Class notes: Advanced Topics in Macroeconomics

Topic: Computing an Empirically-Relevant Model

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I. Empirically Relevant

What does it mean for a model to be empirically-relevant? In macro, one would like to have predictions for aggregate variables in the National Income and Product Accounts (NIPA) that are published by the Bureau of Economic Analysis (BEA).

Before getting into specifics, let's start with the NIPA data for 2018 downloaded from Table 1.1.5 (Gross Domestic Product) and Table 1.10 (Gross Domestic Income), both reported in billions of dollars in Table 1. Let C_t be the sum of consumption for nondurable goods and services (totaling 0.61 times GDP) for t equal to 2018. In later classes, we will also talk about adding imputed services for consumer durables and government fixed assets and subtracting taxes on consumption. Let X_t be the sum of durable consumption, gross private domestic investment, and net exports (since we'll assume a closed economy for now). Summing these categories gives us X_t equal to 0.22 times GDP. The final category is government purchases of goods and services, which we denote by G_t and accounts for 0.17 times GDP.

On the income side, we can group incomes that are payments to labor and those that are payments to capital. If we assume proprietors rent rather than own capital (which is a good approximation), then labor income $w_t H_t$ is the sum of proprietors income and compensation and equal to 0.61 times GDP. Rents to capital $r_t K_t$ are the sum of the remaining categories and equal to 0.39 times GDP. These rents include net interest, rental income, corporate profits, surplus of government enterprises, consumption of fixed capital and—and, if we adjust products, imputed services for durables and government fixed assets, less taxes on consumption. We can also split the rents into accounting profits, $(r_t - \delta)K_t$, the consumption of fixed capital δK_t.

Next, consider the simplest version of a model that we could match up to these NIPA data—where, by simple, I mean one that does not include taxes or different types of capital

Table 1. GDP and GDI, 2018

Gross domestic product	20,580.2
Personal consumption expenditures	13,998.7
Durable goods	1,475.6
Nondurable goods	2,889.2
Services	9,633.9
Gross private domestic investment	3,628.3
Fixed investment	3,573.6
Change in private inventories	54.7
Net exports of goods and services	−638.2
Government consumption expenditures and gross investment	3,591.5
Gross domestic income	20,569.4
Compensation of employees	10,941.4
Taxes on production and imports	1,441.8
Less: Subsidies	64.4
Net interest	893.5
Business current transfer payments	153.7
Proprietors' income	1,588.8
Rental income	756.8
Corporate profits	1,573.0
Taxes on corporate income	219.8
Net dividends	466.3
Undistributed profits	886.9
Current surplus of government enterprises	−6.5
Consumption of fixed capital	3,291.4
Statistical discrepancy	10.8

incomes. Assume that households choose paths for per capita consumption c_t and leisure ℓ_t to solve:

$$\max_{\{c_t,\ell_t,s_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_t,\ell_t\right) N_t$$

$$\text{subject to} \sum_{t=0}^{\infty} p_t\{c_t + v_t\left(s_{t+1}-s_t\right)\} \leq \sum_{t=0}^{\infty} p_t\{d_t s_t + w_t h_t + \kappa_t\}$$

where $N_t = (1+\gamma_n)^t$ is the size of the population, v_t is the price of firm shares, s_t is the quantity of shares held, d_t is the amount of per capita distributions paid per share to the shareholders (i.e., households), w_t is the wage rate paid to labor, $h_t = 1 - \ell_t$, and κ_t is government transfers less taxes. The price used when summing expenditures and incomes is p_t, which is the Arrow-Debreu price (and equal to the household marginal utility in equilibrium and the rate at which firms discount future distributions).

Businesses maximize the present value of aggregate distributions D_t to households:

$$
\begin{aligned}
&E_0 \sum_{t=0}^{\infty} p_t D_t \\
&\text{s.t. } K_{t+1} = (1-\delta) K_t + X_t \\
&\qquad D_t = F(K_t, Z_t H_t) - w_t H_t - X_t
\end{aligned}
$$

where, again, p_t, is the discount factor for the firm shareholders, K_t is the capital stock, X_t is gross investment, H_t is the total labor input, and $Z_t = z_t(1+\gamma_z)^t$ is the technology parameter with

$$\log z_{t+1} = \rho \log z_t + \epsilon_{t+1}$$

and $\epsilon \sim N(0, \sigma^2)$.

If we compute an equilibrium for this economy, we could compare $N_t c_t$ to the NIPA personal consumption expenditures, X_t to the NIPA investments, $w_t H_t$ to labor income, and $F(K_t, Z_t H_t) - w_t H_t$ to NIPA capital income. If we assume that government purchases plus transfers are financed by lump-sum taxes, we would compare $N_t \kappa_t$ to G_t. We can also compare subcategories of accounting profits in the model and the data. Specifically, we can compare D_t to net dividends and and can compare $K_{t+1} - K_t$ to undistributed profits.

We can also compare this economy to data in the Flow of Funds. Specifically, we can compare the total value (or market capitalization) of firms, given here by v_t (if the supply of shares is normalized by 1) to the market value of U.S. firms. We can also compare this to the end of period reproducible cost (or current cost) of capital K_{t+1} found in the BEA's fixed asset tables, because Tobin's Q in this model is equal to 1 here. (Next class, show a picture of market valuations for the United States versus BEA stocks as in McGrattan and Prescott (2005), Figure 1.)

II. Computing Equilibria

The first homework asks students to compute equilibria for the model in Section I by first mapping to a planning problem that is much easier to solve, namely,

$$
\begin{aligned}
&\max_{\{c_t,\ell_t,x_t\}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t,\ell_t) N_t \\
&\text{subject to} \quad c_t + x_t = F\left(k_t, (1+\gamma_z)^t z_t h_t\right) \\
&\qquad N_{t+1} k_{t+1} = [(1-\delta) k_t + x_t] N_t \\
&\qquad \log z_t = \rho \log z_{t-1} + \epsilon_t, \epsilon \sim N\left(0,\sigma^2\right) \\
&\qquad h_t + \ell_t = 1 \\
&\qquad c_t, x_t \geq 0 \quad \text{in all states.}
\end{aligned}
$$

Notice that here we do not get into discussions about transactions and ownership. Instead, we solve for allocations and then use the solution to construct dividends and firm valuations.

Method 1.

The first method is a "brute force" method to compute a value function that satisfies the following Bellman equation:

$$V\left(\hat{k}_t, z_t\right) = \max_{\hat{c}_t, h_t, \hat{k}_{t+1}} \left\{U(\hat{c}_t, 1-h_t) + \beta(1+\gamma_n) E_t V\left(\hat{k}_{t+1}, z_{t+1}\right), \right\}$$

where the maximization is subject to

$$
\begin{aligned}
\hat{c}_t + \hat{x}_t &= \hat{k}_t^{\theta} (z_t h_t)^{1-\theta} \\
\hat{k}_{t+1} &= \left[(1-\delta)\hat{k}_t + \hat{x}_t\right] / [(1+\gamma_z)(1+\gamma_n)] \\
\log z_{t+1} &= \rho \log z_t + \epsilon_{t+1}
\end{aligned}
$$

Lower-case letters are used to indicate that the variable is per capita and the hat further indicates that it has been divided by the growth in technology (e.g, $k_t = K_t/N_t$, $\hat{k}_t = k_t/(1+\gamma_z)^t$). Here, I am assuming that the choice of utility function is consistent with balanced growth so that I can replace c_t by $\hat{c}_t$ without consequence.

Consider two ways to deal with the expectation operator. The first is to treat z_t as a continuous state variable and use a quadrature method to compute the integral related to the expectation of z next period:

$$\begin{aligned} E\left[V\left(\hat{k}', z'\right)|z\right] &= \int V\left(\hat{k}', \rho z + \epsilon\right) f(\epsilon)\, d\epsilon \\ &\approx \sum_i \omega_i V\left(\hat{k}', \rho z + \epsilon_i\right) f(\epsilon_i) \end{aligned}$$

where $f(\cdot)$ is the density function of a normally distributed random variable. The second is to treat z_t as a Markov chain and replace the expectation with a sum:

$$E\left[V\left(\hat{k}', z_j\right)|z_i\right] \approx \sum_j \text{prob}\left(z_j|z_i\right) V\left(\hat{k}', z_j\right)$$

where z_i and z_j are the exogenous states today and tomorrow, respectively.

An easy (but extremely tedious) way to compute an approximate solution for the value function and policy functions is to guess an initial function V (say, a piecewise linear or bi-linear function over a grid for the states), solve the right hand side maximization problem for all possible states $(\hat{k}, z)$—say, by checking all possible triplets of $\hat{c}, h, \hat{k}'$ until a maximum value is found—and then updating the guess for V.