Class notes: Advanced Topics in Macroeconomics

Topic: Application of Vaughan to BCA

Date: October 4, 2021

Let's continue the discussion from last week and consider a simpler version of Homework 2, which will be a version used in the paper "Business Cycle Accounting" (*Econometrica*, 2007).

Recall that problem is to compute equilibria of the following growth model:

$$\max_{\{c_t, x_t, \ell_t\}} E \sum_{t=0}^{\infty} \beta^t \{\left(c_t \ell_t^{\psi}\right)^{1-\sigma} / (1-\sigma)\} N_t$$

$$\begin{aligned}
\text{subj. to}\quad & c_t + (1+\tau_{xt})\, x_t = r_t k_t + (1-\tau_{ht})\, w_t h_t + \kappa_t \\
& N_{t+1} k_{t+1} = [(1-\delta)\, k_t + x_t]\, N_t \\
& h_t + \ell_t = 1 \\
& S_t = P S_{t-1} + Q \epsilon_t, \quad S_t = [\log z_t, \tau_{ht}, \tau_{xt}, \log g_t] \\
& c_t, x_t \geq 0 \quad \text{in all states,}
\end{aligned}$$

where $N_t = (1+\gamma_n)^t$ and firm technology is $Y_t = K_t^{\theta} (Z_t L_t)^{1-\theta}$. Factors are paid their marginal products r and w, and revenues in excess of government purchases of goods and services, $N_t g_t$, are lump-sum transferred to households in amount κ_t. The stochastic shocks hitting this economy affect technology, tax rates, and government spending and the stochastic processes are modeled as a VAR(1) process. The resource constraint in this economy is $Y_t = N_t (c_t + x_t + g_t)$.

In this case, the (detrended) first order conditions are:

$$\begin{aligned}
& \hat{c}_t + (1+\gamma_z)(1+\gamma_n)\, \hat{k}_{t+1} - (1-\delta)\, \hat{k}_t + \hat{g}_t = \hat{y}_t = \hat{k}_t^{\theta} (z_t h_t)^{1-\theta} \\
& \psi \hat{c}_t / (1-h_t) = (1-\tau_{ht})(1-\theta) \left(\hat{k}_t / h_t\right)^{\theta} z_t^{1-\theta} \\
& \hat{c}_t^{-\sigma} (1-h_t)^{\psi(1-\sigma)} (1+\tau_{xt}) \\
& \qquad = \beta (1+\gamma_z)^{-\sigma} E_t \hat{c}_{t+1}^{-\sigma} (1-h_{t+1})^{\psi(1-\sigma)} \left(\theta \hat{k}_{t+1}^{\theta-1} (z_{t+1} h_{t+1})^{1-\theta} + (1-\delta)(1+\tau_{xt+1})\right)
\end{aligned}$$

If we substitute for $\hat{c}_t$ using the resource constraint and then log-linearize the conditions around the steady state, we get:

$$
\begin{aligned}
0 &= E_t\{a_1\tilde{k}_t + a_2\tilde{k}_{t+1} + a_3\tilde{h}_t + a_4\tilde{z}_t + a_5\tilde{\tau}_{ht} + a_6\tilde{g}_t\} \\
0 &= E_t\{b_1\tilde{k}_t + b_2\tilde{k}_{t+1} + b_3\tilde{k}_{t+2} + b_4\tilde{h}_t + b_5\tilde{h}_{t+1} + b_6\tilde{z}_t + b_7\tilde{\tau}_{xt} \\
&\qquad + b_8\tilde{g}_t + b_9\tilde{z}_{t+1} + b_{10}\tilde{\tau}_{xt+1} + b_{11}\tilde{g}_{t+1}\},
\end{aligned}
$$

where $\tilde{k}_t = \log\hat{k}_t/\log\hat{k}_{ss}$, $\tilde{h}_t = \log h_t/\log h_{ss}$, $\tilde{z}_t = \log z_t/\log z_{ss}$, $\tilde{\tau}_{ht} = \tau_{ht}/\tau_{hss}$, $\tilde{\tau}_{xt} = \tau_{ht}/\tau_{hss}$, and $\tilde{g}_t = \log\hat{g}_t/\log\hat{g}_{ss}$. These equations can be stacked up as follows:

$$
\begin{aligned}
0 = E_t\Bigg\{ & \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_3 & b_5 \end{bmatrix}}_{=A_1} \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{k}_{t+2} \\ \tilde{h}_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_4 \end{bmatrix}}_{=A_2} \begin{bmatrix} \tilde{k}_t \\ \tilde{k}_{t+1} \\ \tilde{h}_t \end{bmatrix} \\
& + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & 0 & a_6 & 0 & 0 & 0 & 0 \\ b_6 & 0 & b_7 & b_8 & b_9 & 0 & b_{10} & b_{11} \end{bmatrix} \begin{bmatrix} S_t \\ S_{t+1} \end{bmatrix} \Bigg\}
\end{aligned}
$$

For this problem, we are looking for a solution of the form:

$$
\begin{aligned}
\tilde{k}_{t+1} &= A\tilde{k}_t + BS_t \\
Z_t &= CX_t + DS_t \\
S_t &= PS_{t-1} + Q\epsilon_t
\end{aligned}
$$

where $Z_t = [\tilde{k}_{t+1}, \tilde{h}_t]'$ and S_t are the stochastic exogenous variables.

Applying Vaughan, we compute the *generalized* eigenvalues and eigenvectors for matrices A_1 and $-A_2$ because A_1 is not invertible. (Note that if A_1 were invertible, then we would compute eigenvalues and eigenvectors of $-A_1^{-1}A_2$ as we did in the LQ problem.) Let V be the eigenvectors and Λ is a diagonal matrix with eigenvalues, so that

$$A_2V = -A_1V\Lambda.$$

Sort the eigenvalues in Λ and the associated columns in the matrix of eigenvectors V so that the eigenvalue inside the unit circle is in the (1,1) position of Λ. Then,

$$
\begin{aligned}
A &= V_{11}\Lambda(1,1)V_{11}^{-1} \\
C &= V_{21}V_{11}^{-1}
\end{aligned}
$$

Note that A is 1×1 and C is 2×1.

Once we have A and C, we plug them into the system above—along with $S_{t+1} = PS_t + Q\epsilon_{t+1}$—and we can form a linear system in B and D. Since the policy function for the first element of Z_t is the same as A, B, we'll just consider the policy rule for $\tilde{h}_t$, namely:

$$\tilde{h}_t = C_2 \tilde{k}_t + D_2 S_t$$

Plugging the policy rules into the first-order equations yields the following:

$$
\begin{aligned}
0 &= (a_1 + a_2 A + a_3 C_2)\, \tilde{k}_t + (a_2 B + a_3 D_2 + [a_4, a_5, 0, a_6])\, S_t \\
0 &= \left(b_1 + b_2 A + b_3 A^2 + b_4 C_2 + b_5 C_2 A\right) \tilde{k}_t \\
&\quad + (b_2 B + b_3 AB + b_3 BP + b_4 D_2 + b_5 C_2 B + b_5 BP + [b_6, 0, b_7, b_8] + [b_9, 0, b_{10}, b_{11}]\, P)\, S_t
\end{aligned}
$$

We can check that the coefficients on $\tilde{k}_t$ are zero at the solution. (Notice that they do not depend on B or D_2.) That leaves eight unknowns, namely four elements of B and four elements of D_2 and eight linear equations, namely the coefficients on S_t, which must be set equal to zero at the solution:

$$
\begin{aligned}
0 &= a_2 B + a_3 D_2 + [a_4, a_5, 0, a_6] \\
0 &= b_2 B + b_3 AB + b_3 BP + b_4 D_2 + b_5 C_2 B + b_5 BP + [b_6, 0, b_7, b_8] + [b_9, 0, b_{10}, b_{11}]\, P.
\end{aligned}
$$

We just have to stack these eight equations and solve the linear system. To do this, we'll need to use the fact that $vec(FGH) = (H' \otimes F) vec(G)$ for any matrices F, G, and H.

One last thing to note. If we set $\tau_{ht} = 0$, $\tau_{xt} = 0$, and $g_t = 0$, we are back to the simple case of Homework 1. Thus, we have a test case for the codes.