Math 164: HW 1

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1 Solution to Exercise 8.12

Let x^* be a minimizer of f. Define an open interval $I := (\frac{1}{\sqrt{3}} - \beta, \frac{1}{\sqrt{3}} + \beta)$ for some $\beta > 0$. Suppose $x^0 \in I$. We will show $x^k \to x^*$ if α satisfies the following:

$$0 < \alpha < \frac{2}{\max_{z \in I} f''(z)} \tag{1.1}$$

For the sake of contradiction, suppose $\alpha \leq 0$ or $\alpha \geq \frac{2}{\max_{z \in I} f''(z)}$. Then,

$$\begin{aligned} x^{k+1} &= x^k - \alpha f'(x^k) \\ x^{k+1} - x^* &= x^k - x^* - \alpha(f'(x^k) - f'(x^*)) \qquad \text{(By SONC, } f'(x^*) = 0) \end{aligned}$$

By the mean value theorem, we can find z between x^k and x^* such that $f'(x^k) - f'(x^*) = f''(z)(x^k - f^*)$. Thus,

$$\begin{aligned} x^{k+1} - x^* &= x^k - x^* - \alpha f''(z)(x^k - x^*) \\ x^{k+1} - x^* &= (1 - \alpha f''(z))(x^k - x^*) \\ \|x^{k+1} - x^*\| &= \|(1 - \alpha f''(z))(x^k - x^*)\| \\ &= |1 - \alpha f''(z)| \|x^k - x^*\| \\ &= \cdots \\ &= |1 - \alpha f''(z)|^{k+1} \|x^0 - x^*\| \end{aligned}$$

Since $\alpha \leq 0$ or $\alpha \geq \frac{2}{\max_{z \in I} f''(z)}$,

$$|1 - \alpha f''(z)| \geq 1 \tag{1.2}$$

Thus,

$$\|x^{k+1} - x^*\| \geq \|x^0 - x^*\| \tag{1.3}$$

for all k. This shows x^k cannot converge to x^*.

By the above claim, x^k converges to x^* if α satisfies (1.1). Now let's find $\max_{z \in I} f''(z)$.

$$\max_{z \in I} f''(z) = \max_{z \in I} 6z = 6\left(\frac{1}{\sqrt{3}} + \beta\right) \tag{1.4}$$

Thus, the range of α is

$$0 < \alpha < 1 \Big/ 3\left(\frac{1}{\sqrt{3}} + \beta\right) \tag{1.5}$$