# 1 Economies with Idiosyncratic Risk and Incomplete Markets: Stationary Equilibrium

We are interested in building a class of models whose equilibria feature a nontrivial endogenous distribution of income and wealth across agents in order to analyze questions such as:

1. What is the fraction of aggregate savings due to the precautionary motive?

2. How much of the observed wealth inequality can one explain through uninsurable earnings variation across agents?

3. What are the redistributional implications of various fiscal policies? How are inequality and welfare affected by such policies? It is key to have an equilibrium model to answer policy questions, because changes in policy affect equilibrium prices.

4. Can we generate a reasonable equity premium (i.e., excess return of stocks over a risk-free bonds), once we introduce a risky asset? This model has an additional source of uninsurable risk, which is idiosyncratic, and if this risk comoves with aggregate risk households require an even larger risk premium to hold stocks.

5. How large are the welfare losses from individual-level labor market risk (e.g., unemployment)?

The model is constructed around three building blocks: 1) the "income-fluctuation problem", 2) the aggregate neoclassical production function, and 3) the equilibrium of the asset market. We focus on the *stationary equilibrium*, for now, i.e., an economy without aggregate shocks.

**Income fluctuation problem**– This is the problem we studied in the previous chapter. Individuals are subject to exogenous income shocks. These shocks are not fully insurable because of the lack of a complete set of Arrow-Debreu contingent claims. There is only a risk-free asset (i.e., and asset with non-state contingent rate of return) in which the individual can save/borrow, and that the individual faces a borrowing (liquidity) constraint. A continuum of such agents subject to different shocks will give rise to a wealth distribution. Integrating wealth holdings across all agents will give rise to an *aggregate supply of capital.*

**Aggregate production function:** profit maximization of the competitive representative firm operating a CRS technology will give rise to an *aggregate demand for capital.*

**Equilibrium in the asset market:** When we let demand and supply interact in an asset market, an *equilibrium interest rate* will arise endogenously. Notice that if a full set of Arrow-Debreu contingent claims were available, the economy would collapse to a representative agent model with a stationary amount of savings such that $(1+r)\beta = 1$. With uninsurable risk, the supply of savings is larger ($r$ is lower) because of precautionary saving, and consequently $(1+r)\beta < 1$. We like this because we know that this is a necessary condition for the income-fluctuation problem to have a bounded consumption sequence as solution.

## 1.1 The Economy

**Demographics:** the economy is populated with a continuum of measure one of infinitely lived, ex-ante identical agents.

**Preferences:** the individual has time-separable preferences over streams of consumption

$$U(c_0, c_1, c_2, ...) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where the period utility function $u(c_t)$ satisfies $u' > 0, u'' < 0$ and the discount factor $\beta \in (0,1)$. The expectation is over future sequences of shocks, conditional to the realization at time 0. The individual supplies labor inelastically.

**Endowment:** each individual has a stochastic endowment of efficiency units of labor $\varepsilon_t \in E \equiv \{\varepsilon^1, \varepsilon^2, ..., \varepsilon^{N-1}, \varepsilon^N\}$. The shocks follow a Markov process with transition probabilities $\pi(\varepsilon', \varepsilon) = \Pr(\varepsilon_{t+1} = \varepsilon' \mid \varepsilon_t = \varepsilon)$. Shocks are *iid* across individuals. We assume a law of large numbers to hold, so that $\pi(\varepsilon', \varepsilon)$ is also the fraction of agents in the population subject to this particular transition.[1] We assume that the Markov transition is well-behaved, so there is a unique invariant distribution $\Pi^*(\varepsilon)$. As a result, the aggregate

[1] There are some tricky issues with laws of large numbers in this setting. Please, refer to Judd (1985) and Uhlig (1996) for a discussion.

endowment of efficiency units

$$H_t = \sum_{i=1}^{N} \varepsilon_i \Pi^* \left( \varepsilon_i \right), \text{ for all } t$$

is constant over time, i.e., there is no aggregate uncertainty. Note in particular, that $H_t$ is exogenously determined.

**Budget constraint:** For individual $i$ at time $t$, the budget constraint reads

$$c_t + a_{t+1} = \left(1 + r_t\right) a_t + w_t \varepsilon_t,$$

where $c_t$ is current consumption, $a_{t+1}$ is next period wealth, $(1 + r_t)$ is the gross interest rate and $w_t$ is the wage rate at period $t$. Wealth is held in the form of a one-period risk-free bond whose price is one and whose return, next period, will be $(1 + r_{t+1})$, independently of the individual state (i.e., $r_{t+1}$ does not depend on the realization of $\varepsilon_{t+1}$). In this sense, the asset $a$ is non state-contingent.

**Liquidity constraint:** At every $t$, agents face the borrowing limit

$$a_{t+1} \geq -b$$

where $b$ is exogenously specified. Alternatively, we could assume agents face the "natural" borrowing constraint, which is the present value of the lowest possible realization of her future earnings.

**Technology:** The representative competitive firm produces with CRS production function $Y_t = F(K_t, H_t)$ with decreasing marginal returns in both inputs and standard Inada conditions. Physical capital depreciates geometrically at rate $\delta \in (0, 1)$.

**Market structure:** final good market (consumption and investment goods), labor market, and capital market are all competitive.

**Aggregate resource constraint:** The aggregate feasibility condition in this economy reads:

$$F(K_t, H_t) = C_t + I_t = C_t + K_{t+1} - (1 - \delta) K_t,$$

where capital letters denote aggregate variables.

## 1.2 Stationary Equilibrium

We are now ready to define the stationary equilibrium of this economy through the concept of *Recursive Competitive Equilibrium* (RCE). Most of the requirement of this RCE definition will be standard (agents optimize, markets clear). Moreover, in the stationary equilibrium of this economy we require the distribution of agents across states to be invariant.[2] This probability measure will permanently reproduce itself. It is in this sense that the economy is in a rest-point, i.e., a steady state.

### 1.2.1 Some Mathematical Preliminaries

The individual is characterized by the pair $(a, \varepsilon)$ –the individual states. Let $\lambda$ be the distribution of agents over states. We would like this object to be a *probability measure,* so we need to define an appropriate mathematical structure. Let $\bar{a}$ be the maximum asset holding in the economy, and for now assume that such upper bound exists. Define the compact set $A \equiv [-b, \bar{a}]$ of possible asset holdings, and the countable set $E$ as above.

Let the the state space $S$ be the Cartesian product $A \times E$, and let the $\sigma$-algebra $\Sigma_s$ be defined as $B_A \otimes P(E)$ where $B_A$ is the Borel sigma-algebra on $A$ and and $P(E)$ is the power set of $E$. The space $(S, \Sigma_s)$ is a measurable space. Let $\mathcal{S} = (\mathcal{A} \times \mathcal{E})$ be the typical subset of $\Sigma_s$. For any element of the sigma algebra $\mathcal{S} \in \Sigma_s$, $\lambda(\mathcal{S})$ is the measure of agents in the set $\mathcal{S}$.

How can we characterize the way individuals transit across states over time? I.e. how do we obtain next period distribution, given this period distribution? We need a transition function. Define $Q((a, \varepsilon), \mathcal{A} \times \mathcal{E})$ as the (conditional) probability that an individual with current state $(a, \varepsilon)$ transits to the set $\mathcal{A} \times \mathcal{E}$ next period, formally $Q : S \times \Sigma_s \to [0, 1]$, and

$$Q((a, \varepsilon), \mathcal{A} \times \mathcal{E}) = I_{\{a'(a,\varepsilon) \in \mathcal{A}\}} \sum_{\varepsilon' \in \mathcal{E}} \pi(\varepsilon', \varepsilon) \tag{1}$$

where $I_{\{.\}}$ is the indicator function, and $a'(a, \varepsilon)$ is the optimal saving policy. Then $Q$ is our transition function and the associated $T^*$ operator yields

$$\lambda_{n+1}(\mathcal{A} \times \mathcal{E}) = T^*(\lambda_n) = \int_{A \times E} Q((a, \varepsilon), \mathcal{A} \times \mathcal{E}) \, d\lambda_n. \tag{2}$$

[2] However, individuals move up and down in the earnings and wealth distribution, so "social mobility" can be meaningfully defined. Recall that with complete markets, there is no social mobility: initial rankings persist forever.

where I have used the notation $d\lambda_n$ as short for $\lambda_n(da, d\varepsilon)$.

Let us now re-state the problem of the individual in recursive form, i.e. through dynamic programming

$$\begin{aligned} v(a,\varepsilon;\lambda) &= \max_{c,a'} \left\{ u(c) + \beta \sum_{\varepsilon' \in E} \pi(\varepsilon', \varepsilon) v(a', \varepsilon'; \lambda) \right\} \\ & s.t. \\ c + a' &= R(\lambda)\, a + w(\lambda)\, \varepsilon \\ a' &\geq -b \end{aligned} \tag{3}$$

where, for clarity, we have made explicit the dependence of prices from the distribution of agents (although, strictly speaking this dependence is *redundant* in a stationary environment and it can be omitted since the probability measure $\lambda$ is constant). We are now ready to proceed to the definition of equilibrium.

### 1.2.2 Definition of Stationary RCE

A **stationary recursive competitive equilibrium** is a value function $v : S \to \mathbb{R}$; policy functions for the household $a' : S \to \mathbb{R}$, and $c : S \to \mathbb{R}_+$; firm's choices $H$ and $K$; prices $r$ and $w$; and, a stationary measure $\lambda^*$ such that:

- given prices $r$ and $w$, the policy functions $a'$ and $c$ solve the household's problem (3) and $v$ is the associated value function,
- given $r$ and $w$, the firm chooses optimally its capital $K$ and its labor $H$, i.e., $r + \delta = F_K(K, H)$ and $w = F_H(K, H)$,
- the labor market clears: $H = \int_{A \times E} \varepsilon d\lambda^* = \sum_{i=1}^{N} \varepsilon_i \Pi^*(\varepsilon_i)$,
- the asset market clears: $K = \int_{A \times E} a'(a, \varepsilon) d\lambda^*$,
- the goods market clears:[3] $\int_{A \times E} c(a, \varepsilon) d\lambda^* + \delta K = F(K, H)$,
- for all $(\mathcal{A} \times \mathcal{E}) \in \Sigma_s$, the invariant probability measure $\lambda^*$ satisfies

  $$\lambda^*(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} Q((a, \varepsilon), \mathcal{A} \times \mathcal{E})\, d\lambda^*,$$

  where $Q$ is the transition function defined in (1).

[3] This condition is redundant by Walras law.

## 1.3 Existence and Uniqueness of the Stationary Equilibrium

Characterizing the conditions under which an equilibrium exists and is unique boils down, like in every general equilibrium model, to show that the excess demand function (i.e., a function of the price) in each market is continuous, strictly monotone and intersects "zero". Equilibrium in the labor market is trivial: aggregate labor supply is exogenous and labor demand is strictly decreasing in wages. By Walras law, if we prove that the equilibrium in the asset market exists and is unique, we are done.

**Demand for capital**– Consider first the aggregate demand of capital. From the optimal choice of the firm, we obtain

$$K(r) = F_k^{-1}(r+\delta).$$

Note that for $r=-\delta$, then $K \to +\infty$, while for $r \to +\infty$, $K \to 0$. Moreover, the supply of capital is a continuous, strictly decreasing function of the interest rate $r$. For example, if $F(K,H) = K^{\alpha}H^{1-\alpha}$, then

$$K(r) = \left(\frac{\alpha H}{\delta + r}\right)^{\frac{1}{1-\alpha}}.$$

**Supply of capital**– If we could show that the aggregate supply function

$$A(r) = \int_{A\times E} a'(a,\varepsilon;r)d\lambda_r^*$$

is continuous in $r$ and crosses the aggregate demand function, then we would prove existence. Suppose first $(1+r)\beta = 1$, i.e. $r = \frac{1}{\beta} - 1$, then we know by the super-martingale converge theorem that the aggregate supply of assets goes to infinity, i.e. $A(\frac{1}{\beta}-1) \to \infty$. For $r=-1$ the individual would like to borrow until the limit, as every unit of capital saved will vanish, so $A(-1) \to -b$.[4]

Thus, if $A(r)$ is continuous, it will cross the aggregate demand and we would have an existence result.

Standard results in dynamic programming ensures us that, if $u$ is continuous, $u' > 0$ and $u'' < 0$, the solution to the household problem is unique and that the policy function $a'(a,\varepsilon;r)$ is continuous in $r$ (by the Theorem of the Maximum). To establish continuity

[4]For values of the interest rate $r < 0$, the agent may still want to hold some wealth for precautionary reasons. It depends on the exact parametrization.

of $\lambda_r^*$ in $r$ we would like to apply Theorem 12.13 in SLP. However, the Theorem requires existence and uniqueness of an invariant distribution for given $r$.

To establish the existence and uniqueness of the invariant distribution, from Theorem 12.12 in SLP, we know that we need to verify four properties: compactness of the state space and $Q$ with Feller property imply existence. $Q$ satisfying monotonicity and the monotone-mixing condition (MMC) yields uniqueness. We verify these properties one at the time.

- *Compactness:* When $\beta(1+r) < 1$ and preferences display decreasing absolute risk aversion (or asymptotitcally bounded relative risk aversion) we showed that an upper bound $\bar{a}$ on the asset space exists–recall our discussion in the previous chapter– so the state space is a compact subset of $\mathbb{R}^2$.

- *Feller property of $Q$:* The Feller property requires that the associated operator $T$ maps continuous and bounded functions into themselves. For $Q$ it is easily verified, because $a'(a, \varepsilon)$ is continuous and bounded since the domain of the asset space is compact. In particular, we can apply Theorem 9.14 in SLP which states that if $E$ is countable and $P(E)$ is the sigma-algebra on $E$, $A$ is compact and $a'$ is continuous, then $Q$ has the Feller property.

- *Monotonicity of $Q$:* Monotonicity of $Q$ requires that for every increasing function $f$, the function $Tf$ is also increasing. Suppose that the Markov process has two possible states, $E \equiv (\varepsilon_L, \varepsilon_H)$. Assume that $\pi(\varepsilon_H, \varepsilon_H) \geq \pi(\varepsilon_H, \varepsilon_L)$ and $\pi(\varepsilon_L, \varepsilon_L) \geq \pi(\varepsilon_L, \varepsilon_H)$, i.e. the Markov chain is monotone.[5] Recall that $a'(a, \varepsilon)$ is an increasing function. Then it is easy to see that $Q$ is monotone. Let $f(a', \varepsilon')$ be an increasing function. Applying the definition in SLP, we want to show that the conditional expectation
$$h(a, \varepsilon) = Tf = \sum_{\varepsilon' \in E} \int_A f(a', \varepsilon') Q((a, \varepsilon), da' \times \varepsilon')$$

[5] A Markov chain $\varepsilon$ is monotone iff, for any increasing function $f(\varepsilon')$ the conditional expectation
$$\mathbb{E}[f(\varepsilon') | \varepsilon] = \sum_{\varepsilon'} f(\varepsilon') \pi(\varepsilon', \varepsilon) d\varepsilon'$$
is increasing in $\varepsilon$. Note that this restriction on the Markov chain, with multiple states, corresponds to positive autocorrelation in the income process.

is monotonically increasing. This is easy to see. Intuitively, a higher pair $(a, \varepsilon)$ increases the probability of being in state $(a', \varepsilon') > (a, \varepsilon)$ next period. Thus, more weight is put on the region of the domain where $f$ is high (since $f$ is increasing).

- *MMC:* Suppose the household starts from $(\bar{a}, \varepsilon_{\max})$ and receives a long stream of the worst realization of the shock $\varepsilon_{\min}$. If the $\varepsilon$ process is stationary (i.e., mean reverting) then, she will keep decumulating wealth until she reaches some neighborhood of the lower bound. The reason for decumulation is that the household knows that this income realization is well below average, his permanent income is higher and consumption is dictated by permanent income. Suppose now that the household starts with $(-b, \varepsilon_{\min})$ and receives a long stream of the best shock $\varepsilon_{\max}$. Then, she will accumulate wealth until she reaches some neighborhood of the upper bound. The reason for accumulation is similar: the household realizes that this good realization is "transitory" and her expected income is below the current income, so she saves a fraction of these lucky draws.

At this point, we can apply Theorem 12.13 in SLP. This proves existence of the equilibrium.

If, in addition, we could show that $A(r)$ is strictly increasing, we would prove uniqueness. Unfortunately, there are no results on the monotonicity of the aggregate supply of capital with respect to $r$, so uniqueness is never guaranteed. Intuitively, a higher $r$ has both income and substitution effects on savings: the relative dominance between the two could switch at a certain level of assets, so $a'(a, \varepsilon; r)$ may not be monotone. Even if we make sure that preferences are such that one of the two effects always dominates, it is very hard to assess what a change in $r$ does to the distribution of assets.

One can use the computer to plot aggregate supply as a function of the interest rate on a fine grid for a reasonably large range of values of $r$ to check its slope.

*FIGURE*

This figure plots the equilibrium in the asset market graphically.

## 1.4 An Algorithm for the Computation of the Equilibrium

How do we compute, in practice, this equilibrium? The algorithm that can be used is a fixed point algorithm over the interest rate.

1. Fix an initial guess for the interest rate $r^0 \in \left(-\delta, \frac{1}{\beta} - 1\right)$, where these bounds follow from our previous discussion. The interest rate $r^0$ is our first candidate for the equilibrium (the superscript denotes the iteration number).

2. Given the interest rate $r^0$, obtain the wage rate $w(r^0)$ using the CRS property of the production function (recall that $H$ is given exogenously with inelastic labor supply).

3. Given prices $(r^0, w(r^0))$, you can now solve the dynamic programming problem of the agent (3) to obtain $a'(a, \varepsilon; r^0)$ and $c(a, \varepsilon; r^0)$. We described several solution methods.

4. Given the policy function $a'(a, \varepsilon; r^0)$ and the Markov transition over productivity shocks $\pi(\varepsilon', \varepsilon)$, we can construct the transition function $Q(r^0)$ and, by successive iterations over (2), we obtain the fixed point distribution $\lambda(r^0)$, conditional on the candidate interest rate $r^0$.

    (a) The easiest method to implement this step, in practice, is by simulation of a large number of households $N$ (say 10,000) and track them over time, like survey data do. Initialize each individual in the sample with a pair $(a_0, \varepsilon_0)$ and, using the decision rule $a'(a, \varepsilon)$ and a random number generator that replicates the Markov chain $\pi(\varepsilon', \varepsilon)$, update their pair of individual states at every period $t$.

    (b) For every $t$, compute a set of cross-sectional moments $J_t^N$ which summarize the distribution of assets (e.g., mean, variance, various percentiles). Stop when $J_t^N$ and $J_{t-1}^N$ are close enough. At that point, the cross-sectional distribution has converged. We know that for any given $r$, a unique invariant distribution will be reached for sure.

5. Compute the aggregate demand of capital $K(r^0)$ from the optimal choice of the firm who takes as given $r^0$, i.e.

$$K(r^0) = F_k^{-1}(r^0 + \delta)$$

6. Compute the integral

$$A\left(r^{0}\right)=\int_{A\times E}a'(a,\varepsilon;r^{0})d\lambda(a,\varepsilon;r^{0})$$

which gives the aggregate supply of assets. Clearly, this can be easily done by exploiting the model-generated data from the invariant distribution obtained in step 4. It's just an average over the artificial cross-section of households.

7. Compare $K\left(r^{0}\right)$ with $A\left(r^{0}\right)$ to verify the asset market clearing condition. If $A\left(r^{0}\right) > (<)\, K\left(r^{0}\right)$, then the next guess of the interest rate should be lower (higher), i.e. $r^{1} < (>)\, r^{0}$. To obtain the new candidate $r^{1}$ a good choice is, for example,

$$r^{1}=\frac{1}{2}\left\{r^{0}+\left[F_{K}(A\left(r^{0}\right),H)-\delta\right]\right\}$$

This method is called bi-section method. Note that $r^{0}$ and $F_{K}(A\left(r^{0}\right),H)-\delta$ are, by construction, on opposite sides of the steady-state interest rate $r^{*}$.

8. Update your guess to $r^{1}$ and go back to step 1). Keep iterating until one reaches convergence of the interest rate, i.e., until

$$\left|r^{n+1}-r^{n}\right|<\varepsilon,$$

for $\varepsilon$ small. Typically, you need less than 10 iterations for convergence.

9. All the equilibrium statistics of interest, like aggregate savings, inequality measures, etc. can be computed from the simulated data in step 4.

## 1.5 Calibration of the model

To solve the model numerically, one needs first to choose values for the parameters. Here's some guidance on how to pick values. Suppose you set the model's period to one year.

**Technology**– With Cobb-Douglas production function, pick the capital share $\alpha$ to be equal to 1/3. Set the depreciation rate $\delta$ to 6%.

**Preferences**– Typically, we work with CRRA utility. Let $\gamma$ be the coefficient of relative risk aversion. Typical values, in this type of applications, range between 1 and 5. As for the discount rate $\beta$, it should be chosen so that the aggregate wealth-income ratio replicates the one for the U.S. economy which is around 3.[6] However, this means that the parameter is not calibrated externally, but internally which is computationally painful. So, you could do the following. Imagine that you're in complete markets, then you know that

$$\begin{aligned}
\alpha K^{\alpha-1}H^{1-\alpha}-\delta &= \left(\frac{1}{\beta}-1\right)\Rightarrow\alpha\left(\frac{Y}{K}\right)-\delta=\frac{1}{\beta}-1\Rightarrow \\
\beta &= \frac{1}{1+\alpha\left(\frac{Y}{K}\right)-\delta}=\frac{1}{1+0.33\,(0.33)-0.06}=0.951.
\end{aligned}$$

In other words, this value for $\beta$ would give you a $K/Y$ ratio of 3 in complete markets. With incomplete markets the same $\beta$ gives you a slightly larger capital-output ratio because of the extra precautionary capital accumulation, so one should set $\beta$ slightly smaller.

**Labor income process**– We want to calibrate the labor endowment shocks to replicate the typical dynamics of individual earnings in the U.S. economy. The right source of data for this purpose are panel-data with information on labor income, such as the *Panel Study of Income Dynamics*. A decent approximation to U.S. individual earnings dynamics is an AR(1) process like

$$\ln y_t=\rho\ln y_{t-1}+\varepsilon_t,\qquad \text{with } \varepsilon_t\sim N\left(0,\sigma_\varepsilon\right)$$

where the autocorrelation coefficient is $\rho=0.95$ and the standard deviation of the shocks is $\sigma_\varepsilon=0.20$ at a annual frequency. More sophisticated estimates include a transitory component to capture measurement error, as well as less persistent shocks, and a fixed individual component to capture the effect of education, ability, etc.

**Borrowing constraint**– If the natural borrowing constraint is not a good choice for the problem at hand, one could calibrate the borrowing constraint in order to match, say, the fraction of agents with negative net worth which is around 13% in the U.S. economy. The difficulty is that this strategy requires, again, an internal calibration.

## 1.6 Notes

This class of model with a continuum if agents facing individual income shocks and trading a risk-free asset (money, in the original model) was introduced by Bewley (1986).

[6] To be precise, the size of the $K/Y$ ratio depends on whether we include residential capital. If we do, then the $K/Y$ ratio is closer to 4.

Laitner (1992) studies one version of this economy with altruism. Huggett (1993) analyzes the equilibrium in an endowment economy where agents trade an asset in zero net supply. Aiyagari (1994) generalizes the model to a production economy with an aggregate production function (the model described here). Huggett (1996) presents an OLG version of this model. The chapter by Rios-Rull in the book edited by Cooley (1995) contains a careful description of how to compute equilibria in this class of economies. So does Section 7 in Herr and Maussner.