant n) \end{array}\right).$$

Applying the Holonomic Ansatz to our Problem

We expand about the **first column** and this gives us the following three identities that we need to prove for all $n > s$:

$$\begin{aligned}
\sum_{i=2}^{n} \frac{1}{\mu+i+s-2} \cdot c_{n,i} &= -1, \\
\sum_{i=2}^{n} \binom{\mu+i+j+s-5}{j-3} \cdot c_{n,i} &= \pm c_{n,j-s}, \qquad (3 \leqslant j \leqslant n), \\
\sum_{i=2}^{n} \frac{-1}{(\mu+i+s-4)_2} \cdot c_{n,i} &= R^{\mu}_{s,-1}(n),
\end{aligned}$$

where $c_{n,j-s} = 0$ for $j \leqslant s$ and

$$R^{\mu}_{s,-1}(n) := \frac{2s(n-1)(\mu-3)(\mu+n+s-2)}{\mu(n+s)(\mu+n-3)(\mu+s-2)}.$$

Relationships

$$E^{\mu}_{s,0}(n) = D^{\mu+3}_{s-1,0}(n-1)$$
$$D^{\mu}_{s,0}(n) = E^{\mu+3}_{s-1,0}(n-1)$$

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Relationships

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Applying the Holonomic Ansatz to our Problem

We change the matrix $\mathcal{R}_n$ slightly:

$$\mathcal{L}_n := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{R}_n := \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Suppose $s = 1 + \varepsilon, t = -1 + \varepsilon$ and $n = 2m + 1$. Multiplying

$$\mathcal{L}_{2m+1} \cdot \left(\binom{\mu+i+j-4+2\varepsilon}{j-1+\varepsilon} \pm \delta_{i-1,j-1} \right)_{1 \leqslant i,j \leqslant 2m+1} \cdot \tilde{\mathcal{R}}_{2m+1},$$

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we can extract:

$$\left(\begin{array}{c|c|c} 1 & \frac{1}{1-\mu} & \binom{\mu+j-2}{j-2} - 1 \\ & & (3 \leqslant j \leqslant 2m+1) \\ \hline 0 & \frac{1}{(\mu-1)_2} & \binom{\mu+j-2}{j-3} + \delta_{1,j-2} \\ & & (3 \leqslant j \leqslant 2m+1) \\ \hline 0 & \frac{1}{(\mu+i-3)_2} & \mathcal{D}_{1,0}^{\mu+3}(2m-1) \\ & (3 \leqslant i \leqslant 2m+1) & \end{array}\right).$$

Getting Closed Forms from the “Relationships”

$$\lim_{\varepsilon \to 0} \left(\frac{D^{\mu}_{2r+\varepsilon,-1+\varepsilon}(2m)}{E^{\mu+3}_{2r-1+\varepsilon,-1+\varepsilon}(2m-1)} \right) = \text{nice}$$

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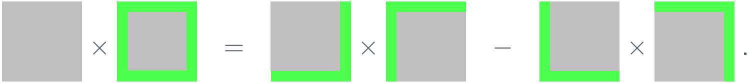
K and T Conjecture 21: $D^{\mu}_{-1,2r}(2m)$

E-Analog to Conjecture 21: $E^{\mu}_{-1,2r-1}(2m-1)$

Other Methods

- **Desanont-Jacobi-Dodgson Identity (DJD):**
  Suppose $(m_{i,j})_{i,j\in\mathbb{Z}}$ is a doubly infinite sequence and $M_{s,t}(n)$ is the determinant of the $n \times n$-matrix $(m_{i,j})_{s\leqslant i<s+n,t\leqslant j<t+n}$, then

$$M_{s,t}(n)M_{s+1,t+1}(n-2) = M_{s,t}(n-1)M_{s+1,t+1}(n-1) - M_{s+1,t}(n-1)M_{s,t+1}(n-1).$$



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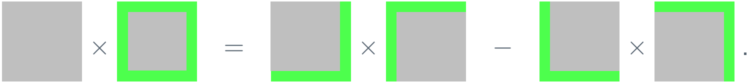
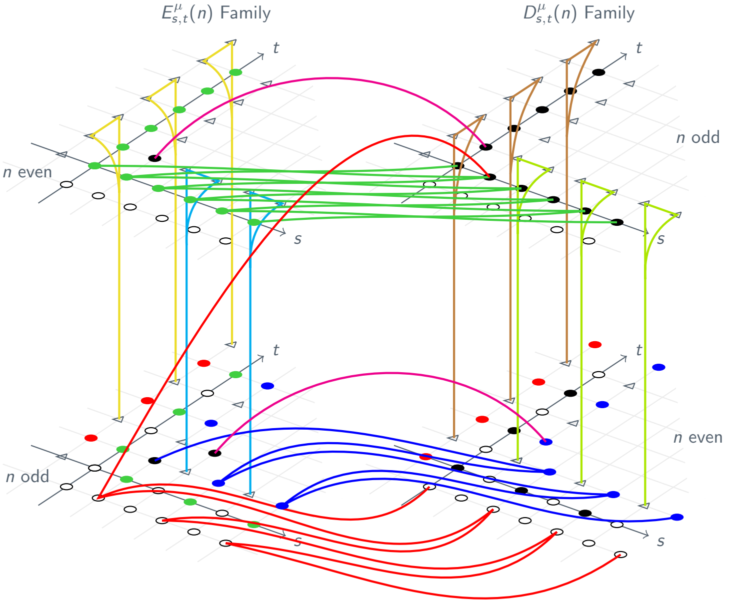


- **Switching:**
  Let $\mathcal{A}^{\mu}_{s,t}(n)$ be either $\mathcal{D}^{\mu}_{s,t}(n)$ or $\mathcal{E}^{\mu}_{s,t}(n)$, and $A^{\mu}_{s,t}(n)$ its corresponding determinant. For μ indeterminate, real numbers $s, t \notin \{-1, -2, \ldots\}$ with $s < t$ and $n \in \mathbb{Z}^+$,

$$A^{\mu}_{s,t}(n) = \prod_{i=0}^{t-s-1} \frac{(\mu+s+i-1)_n}{(i+s+1)_n} \cdot A^{\mu}_{t,s}(n).$$

All Relationships



Website

For more information, please visit:
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Thank you!