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Creative Telescoping on Multiple Sums

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ÖAW RICAM

Our Problem

We would like to prove that

$\max(k-m,0)$

$$G_s(x) := \sum_{k=1}^{m+s-1} \left(\sum_{r=1}^{s} \binom{s}{r} \binom{k-1}{r-1} \frac{b-1}{(-b)^r} \sum_{i=0}^{r-1-c_m(k)} (-b)^i \binom{r-1}{i} \right) (bx)^k$$

is not positive for all $b, m, s \in \mathbb{N}$, $b \geqslant 2$, and $x \in [0, 1)$.

Our Strategy

We would like to prove that

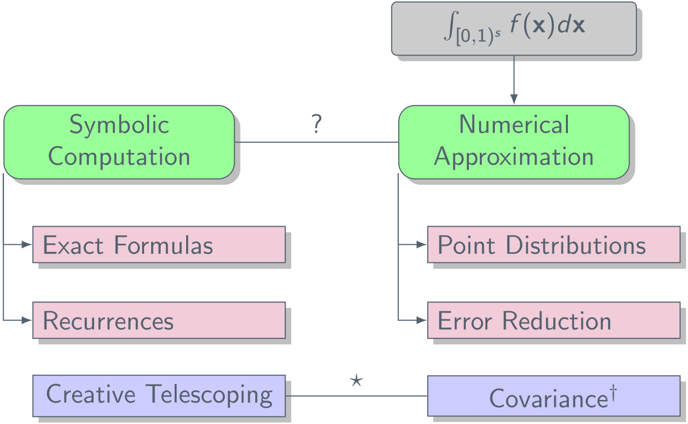
$\boxed{\max(k-m,0)}$

$$G_s(x) := \sum_{k=1}^{m+s-1} \left(\sum_{r=1}^{s} \binom{s}{r} \binom{k-1}{r-1} \frac{b-1}{(-b)^r} \sum_{i=0}^{r-1-c_m(k)} (-b)^i \binom{r-1}{i} \right) (bx)^k$$

is not positive for all $b, m, s \in \mathbb{N}$, $b \geqslant 2$, and $x \in [0,1)$.

1. We deduce a recurrence for $G_s(x)$.
   (HolonomicFunctions.m, C. Koutschan 2010)
2. We solve it and achieve a sign-equivalent closed form expression.
   (Sigma.m, C. Schneider 2007)
3. We simplify our closed form and use it to deduce that $G_s(x) \leqslant 0$.

Multivariate Integration



† C. Lemieux, *Negative dependence, scrambled nets, and variance bounds.* Mathematics of Operations Research 43. (2017) 228–251.

⋆ J. Wiart and E.W., *Walsh functions, scrambled* $(0, m, s)$*-nets, and negative covariance: applying symbolic computation to quasi-Monte Carlo integration.* Mathematics and Computers in Simulation. (2020) To appear.

Result for Part 1.

We would like to deduce a recurrence for

$\boxed{\max(k-m,0)}$

$$G_s(x) := \sum_{k=1}^{m+s-1}\left(\sum_{r=1}^{s}\binom{s}{r}\binom{k-1}{r-1}\frac{b-1}{(-b)^r}\sum_{i=0}^{r-1-c_m(k)}(-b)^i\binom{r-1}{i}\right)(bx)^k$$

for all $b, m, s \in \mathbb{N}$, $b \geqslant 2$, and $x \in [0,1)$.

Theorem (C. Koutschan, E.W.)

For $b, m, s \in \mathbb{N}, b \geqslant 2$ and $x \in [0,1)$, G_s satisfies the recurrence

$$\begin{aligned}
&(s+2)(bx-1)\cdot G_{s+3}\\
&+(m(bx-1)(x-1)+bsx(x-2)+bx(x-3)-s(2x-3)-3x+5)\cdot G_{s+2}\\
&-(x-1)(bmx+bsx+bx+mx-2m+sx-3s+x-4)\cdot G_{s+1}\\
&+(x-1)^2(m+s+1)\cdot G_s=0.
\end{aligned}$$

Creative Telescoping

$$\sum_k \text{summand} = ?$$

Creative telescoping outputs operators P and Q such that:

$$\sum_k (P - (S_k - 1) \cdot Q) \cdot \text{summand} = 0,$$

$$P \cdot \sum_{k=m}^{n} \text{summand} - \underbrace{\left(Q \cdot \text{summand}\,\Big|_{k=m}^{k=n+1}\right)}_{=0} = 0,$$

$$P \cdot \sum_k \text{summand} = 0.$$

Understand: This is not some magic black box!

Understand: The above occurs under ideal circumstances.

Toy Example

$$\mathrm{SUM}(n) := \sum_{k=5}^{n} \binom{n}{k}$$

Creative telescoping computes $P = S_n - 2$ and $Q = \frac{k}{k-n-1}$.

Conclusion: $\mathrm{SUM}(n)$ satisfies the recurrence

$$c(n+1) - 2c(n) = 0,$$

with the initial value $c(5) = 1$.

This is WRONG.

Toy Example

$$\mathrm{SUM}(n) := \sum_{k=5}^{n} \binom{n}{k}$$

Creative telescoping computes $P = S_n - 2$ and $Q = \frac{k}{k-n-1}$.

- Singularities in the certificates.

$$\sum_{k=5}^{n-1} (S_n - 2)\binom{n}{k} - \underbrace{\left(\frac{k}{k-n-1}\binom{n}{k} \Bigg|_{k=5}^{k=n} \right)}_{\text{inhomogeneous part}} = 0$$

- Boundary limits depend on n.

$$(S_n - 2)\sum_{k=5}^{n} \binom{n}{k} \underbrace{- n - (\text{inhomogeneous part})}_{\text{updated inhomogeneous part}} = 0$$

The operator for the recurrence corresponding to the "updated inhomogeneous part" can be computed pretty easily in this case.

Example

- Complicated inhomogeneous expressions require more manipulation.

$$
\begin{aligned}
&-\frac{(b-1)(m+s+1)(bx)^{m+s+1}}{b^2x}\cdot {}_2F_1\left(\begin{matrix}1-s,-m-s\\2\end{matrix}\,\middle|\,\frac{b-1}{b}\right)\\
&\quad+\frac{(b-1)(m+bs)(bx)^{m+s}}{b^2}\cdot {}_2F_1\left(\begin{matrix}1-s,1-m-s\\2\end{matrix}\,\middle|\,\frac{b-1}{b}\right)\\
&\frac{(b-1)(s+1)(bx-1)(bx)^{m+s}}{b}\cdot {}_2F_1\left(\begin{matrix}-s,1-m-s\\2\end{matrix}\,\middle|\,\frac{b-1}{b}\right).
\end{aligned}
\tag{1}
$$

By observation, we see that selecting the operator

$$
\frac{(b-1)(m+s+1)}{b^2x(bx)}S_m^2-\frac{(b-1)(m+bs)}{b^2(bx)}S_m+\frac{(b-1)(s+1)(bx-1)}{b(bx)}S_s
$$

and "applying" it to $(bx)^{m+s}\cdot {}_2F_1\left(\begin{matrix}1-s,2-m-s\\2\end{matrix}\,\middle|\,\frac{b-1}{b}\right)$ gives (1).

View One: $G_s(x)$ as a Split Sum

$\max(k-m,0)$

$$G_s(x) := \sum_{k=1}^{m+s-1} \left(\sum_{r=1}^{s} \binom{s}{r} \binom{k-1}{r-1} \frac{b-1}{(-b)^r} \sum_{i=0}^{r-1-c_m(k)} (-b)^i \binom{r-1}{i} \right) (bx)^k$$

$$G_s(x) = G_s^{(1)} + G_s^{(2)}$$

$$G_s^{(1)} := -\sum_{k=1}^{m+s-1} \sum_{r=1}^{s} \binom{s}{r} \binom{k-1}{r-1} \left(\frac{b-1}{b} \right)^r (bx)^k$$

$$G_s^{(2)} := \sum_{k=m+1}^{m+s-1} \sum_{r=1}^{s} \binom{s}{r} \binom{k-1}{r-1} \frac{1-b}{(-b)^r} \sum_{i=r-(k-m)}^{r-1} (-b)^i \binom{r-1}{i} (bx)^k$$

Here is the operator corresponding to the recurrence for the inhomogeneous part of $G_s^{(2)}$:

$$(11m^6s + 6m^6 + 185m^5s + 78m^5 + 1199m^4s + 402m^4 + 3863m^3s + 1050m^3 + 6554m^2s + 1464m^2 + 5564ms + 1032m + 4916s^2 + 1848s + 288 + \cdots \textit{920 terms} \cdots + 33b^2m^4s^5x^6 + 63b^2m^3s^6x^6 + 66b^2m^2s^7x^6 + 961b^2m^2s^6x^6 + 36b^2ms^8x^6 + 596b^2ms^7x^6 + 4174b^2ms^6x^6 + 8b^2s^9x^6 + 148b^2s^8x^6 + 1168b^2s^7x^6 + 5128b^2s^6x^6)S_s^2 + (3294m^3x - 2724bm^4x - 5934bm^3x - 7398bm^2x - 4860bmx - 1296bx + 1788m^4x + 3450m^2x + 1908mx + 432x + \cdots \textit{1326 terms} \cdots - 272b^3m^2s^7x^7 - 32b^3s^9x^7 - 144b^3ms^8x^7 - 2464b^3ms^7x^7 - 608b^3s^8x^7 - 4920b^3s^7x^7 + 16b^2s^9x^7 + 140b^2m^2s^7x^7 + 72b^2ms^8x^7 + 1232b^2ms^7x^7 + 304b^2s^8x^7)S_s + (6m^6x^2 + 90m^5x^2 + 534m^4x^2 + 5325m^3sx^2 + 1566m^3x^2 + 9468m^2sx^2 + 2268m^2x^2 + 7344msx^2 + 1296mx^2 + 1296sx^2 + \cdots \textit{904 terms} \cdots + 33b^2m^4s^5x^8 + 63b^2m^3s^6x^8 + 66b^2m^2s^7x^8 + 1035b^2m^2s^6x^8 + 36b^2ms^8x^8 + 636b^2ms^7x^8 + 4734b^2ms^6x^8 + 8b^2s^9x^8 + 156b^2s^8x^8 + 1296b^2s^7x^8 + 5976b^2s^6x^8)$$

View Two: $G_s(x)$ as one Triple Sum

$$\sum_{k=1}^{m+s-1} \sum_{r=1}^{s} \sum_{i=0}^{r-1-(k-m)} \underbrace{\binom{s}{r}\binom{k-1}{r-1}\binom{r-1}{i}\frac{b-1}{(-b)^{r-i}}(bx)^k}_{\text{summand}}.$$

Factor in Summand	Nonzero Range	Summation Bounds
$\binom{s}{r}$	$0 \leqslant r \leqslant s$	$1 \leqslant r \leqslant s$
$\binom{k-1}{r-1}$	$r-1 \leqslant k-1$	$1 \leqslant k \leqslant m+s-1$
$\binom{r-1}{i}$	$i \leqslant r-1$	$0 \leqslant i \leqslant r-1-(k-m)$

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$\binom{s}{r}$	$0 \leqslant r \leqslant s$	$1 \leqslant r \leqslant s$
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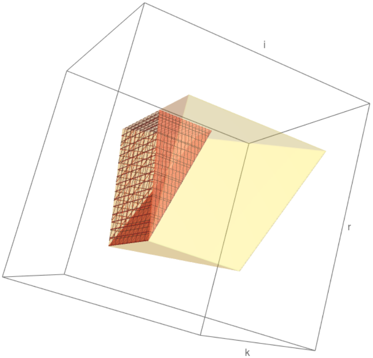
$\binom{r-1}{i}$	$i \leqslant r-1$	$0 \leqslant i \leqslant r-1-(k-m)$

Recall that $\Gamma(k)$ has an infinite number of poles at $k = 0, -1, -2, \ldots$.
We can use this to remove those troublesome nonzero terms.

View Two: $G_s(x)$ as one Triple Sum

$$\sum_{k=1}^{m+s-1} \sum_{r=1}^{s} \sum_{i=0}^{r-1-(k-m)} \underbrace{\text{summand} \cdot \frac{\Gamma(k+\epsilon)}{\Gamma(k)} \cdot \frac{\Gamma(r-i-(k-m)+\epsilon)}{\Gamma(r-i-(k-m))}}_{\text{new summand}}.$$



Summary of Results

View 1	corrections	55 days
	closure properties	30 hours
	substitution speedup	1.4 hours
View 2	corrections	9 minutes
	gamma insertion	30 seconds

Theorem (C. Koutschan, E.W.)

For $b, m, s \in \mathbb{N}$, $b \geqslant 2$ and $x \in [0,1)$, G_s satisfies the recurrence

$$
\begin{aligned}
&(s+2)(bx-1)\cdot G_{s+3}\\
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&-(x-1)(bmx+bsx+bx+mx-2m+sx-3s+x-4)\cdot G_{s+1}\\
&+(x-1)^2(m+s+1)\cdot G_s=0.
\end{aligned}
$$

Future Work

1. We believe all of these issues could (and should) be resolved automatically.
2. We believe there are other numerical applications out there of a similar nature that could benefit as a result.
3. For more information, please visit:
   https://wongey.github.io/digital-nets-walsh/

Thank you!