Problem Set 1: Quantitative Economics (ECON 8185-001)

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Stochastic Growth Model

We want to compute the equilibria of the following growth model:

$$\max_{c_t,x_t,l_t} E\sum_{t=0}^{\infty}\beta^t[\log(c_t)+\psi\log(l_t)]N_t$$

$$\text{s.t. } c_t+x_t=k_t^{\theta}\Big((1+\gamma_z)^t z_t h_t\Big)^{1-\theta}$$

$$N_{t+1}k_{t+1}=[(1-\delta)k_t+x_t]N_t$$

$$\log(z_t)=\rho\log(z_{t-1})+\epsilon_t, \epsilon\sim N(0,\sigma^2)$$

$$h_t+l_t=1$$

$$c_t,x_t\geq 0,$$

where $N_t=(1+\gamma_n)^t$. First, we detrend the technological progress. The resource constraint can be written as

$$c_t+x_t=(1+\gamma_z)^t\Big(\frac{k_t}{(1+\gamma_z)^t}\Big)^{\theta}(z_t h_t)^{1-\theta}.$$

Then, define $\hat{c}_t=\frac{c_t}{(1+\gamma_z)^t}$, $\hat{x}_t=\frac{x_t}{(1+\gamma_z)^t}$, $\hat{k}_t=\frac{k_t}{(1+\gamma_z)^t}$, and $\hat{\beta}=\beta(1+\gamma_n)$ and rewrite our original model to be

$$\max_{\hat{c_t},\hat{k_{t+1}},h_t} E\sum_{t=0}^{\infty}\hat{\beta}^t[\log((1+\gamma_z)^t\hat{c}_t)+\psi\log(1-h_t)]$$

$$\text{s.t. } \hat{c}_t+\hat{x}_t=k_t^{\theta}(e^{z_t}h_t)^{1-\theta}$$

$$(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}=(1-\delta)\hat{k}_t+\hat{x}_t$$

$$z_t=\rho z_{t-1}+\epsilon_t, \epsilon\sim N(0,\sigma^2)$$

$$c_t,x_t\geq 0.$$

By substituting in c_t and x_t, we obtain the following Bellman's equation:

$$V(\hat{k}_t,z_t)=\max_{\hat{k}_{t+1},h_t}\Big\{\log\Big((\hat{k}_t)^{\theta}(e^{z_t}h_t)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t\Big)\Big\}+\psi\log(1-h_t)+\hat{\beta}\mathbb{E}\big[V(\hat{k}_{t+1},z_{t+1})\big].$$

Given this Bellman's equation, we can write the FOCs and envelope condition as:

$$[\hat{\boldsymbol{k}}_{\boldsymbol{t+1}}]: \frac{-(1+\gamma_n)(1+\gamma_z)}{(\hat{k}_t)^{\theta}(e^{z_t}h_t)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t}+\hat{\beta}\mathbb{E}\Big[\frac{\partial V(\hat{k}_{t+1},z_{t+1})}{\partial\hat{k}_{t+1}}\Big]=0$$

$$[\boldsymbol{h_t}]: \frac{(1-\theta)\hat{k}_t^{\theta}e^{z_t(1-\theta)}h_t^{-\theta}}{(\hat{k}_t)^{\theta}(e^{z_t}h_t)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t}-\frac{\psi}{(1-h_t)}=0$$

$$[\boldsymbol{ENV}]: \frac{\partial V(\hat{k}_{t+1},z_{t+1})}{\partial\hat{k}_{t+1}}=\frac{\theta\hat{k}_{t+1}^{\theta-1}(e^{z_{t+1}}h_{t+1})^{1-\theta}+1-\delta}{(\hat{k}_{t+1})^{\theta}(e^{z_{t+1}}h_{t+1})^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+2}+(1-\delta)\hat{k}_{t+1}}.$$

Combining the FOC for $\hat{k}_{t+1}$ and the envelope condition, we have the following Euler equation:

$$\frac{(1+\gamma_n)(1+\gamma_z)}{(\hat{k}_t)^\theta(e^{z_t}h_t)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t}=\hat{\beta}\mathbb{E}\Big[\frac{\theta\hat{k}_{t+1}^{\theta-1}(e^{z_{t+1}}h_{t+1})^{1-\theta}+1-\delta}{(\hat{k}_{t+1})^\theta(e^{z_{t+1}}h_{t+1})^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+2}+(1-\delta)\hat{k}_{t+1}}\Big].$$

Also, notice that the FOC for h_t governs the labor-consumption choices. Therefore, we can solve for capital and labor in steady state with the following two equations:

$$\hat{\beta}(\theta\hat{k}_{ss}^{\theta-1}(e^0h_{ss})^{1-\theta}+1-\delta)-(1+\gamma_n)(1+\gamma_z)=0$$

$$\frac{(1-\theta)\hat{k}_{ss}^\theta e^0 h_{ss}^{-\theta}}{(\hat{k}_{ss})^\theta(e^0h_{ss})^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{ss}+(1-\delta)\hat{k}_{ss}}-\frac{\psi}{(1-h_{ss})}=0$$

With the following calibration, we can compute the steady state of this economy:

Parameter	Value
θ	0.35
δ	0.0464
γ_z	0.016
γ_n	0.015
δ	0.0464
β	0.9722
ψ	2.24
ρ	0.2
σ	0.5

Variable	Steady-State Value
k_{ss}	2.304
h_{ss}	0.292
l_{ss}	0.708
c_{ss}	0.423

Value Function Iteration

For the Value Function Iteration, I constructed 1000 grids of capital around steady states level of capital, with the range $[0.5*k_{ss}, 1.5*k_{ss}]$, and 5 grids for production shocks, which are discretized using Tauchen's method.

I first solved the static problem from Intratemporal equation with Newton Root method to find optimal level of labor choice over the multidimensional grid $(k \times z \times k')$ to speed up the algorithm. The optimal policy functions for capital, consumption, and labor are plotted below:

Linear Quadratic Approximation

Here, I assumed that my return function depends on hours (h_t), capital today (k_t), and capital tomorrow (k_{t+1}).

Step 1: Compute the steady state level of variables with Nonlinear solver. This step is done in the same manner as in VFI.

Step 2: Express the return function with Linear-Quadratic

$$r\left(X_t=\begin{bmatrix}\hat{k}_t\\ \hat{z}_t\\ 1\end{bmatrix}, u_t=\begin{bmatrix}\hat{k}_{t+1}\\ h_t\end{bmatrix}\right)=\log\Big((e^{\hat{z}_t}\hat{k}_t)^\theta(h_t)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t\Big)+\psi\log(1-h_t)$$

$$\text{s.t.}\quad\begin{bmatrix}\hat{k}_{t+1}\\ \hat{z}_{t+1}\\ 1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&\rho&0\\0&0&1\end{bmatrix}\begin{bmatrix}\hat{k}_t\\ \hat{z}_t\\ 1\end{bmatrix}+\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}\begin{bmatrix}\hat{k}_{t+1}\\ h_t\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}\epsilon_{t+1}.$$

Next, I applied Kydland and Prescott's method to obtain matrices R, Q, and W by implementing second order linearization around steady state. So now we can express the above problem in the following set-up:

$$\max_{\{u_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \hat{\beta}^t \{X_t' Q X + u_t' R u_t + 2X_t' W u_t | X_0\} \right]$$

$$\text{s.t. } X_{t+1} = AX_t + Bu_t + C\epsilon_t,$$

$$X_0 \text{ is given.}$$

Then we map this problem into undiscounted problem using below transformations:

$$\tilde{X}_t = \beta^{\frac{t}{2}} X_t$$

$$\tilde{u}_t = \beta^{\frac{t}{2}} (u_t + R^{-1} W' X_t)$$

$$\tilde{A} = \sqrt{\beta}(A - BR^{-1}W')$$

$$\tilde{B} = \sqrt{\beta} B$$

$$\tilde{Q} = Q - WR^{-1}W'.$$

Step 3: Obtain a policy function by using convergence of Riccati equation

Given the initial P_0 and F_0,

1) Update P_n and F_n with the following update rules

$$P_{n+1} = \tilde{Q} + \tilde{A}' P_n \tilde{A} = \tilde{A}' P_n \tilde{B} (R + \tilde{B}' P_n \tilde{B})^{-1} \tilde{B}' P_n \tilde{A}$$
$$F_{n+1} = (R + \tilde{B}' P_n \tilde{B})^{-1} \tilde{B}' P_n \tilde{A}.$$

2) Iterate until both equations satisfy their convergence criteria at the same time.

3) Set $F = F_n + R^{-1} W'$ and $P = P_n$.

4) Lastly, our optimal policy function is

$$u_t = -F \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \\ 1 \end{bmatrix}$$

Below I plotted the optimal capital and labor policy functions, with three different labor productivity shocks (high, low, steady-state).

Vaughan's Method

Next we will implement Vaughan's method. All the steps are the same as in LQ method up to obtaining the undiscounted problem. Then we define the matrix H as

$$H = \begin{bmatrix} \tilde{A}^{-1} & \tilde{A}^{-1} \tilde{B} R^{-1} \tilde{B}' \\ \tilde{Q} \tilde{A}^{-1} & \tilde{Q} \tilde{A}^{-1} \tilde{B} R^{-1} \tilde{B}' + \tilde{A}' \end{bmatrix}$$

Then, we implement Eigenvalue decomposition on the matrix H with adjustment of position to locate eigenvalue inside of the unit circle as Γ. That is,

$$H = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma^{-1} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1}.$$

Lastly, we obtain

$$P = V_{21} V_{11}^{-1}$$

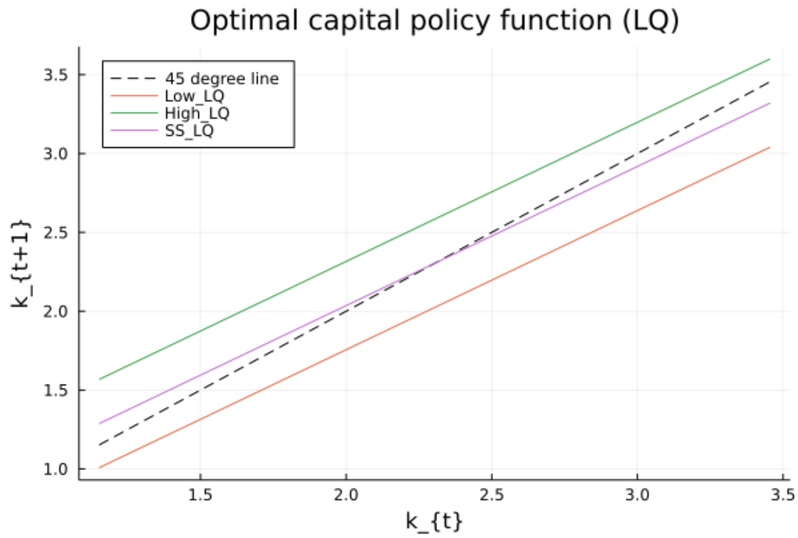

Figure 1: Optimal policy function for capital from LQ method

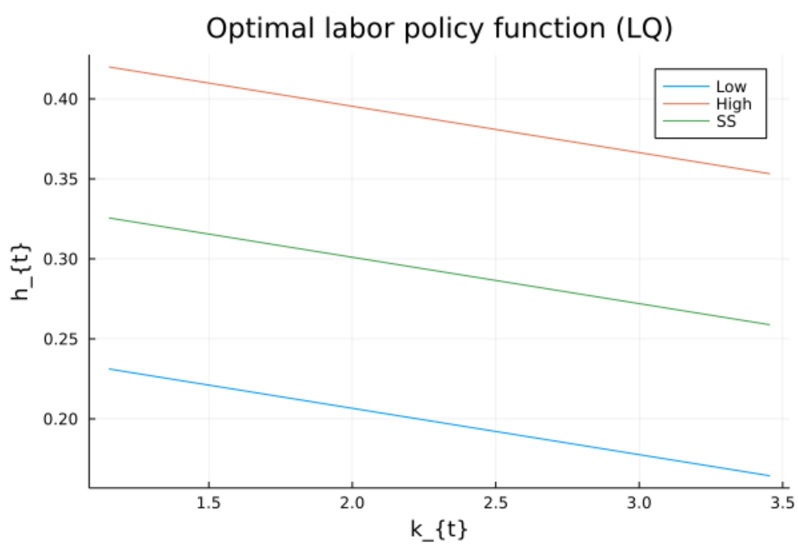

Figure 2: Optimal policy function for labor from LQ method

$$F = \left(R + \tilde{B}' P \tilde{B}\right)^{-1} \tilde{B}' P \tilde{A} + R^{-1} W'.$$

I also plotted the optimal policy functions for capital and labor from Vaughan's method, but notice here that since the matrices F are the same for both methods, the optimal policy functions are the same.

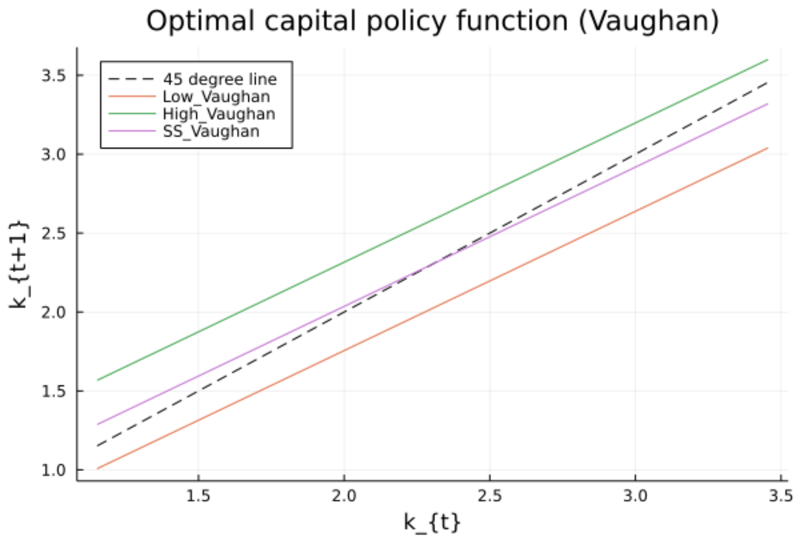

Figure 3: Optimal policy function for capital from Vaughan method

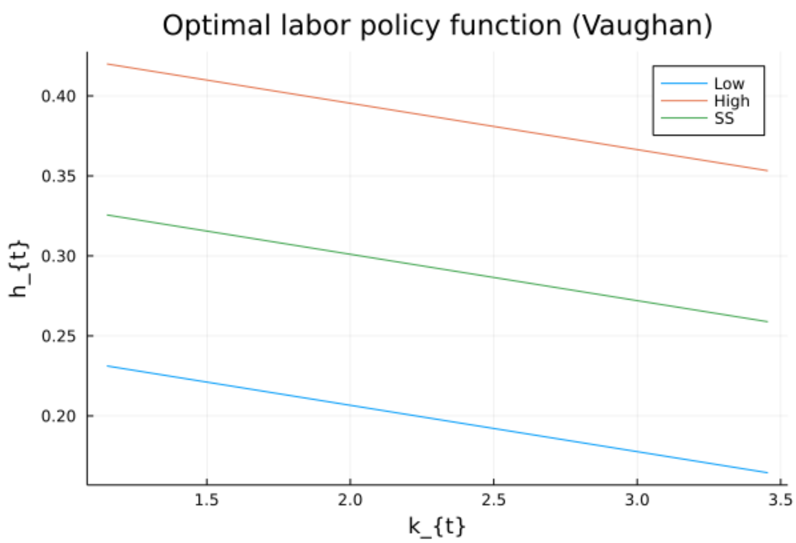

Figure 4: Optimal policy function for labor from Vaughan method