Class notes: Advanced Topics in Macroeconomics

Topic: Finite Element Method

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Last class, we considered the class of weighted residual methods. Today, we will work with basis functions that are nonzero on only small regions of the domain of x. The resulting representations of d^n will be piecewise functions (for example, piecewise linear, piecewise quadratic). In the terminology of numerical analysts, we will be applying a *finite element method*. Finite element methods use basis functions that are only nonzero on small regions of the domain of x (for example, the tent functions drawn in Figure 2 of the Marimon-Scott chapter).

The idea behind the finite element method is to break up the domain of x into smaller pieces, use low-order polynomials to get good local approximations for the function d, and then piece the local approximations together to get a good global approximation. In effect, one can think of the finite element method as a piecewise application of a weighted residual method. Thus, to apply a finite element method, we first divide the domain into smaller nonoverlapping subdomains. On each of the subdomains, we construct a local approximation to the function d. For the problem described last class, Ω is one-dimensional, and therefore, division of Ω means coming up with some partition, say, $[x_1, x_2, \ldots, x_n]$ on $\mathbb{R}$. Each subinterval $[x_i, x_{i+1}]$ is called an *element*.

Suppose, for example, that we want to represent d as a piecewise linear function; that is, over each element, we assume that the approximation is of the form $a + bx$. Suppose also that we want the function d to be continuous on the whole domain Ω. How would we construct basis functions $\psi_i(x)$ so that we can write d^n as:

$$d^n(x;\theta) = \psi_0(x) + \sum_{i=1}^{n} \theta_i \psi_i(x). \tag{1}$$

The first step is to assign *nodes* on the element. For the finite element method, nodes are points on an element that are used to define the geometry of the element and to uniquely define the order of the polynomial being used to approximate the true solution over the

element. Since we are assuming that an element is some interval $[x_i, x_{i+1}]$, two nodes – in particular, the two endpoints x_i and x_{i+1} – are needed to define the geometry. And only two points are needed to uniquely define a linear function. Therefore, the nodes on a one-dimensional element with linear bases are the two endpoints of the element.

The second step in constructing the basis functions is to assume that the undetermined coefficients are equal to the approximate solution at the nodal points. Assume that the numbering of elements and nodes is such that element i is the interval $[x_i, x_{i+1}]$: the first element is $[x_1, x_2]$, the second element is $[x_2, x_3]$, and so on. Assume also that the approximate solution on element i, $d_i^n(x;\theta)$, satisfies $d_i^n(x_i) = \theta_i$ and $d_i^n(x_{i+1}) = \theta_{i+1}$. In other words, assume that the undetermined coefficients represent the solution at the nodes. The approximation of d on element i, d_i^n, is therefore uniquely given by

$$d_i^n(x;\theta) = \theta_i \psi_i(x) + \theta_{i+1}\psi_{i+1}(x), \quad x \in [x_i, x_{i+1}], \tag{2}$$

where the basis functions are piecewise linear and drawn in Figure 2 of the chapter. Since elements are connected to each other at nodal points on the element boundaries, this choice of basis functions guarantees that the approximation is continuous across elements. Notice also that any linear function (and, hence, any continuous piecewise linear d^n) can be represented with the basis functions.

More specifically, Let the approximate solution be of the form

$$d^n(x;\theta) = \sum_{i=1}^{n} \theta_i \psi_i(x),$$

with $\psi_i(x)$, $i = 1, \ldots, n$ given by linear basis functions:

$$\psi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

To impose the boundary condition $d^n(0;\theta) = 1$, we need to set θ_1 to one. Let's apply a Galerkin method. Therefore, the weight functions are given by the bases $\psi_i(x)$, $i = 1, \ldots, n$.

Suppose that there are three elements with nodes at 0, 1, 3, and 6. Then the residual equation is given by

$$
\begin{aligned}
R(x;\theta) &= \sum_{i=1}^{4} \theta_i \left(\psi_i'(x) + \psi_i(x)\right) \\
&= \begin{cases} \theta_1 \ (-x) + \theta_2 \ (1+x) & \text{if } x \in [0,1] \\ \theta_2 \left(1 - \frac{1}{2}x\right) + \theta_3 \left(\frac{1}{2}x\right) & \text{if } x \in [1,3] \\ \theta_3 \left(\frac{5}{3} - \frac{1}{3}x\right) + \theta_4 \left(-\frac{2}{3} + \frac{1}{3}x\right) & \text{if } x \in [3,6]. \end{cases}
\end{aligned} \tag{4}
$$

If we substitute the residual (4) into the weighted integral:

$$
\int_{\Omega} w(x)\, R(x;\theta)\, dx = 0, \tag{5}
$$

with $\phi_i(x) = \psi_i(x)$, then we get the following system of equations:

$$
\begin{aligned}
&\left\{ \int_0^1 \begin{bmatrix} 1-x \\ x \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -x & 1+x & 0 & 0 \end{bmatrix} dx \right. \\
&+ \int_1^3 \begin{bmatrix} 0 \\ \frac{3}{2} - \frac{1}{2}x \\ -\frac{1}{2} + \frac{1}{2}x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 - \frac{1}{2}x & \frac{1}{2}x & 0 \end{bmatrix} dx \\
&\left. + \int_3^6 \begin{bmatrix} 0 \\ 0 \\ 2 - \frac{1}{3}x \\ -1 + \frac{1}{3}x \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{5}{3} - \frac{1}{3}x & -\frac{2}{3} + \frac{1}{3}x \end{bmatrix} dx \right\} \begin{bmatrix} 1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\end{aligned}
$$

or if we compute the integrals,

$$
\begin{bmatrix} -1/6 & 2/3 & 0 & 0 \\ -1/3 & 1 & 5/6 & 0 \\ 0 & -1/6 & 5/3 & 1 \\ 0 & 0 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6}
$$

Note that we need to drop the first equation because we have to impose that $\theta_1 = 1$ for the boundary condition to be satisfied. Recall that the integral equation can be written

as in (5), where in this case, $w(x) = \sum_i \omega_i \psi_i(x)$. The function $w(x)$ must satisfy the homogeneous counterpart of the boundary condition $d(0) = 1$, that is, $w(0) = 0$. For those familiar with the calculus of variations, w is like the variation of the solution and thus must satisfy the homogeneous counterparts of boundary conditions for d. Enforcing the condition $w(0) = 0$ is equivalent to dropping the first equation in (6). Therefore, the system of equations reduces to

$$\begin{bmatrix} 1 & 5/6 & 0 \\ -1/6 & 5/3 & 1 \\ 0 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix},$$

with three equations and three unknowns. In Figure 7 of the Marimon-Scott chapter, I plot the finite element approximation and the exact solution. By construction, the approximate function is piecewise linear.

What is involved if we instead apply the method to the simplest deterministic growth model with inelastic labor? In that case, the decision function is consumption $c^n(k;\theta)$ and the residual is

$$R(k;\theta) = 1 - \beta \frac{U\left(c^n\left(F(k) + (1-\delta)k - c^n(k;\theta)\right)\right)}{U\left(c^n(k;\theta)\right)} \cdot \left(F_k\left(F(k) + (1-\delta)k - c^n(k;\theta)\right) + 1 - \delta\right).$$

Unlike the problem above, the final set of equations will not turn out to be linear but the procedure up to the point of setting up the problem is no different. The weighted residuals are then given by

$$\int \psi_i(k) R(k;\theta)\, dk = 0$$

for $i = 1, \ldots, n$, which can be stacked into a system of equations in the vector of unknowns $\vec{\theta} = [\theta_1, \ldots, \theta_n]'$:

$$G\left(\vec{\theta}\right) = 0$$

where G is n dimensional. From here, a Newton update can be applied:

$$\vec{\theta}^{j+1} = \vec{\theta}^{j} - \left[\frac{\partial G\left(\vec{\theta}\right)}{\partial \vec{\theta}}|_{\vec{\theta}=\vec{\theta}^j}\right]^{-1} G\left(\vec{\theta}^j\right)$$

starting from a guess of $\vec{\theta}^0$ based on our linear or log-linear approximations.