Note on the Heterogeneous Agent Model: Aiyagari (1994)

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1 Introduction

The purpose of this note is to explain the details of the algorithm of the standard heterogenous agents model developed by, among others, Aiyagari (1994). Since his model is a general equilibrium version of Bewley's model, the model is also called Bewley model or Bewley-Aiyagari model. This note restricts attention to the very basic model. Good sources of information are Ríos-Rull (1999) and Heer and Maussner (2005).

First of all, let me summarize the main features of the basic Aiyagari model:

1. There are mass of agents. Each agent is atomless and thus a price taker.
2. Agents are ex-ante homogeneous but ex-post heterogeneous, depending on the history of realizations of idiosyncratic shocks (Aiyagari's model has only shocks to labor income).
3. The set of assets which are traded is exogenously determined. In particular, there is only one asset (risk-free asset or capital) allowed to be traded.
4. Therefore, agents cannot fully insure away their idiosyncratic risks. They can only self-insure by saving.
5. Only the steady state equilibrium is studied.
6. Prices (wage and interest rate) are determined competitively.

2 The Model

Let's start from the general environment of the model.

Model 1 (Aiyagari (1994))

1. *Time is discrete ($t = 0, 1, ...$). There are continuum of agents. Total measure of agents is normalized to one. Each agent has the following preference:*

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

2. *Agents are endowed with capital a_0 initially and one unit of time in each period. Agents spend all of their time in working, since leisure is not valued.*
3. *Agents can hold capital $a_t \in A = [\underline{a}, \infty)$ which yields return r_t in period t. An agent's labor income in period t is $w_t\ s_t$ where w_t is the wage for efficiency unit of labor. s_t follows a Markov chain (S, Π) ($S = \{s_1 < s_2 < ... < s_n\}$. An element of Π, $\pi_{ii'}$, represents $p_{ii'} = prob(s_{t+1} = s'_i | s_t = s_i)$). s_t of each agent is independent of others' s_t.*

4. *There are continuum of firms which have access to the following CRS technology:*

$$Y_t = F(K_t, L_t)$$

where K_t is capital input and L_t is labor input. CRS technology means that the size of the firms does not matter, or we can assume a single representative firm, without loss of generality. Firms rent inputs in competitive markets. capital depreciates at a constant rate δ.

Aiyagari (1994) makes the following assumptions regarding the functional forms:

1. The utility function is of CRRA:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

where σ is the coefficient of relative risk aversion. Intertemporal elasticity of substitution is $\frac{1}{\sigma}$

2. Production function is Cobb-Douglas:

$$Y = K^{\theta} L^{1-\theta}$$

3 Steady State Equilibrium

In general, this problem is a really hard problem. Agents need to know the future path of the prices, which depend on the future path of the type distribution of agents. Therefore, there is a high dimensional fixed-point problem here. Instead of solving the general problem, Aiyagari (1994) restricts attention to the steady state equilibrium, where the type distribution of agents is stationary (not changing over time). This implies that the prices, which depend on the type distribution, are constant over time as well.

In this note the equilibrium is called the *steady state equilibrium* but be careful. The name might be a misleading one. Even if the type distribution, aggregate statistics, and the prices are constant over time, at the individual level, there are a lot of movements going on. Individual agents are hit by idiosyncratic shocks every period and adjusting their capital stock holding accordingly. This individual level dynamics is what makes the model extremely exciting.

Suppose we focus on the steady state equilibrium. First, we need to know that a stationary distribution exists. It might be even better if the stationary distribution is unique, and the stationary distribution is a limiting distribution no matter what kind of initial distribution we start from. I will not go into details but interested readers should read Hopenhayn and Prescott (1992). They show the conditions for existence, uniqueness, and global stability (any distribution converges to the stationary distribution) of the stationary distribution (Theorem 2).

There are three crucial conditions for their theorem (Theorem 2) to hold:

1. The transition function P is increasing,

2. Space for individual state is compact,
3. Monotone Mixing Condition (MMC): roughly speaking, there exists n and s^* such that there is a strictly positive probability that an agent who starts from the worst state reaches above s^* after n periods, and that another agent who starts from the best state reaches below s^* after n periods.

The MMC guarantees that there is a sufficient mixing even among the agents in the best and the worst state. Victor calls the condition as the "American dream and American Nightmare" condition.

Using a similar model with private IOUs instead of production, Huggett (1993) shows that all the conditions above are satisfied, thus the stationary distribution exists, is unique, and globally stable.

As we will see, there are two prices to be solved in the model. Therefore, by restricting our attention to the steady state equilibrium, we only need to find two numbers (prices) instead of solving for the future entire path of prices. This makes the problem substantially easier (but still not that easy). In addition, in the simple model of Aiyagari (1994), the two prices can be computed from one number (K, capital stock), because the aggregate labor supply is constant over time, and both prices are functions of only the aggregate capital stock and the aggregate labor supply. Therefore, we only need to find one real number (one of K, r, or w) to compute the steady state equilibrium of the model.

Naturally, the model has been extended so that we can also compute equilibria with time-varying prices. We will see those models later.

Now, let's assume that the prices in the model are constant; the interest rate r and the wage per efficiency unit of labor w are taken as constant. So we drop the time script from the description of the model. The individual agent's problem can be formulated recursively as follows:

Problem 1 (Recursive formulation of agent's problem)

$$V(s,k) = \max_{c,k'} \left\{ u(c) + \beta \sum_{s'} p_{ss'} V(s',k') \right\}$$

subject to

$$k(1+r) + ws = k' + c$$
$$c \geq 0$$
$$k' \geq \underline{k}$$
$$r \text{ and } w \text{ are taken as given}$$

A couple of remarks below:

1. $\underline{k} = 0$ means there is no borrowing allowed. But agents still can save and prepare for the future risks.
2. What is the laxest $\underline{k}$? The laxest $\underline{k}$ should be the level such that the agent can keep consuming positive amount even if all the future realization is the worst one (s_1), assuming that there is a positive probability that agent with any current s will keep receiving s_1 in the entire future.

Look at the budget constraint. If an agent keeps receiving s_1 and keeps borrowing the maximum amount, the agent should keep borrowing exactly $\underline{k}$ in every period and still can sustain positive consumption. Therefore:

$$\underline{k}(1+r) + ws_1 = \underline{k} + c$$

If we solve the equation for c and assume $c \simeq 0$

$$c = \underline{k}r + ws_1 = 0$$

If we solve the inequality for $\underline{k}$ we get:

$$\underline{k} = -\frac{ws_1}{r}$$

Now we have the formula for $\underline{k}$ which only depends on w, s_1 and r. As long as the agent avoids a non-zero probability of consuming negative consumption, the borrowing must be restricted to be above $\underline{k}$ as characterized above. This is called the *natural borrowing limit* by Aiyagari. Aiyagari calls any borrowing limit above this natural borrowing limit as an ad-hoc borrowing limit.

3. If we solve the problem, we can obtain the optimal decision rules $k' = g_k(s,k)$ and $c = g_c(s,k)$. These decision rules are valid for all agents, because agents are characterized by (s,k) and thus the problem is shared by all the agents (possibly with different (s,k)).

4. When we solve the agent's optimization problem, and we prove the existence of the stationary distribution, we need to set an upperbound to the choice k'. Let's call the bound $\overline{k}$. What we should do is to set an arbitrary ad-hoc $\overline{k}$ before solving the problem, and make sure that the bound is not binding for any type of agent. If that's the case, having an ad-hoc arbitrary upperbound $\overline{k}$ does not make any difference from the problem without an upperbound. If it turns out that $\overline{k}$ is binding for some type of agent (typically, the one with highest k and the highest s which is s_n), try higher $\overline{k}$ and solve the optimization problem again. You can start from really high $\overline{k}$ but that can be a loss of precious computer resources.

5. At the same time, it is shown that if there exists an upperbound of the space of capital $\overline{k}$ that is not binding, it has to be the case that $\beta(1+r) < 1$. If either β or r is too high, agents optimally keep accumulating the capital stock, and there will be no non-binding upperbound. It can be also shown that actually it is the case in equilibrium. If the supply of capital explodes, it is impossible that the capital market clears.

Now we are ready to define the steady state equilibrium for the model.

Definition 1 (Steady state recursive competitive equilibrium)
A recursive competitive equilibrium consists of prices r, w, value function $V(s,k)$, optimal decision rules $g_k(s,k)$, $g_c(s,k)$, type distribution of agents, $x(s,k)$, and the aggregate capital stock K and labor supply L, such that:

1. ***Agents' optimization:*** *Given prices r and w, the value function $V(s,k)$ is a solution to the agent's optimization problem, and $g_k(s,k)$ and $g_c(s,k)$ are the associated optimal decision rules.*

2. ***Firm's optimization:*** *Prices r and w satisfy the following marginal conditions:*

$$r = F_K(K, L) - \delta$$

$$w = F_L(K, L)$$

3. ***Consistency:*** *$x(s, k)$ is the stationary distribution associated with the transition function implied by the optimal decision rule $g_k(s, k)$ and the law of motion for s.*

4. ***Aggregation:*** *Aggregate capital stock and labor supply are consistent with the stationary distribution $x(s, k)$, or:*

$$K = \int_X k \, dx$$

$$L = \int_X s \, dx$$

4 Solving Agent's Optimization Problem

There are a lot of ways to solve the problem of the agent, but let me present two ways, one using the value function iteration, and the other using the policy function equation (implied by the Euler equation).

Algorithm 1 (Solving agent's problem: value function iteration)

1. *Set an arbitrary upperbound for the space of capital $\overline{k}$ to make the domain of the value function compact. It is necessary to avoid using extrapolation (which is usually problematic).*

2. *Choose a method to approximate the value function. Candidates are* (i) *finite element methods,* (ii) *weighted residual methods, and* (iii) *discretization. Please see notes for each of the methods. As an example, suppose we use piecewise-linear interpolation with m knots. The value function can be stored as n times m* matrix, *$V(s_i, k_j)$.*

3. *Set the tolerance parameter ϵ.*

4. *Set an initial guess for the value function. Denote it as $V^0(s, k)$. In the case of piecewise-linear approximation, what is actually guessed is a matrix $V^0(s_i, k_j)$.*

5. *For $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$, solve the Bellman equation, using $V^0(s, k)$ for the future value, and assign the value on the left hand side of the Bellman equation as $V^1(s_i, k_j)$. You can use one-dimensional optimization algorithm, like the golden section search.*

6. *Compare the initial guess and the updated value. One way is to check if the following holds:*

$$\text{max error} = \max_{i,j} |V^1(s_i, k_j) - V^0(s_i, k_j)| < \epsilon$$

   If the above inequality holds, we are done. Otherwise update the value by $V^0(s_i, k_j) = V^1(s_i, k_j)$ for all i and j, and go back to step 5.

A couple of remarks:

1. You can use various tricks to speed up the algorithm, most importantly, Howard policy iteration algorithm.

2. If $r \geq \frac{1}{\beta} - 1$, it might not be the case that there is a non binding upperbound for the agent's problem. Be careful not to guess a high r.

Algorithm 2 (Solving agent's problem: policy function iteration)

1. *Set an arbitrary upperbound for the space of capital $\overline{k}$ to make the domain of the value function compact. It is necessary to avoid using extrapolation (which is usually problematic).*

2. *Choose a method to approximate the optimal decision rule. Candidates are* (i) *finite element methods,* (ii) *weighted residual methods. We cannot use discretization because limiting the choice to the finite set of grid points means we cannot find a root of the Euler Equation almost surely. Please see notes for each of the methods. As an example, suppose we use piecewise-linear interpolation with m knots. The optimal decision function for the future capital can be stored as n times m matrix, $g_k(s_i, k_j)$.*

3. *Set the tolerance parameter ϵ.*

4. *Set an initial guess for $g_k(s,k)$. Denote it as $g_k^0(s,k)$. In the case of piecewise-linear approximation, what is actually guessed is a matrix $g_k^0(s_i,k_j)$.*

5. *For $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$, find the capital stock k' which solves the Euler equation combined with $g_k^0(s,k)$ for the savings decision for the next period. k' is the updated value for the optimal savings $g_k^1(s_i,k_j)$. In particular, this step contains the following sub-steps:*

   (a) *Define the function $\zeta(k')$ as follows:*

   $$\zeta(k') = u'(s_i w + (1+r)k_j - k') - \beta \sum_{s'} p_{s_i s'} u'(s'w + (1+r)k' - g_k^0(s', k'))(1+r)$$

   This is basically the Euler equation, with $g_k^0(s,k)$ is used for the savings decision in the next period (that's why we do not have k'' in the equation). Notice that, given s_i and k_j, this is an equation with one unknown (k').

   (b) *In case $\zeta(\underline{k}) > 0$, the solution should be at the corner. Assign $g_k^1(s_i,k_j) = \underline{k}$ and done.*

   (c) *In case $\zeta(\overline{k}) < 0$, the upperbound is binding, which is inappropriate. Stop.*

   (d) *Otherwise, find a root for $\zeta(k') = 0$, and assign the root as the new optimal saving $g_k^1(s_i,k_j)$. We can use a variety of one-dimensional root-finding subroutine, like bisection or false positive method to find the root of $\zeta(k')$.*

6. *Compare the initial guess and the updated value. One way is to check if the following holds:*

   $$\text{max error} = \max_{i,j} |g_k^1(s_i,k_j) - g_k^0(s_i,k_j)| < \epsilon$$

   If the above expression is true, we are done. Otherwise update the decision rule function using $g_k^0(s_i,k_j) = g_k^1(s_i,k_j)$ for all i and j, and go back to step 5.

Regarding the comparison between the two methods:

1. Root-finding can be more efficient than optimization in one-dimensional case.
2. However, you have to have differentiability of the value function in order to be able to use the Euler equation. For many problems with discrete choice set, that's not the case, so the methods which rely on the differentiability of the value function are not available.

5 Finding a Steady State Equilibrium

5.1 Using the Excess Demand Function

Now we can solve for the optimal decision rules, given prices r and w. In addition, we can find the stationary distribution associated with an optimal decision rule $g_k(s,k)$ and the exogenously given law of motion for idiosyncratic shocks. The last piece of the algorithm is a routine to find the market clearing prices (r and w).

First of all, there is a property of the model which makes the algorithm simpler. That is, the aggregate labor supply can be computed exogenously in the stationary equilibrium. If we store the stationary distribution regarding the idiosyncratic shocks as $\{\mu_i\}_{i=1}^n$, the aggregate labor supply in the stationary distribution, L, can be computed as follows:

$$L = \sum_{i=1}^{n} s_i \mu_i$$

Also remember that both r and w are determined only by the aggregate labor supply L and the aggregate capital stock K. It means that, if we have a guess for r, we can back up the capital demand K which is consistent with the current guess r. Furthermore, we can compute w which is consistent with the fixed L and the guess r (because we can back up K from r). In other words, we only need to iterate on either one of K, r, or w, instead of iterating for both prices.

How do we iterate on r? Let us start by defining the excess supply function $\Phi(r)$.

Definition 2 (Excess demand function $\Phi(r)$)

1. *Given r, the function does the followings.*
2. *Compute the aggregate capital demand K^d, which satisfies the following:*

$$r = F_K(K^d, L) - \delta$$

where L is the aggregate labor supply which we have already computed independently.

3. *Compute w which is consistent with r and K^d as follows:*

$$w = F_L(K^d, L)$$

4. *Given r and w, solve the agents' optimization problem. Denote the optimal decision rules as $g_k(s,k)$ and $g_c(s,k)$.*

5. *Using the obtained optimal decision rule $g_k(s,k)$ and the exogenous law of motion for the idiosyncratic shock s, find the stationary distribution $x(s,k)$.*

6. *Compute the aggregate supply of capital K^s as follows:*

$$K^s = \int_X k \, dx$$

7. *Excess demand can be defined as:*

$$\Phi(r) = K^d - K^s$$

In an equilibrium, it has to be the case that $\Phi(r) = 0$. In other words, the equilibrium interest rate r^* is the root of the function $\Phi(r)$. Finding an equilibrium (equilibrium prices) boils down to finding a root of the excess demand function $\Phi(r)$.

To find a root, we can use various root-finding routines. Intuitive and robust routines, such as bisection method and false positive method, requires a bracket, which is a closed interval which contains a root of the function inside. What are the candidates for the bounds of r?

As for the lowerbound, $-\delta$ is the lowerbound, as the rate of return of capital (after depreciation) approaches to $-\delta$ as K^d goes to infinity. So we know that the excess demand function is for sure a large positive if r is close to $-\delta$. Therefore, we can set our lowerbound as a number above $-\delta$.

as for the upperbound, a slightly smaller number that $\frac{1}{\beta} - 1$ is the upperbound. It is generally true that agents in the model accumulate asset without bound if $\beta(1+r) = 1$, or $r = \frac{1}{\beta} - 1$. It means that the excess demand function goes to negative infinity when r approaches to $\frac{1}{\beta} - 1$. Therefore, we can set an upperbound for r as a number slightly lower than $\frac{1}{\beta} - 1$.

Now we have the function for which we want to find a root, and upper and lower bounds which contains a root. So all we need to do is to throw the function $\Phi(r)$ together with the bracket into one of the one-dimensional root-finding algorithm.

5.2 Iteration over the Space of Capital

There is an equivalent, but possibly more intuitive way to find the equilibrium of the model. The method iterates over the capital stock levels instead of interest rates.

Algorithm 3 (Computing the steady state equilibrium)

1. *Set the tolerance parameter ϵ.*

2. *Guess the aggregate capital demand K^0.*

3. *Compute the interest rate and the wage using the following marginal conditions:*

$$r = \theta K^{0\theta-1} L^{1-\theta} - \delta$$

$$w = (1-\theta) K^{0\theta} L^{-\theta}$$

4. *Using r and w that are computed, solve the problem of the agent, and obtain the optimal decision rule for capital stock holding $g_k(s,k)$.*

5. *Using $g_k(s,k)$ and the law of motion for s, find the stationary distribution $x(s,k)$.*

6. *Compute the aggregate capital supply K^1 as follows:*

$$K^1 = \int_X k \; dx$$

7. *Compare K^0 and K^1. If $|K^0 - K^1| < \epsilon$, done. Otherwise, update the guess for the capital stock demand using:*

$$K^0_{new} = \lambda K^1 + (1-\lambda)K^0$$

with $\lambda \in (0,1]$. Even though makes the algorithm faster, it is risky to use $\lambda = 1$. The guess for K might oscillate between two values.

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