Problem Set 4: Quantitative Economics (ECON 8185-001)

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Aiyagari Model with Finite Element Method

Our problem is

$$\max \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[c_t^{1-\mu} - 1\right] / (1-\mu)$$

$$\text{s.t. } c_t + a_{t+1} = wl_t + (1+r)a_t$$

$$c_t \geq 0, a_t \geq 0,$$

where l_t is assumed to be i.i.d. and well approximated by a Markov chain. In this note, we assume

$$l_{t+1} = \begin{cases} l_{low} = 0.2, \pi = 1/2 \\ l_{high} = 0.9, \pi = 1/2 \end{cases}$$

$$\Pi_{ij} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix},$$

Parameter	Value
β	0.96
μ	3
w	1
$1+r$	1.02
ζ	50000

The recursive formulation of our problem is

$$V(a_t) = \max_{a_{t+1}} \frac{(wl_{i,t} + (1+r)a_t - a_{t+1})^{1-\mu} - 1}{1-\mu} + \beta \mathbb{E}[V(a_{t+1})]$$

and the FOC and envelope condition imply

$$(wl_t + (1+r)a_t - a_{t+1})^{-\mu} = \beta(1+r)\mathbb{E}[(wl_{t+1} + (1+r)a_{t+1} - a_{t+2})^{-\mu}]$$

$$(wl_t + (1+r)a_t - a_{t+1})^{-\mu} - \frac{\beta(1+r)}{2}\left[(wl_{t+1}^L + (1+r)a_{t+1} - a_{t+2})^{-\mu} + (wl_{t+1}^H + (1+r)a_{t+1} - a_{t+2})^{-\mu} + \zeta \min(a_{t+1}, 0)^2\right] = 0$$

First we want to represent the saving policy function with linear basis functions. Specifically, we write

$$g^a(a_t, l_t; \theta) = \sum_{i=1}^{N} \theta_i^l \psi_i(a) = \begin{cases} \theta_1^L \psi_1(a) + ... + \theta_N^L \psi_N(a), \text{ for low} \\ \theta_1^H \psi_1(a) + ... + \theta_N^H \psi_N(a), \text{ for high,} \end{cases}$$

where N is the number of elements and each ψ represents a small tent defined in adjacent points of asset grids. That is, for $i \in \{2, 3, ..., N-1\}$, we have

$$\psi_i(a) = \begin{cases} \frac{a - a_{i-1}}{a_i - a_{i-1}} & \text{if } a_{i-1} \leq a \leq a_i \\ \frac{a_{i+1} - a}{a_{i+1} - a_i} & \text{if } a_i \leq a \leq a_{i+1} \\ 0 & \text{else .} \end{cases}$$

For elements at the boundary, we have

$$\psi_1(a) = \begin{cases} \frac{a_2 - a}{a_2 - a_1} & \text{if } a_1 \le a \le a_2 \\ 0 & \text{else ,} \end{cases}$$

$$\psi_N(a) = \begin{cases} \frac{a - a_N}{a_N - a_{N-1}} & \text{if } a_{N-1} \le a \le a_N \\ 0 & \text{else .} \end{cases}$$

Below is the plot these linear bases for the grid $A = [0, 1, 3, 6]$:

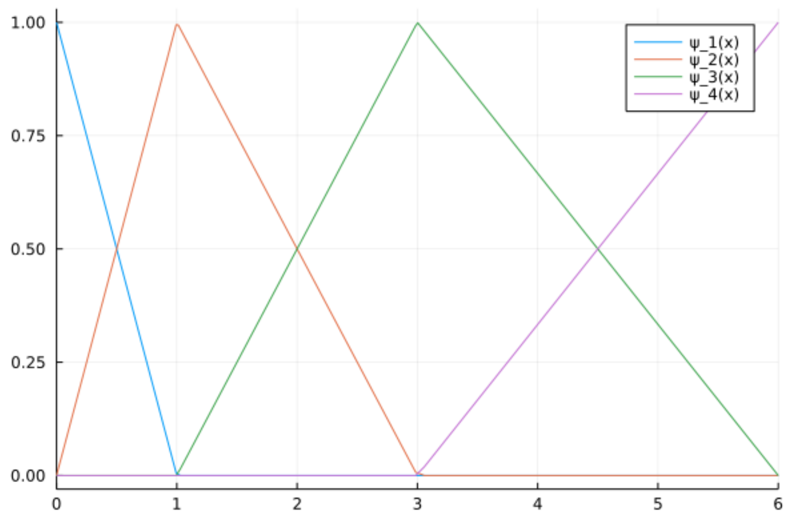

Figure 1: Piecewise Linear Basis Functions

With this saving function, we can also obtain consumption policy function as

$$g^c(a_t, l_t; \theta) = Ra_t + wl_t - g^a(a_t, l_t; \theta).$$

From the Euler equation above, we can write the residual function as

$$\begin{aligned} R(a_t, l_t|\theta) = & (wl_t + (1+r)a_t - g^a(a_t, l_t|\theta))^{-\mu} \\ & - \frac{\beta(1+r)}{2}\Big[(wl^L_{t+1} + (1+r)g^a(a_t, l_t|\theta) - g^a(g^a(a_t, l_t|\theta), l^L_{t+1}|\theta))^{-\mu} \\ & + (wl^H_{t+1} + (1+r)g^a(a_t, l_t|\theta) - g^a(g^a(a_t, l_t|\theta), l^H_{t+1}|\theta))^{-\mu} + \zeta \min(g^a(a_t, l_t|\theta), 0)^2\Big]. \end{aligned}$$

Then we use the Galerkin choice of weight functions to evaluate the weighted residual of the residual and set it equals to zero. That is,

$$\int \phi_i(a)R(a, l|\theta)da = \int \psi_i(a)R(a, l|\theta)da = 0, \qquad \text{where } i = 1, ..., n, l = \{\text{low}, \text{high}\}$$

Algorithm and Results

To summarize, the main algorithm is as follows:

1) Make a “proper” guess of $\theta^j = \{\theta^L_1, ..., \theta^L_n, \theta^H_1, ..., \theta^L_n\}$.

2) Over the 2-dimensional grids of a and l, evaluate $g^a(a, l; \theta)$ and $R(a, l|\theta)$.

3) Stack all numerical integrals $\int \psi_i(a)R(a, l|\theta)da$ into a system of equations $G(\theta^j) = 0$ and apply Newton update until θ converges.

Note that I construct 8 points on the asset grid between 0 and 6, with more points clustered near zero to capture binding borrowing constraint. The plot of the estimated saving policy function for two states of labor is as follows:

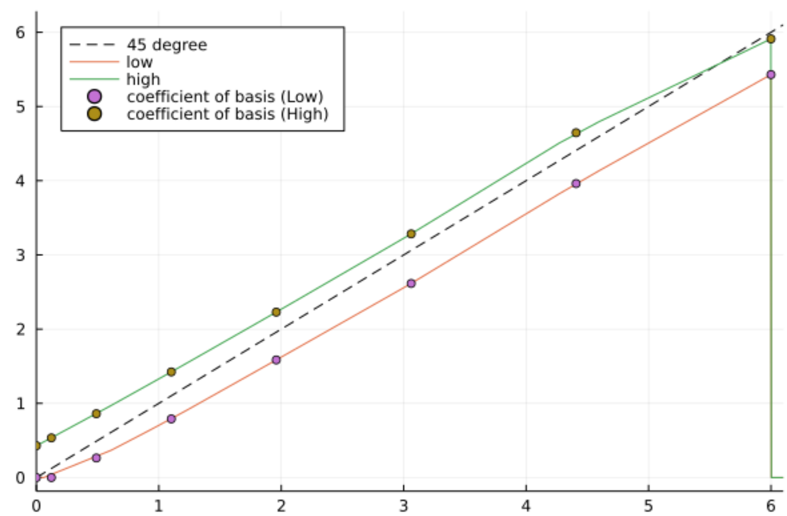

Figure 2: Optimal Saving Function with FEM