Class notes: Advanced Topics in Macroeconomics

Topic: Mapping to a LQ Framework

Date: September 15, 2021

II. Computing Equilibria (continued)

In class, we continued the discussion of computing equilibria for the very simplest (but empircally-relevant) growth model. The first method was brute force but achieved a non-linear solution. We turn next to a near-linear method that relies importantly on mapping our non-linear problem to a standard LQ problem with linear constraints and a quadratic objective.

Method 2.

Before doing the mapping, consider a general maximization problem with state vector X and control vector u:

$$\begin{aligned} &\max_{\{u_t\}_{t=0}^{\infty}} \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t r\left(X_t, u_t\right) \mid X_0\right] \\ \text{subject to} \quad & X_{t+1} = g\left(X_t, u_t, \epsilon_{t+1}\right) \\ & X_0 \text{ given.} \end{aligned}$$

Here, r is the objective function which is known, g governs the evolution of the state vector and is also known, and ϵ is a vector of shocks affecting this evolution which we'll assume to be iid. During class, we considered a nested version of Homework 1 with inelastic labor and full depreciation of capital so that we could get very concrete about what we are trying to do.

We can approximate the nonlinear prototype problem with a near-linear related problem:

$$\begin{aligned} &\max_{\{u_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left(X_t' Q X_t + u_t' R u_t + 2 X_t' W u_t\right) \\ \text{subject to} \quad & X_{t+1} = A X_t + B u_t + C \epsilon_{t+1} \\ & X_0 \text{ given} \end{aligned} \tag{1}$$

where

$$
\begin{aligned}
r\left(X_t, u_t\right) &\simeq X_t' Q X_t + u_t' R u_t + 2 X_t' W u_t \\
g\left(X_t, u_t, \epsilon_{t+1}\right) &\simeq A X_t + B u_t + C \epsilon_{t+1},
\end{aligned} \tag{2}
$$

with Q and R symmetric. That is, we solve a problem with a quadratic objective function and linear constraints. Note that implicit in our formulation of (1) are the assumptions that X_t is contained in the agents' information sets at time t and that the agents know the objective function and transition functions for all variables.

To obtain the functions in (2), we take a second and first-order Taylor expansion of the corresponding nonlinear functions around the steady state of the system. Thus, when evaluated at the stationary point, the original and approximated functions have the same value.

To find the steady state of the system, we first set the disturbance term ϵ_t to its unconditional mean. Without loss of generality, assume the mean is zero. We then find the first order conditions of the resulting nonstochastic version of the model:

$$
\begin{gathered}
\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r\left(X_t, u_t\right) \\
\text{subject to} \quad X_{t+1} = g\left(X_t, u_t, 0\right)
\end{gathered} \tag{3}
$$

and X_0 given. Formulating the Lagrangian:

$$
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{ r\left(X_t, u_t\right) - \lambda_{t+1}' \left(X_{t+1} - g\left(X_t, u_t, 0\right)\right) \} \tag{4}
$$

and taking derivatives with respect to u_t and X_{t+1}, we obtain the following first-order conditions

$$
\begin{aligned}
\frac{\partial r\left(X_t, u_t\right)}{\partial u_t} + \frac{\partial g\left(X_t, u_t, 0\right)'}{\partial u_t} \lambda_{t+1} &= 0 \\
\beta \frac{\partial r\left(X_{t+1}, u_{t+1}\right)}{\partial X_{t+1}} - \lambda_{t+1} + \beta \frac{\partial g\left(X_{t+1}, u_{t+1}, 0\right)'}{\partial X_{t+1}} \lambda_{t+2} &= 0
\end{aligned} \tag{5}
$$

for $t \geq 0$, where $\{\lambda_t\}$ is a sequence of Lagrange multipliers. Eliminating time subscripts from (5) and the constraint in (3), we then get the following set of nonlinear equations:

$$
\frac{\partial r\left(X, u\right)}{\partial u} + \frac{\partial g\left(X, u, 0\right)'}{\partial u} \lambda = 0
$$

$$\beta \frac{\partial r(X,u)}{\partial X} - \lambda + \beta \frac{\partial g(X,u,0)'}{\partial X} \lambda = 0$$
$$X - g(X,u,0) = 0 \tag{6}$$

This is a set of $2m+n$ equations with $2m+n$ unknowns, X, u, λ. The fixed point of this system is the steady state, say $\bar{X}, \bar{u}, \bar{\lambda}$, around which we take first and second-order Taylor expansions of g and r. Thus, we have the problem given by (1).

Thus far, we have derived the first order conditions for the original nonlinear problem that imply a set of equations for finding the steady state (or more precisely, the balanced growth path). We take a second order Taylor expansion of the objective function ($r(X,u)$) around the steady state to get matrices Q, R, and W. We take a first-order Taylor expansion of the constraints ($g(X,u,\epsilon)$) around the steady state to get A, B, C.

Next, we need to put some conditions on these matrices to ensure that the optimal solution to our problem yields a stable system (and that we are maximizing, not minimizing). The relevant conditions are usually stated in terms of a problem with $\beta = 1$ and $W = 0$. We can reformulate our problem so that there is no discounting or cross-products as follows. Let

$$\begin{aligned}
\tilde{X}_t &= \beta^{\frac{t}{2}} X_t \\
\tilde{u}_t &= \beta^{\frac{t}{2}} \left(u_t + R^{-1}W'X_t\right) \\
\tilde{A} &= \sqrt{\beta}\left(A - BR^{-1}W'\right) \\
\tilde{B} &= \sqrt{\beta}B \\
\tilde{Q} &= Q - WR^{-1}W'.
\end{aligned}$$

Assume that $\tilde{Q}$ and R are negative definite matrices (which is an assumption that can be weakened) and assume that there exists a matrix $\tilde{F}$ such that $\tilde{A} - \tilde{B}\tilde{F}$ has eigenvalues inside the unit circle. In this case, the system is stable and, in the language of control theorists, $(\tilde{A}, \tilde{B})$ is stabilizable. The matrix $\tilde{F}$ that is relevant for us is the matrix governing the optimal solution, namely, $\tilde{u}_t = -\tilde{F}\tilde{X}_t$.

If the conditions above are satisfied, then the optimal policy function for the original optimization problem is the time-invariant linear rule:

$$u_t = -FX_t, \qquad F = \left(R + \beta B'PB\right)^{-1}\left(\beta B'PA + W'\right)$$

$$
\begin{aligned}
&= \left(R + \tilde{B}'P\tilde{B}\right)^{-1} \tilde{B}'P\tilde{A} + R^{-1}W' \\
&\tilde{F} + R^{-1}W'.
\end{aligned} \tag{7}
$$

The matrix P in (7) is the steady-state solution to the matrix Riccati difference equation

$$
\begin{aligned}
P_t &= Q + \beta A'P_{t+1}A - (\beta A'P_{t+1}B + W)\left(R + \beta B'P_{t+1}B\right)^{-1}(\beta B'P_{t+1}A + W') \\
&= \tilde{Q} + \tilde{A}'P_{t+1}\tilde{A} - \tilde{A}'P_{t+1}\tilde{B}\left(R + \tilde{B}'P_{t+1}\tilde{B}\right)^{-1}\tilde{B}'P_{t+1}\tilde{A}
\end{aligned} \tag{8}
$$

as $t \to -\infty$, with terminal condition $P_T \leq 0$.

There have been many algorithms developed for the solution of the discrete-time Riccati equation. In all cases, we take as given the matrices A, B, Q, R, W and scalar β (or equivalently $\tilde{A}$, $\tilde{B}$, $\tilde{Q}$, and R), tolerance criteria γ_1 and γ_2, and a matrix norm $\|\cdot\|$. The simplest method is simply direct iteration. To do this, set an initial symmetric Riccati matrix, $P^0 \leq 0$.

a) At iteration n, we compute P^{n+1} and $\tilde{F}^n$ to be

$$
\begin{aligned}
P^{n+1} &= \tilde{Q} + \tilde{A}'P^n\tilde{A} - \tilde{A}'P^n\tilde{B}\left(R + \tilde{B}'P^n\tilde{B}\right)^{-1}\tilde{B}'P^n\tilde{A} \\
\tilde{F}^n &= \left(R + \tilde{B}'P^n\tilde{B}\right)^{-1}\tilde{B}'P^n\tilde{A}
\end{aligned}
$$

b) If $\|P^{n+1} - P^n\| < \gamma_1\|P^n\|$ and $\|\tilde{F}^{n+1} - \tilde{F}^n\| < \gamma_2\|\tilde{F}^n\|$, go to (c); otherwise, increase n by one and return to (a).

c) Set $F = \tilde{F}^n + R^{-1}W'$, $P = P^n$.

With a steady-state solution to the Riccati matrix, we can use (7) to compute F and the law of motion for the state variables:

$$
X_{t+1} = (A - BF)\,X_t + C\epsilon_{t+1} \tag{9}
$$

Furthermore, given an initial condition for the states, X_0, and a realization of the shocks, $\epsilon_t,\ t \geq 0$, we can generate time-series for X_t via (9) and u_t via (7).