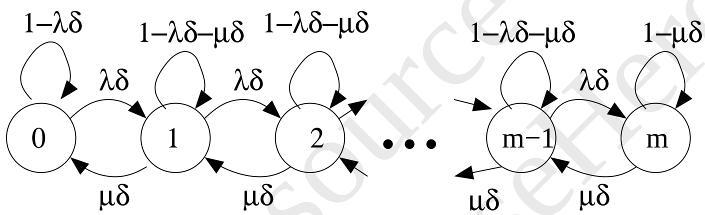
Faculty of Arts and Science
University of Toronto
CSC 358 - Introduction to Computer Networks, Spring 2013

Solutions for Assignment 3

Question 1 (10 Points):

In class, we discussed queues with infinite buffer space. However, in reality, buffers are not infinite but finite. We consider this more realistic situation in this question, where we analyze the $M/M/1/m$ queue. The $M/M/1/m$ queueing system is the same as the $M/M/1$ system, except that there can be no more than m packets in the system (that is waiting in the buffer or in service), and packets arriving when the system is full are dropped and lost. Packets arrive according to a Poisson process with rate λ and are served at rate μ.

(a) Draw the state-transition diagram for the $M/M/1/m$ queue.



(b) Derive the steady-state probabilities p_n, $n = 0, 1, ..., m$, that there are n packets in the queue.

Note that we have

$$\lambda\delta p_{n-1} = \mu\delta p_n, \qquad n = 1, ..., m$$

or

$$p_n = \frac{\lambda}{\mu} p_{n-1}, \qquad n = 1, ..., m$$

Combining the above equations, we obtain for $\rho = \frac{\lambda}{\mu}$,

$$p_n = p_0 \rho^n, \qquad n = 0, 1, ..., m.$$

Using the condition that $\sum_{n=0}^{m} p_n = 1$, we obtain that

$$\sum_{n=0}^{m} p_n = p_0 \sum_{n=0}^{m} \rho^n = p_0 \frac{1-\rho^{m+1}}{1-\rho} = 1,$$

or

$$p_0 = \frac{1-\rho}{1-\rho^{m+1}}.$$

It follows that

$$p_n = \frac{1-\rho}{1-\rho^{m+1}}\rho^n, \qquad n = 0, ..., m.$$

Note that the above results hold for any $\rho \geq 0$ and for $\rho = 1$, we obtain that

$$p_n = \frac{1}{m+1}, \qquad n = 0, 1, ..., m.$$

(c) Find the probability that a new packet is lost and the rate at which packets are dropped.

A new packet is dropped and lost when there are already m packets in the system. The probability that a packet is lost is then given by

$$p_m = \frac{1-\rho}{1-\rho^{m+1}}\rho^m.$$

Therefore, probability that a new packet is lost is equal to p_m and the rate at which packets are dropped is equal to $p_m\lambda$.

(d) Compute the throughput of the system, i.e. the average rate (in packets per unit time) at which packets leave the system. What is the relation between the packet arrival rate, the throughput, and the rate at which packets are dropped? The throughput is equal to the arrival rate minus the loss rate, *i.e.*

$$\text{Throughput} = \lambda - p_m\lambda = \lambda(1-p_m).$$

(e) Assume that $\rho^m << 1$ (when is this the case?), and redo part (b). When $\rho^{m+1} << 1$, then we have

$$p_m \approx (1-\rho)\rho^m.$$

(f) Using Little's formula, find the expected delay (queuing plus transmission delay) of a packet that enters the system. Using Little's formula we have

$$T = \frac{N}{\lambda'},$$

where

$$\lambda' = (1-p_m)\lambda.$$

Note that

$$N = \sum_{n=1}^{m} np_n,$$

where the steady state probabilities are as given above. It follows that

$$T = \frac{\sum_{n=1}^{m} np_n}{\lambda(1-p_m)},$$

(g) Find the expected delay (queuing plus transmission delay) of a packet that enters the system directly by using the same approach as in Question 1 of Tutorial 5. Show all steps of your derivation. Let $\hat{p}_n$ be the probability that a packet that enters the system has n packets ahead of it. Then we have that

$$T = \sum_{n=0}^{m-1} \frac{1+n}{\mu} \hat{p}_n.$$

Note that $\hat{p}_n \neq p_n$, to see this observe that

$$\sum_{n=0}^{m-1} \hat{p}_n = 1$$

and

$$\sum_{n=0}^{m-1} p_n < 1.$$

The probability $\hat{p}_n$ is equal to the probability that an arriving packet finds n packets in the system conditioned on the fact that it enters the system, *i.e.* we have

$$\hat{p}_n = P(A_n \mid E),$$

where A_n is the event that n packets are in the system and E is the event that the new packet enters the system. Note that

$$P(E) = \sum_{n=0}^{m-1} p_n = 1 - p_m.$$

It follows that

$$\hat{p}_n = \frac{P(A_n \cap E)}{P(E)}$$

where

$$P(A_n \cap E) = \begin{cases} p_n, & 0 < n \leq m-1 \\ 0, & \text{otherwise} \end{cases}$$

It follows that

$$T = \sum_{n=0}^{m-1} \frac{n+1}{\mu} \frac{p_n}{1-p_m},$$

or

$$T = \sum_{n=1}^{m} \frac{n}{\mu} \frac{p_{n-1}}{1-p_m}.$$

From the state transition diagram we have

$$p_{n-1}\lambda = p_n \mu, \qquad n = 1, ..., m,$$

and we obtain

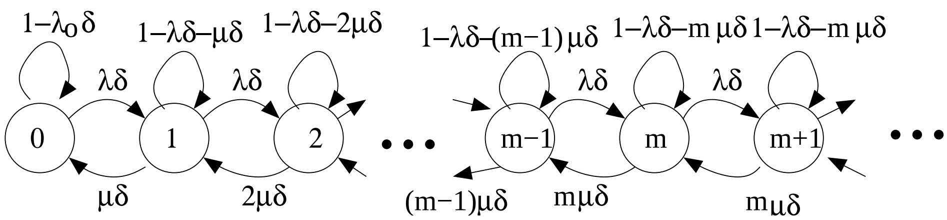
$$T = \sum_{n=1}^{m} \frac{n p_n}{\lambda(1-p_m)}.$$

Note that this result is the same as the one obtained in (f).

Question 2 (10 Points):

In this question, we analyze the $M/M/m$ queueing system which is identical to the $M/M/1$ system, except that there m servers. Packets arrive according to a Poisson process with rate λ, and each server serves packets at rate μ, independently from the other servers. A packet at the head of the buffer is routed to any server that is currently not busy, or to the first server that becomes available.

(a) Draw the state-transition diagram for the $M/M/m$ queue.



(b) Find the steady-state probabilities p_n, $n = 0, 1, 2, ...$, that there are n packets in the system.

Using the above state transition diagram, we obtain the following balance equations,

$$\lambda p_{n-1} = n\mu p_n, \qquad n \leq m$$
$$\lambda p_{n-1} = m\mu p_n, \qquad n > m$$

Setting $\rho = \frac{\lambda}{m\mu}$, we obtain from these equations,

$$p_n = \begin{cases} p_0 \frac{(m\rho)^n}{n!}, & n \leq m \\ p_0 \frac{m^m \rho^n}{m!}, & n > m. \end{cases}$$

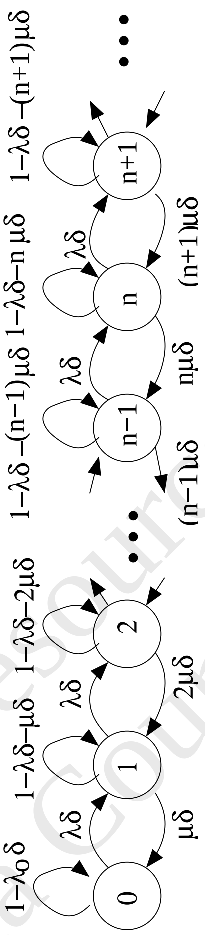
where

$$p_0 = \frac{1}{1 + \sum_{n=1}^{m} \frac{(m\rho)^n}{n!} + \sum_{n=m+1}^{\infty} \frac{m^m \rho^n}{m!}}.$$

Question 3 (10 points total)

Consider a $M/M/\infty$ queue, i.e. a queue with an infinite number of servers. Packes arrive to the queue according to a Possion process with rate λ. Whenever a packet arrives, it is immediately served with service rate μ by a server that is currently idle.

(a) Draw the state-transition diagram for the system.



(b) Find the steady-state probabitlities p_n, $n = 0, 1, 2, ...$, that n packets are in the system.

Using the state-transition diagram from a), we obtain the following balance equations,

$$\lambda p_{n-1} = n\mu p_n, \qquad n \geq 1$$

or

$$p_n = \frac{1}{n}\frac{\lambda}{\mu}p_{n-1}.$$

Setting $\rho = \frac{\lambda}{\mu}$, we obtain from these equations,

$$p_n = p_0\frac{(\rho)^n}{n!}, \qquad n \geq 1.$$

Using the condition that $\sum_{n=0}^{\infty} p_n = 1$, we obtain that

$$p_0 = \left[1 + \sum_{n=1}^{\infty}\frac{\rho^n}{n!}\right]^{-1} = e^{-\rho},$$

and

$$p_n = \rho^n\frac{e^{-\rho}}{n!}.$$

(c) Compute the average number of packets $E[N]$ in the system and the total expected delay $E[T]$ of a packet.

As each packet is immediately served by an idle server, the total expected delay is equal to

$$E[T] = \frac{1}{\mu}.$$

Using Little's theorem, we obtain that

$$E[N] = \frac{\lambda}{\mu}.$$

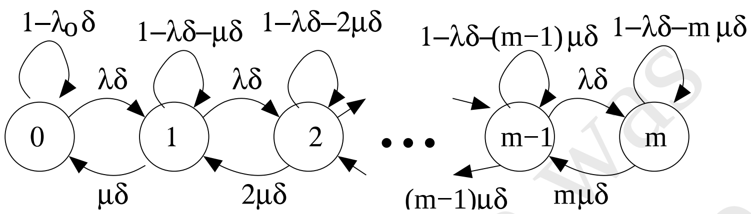
Question 4 (10 points total)

Consider a $M/M/\infty$ queue as given in Question 3; however now the servers are numbered 1,2,.... A new packet that arrives to the system will choose the lowest-numbered server that is idle and immediately enter service.

Again, packes arrive to the queue according to a Possion process with rate λ, and packets are served at rate μ.

(a) In steady-state, what is the probability p_m, $m \geq 1$, that all of the first m servers (that is server 1 to m) are busy? Hint: For a fixed value m, find the state-transition diagram that allows you to compute p_m.

Note that this probability is equal to the probability that all servers are busy in a $M/M/m/m$ queue (queue with m servers and no buffer space). The state-transition diagram is as follows. Using the above state transition diagram, we obtain the



following balance equations,

$$\lambda p_{n-1} = n\mu p_n, \qquad n = 0, 1, ..., m.$$

From these equations, we obtain

$$p_n = p_0 \frac{(\lambda/\mu)^n}{n!}, \qquad n = 0, 1, ..., m.$$

where

$$p_0 = \frac{1}{\sum_{n=0}^{m} \frac{(\lambda/\mu)^n}{n!}}.$$

Therefore, the probability p_m is equal to

$$p_m = \frac{(\lambda/\mu)^m/m!}{\sum_{n=0}^{m} (\lambda/\mu)^n/n!}.$$

(b) Let p_m, $m \geq 1$, be the answer to (a). In steady-state, what is then the arrival rate λ_m of packets to server m, $m \geq 1$? Hint: For a given m, what is the total arrival rates to servers $(m+1)$ and higher?

Let r_m be the arrival rate to servers $(m+1)$ and above, and let λ_m be the arrival rate to server m. We then have the following relations

$$r_m = p_m \lambda$$
$$\lambda_m = r_{m-1} - r_m = (p_{m-1} - p_m)\lambda.$$

(c) (10 points) Let λ_m be the answer to (b), what is then the probability that server m, $m \geq 1$, is busy?

Using Little's theorem, the probability (fraction of time) that server m is busy is then given by

$$b_m = \frac{\lambda_m}{\mu}.$$