Class notes: Advanced Topics in Macroeconomics

Topic: FEM Test Cases for Aiyagari-McGrattan

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In class, we considered two test cases (which were fully described in the technical appendix for Aiyagari and McGrattan (2003) available on canvas). In the first test case, we assume that the household solves

$$\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{subject to} \quad c_t + a_{t+1} = (1+r)\, a_t + w.$$

This specification assumes that there is no uncertainty ($e_t = 1$); therefore, wages are constant. The dynamic program for this example involves the following form for Bellman's equation:

$$v(x) = \max_{0 \le y \le Rx+w} \{u(Rx + w - y) + \beta v(y)\}, \tag{1}$$

where $y_t = x_{t+1}$ is the asset position next period and $R = 1 + r$ is the gross return. A conjectured solution is as follows:

$$\begin{cases} y = 0 & \text{if } x \in [0, m_1], \\ y = \frac{-m_1^2}{m_2 - m_1} + \frac{m_1}{m_2 - m_1}\, x & \text{if } x \in (m_1, m_2]\,, \\ y = \frac{\left(m_3 m_1 - m_2^2\right)}{m_3 - m_2} + \frac{m_2 - m_1}{m_3 - m_2}\, x & \text{if } x \in (m_2, m_3]\,, \\ \vdots \end{cases} \tag{2}$$

where m_j, $j = 1, 2, \ldots$, will be calculated below. Note that the solution assumes that if $x = m_{j+1}$, then $y = m_j$.

The Lagrangian for the maximization in the right-hand side of Eq. (1) is given by

$$L = u(xR + w - y) + \beta v(y) + p(Rx + w - y) + qy,$$

where p and q are multipliers. The first-order conditions for this problem are

$$\begin{aligned} &-u'(Rx + w - y) + \beta v'(y) - p + q = 0, \\ &p \ge 0, \quad Rx + w - y \ge 0, \quad p(Rx + w - y) = 0, \\ &q \ge 0, \quad y \ge 0, \quad qy = 0. \end{aligned} \tag{3}$$

If we assume that the conjecture above is correct, then when $x \in [0, m_1)$ we have the $y \geq 0$ constraint binding. Therefore, if we assume that $Rx + w > 0$, then it must be true that $y < Rx + w$, $p = 0$, and

$$v'(0) = \frac{1}{\beta} u'(Rx + w) - \frac{q}{\beta} \leq \frac{1}{\beta} u'(Rx + w).$$

Furthermore, from Bellman's equation, we get

$$v(x) = u(Rx + w) + \beta v(0), \quad \text{for } x \in (0, m_1) \quad \text{and } v(0) = \frac{\beta}{1-\beta} u(w),$$

and taking derivatives, we get

$$v'(x) = Ru'(Rx + w) < \frac{1}{\beta} u'(Rx + w), \tag{4}$$

since $\beta R < 1$.

Consider next the interval $(m_1, m_2]$. The conjectured solution is such that in this interval, the constraint $y \geq 0$ is not binding. If we assume that $Rx + w > y$, then the first-order conditions imply

$$v'(y) = \frac{1}{\beta} u'(Rx + w - y). \tag{5}$$

If $y = 0$ at $x = m_1$, then

$$v'(0) = \frac{1}{\beta} u'(Rm_1 + w). \tag{6}$$

Using Eq. (4) evaluated at $x = 0$ and Eq. (6), we get

$$u'(Rm_1 + w) = \beta R u'(w),$$

which gives us an equation for m_1. For example, if $u(c) = c^{1-\mu}/(1-\mu)$, then

$$m_1 = \frac{w\left(1 - (\beta R)^{\frac{1}{\mu}}\right)}{(\beta R)^{\frac{1}{\mu}} R},$$

and $y = 0$ in the interval $[0, m_1]$.

Now we want to compute the asset function for the next interval $(m_1, m_2]$. If the conjecture in Eq. (2) is correct, then Eq. (5) holds, as does

$$v'(x) = Ru'(Rx + w - g(x)), \tag{7}$$

which is the derivative of the value function once y is replaced by the optimal policy $y = g(x)$. The conjecture assumes that $y = m_1$ when $x = m_2$, and by Eq. (5) and Eq. (7), we have,

$$u'(Rm_2 + w - m_1) = \beta Ru'(Rm_1 + w).$$

Note that this equation can be solved for m_2. If we follow the same logic for the remaining m's, we find that, in general,

$$u'(Rm_{j+1} + w - m_j) = \beta Ru'(Rm_j + w - m_{j-1}), \quad j = 1, \ldots, \text{ and } m_0 = 0. \tag{8}$$

Thus, given m_1 and m_2, we can compute m_3 and so on.

What we have done is conjectured a solution and derived the functions analytically. It is easy to show that the solution is, in fact, piecewise linear and that the conjecture is correct.

Now we consider the finite element approximation. Let $\beta = 0.95$, $w = 1.0$, $r = 0.02$, $u(c) = c^{1-\mu}/(1-\mu)$, and $\mu = 3$. We can use the formula in Eq. (8) to derive the exact solution. In a series of figures (see the document online), we plot the true solution and the finite element approximations. We first use a grid that can produce an exact match to the true solution. Then we vary the grids and show how the approximation looks when the grids don't match. We also show results with and without the imposition of the boundary condition.

The second test case is relevant for computing the distribution of assets, which has discontinuities throughout because of the Markov chain used for modeling the idiosyncratic shocks. To mimic this, we suppose that the productivities can take on two possible values and the decision functions are given by

$$\alpha(x, i) = \begin{cases} \max(0, -0.25 + x), & \text{if } i = 1 \\ 0.5 + 0.5x, & \text{if } i = 2, \end{cases}$$

with $\pi_{1,1} = \pi_{2,2} = 0.8$. Recall that we want to compute $H(x, i) = Pr(x_t < x \,|\, e_t = e(i))$, which solves:

$$H\left(x, i\right) = \sum_{j=1}^{m} \pi_{j,i} H\left(\alpha^{-1}\left(x, j\right), j\right) I\left(x \geq \alpha\left(0, j\right)\right), \tag{9}$$

where π is the transition matrix for the Markov chain governing earnings and I is an indicator function (i.e., $I(x > y)$ is equal to one if $x > y$ and is equal to zero otherwise).

It is relatively easy to show that the following equations must hold for this simple example:

$$\begin{aligned}
H\left(0, 1\right) &= 0.8H\left(0.25, 1\right), \\
H\left(0, 2\right) &= 0.2H\left(0.25, 1\right), \\
H\left(0.25, 1\right) &= 0.8H\left(0.5, 1\right), \\
H\left(0.25, 2\right) &= 0.2H\left(0.5, 1\right), \\
H\left(0.5, 1\right) &= 0.8H\left(0.75, 1\right) + 0.2H\left(0, 2\right), \\
H\left(0.5, 2\right) &= 0.2H\left(0.75, 1\right) + 0.8H\left(0, 2\right), \\
H\left(0.75, 1\right) &= 0.8H\left(1.0, 1\right) + 0.2H\left(0.5, 2\right), \\
H\left(0.75, 2\right) &= 0.2H\left(1.0, 1\right) + 0.8H\left(0.5, 2\right), \\
H\left(0.875, 1\right) &= 0.8H\left(1.125, 1\right) + 0.2H\left(0.75, 2\right), \\
H\left(0.875, 2\right) &= 0.2H\left(1.125, 1\right) + 0.8H\left(0.75, 2\right), \\
H\left(1.0, 1\right) &= 0.5, \\
H\left(1.0, 2\right) &= 0.5.
\end{aligned}$$

If we assume that $H(x, i) = 0.5$ for $x > 1$, then the above expressions can easily be solved. We can first determine $H(0, j)$, $H(0.25, j)$, $H(0.5, j)$, and $H(0.75, j)$ for j =1,2 by solving $Ax = b$, where

$$A = \begin{bmatrix} 1 & -.8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -.8 & -.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -.2 & 0 \\ 0 & -.2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -.2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -.2 & -.8 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -.8 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ .4 \\ 0 \\ 0 \\ 0 \\ .1 \end{bmatrix}.$$

The solution is $H(0,1) = 0.225$, $H(0.25,1) = 0.282$, $H(0.5,1) = 0.352$, $H(0.75,1) = 0.426$, $H(0,2) = 0.056$, $H(0.25,2) = 0.070$, $H(0.5,2) = 0.130$, and $H(0.75,2) = 0.204$. Note that we can back out the other points from these solutions by applying the formula in Eq. (9). Again, the technical appendix online shows comparisons of the exact and approximate solutions for different choices of the grid and the order of the approximation.