Duality

Suppose a linear programming problem

$$\begin{pmatrix} \text{minimize} & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{pmatrix} \quad \text{Primal problem}$$

$$\begin{pmatrix} \text{maximize} & \lambda^T b \\ \text{s.t.} & \lambda^T A \leq c^T \\ & \lambda \geq 0 \end{pmatrix} \quad \text{Dual problem}$$

$\lambda \in \mathbb{R}^m$: dual vector

How to derive.

Lagrangian $\mathcal{L}(x, \lambda) = c^T x + \lambda^T (b - Ax)$

$$\inf_x \sup_\lambda c^T x + \lambda^T (b - Ax) \quad \text{with } \lambda \geq 0$$

$$= \inf_x \sup_\lambda c^T x + \lambda^T b - \lambda^T (Ax)$$

$$= \inf_x \sup_\lambda c^T x + \lambda^T b - x^T (A^T \lambda)$$

$$= \sup_\lambda \inf_x \lambda^T b + x^T (c - A^T \lambda)$$

$$= \sup_\lambda \left(\lambda^T b + \inf_x x^T (c - A^T \lambda) \right)$$

$$\Rightarrow \quad \begin{array}{ll} \max & \lambda^T b \\ \text{s.t.} & c - A^T \lambda \geq 0 \\ & \lambda \geq 0 \end{array}$$

Example 17.1

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

Lagrangian: $\mathcal{L}(x, \lambda) = c^T x + \lambda^T (Ax - b) \qquad \lambda \geq 0$

$$\inf_x \sup_\lambda c^T x + \lambda^T A x - \lambda^T b$$

$$= \sup_\lambda -\lambda^T b + \inf_x (c + A^T \lambda)^T x$$

$$\begin{array}{ll} \text{maximize} & -\lambda^T b \\ \text{s.t.} & -\lambda^T A = c^T \\ & \lambda \geq 0 \end{array}$$

Replace λ with $-\lambda$,

$$\begin{cases} \text{maximize} & \lambda^T b \\ \text{s.t.} & \lambda^T A = c^T \\ & \lambda \le 0 \end{cases}$$

$\square$

Example 17.3

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 5x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 - x_2 + 7x_3 \le 6 \\ & x_1 + 3x_2 + 4x_3 \le 9 \\ & 3x_1 + 6x_2 + x_3 \le 3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Find the corresponding dual problem and solve it.

$\Rightarrow$ Change the problem to minimization problem

$$\begin{cases} \min & -2x_1 - 5x_2 - x_3 \\ \text{s.t.} & -2x_1 + x_2 - 7x_3 \ge -6 \\ & -x_1 - 3x_2 - 4x_3 \ge -9 \qquad x_1, x_2, x_3 \ge 0 \\ & -3x_1 - 6x_2 - x_3 \ge -3 \end{cases}$$

$$\begin{cases} \min & c^T x \\ \text{s.t.} & Ax \ge b \\ & x \ge 0 \end{cases}$$

$\Updownarrow$

$$\begin{cases} \max & \lambda^T b \\ \text{s.t.} & A^T \lambda \le c \\ & \lambda \ge 0 \end{cases} \Rightarrow \begin{aligned} \text{maximize} \quad & -6\lambda_1 - 9\lambda_2 - 3\lambda_3 \\ \text{s.t.} \quad & -2\lambda_1 - \lambda_2 - 3\lambda_3 \le -2 \\ & \lambda_1 - 3\lambda_2 - 6\lambda_3 \le -5 \\ & -7\lambda_1 - 4\lambda_2 - \lambda_3 \le -1 \end{aligned}$$

Add slack variables $\lambda_4, \lambda_5, \lambda_6$

$$\begin{aligned} \text{minimize} \quad & 6\lambda_1 + 9\lambda_2 + 3\lambda_3 \\ \text{s.t.} \quad & 2\lambda_1 + \lambda_2 + 3\lambda_3 - \lambda_4 = 2 \\ & -\lambda_1 + 3\lambda_2 + 6\lambda_3 - \lambda_5 = 5 \\ & 7\lambda_1 + 4\lambda_2 + \lambda_3 - \lambda_6 = 1 \\ & \lambda_1, \lambda_2, \cdots, \lambda_6 \ge 0 \end{aligned}$$

No obvious basic feasible solution. $\Rightarrow$ 2-phase simplex method.

Phase 1. Add slack variables $\lambda_7, \lambda_8, \lambda_9$.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	c
	2	1	3	-1			1			2
	-1	3	6		-1			1		5
	7	4	1			-1			1	1
Cost	0	0	0	0	0	0	1	1	1	0

$\downarrow$

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	c
	2	1	3	-1			1			2
	-1	3	6		-1			1		5
	7	4	1			-1			1	1
Cost	-8	-8	-10	1	1	1	0	0	0	-8

$\downarrow$

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	c
	$\frac{2}{3}$	$\frac{1}{3}$	1	$-\frac{1}{3}$			$\frac{1}{3}$			$\frac{2}{3}$
	-5	1	0	2	-1		-2	1		1
	$\frac{19}{3}$	$\frac{11}{3}$	0	$\frac{1}{3}$		-1	$-\frac{1}{3}$		1	$\frac{1}{3}$
Cost	$-\frac{4}{3}$	$-\frac{14}{3}$	0	$-\frac{2}{3}$	1	1	$\frac{10}{3}$	0	0	$-\frac{4}{3}$

$\downarrow$ Continue the process . . .

$\triangle$

§ 17.2 Properties of Dual Problems

Lemma 17.1 (Weak Duality Lemma)

Suppose x and λ are feasible solutions to primal and dual LP problems, respectively. Then $\lambda^T b \le c^T x$

(proof)

Asymmetric : Since x and λ are feasible,

$$Ax = b, \quad x \ge 0 \qquad \text{and} \qquad \lambda^T A \le c^T$$

Multiply by x,

$$\lambda^T A x \le c^T x$$

$$\lambda^T b \le c^T x$$

Symmetric : x and λ are feasible.

$$Ax \ge b, \quad x \ge 0 \qquad \text{and} \qquad \lambda^T A \le c^T, \quad \lambda \ge 0$$

Multiply by x,

$\lambda^T A x \le c^T x$ and $\lambda^T A x \ge \lambda^T b$

$\Rightarrow \quad \lambda^T b \le c^T x$ $\square$

The cost of dual $\le$ The cost of primal

Example 17.4

minimize x s.t. $x \le 1$ $\Rightarrow$ minimize cx s.t. $Ax \le 1$, $c = 1$, $A = 1$

The minimizer of the problem is $x^* = -\infty$. which is unbounded.

From example 17.1, the dual problem is

$$\begin{cases} \text{maximize} & \lambda \\ \text{s.t.} & \lambda = 1 \\ & \lambda \le 0 \end{cases}$$

$\Rightarrow$ The feasible λ is empty. $\square$

Thm 17.1 Suppose x_0 and λ_0 are feasible solutions to primal and dual.

$c^T x_0 = \lambda_0^T b \iff x_0, \lambda_0$ are optimal solutions.

Exercise 17.3

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 4 \\ & 2x_1 + x_2 \le 5 \\ & x_1, x_2 \ge 0 \end{aligned}$$

(b)

$$\begin{cases} \text{minimize} & -2x_1 - 3x_2 \\ \text{s.t.} & \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ge \begin{pmatrix} -4 \\ -5 \end{pmatrix} \\ & x_1, x_2 \ge 0 \end{cases}$$

$\downarrow$

$$\begin{cases} \text{maximize} & \lambda^T \begin{pmatrix} -4 \\ -5 \end{pmatrix} \\ \text{s.t.} & \lambda^T \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \le \begin{pmatrix} -2 \\ -3 \end{pmatrix}^T \\ & \lambda \ge 0 \end{cases}$$

$$\begin{cases} \text{minimize} & 4\lambda_1 + 5\lambda_2 \\ \text{s.t.} & \lambda_1 + 2\lambda_2 \ge 2 \\ & 2\lambda_1 + \lambda_2 \ge 3 \end{cases}$$

$\triangle$

Exercise 17.4

$$\begin{cases} \text{minimize} & 4x_1 + 3x_2 \\ \text{s.t.} & 5x_1 + x_2 \geq 11 \\ & 2x_1 + x_2 \geq 8 \\ & x_1 + 2x_2 \geq 7 \\ & x_1, x_2 \geq 0 \end{cases}$$

$$\begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{cases} \quad \text{with} \quad c = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 11 \\ 8 \\ 7 \end{pmatrix}$$

dual :
$$\begin{cases} \text{maximize} & \lambda^T b \\ \text{s.t.} & \lambda^T A \leq c^T \\ & \lambda \geq 0 \end{cases}$$

$$\Rightarrow \quad \begin{array}{ll} \text{maximize} & 11\lambda_1 + 8\lambda_2 + 7\lambda_3 \\ \text{s.t.} & 5\lambda_1 + 2\lambda_2 + \lambda_3 \leq 4 \\ & \lambda_1 + \lambda_2 + 2\lambda_3 \leq 3 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}$$

$\triangle$

Exercise 17.5

(a)

$$\begin{cases} \text{maximize} & x_1 + 2x_2 \\ \text{s.t.} & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + 2x_2 + x_4 = 7 \\ & x_1 + x_5 = 3 \\ & x_i \geq 0 \quad i = 1, \dots, 5 \end{cases}$$

$$\begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases} \quad c = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$$

Asymmetric
$\Rightarrow$ dual :
$$\begin{cases} \text{maximize} & \lambda^T b \\ \text{s.t.} & \lambda^T A \leq c^T \end{cases}$$

$$\begin{cases} \text{maximize} & 2\lambda_1 + 7\lambda_2 + 3\lambda_3 \\ \text{s.t.} & -2\lambda_1 - \lambda_2 + \lambda_3 \leq -1 \\ & \lambda_1 + 2\lambda_2 \leq -2 \\ & \lambda_1 \leq 0 \\ & \lambda_2 \leq 0 \\ & \lambda_3 \leq 0 \end{cases}$$

$\triangle$