$$\begin{cases} \text{maximize} & f(x) \\ \text{s.t.} & h(x) = 0 \\ & g(x) \le 0 \end{cases}$$

KKT condition

1. $\mu^* \ge 0$
2. $-Df(x^*) + \lambda^{*T} Dh(x^*) + \mu^{*T} Dg(x^*) = 0^T$
3. $\mu^{*T} g(x^*) = 0$
4. $h(x^*) = 0$
5. $g(x^*) \le 0$

Thm 21.3 SOSC Suppose that $f, g, h \in C^2$ and there exists a feasible point $x^* \in \mathbb{R}^n$ and vectors $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that

1. $\mu^* \ge 0$, $Df(x^*) + \lambda^{*T} Dh(x^*) + \mu^{*T} Dg(x^*) = 0^T$, $\mu^{*T} g(x^*) = 0$
2. $\forall y \in \tilde{T}(x^*, \mu^*)$, $y \neq 0$, we have $y^T L(x^*, \lambda^*, \mu^*) y > 0$

Then x^* is a strict local minimizer of f s.t. $h(x) = 0$, $g(x) \le 0$.

$$\begin{cases} \tilde{T}(x^*, \mu^*) = \{ y : Dh(x^*) y = 0 \quad Dg_i(x^*) y = 0,\ i \in \tilde{J}(x^*, \mu^*) \} \\ \tilde{J}(x^*, \mu^*) = \{ i : g_i(x^*) = 0,\ \mu_i^* > 0 \} \end{cases}$$

Exercise 21.7

(a) Suppose it is minimization.

$$\begin{aligned} \min \quad & (x_1 - 2)^2 + (x_2 - 1)^2 \\ \text{s.t.} \quad & x_1^2 - x_2 \le 0 \\ & x_1 + x_2 - 2 \le 0 \\ & -x_1 \le 0 \end{aligned}$$

$$Df(x^*) + \lambda^{*T} Dh(x^*) + \mu^{*T} Dg(x^*) = \begin{pmatrix} 2x_1 - 4 \\ 2x_2 - 2 \end{pmatrix} + \mu_1^* \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} + \mu_2^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mu_3^* \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0$$

$$(\mu_1^* \ \mu_2^* \ \mu_3^*) \begin{pmatrix} x_1^2 - x_2 \\ x_1 + x_2 - 2 \\ -x_1 \end{pmatrix} = 0$$

$$\rightarrow \begin{cases} 2x_1 - 4 + 2\mu_1^* x_1 + \mu_2^* - \mu_3^* = 0 \\ 2x_2 - 2 - \mu_1^* + \mu_2^* = 0 \\ \mu_1^*(x_1^2 - x_2) + \mu_2^*(x_1 + x_2 - 2) + \mu_3^*(-x_1) = 0 \end{cases}$$

Let $(x_1, x_2) = (0,0)$. Then

$$\left\{\begin{array}{l} -4 + \mu_2^* - \mu_3^* = 0 \\ -2 - \mu_1^* + \mu_2^* = 0 \\ \quad -2\mu_2^* = 0 \end{array}\right.$$

$\Rightarrow \mu_2^* = 0$, $\mu_1^* = -2$, $\mu_3^* = -4$. Since $\mu_3^*, \mu_1^* < 0$, not feasible.

Consider maximization.

$$\begin{array}{ll} \min & -(x_1-2)^2 - (x_2-1)^2 \\ \text{s.t.} & x_1^2 - x_2 \leq 0 \\ & x_1 + x_2 - 2 \leq 0 \\ & -x_1 \leq 0 \end{array}$$

From KKT,

$$\begin{pmatrix} -2x_1+4 \\ -2x_2+2 \end{pmatrix} + \mu_1^* \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} + \mu_2^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mu_3^* \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0$$

$$\mu_1^*(x_1^2 - x_2) + \mu_2^*(x_1 + x_2 - 2) + \mu_3^*(-x_1) = 0$$

Let $(x_1, x_2) = (0,0)$,

$$\left.\begin{array}{l} 4 + \mu_2^* - \mu_3^* = 0 \\ 2 - \mu_1^* + \mu_2^* = 0 \\ \quad \mu_2^* = 0 \end{array}\right) \Rightarrow \begin{array}{l} \mu_3^* = 4 \\ \mu_1^* = 2 \\ \mu_2^* = 0 \end{array}$$

All positive, thus feasible.

(b.) $J(x^*, \mu^*) = \{i : g_i(x^*) = 0, \mu_i^* > 0\} = \{1, 3\}$

$\tilde{T}(x^*, \mu^*) = \{y : Dh(x^*)y = 0,\ Dg_i(x^*)y = 0,\ i \in \{1,3\}\}$

$= \{y : \begin{pmatrix} 0 \\ -1 \end{pmatrix} y = 0,\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} y = 0\} = \{0\}$

From (a), SOSC (1) holds. Since $\tilde{T}(x^*, \mu^*) = \{0\}$, SOSC (2) is vacuously true.

$\square$

Exercise 21.12

$$\begin{array}{ll} \min & \frac{1}{2} x^T Q x \\ \text{s.t.} & Ax \leq b \end{array}$$

where $Q = Q^T > 0$, $A \in \mathbb{R}^{m \times n}$, $b \geq 0$. Find all points satisfying the KKT condition.

$$\begin{cases} Df(x^*) + \lambda^{*T} Dh(x^*) + \mu^{*T} Dg(x^*) = 0 \\ \mu^{*T} g(x^*) = 0 \end{cases}$$

$f(x) + \mu^T (Ax - b)$

$\nabla f(x) + A^T \mu$

$$Qx^* + A^T \mu^* = 0$$

$$\mu^{*T} (Ax^* - b) = 0$$

$$\begin{cases} x^* = -Q^{-1} A^T \mu^* \\ \mu^{*T} (-AQ^{-1} A^T \mu^* - b) = 0 \end{cases}$$

Exercise 21.19

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & 4 - x_1 - x_2^2 \le 0 \\ & -x_1 + 3x_2 \le 0 \\ & -x_1 - 3x_2 \le 0 \end{aligned}$$

From KKT,

$$\begin{cases} \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} + \mu_1^* \begin{pmatrix} -1 \\ -2x_2 \end{pmatrix} + \mu_2^* \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \mu_3^* \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 0 \\ \mu_1^* (4 - x_1 - x_2^2) + \mu_2^* (-x_1 + 3x_2) + \mu_3^* (-x_1 - 3x_2) = 0 \end{cases}$$

From Figure 22.3, $(3,-1)$ $(3,1)$ are minimums.

Let $x^* = (3,1)$. Then

$$\begin{cases} \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \mu_1^* \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \mu_2^* \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \mu_3^* \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 0 \\ \mu_1^* (-6) - 6\mu_3^* = 0 \\ \mu_1^* + \mu_3^* = 0 \end{cases}$$

$$\begin{cases} 6 - \mu_1^* - \mu_2^* - \mu_3^* = 0 \\ 2 - 2\mu_1^* + 3\mu_2^* - 3\mu_3^* = 0 \\ \mu_1^* + \mu_3^* = 0. \end{cases}$$

$\mu_1^* = -\mu_3^* \Rightarrow 6 - \mu_2^* = 0 \Rightarrow \mu_2^* = 6$

$2 - 5\mu_1^* + 18 = 0 \Rightarrow \mu_1^* = 4$

SOSC (1) is satisfied at $(3,1)$.

* Checking SOSC (2)

$$\mathcal{J}(x^*, \mu^*) = \{1, 2\}$$

$$\mathcal{T}(x^*, \mu^*) = \{ y : \begin{pmatrix} -1 \\ -2 \end{pmatrix} y = 0, \begin{pmatrix} -1 \\ 3 \end{pmatrix} y = 0 \} = \{0\}$$

Thus, SOSC (2) is vacuously true. $\square$

§ Chapter 22

Def 22.1 The graph of $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, is the set of points in $\Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$ given by

$$\left\{ \begin{bmatrix} x \\ f(x) \end{bmatrix} : x \in \Omega \right\}$$

Def 22.2 The epigraph of $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $\mathrm{epi}(f)$, is the set of points in $\Omega \times \mathbb{R}$ given by

$$\mathrm{epi}(f) = \left\{ \begin{bmatrix} x \\ \beta \end{bmatrix} : x \in \Omega, \beta \in \mathbb{R}, \beta \geq f(x) \right\}$$

Def 22.3 $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ is convex on Ω iff $\mathrm{epi}(f)$ is a convex set.

Thm 22.1 If $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ is convex on Ω, then Ω is a convex set.

(proof) Suppose Ω is not convex set. Then, there are y_1, y_2 s.t.

$$\alpha y_1 + (1-\alpha) y_2 \notin \Omega \quad \text{for some } \alpha \in (0,1)$$

$$\alpha \begin{bmatrix} y_1 \\ f(y_1) \end{bmatrix} + (1-\alpha) \begin{bmatrix} y_2 \\ f(y_2) \end{bmatrix} = \begin{bmatrix} \alpha y_1 + (1-\alpha) y_2 \\ \alpha f(y_1) + (1-\alpha) f(y_2) \end{bmatrix} \notin \mathrm{epi}(f)$$

since $\alpha y_1 + (1-\alpha) y_2 \notin \Omega$. $\Rightarrow$ $\mathrm{epi}(f)$ is not convex and hence f is not convex which is a contradiction. $\square$

Thm 22.4 Let $f : \Omega \to \mathbb{R}$, $f \in C^1$ be defined on an open convex set $\Omega \subset \mathbb{R}^n$. Then f is convex on Ω iff

$$f(y) \geq f(x) + Df(x)(y - x) \quad \text{for all } x, y \in \Omega.$$

Def: A vector $g \in \mathbb{R}^n$ is said to be a subgradient of f at point $x \in \Omega$

$$f(y) \geq f(x) + g^T (y-x) \quad \text{for all } y \in \Omega$$

Thm 22.8 Let $f: \mathbb{R}^n \to \mathbb{R}$, $f \in C^1$ be a convex function on the set of feasible points

$$\Omega = \{x \in \mathbb{R}^n : h(x) = 0\}$$

where $h: \mathbb{R}^n \to \mathbb{R}^m$, $h \in C^1$, and Ω is convex. Suppose there exist $x^* \in \Omega$ and $\lambda^* \in \mathbb{R}^m$ such that

$$Df(x^*) + \lambda^{*T} Dh(x^*) = 0^T$$

Then, x^* is a global minimizer of f over Ω.

Exercise 22.12

$$\min \ \frac{1}{2} x^T Q x$$

$$\text{s.t. } Ax = b$$

where $Q \in \mathbb{R}^{n \times n}$, $Q = Q^T > 0$, $A \in \mathbb{R}^{m \times n}$ and $\text{rank}(A) = m$.

a) Find all points satisfying the Lagrange condition.

$$Df(x^*) + \lambda^{*T} Dh(x^*) = 0^T$$

$$Qx^* + A^T \lambda^* = 0 \implies x^* = -\underbrace{Q^{-1}}_{Q > 0 \text{, thus invertible}} A^T \lambda^*$$

$$Ax^* = b \implies -\underbrace{AQ^{-1}A^T}_{\text{rank}(A) = m \text{ and } Q > 0 \text{, thus } AQ^{-1}A^T \text{ is invertible}} \lambda^* = b$$

$$\implies \lambda^* = -(AQ^{-1}A^T)^{-1} b$$

$$\implies x^* = -Q^{-1}A^T (AQ^{-1}A^T)^{-1} b$$

b) f is a convex function and $\Omega = \{x \in \mathbb{R}^n : h(x) = 0\}$ is convex.

By Thm 22.8, x^* is a global minimizer of f over Ω. $\square$

Exercise 22.19

$\Omega \subset \mathbb{R}^n$ nonempty closed convex set and $z \in \mathbb{R}^n$, $z \notin \Omega$.

Consider
$$\min \ \|x - z\|$$
$$\text{s.t. } x \in \Omega$$

- Show the problem has an optimal solution.

Define $d = \inf_{x \in \Omega} \|x - z\|$.

Since Ω is closed, there exists $x^* \in \Omega$ that satisfies

$$\|x^* - z\| = \inf_{x \in \Omega} \|x - z\|$$

- Show $x^* \in \partial\Omega$. ← ※ You can ignore this part for the proof.

Suppose x^* is in the interior of Ω. There exists $r > 0$ such that

$$x \in \Omega \quad \text{if} \quad B_r(x^*) = \{x \in \mathbb{R}^d : \|x - x^*\| < r\}$$

We can find $\alpha > 0$ such that $\alpha x^* + (1-\alpha) z \in B_r(x^*)$.

$$\|(\alpha x^* + (1-\alpha) z) - z\| = \|\alpha x^* - \alpha z\| = \alpha \|x^* - z\| < \|x^* - z\|$$

This is a contradiction to x^* being the minimizer. $\square$

- Show the solution is unique.

Suppose x_1 and x_2 are two distinct solutions.

$$\|x_1 - z\| = \|x_2 - z\|$$

Since Ω is convex, $\frac{x_1 + x_2}{2} \in \Omega$. Using the triangle inequality,

$$\left\|\frac{x_1 + x_2}{2} - z\right\| = \left\|\frac{x_1 - z}{2} + \frac{x_2 - z}{2}\right\| \le \frac{1}{2}\|x_1 - z\| + \frac{1}{2}\|x_2 - z\| = \|x_1 - z\|$$

Since $\|x_1 - z\|$ is the minimum, it follows that

$$\left\|\frac{x_1 + x_2}{2} - z\right\| = \|x_1 - z\|.$$

From the hint, the equality holds iff $x = \alpha y$ for some $\alpha \ge 0$ or $x = 0$ or $y = 0$.

Let $x = \frac{x_1 - z}{2}$, $y = \frac{x_2 - z}{2}$.

$x \ne 0$ $y \ne 0$ since $z \notin \Omega$.

Suppose $x = \alpha y$ for some $\alpha \ge 0$.

Then $\frac{x_1 - z}{2} = \alpha \frac{x_2 - z}{2} \implies x_1 - z = \alpha x_2 - \alpha z.$

$\|x_1 - z\| = \alpha \|x_2 - z\|$

Since $\|x_1 - z\| = \|x_2 - z\|$, $\alpha = 1$.

If $\alpha = 1$, then $x_1 - z = x_2 - z$

$\Rightarrow x_1 = x_2$ which is a contradiction.

Thus, the minimizer is unique. $\square$