Class notes: Advanced Topics in Macroeconomics

Topic: Vaughan's Insight

Date: September 20, 2021

II. Computing Equilibria (continued)

In class, we talked about how to check results of our codes. For most problems, there is an easy check: look at the residuals of the first-order conditions. We then continued our discussion of near-linear methods.

Method 3.

We use the insights of Vaughan (1970) to exploit certain properties of the first-order conditions of the LQ problem defined above. Vaughan assumes no discounting or cross-product terms, so we'll map the variables and coefficients to $\tilde{X}$, $\tilde{u}$, $\tilde{A}$, $\tilde{B}$, and $\tilde{Q}$ as shown earlier. Also note that because the solution does not depend on the variances and covariances of ϵ, we can abstract from the uncertainty for now. Writing out the Lagrangian, we have

$$\mathcal{L} = \sum_{t=0}^{\infty} \{\tilde{X}_t' \tilde{Q} \tilde{X}_t + \tilde{u}_t' R \tilde{u}_t - \lambda_{t+1}' \left(X_{t+1} - \tilde{A}\tilde{X}_t - \tilde{B}\tilde{u}_t \right)\} \tag{1}$$

Taking derivatives with respect to $\tilde{u}_t$, $\tilde{X}_{t+1}$, and λ_{t+1}, we obtain the following first-order conditions

$$\begin{aligned} 2R\tilde{u}_t + B'\lambda_{t+1} &= 0 \\ \tilde{Q}\tilde{X}_{t+1} - \lambda_{t+1} + \tilde{A}'\lambda_{t+2} &= 0 \\ \tilde{X}_{t+1} - \tilde{A}\tilde{X}_t - \tilde{B}\tilde{u}_t &= 0 \end{aligned} \tag{2}$$

for $t \geq 0$, where $\{\lambda_t\}$ is a sequence of Lagrange multipliers. Eliminating $\tilde{u}_t$ and letting $\tilde{\lambda}_t = 1/2\lambda_t$, we have:

$$\begin{bmatrix} \tilde{X}_t \\ \tilde{\lambda}_t \end{bmatrix} = \begin{bmatrix} \tilde{A}^{-1} & \tilde{A}^{-1}\tilde{B}R^{-1}\tilde{B}' \\ \tilde{Q}\tilde{A}^{-1} & \tilde{Q}\tilde{A}^{-1}\tilde{B}R^{-1}\tilde{B}' + \tilde{A}' \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ \tilde{\lambda}_{t+1} \end{bmatrix}.$$

Let $\mathcal{H}$ be the coefficient matrix on the right hand side. Vaughan showed that this matrix can be decomposed and used directly to obtain the Riccati matrix P (and hence the solution to the LQ problem); that is, he showed that

$$\mathcal{H} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1},$$

where the eigenvalues of Λ are outside of the unit circle. Notice that the eigenvalues come in reciprocal pairs. This is an important property that implies a unique stable solution, one that satisfies the transversality condition and ensures a bounded return.

Using the fact that the Lagrange multiplier is the derivative of the value function ($\tilde{\lambda}_t = P\tilde{X}_t$), it is easy to figure out how to set P so as to get a stationary dynamical system for X. Let $W = V^{-1}$. In this case, it is easy to show that:

$$\tilde{X}_{t+1} = \{V_{11}\Lambda^{-1}\left(W_{11} + W_{12}P\right) + V_{12}\Lambda\left(W_{21} + W_{22}P\right)\}\tilde{X}_t.$$

Since Λ has roots outside the unit circle, it must be the case that $P = -W_{22}^{-1}W_{21}$. Note that since $W = V^{-1}$, this is equivalent to setting $P = V_{21}V_{11}^{-1}$.

In the case that $\tilde{A}$ is not invertible, we can modify the method slightly and use generalized eigenvalues with the following alternative system:

$$\begin{bmatrix} \tilde{A} & 0 \\ -\tilde{Q} & I \end{bmatrix} \begin{bmatrix} \tilde{X}_t \\ \tilde{\lambda}_t \end{bmatrix} = \begin{bmatrix} I & \tilde{B}R^{-1}\tilde{B}' \\ 0 & \tilde{A}' \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ \tilde{\lambda}_{t+1} \end{bmatrix}.$$

Let $\mathcal{H}_1$ be the coefficient matrix for the state and costate in $t+1$, and let $\mathcal{H}_2$ be the coefficient matrix for the state and costate in t. Then, instead of taking eigenvalues of $\mathcal{H}$ as above, we take generalized eigenvalues with the pair $(\mathcal{H}_1, \mathcal{H}_2)$.

Once we have a steady-state solution to the Riccati matrix, we can use the earlier formula to compute F and the law of motion for the state variables:

$$X_{t+1} = \left(A - BF\right)X_t + C\epsilon_{t+1} \tag{3}$$

Furthermore, given an initial condition for the states, X_0, and a realization of the shocks, $\epsilon_t,\ t \geq 0$, we can generate time-series for X_t and u_t.