Problem Set 3: Quantitative Economics (ECON 8185-001)

Teerat Wongrattanapiboon

December 16, 2021

State Space System

Assume that we have the following state space system:

$$y_t = Z\alpha_t + \epsilon_t \tag{1}$$
$$\alpha_t = T\alpha_{t-1} + \eta_t, \tag{2}$$

where ϵ_t, η_t are assumed to be mean 0 white noise. Specifically,

$$\mathbb{E}[\epsilon_t] = 0, \mathbb{E}[\eta_t] = 0, \mathbb{E}[\epsilon_t \eta_s] = 0$$
$$\mathbb{E}[\epsilon_t \epsilon_t'] = H$$
$$\mathbb{E}[\eta_t \eta_t'] = Q.$$

The variables in y_t are observed and will be used to construct estimates, while the variables in α_t are unobserved and need to be estimated. Equation (1) is the **measurement** or observer equation and equation (2) is the **transition** equation for the unobserved state vector.

The first step here is to convert a model of interest into a state-space representation. Once we have done that, the **Kalman Filter** is a system of two equations, the **predicting** equation and the **updating** equation. That is,

$$\hat{\alpha}_{t|t-1} = T\hat{\alpha}_{t-1} \tag{3}$$
$$\hat{\alpha}_{t+1|t} = T\hat{\alpha}_{t|t-1} + K_t v_t \tag{4}$$
$$y_t = \underbrace{Z\hat{\alpha}_{t|t-1}}_{\hat{y}_{t|t-1}} + v_t \tag{5}$$

where $\hat{\alpha}_{t|t-1}$ is an estimate of unobserved state vector. The matrix K_t is **Kalman gain**, and $v_t = y_t - \hat{y}_{t|t-1} = y_t - Z\hat{\alpha}_{t|t-1}$ is an **innovation**, which is the difference between the observable and its prediction. Note that the prediction of y_t (i.e. $\hat{y}_{t|t-1}$) is entirely based on the prediction of state $\alpha_{t|\hat{t-1}}$ as $\hat{y}_{t|t-1} = Z\alpha_{t|\hat{t-1}}$. Hence, Kalman Filter evolves around predicting $\hat{\alpha}_{t|t-1}$ and updating the prediction of the state vector $\hat{\alpha}_{t+1|t}$.

At time t, before we have actually observed y_t, our information set consists of lagged values of $y : (y_0, y_1, \ldots, y_{t-1})$. Based on this information, we begin by making the best guess for the underlying state. From the law of motion of state we have:

$$\hat{\alpha}_{t|t-1} = \mathbb{E}[\alpha_t|(y_0, y_1, \ldots, y_{t-1})] = T\mathbb{E}[\alpha_{t-1}|(y_0, y_1, \ldots, y_{t-1})] + \underbrace{\mathbb{E}[\eta_t|(y_0, y_1, \ldots, y_{t-1})]}_{0}.$$

Define $\hat{\alpha}_{t-1} = \mathbb{E}[\alpha_{t-1}|y_0, y_1, \ldots, y_{t-1}]$. We can then write $\hat{\alpha}_{t|t-1} = T\hat{\alpha}_{t-1}$, which is a function of lagged values of y. At time t, we also have the variance of prediction error as:

$$P_{t-1} = \mathbb{E}[(\alpha_{t-1} - \hat{\alpha}_{t-1})(\alpha_{t-1} - \hat{\alpha}_{t-1})'].$$

The conditional variance of prediction error is given by:

$$\begin{aligned}
P_{t|t-1} &= \mathbb{E}[(\alpha_t - \hat{\alpha}_{t|t-1})(\alpha_t - \hat{\alpha}_{t|t-1})'] \\
&= \mathbb{E}[(T\alpha_{t-1} + \eta_t - T\hat{\alpha}_{t-1})(T\alpha_{t-1} + \eta_t - T\hat{\alpha}_{t-1})'] \\
&= \mathbb{E}[(T(\alpha_{t-1} - \hat{\alpha}_{t-1}) + \eta_t)(T(\alpha_{t-1} - \hat{\alpha}_{t-1}) + \eta_t)'] \\
&= T\mathbb{E}[(\alpha_{t-1} - \hat{\alpha}_{t-1})(\alpha_{t-1} - \hat{\alpha}_{t-1})']T' + \mathbb{E}[\eta_t \eta_t'] \\
&= TP_{t-1}T' + Q,
\end{aligned}$$

which follows from the fact that $(\alpha_{t-1} - \hat{\alpha}_{t-1})$ and η_t are uncorrelated. The conditional variance of the innovation is given by:

$$\begin{aligned} F_t &= \mathbb{E}[v_t v_t'] \\ &= \mathbb{E}[(y_t - Z\hat{\alpha}_{t|t-1})(y_t - Z\hat{\alpha}_{t|t-1})'] \\ &= \mathbb{E}[(Z\alpha_t - Z\hat{\alpha}_{t|t-1})(Z\alpha_t - Z\hat{\alpha}_{t|t-1})'] \\ &= ZP_{t|t-1}Z' + H, \end{aligned}$$

and the covariance of v_t with the estimation error is:

$$\begin{aligned} G_t &= \mathbb{E}[(y_t - \hat{y}_{t|t-1})(\alpha_t - \hat{\alpha}_{t|t-1})'] \\ &= \mathbb{E}[(Z\alpha_t - Z\hat{\alpha}_{t|t-1} + \epsilon_t)(\alpha_t - \hat{\alpha}_{t|t-1})'] \\ &= ZP_{t|t-1}. \end{aligned}$$

Note that before we actually observe y_t, conditional on the information set $(y_0, y_1, \ldots, y_{t-1})$, all the matrices $P_{t-1}, P_{t|t-1}, F_t, G_t$, are known.

For jointly Normal random variable X and Y with the variance-covariance matrix given by:

$$\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{XY} & \Sigma_{YY} \end{bmatrix},$$

and the conditional expectation of Y given X is given by:

$$\mathbb{E}[Y|X] = \mathbb{E}[Y] + \Sigma_{XY}\Sigma_{XX}^{-1}(X - \mathbb{E}[X]),$$

and the conditional variance is given by:

$$Var[Y|X] = \mathbb{E}[(Y - E[Y|X])^2] = \Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}.$$

It then follows from the above formula that:

$$\mathbb{E}[\alpha_t|(y_0, y_1, \ldots, y_{t-1}), y_t] = \mathbb{E}[\alpha_t|(y_0, y_1, \ldots, y_{t-1})] + Cov(\alpha_t, y_t)Var(y_t)^{-1}(y_t - \mathbb{E}[y_t|(y_0, y_1, \ldots, y_{t-1})]).$$

$$\begin{aligned} Cov(\alpha_t, y_t) &= \mathbb{E}[(\alpha_t - E[\alpha_t|(y_0, y_1, \ldots, y_{t-1})])(y_t - E[y_t|(y_0, y_1, \ldots, y_{t-1})])'] \\ &= \mathbb{E}[(\alpha_t - \hat{\alpha}_{t|t-1})(y_t - \hat{y}_{t|t-1})'] \\ &= P_{t|t-1}Z' \\ Var(y_t) &= \mathbb{E}[(y_t - E[y_t|(y_0, y_1, \ldots, y_{t-1})])(y_t - E[y_t|(y_0, y_1, \ldots, y_{t-1})'] \\ &= \mathbb{E}[(y_t - \hat{y}_{t|t-1})(y_t - \hat{y}_{t|t-1})'] \\ &= ZP_{t|t-1}Z' + H. \end{aligned}$$

Hence, the updating equation is given by:

$$\begin{aligned} \hat{\alpha}_t &= \hat{\alpha}_{t|t-1} + P_{t|t-1}Z'(ZP_{t|t-1}Z' + H)^{-1}(y_t - \hat{y}_{t|t-1}) \\ &= \hat{\alpha}_{t|t-1} + G_t'F_t^{-1}(y_t - \hat{y}_{t|t-1}). \end{aligned}$$

Multiplying both sides of previous equation by T we get:

$$\hat{\alpha}_{t+1|t} = T\hat{\alpha}_{t|t-1} + Kv_t$$

where $K = TP_{t|t-1}Z'(ZP_{t|t-1}Z' + H)^{-1} = TG_t'F_t^{-1}$, and $v_t = y_t - \hat{y}_{t|t-1}$. Similarly, using the conditional variance formula, we obtain

$$P_t = P_{t|t-1} - G_t'F_t^{-1}G_t.$$

Kalman Filter

1. Begin with a guess for α_0, P_0. We can set α_0 to be the unconditional mean of state vector, and P_0 to be the stationary P that solves $P = TPT' + Q$. Set $\hat{\alpha}_{1|0} = T\alpha_0$, and $P_{1|0} = TP_0T' + Q$.

2. Compute:

    (a) $\hat{y}_{1|0} = Z\hat{\alpha}_{1|0}$

    (b) $v_1 = y_1 - \hat{y}_{1|0}$

    (c) $F_1 = ZP_{1|0}Z' + H, G_1 = ZP_{1|0}, K_1 = TG_1'F_1^{-1}$

    (d) $\hat{\alpha}_{2|1} = T\hat{\alpha}_{1|0} + K_1 v_1$

    (e) $P_{2|1} = T(P_{1|0} - G_1'F_1^{-1}G_1)T' + Q$

    (f) Go back to step (a) increasing the time order by 1 unit.

3. Continue the recursion in this fashion to get sequence of $\{\hat{\alpha}_{t|t-1}\}$, $\{i_t\}$ and $\{\hat{y}_{t|t-1}\}$.

Parametric Estimation using Log Likelihood

To compute parameters, we maximize the following log-likelihood:

$$\ln L = \sum_t \left\{ -\frac{n}{2}\ln 2\pi - \frac{1}{2}\ln|F_t| - \frac{1}{2}v_t'F_t^{-1}v_t \right\}.$$

Question 1

First, let's write each process in state space representation.

1. AR(1) : $x_t = \rho x_{t-1} + \epsilon_t$
   State: $x_t = [\rho]x_{t-1} + \epsilon_t$
   Observation: $x_t = [1]x_t$

2. AR(2) : $x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2} + \epsilon_t$
   State: $\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \epsilon_t$
   Observation: $x_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix}$

3. MA(1) : $x_t = \epsilon_t + \rho\epsilon_{t-1}$

   State: $\begin{bmatrix} \epsilon_t \\ \epsilon_{t-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{t-1} \\ \epsilon_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \epsilon_t$
   Observation: $x_t = \begin{bmatrix} 1 & \rho \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \epsilon_{t-1} \end{bmatrix}$

4. Random Walk : $x_t = \mu_t + \epsilon_t$, $\mu_t = \mu_{t-1} + \eta_t$
   State: $\mu_t = [1]\mu_{t-1} + \eta_t$
   Observation: $x_t = [1]\mu_t + \epsilon_t$.

Results

AR(1)

We set $\rho = 0.6$ and $\sigma = 0.5$, simulate the AR(1) process for 100 periods, and apply the Kalman filter algorithm to obtain the following plot:

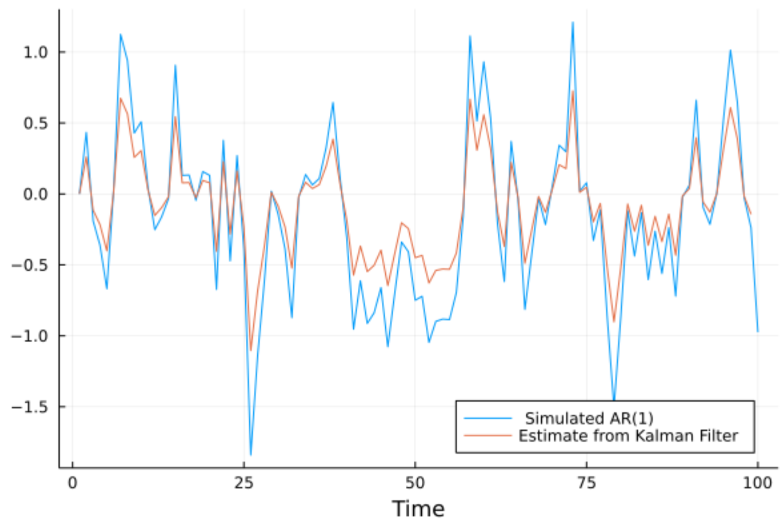

Figure 1: Simulated and Filtered AR(1)

We then use the log-likelihood function given above to compute the likelihood of observing this simulated process, and maximize the likelihood with respect to the AR(1) parameters to obtain their best estimates. The estimated parameters are $\hat{\rho} = 0.61$ and $\hat{\sigma} = 0.48$, which are pretty close to the actual parameters.

AR(2)

We set $\rho_1 = 0.3$, $\rho_2 = 0.4$, $\sigma = 0.5$, simulate the AR(2) process for 100 periods, and apply the Kalman filter algorithm to obtain the following plot:

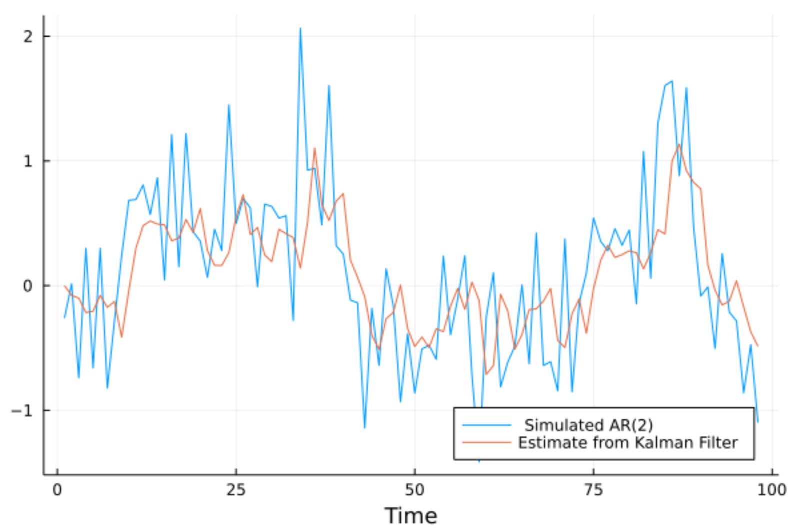

Figure 2: Simulated and Filtered AR(2)

The estimated parameters are $\hat{\rho}_1 = 0.26$ and $\hat{\rho}_2 = 0.48$, and $\hat{\sigma} = 0.54$.

MA(1)

We set $\rho = 0.7$, $\sigma = 0.8$, simulate the MA(1) process for 100 periods, and apply the Kalman filter algorithm to obtain the following plot:

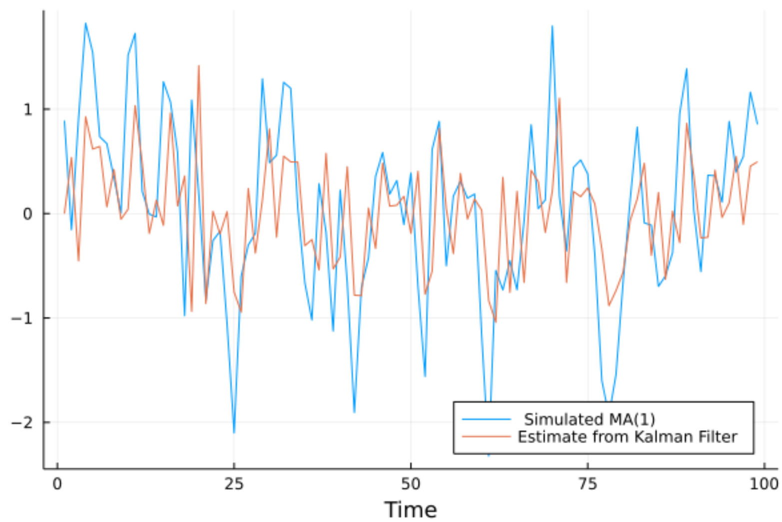

Figure 3: Simulated and Filtered MA(1)

The estimated parameters are $\hat{\rho} = 0.61$ and $\hat{\sigma} = 0.73$.

Random Walk

We set $\sigma_\epsilon = 0.8$, $\sigma_\eta = 0.6$, simulate the random walk process for 100 periods, and apply the Kalman filter algorithm to obtain the following plot:

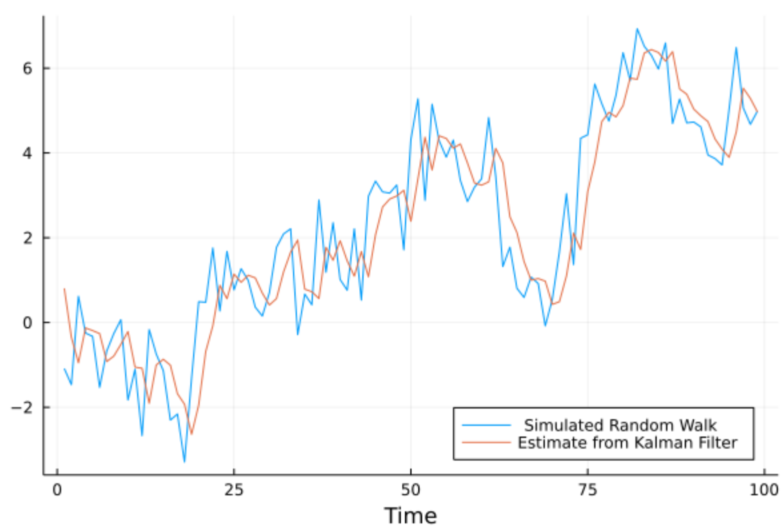

Figure 4: Simulated and Filtered Random Walk

The estimated parameters are $\hat{\sigma}_\epsilon = 0.70$ and $\hat{\sigma}_\eta = 0.65$.