Class notes: Advanced Topics in Macroeconomics

Topic: Adding Fiscal Policies

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Adaptation of LQ Framework

In class, we discussed the possibility of mapping this problem into the LQ framework used earlier (in the first homework). Specifically, how does one deal with the prices? These prices are functions of the aggregate capital K_t and aggregate labor supply H_t. One proposal was to guess a mapping from the pair (K_t, H_t) to (K_{t+1}, H_{t+1}), solve the problem as before, update the guess, and solve the problem again until the candidate solution converges. This idea was quickly dismissed because it involves solving the maximization problem many times.

The alternative is to write out the first order conditions, impose market clearing (e.g., $K_t = N_t k_t$) and construct Vaughan's Hamiltonian as before—although this time, the Q, R, W, A, B matrices will be modified to take into account the distortionary taxes and prices. This is the method laid out in McGrattan (JEDC 1994).

Suppose the original maximization problem has the following general form:

$$\begin{gathered}\max_{\{u_t\}_{t=0}^{\infty}} \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t r\left(X_t, u_t\right) \ \mid X_0\right] \\ \text{subject to} \quad X_{t+1} = g\left(X_t, u_t, \epsilon_{t+1}\right) \\ X_0 \text{ given}\end{gathered}$$

where X_t is the vector of states, u_t is the vector of controls (e.g., decisions and prices), r is the objective function which is known, g governs the evolution of the state vector and is also known, ϵ are shocks affecting this evolution which we'll assume to be iid.

As before, the first step is to map the original problem into the following related problem:

$$\max_{\{u_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left(X_t' Q X_t + u_t' R u_t + 2 X_t' W u_t\right)$$

$$\text{subject to} \quad X_{t+1} = AX_t + Bu_t + C\epsilon_{t+1}$$
$$X_0 \text{ given} \tag{1}$$

where

$$\begin{aligned} r(X_t, u_t) &\simeq X_t' Q X_t + u_t' R u_t + 2X_t' W u_t \\ g(X_t, u_t, \epsilon_{t+1}) &\simeq AX_t + Bu_t + C\epsilon_{t+1}, \end{aligned} \tag{2}$$

with Q and R symmetric. The second step is to map this into an undiscounted problem without cross-products in the objective function as follows. Let:

$$\begin{aligned} \tilde{X}_t &= \beta^{t/2} X_t \\ \tilde{u}_t &= \beta^{t/2} \left(u_t + R^{-1} W' X_t\right) \\ \tilde{Q} &= Q - W R^{-1} W' \\ \tilde{A} &= \sqrt{\beta} \left(A - B R^{-1} W'\right) \\ \tilde{B} &= \sqrt{\beta} B. \end{aligned}$$

When we have distortions, we have to distinguish between aggregate variables (that are arguments of the prices) and individual variables that are choices of consumers or firms (which I'll sometimes refer to as the "big K-little k" problem). When we write out the LQ problem this time, we'll distinguish three types of state variables: individual states (the little k's), the exogenous variables (the taxes and TFP), and the aggregate states (the big K's). In other words, the problem that we'll solve is:

$$\max_{\{\tilde{u}_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \left(\tilde{X}_t' \tilde{Q} \tilde{X}_t + \tilde{u}_t' R \tilde{u}_t\right)$$
$$\text{subject to} \quad \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \end{bmatrix}_{t+1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \end{bmatrix}_t + \begin{bmatrix} \tilde{B}_1 \\ 0 \\ 0 \end{bmatrix} \tilde{u}_t + C \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}_{t+1}. \tag{3}$$

where $\tilde{X}_1$ are the individual states, $\tilde{X}_2$ are the aggregate or exogenous states with known laws of motion, and $\tilde{X}_3$ are the aggregate states with laws of motion that are unknown and need to be computed in equilibrium. Thus, all matrices in (3) are known with the exception of A_{32} and A_{33}.

For convenience, we can stack the states with known laws of motion into a new vector $y_t = [\tilde{X}_{1t}, \tilde{X}_{2t}]'$ and work with the upper partition of (3):

$$y_{t+1} = \tilde{A}_y y_t + \tilde{B}_y \tilde{u}_t + A_z \tilde{X}_{3t}$$

where $\tilde{A}_y$ is the upper left partition of $\tilde{A}$, $\tilde{B}_y$ is the upper partition of $\tilde{B}$, and A_z is the upper right partition of $\tilde{A}$. These matrices are related to the original problem as follows:

$$\begin{aligned}
\tilde{A}_y &= \sqrt{\beta}\left(A_y - B_y R^{-1} W_y'\right) \\
\tilde{A}_z &= \sqrt{\beta}\left(A_z - B_y R^{-1} W_z'\right) \\
\tilde{B}_y &= \sqrt{\beta} B_y.
\end{aligned}$$

where A_y, B_y, and B_z are the analogous partitions of A and B.

If we form the Lagrangian and take first order conditions with respect to $\tilde{u}_t$ and $\tilde{X}_{t+1}$, we get:

$$\begin{aligned}
\tilde{u}_t &= -R^{-1} \tilde{B}_y' \lambda_{t+1} \\
\lambda_{t+1} &= \tilde{A}_y' \lambda_{t+2} + \tilde{Q}_y y_{t+1} + \tilde{Q}_z \tilde{X}_{3t+1}
\end{aligned}$$

where λ_t is the multiplier on the constraints and $\tilde{Q}_y$ and $\tilde{Q}_z$ are the upper right and left partitions of $\tilde{Q}$, respectively. These first order conditions are similar to what we had before except that now we have the aggregate variables X_{3t+1} appearing.

At this point, we impose market clearing conditions: which are given by:

$$X_{3t} = \Theta\left[X_{1t}, X_{2t}\right]' + \Psi u_t$$

in the original problem. For example, equating big K_t to little k_t (where both are per capita). Note that when we map our original problem to the undiscounted problem without cross products, we also have to map Θ and Ψ:

$$\begin{aligned}
\tilde{\Theta} &= \left(I + \Psi R^{-1} W_z'\right)^{-1}\left(\Theta - \Psi R^{-1} W_y'\right) \\
\tilde{\Psi} &= \left(I + \Psi R^{-1} W_z'\right)^{-1} \Psi
\end{aligned}$$

and

$$\tilde{X}_{3t} = \tilde{\Theta} y_t + \tilde{\Psi} \tilde{u}_t.$$

Plugging the expression for $\tilde{X}_{3t}$ equation into the first order conditions and doing some simple algebra yields a system just like before:

$$\begin{bmatrix} y \\ \lambda \end{bmatrix}_t = \begin{bmatrix} \hat{A}^{-1} & \hat{A}^{-1}\hat{B}R^{-1}\tilde{B}'_y \\ \hat{Q}\hat{A}^{-1} & \hat{Q}\hat{A}^{-1}\hat{B}R^{-1}\tilde{B}'_y + \bar{A}' \end{bmatrix} \begin{bmatrix} y \\ \lambda \end{bmatrix}_{t+1} \quad = H \begin{bmatrix} y \\ \lambda \end{bmatrix}_{t+1}$$

where $\hat{A} = \tilde{A}_y + \tilde{A}_z\tilde{\Theta}$, $\hat{Q} = \tilde{Q}_y + \tilde{Q}_z\tilde{\Theta}$, $\hat{B} = \tilde{B}_y + \tilde{A}_z\tilde{\Psi}$, and $\bar{A} = \tilde{A}_y - \tilde{B}_y R^{-1}\tilde{\Psi}'\tilde{Q}'_z$. The matrix H is analogous to Vaughan's Hamiltonian matrix. Note that if $\tilde{\Theta} = 0$ and $\tilde{\Psi} = 0$, H is Vaughan's matrix with eigenvalues that come in reciprocal pairs. When there are distortions, we will not find this pattern but the numerics are just as simple.

Once we have computed the eigenvalues and eigenvectors of H, we can construct the solution:

$$\tilde{u}_t = -\left(R + \tilde{B}'_y P\hat{B}\right)^{-1} \tilde{B}'_y P\hat{A}y_t$$

where $P = V_{21}V_{11}^{-1}$ as before. Recall that V is the eigenvector matrix of the Hamiltonian matrix, $H = V\Lambda V^{-1}$.

The matrix P can also be found by iteratively solving the modified Riccati equation:

$$P_n = \hat{Q} + \bar{A}'\left(P_{n+1}^{-1} + \hat{B}R^{-1}\tilde{B}'_y\right)^{-1}\hat{A}$$

starting from a negative definite matrix (and going backwards in n until convergence). Where does this come from? Using the fact that the multiplier λ_t is the derivative of the value function with respect to the states y_t, we can use either the first-order conditions or, better yet, the elements of the stacked Hamiltonian matrix H to equate coefficients of y_t after filling in $P_n y_t$ for λ_t and $(\hat{A}^{-1} + \hat{A}^{-1}\hat{B}R^{-1}\tilde{B}'_y P_{n+1})^{-1}y_t$ for y_{t+1}. It turns out that the modified Riccati equation can also be written as follows:

$$P_n = \hat{Q} + \bar{A}'P_{n+1}\hat{A} - \bar{A}'P_{n+1}\hat{B}\left(R + \tilde{B}'_y P_{n+1}\hat{B}\right)^{-1}\tilde{B}'_y P_{n+1}\hat{A}$$

thanks to the following identity:

$$\left(A - BD^{-1}C\right)^{-1} = A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)^{-1}CA^{-1}$$

which is easily verified.

Adaptation of Vaughan's Method

The variant of Vaughan's method that we'll use starts with the log-linearized first-order conditions and ends with a solution of the form:

$$\begin{aligned} X_{t+1} &= AX_t + BS_t \\ Z_t &= CX_t + DS_t \\ S_t &= PS_{t-1} + Q\epsilon_t \end{aligned}$$

where X_t are the endogenous state variables, S_t are the exogenous state variables, and Z_t are endogenous decisions or prices that appear in the first-order equations. The first-order conditions of the problem, when stacked, will look like:

$$0 = E_t \left\{ A_1 \begin{bmatrix} X_{t+1} \\ Z_{t+1} \end{bmatrix} + A_2 \begin{bmatrix} X_t \\ Z_t \end{bmatrix} + B_1 S_t + B_2 S_{t+1} \right\}.$$

Because of certainty equivalence, we can break the problem in two: first solve for A and C and then for B and D. Applying Vaughan, we compute the *generalized* eigenvalues and eigenvectors for matrices A_1 and $-A_2$ because A_1 is not invertible. (Note that if A_1 were invertible, then we would compute eigenvalues and eigenvectors of $-A_1^{-1}A_2$ as we did in the LQ problem.) Let V be the eigenvectors and Λ is a diagonal matrix with eigenvalues, so that

$$A_2 V = -A_1 V \Lambda.$$

Sort the eigenvalues in Λ and the associated columns in the matrix of eigenvectors V so that the eigenvalue inside the unit circle is in the (1,1) position of Λ. Then,

$$\begin{aligned} A &= V_{11} \Lambda (1,1) V_{11}^{-1} \\ C &= V_{21} V_{11}^{-1} \end{aligned}$$

The next step is to compute B and D. Given we have A and C, we can write the first-order conditions as:

$$
\begin{aligned}
0 &= A_1 \begin{bmatrix} AX_t + BS_t \\ CAX_t + CBPS_t + DPS_t \end{bmatrix} + A_2 \begin{bmatrix} X_t \\ CX_t + DS_t \end{bmatrix} + B_1 S_t + B_2 P S_t \\
&= F\left(B, D\right) X_t + G\left(B, D\right) S_t
\end{aligned}
$$

Notice that elements of F and G are linear in the elements of B and D. Therefore, to solve B and D we simply solve a system of linear equations with as many unknowns as there are elements of B and D.