Theorem 5.3 Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L, \quad \text{for all } (t, y) \in D, \tag{5.1}$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L. ■

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ex) $y'(t) = y^2 e^t \quad (t, y) \in [0,1] \times [0,2]$

$f(t,y) = y^2 e^t$ ← Show f satisfies Lip condition in the var y.

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = \left|2y e^t\right| \leq 2 \cdot 2 e^1 = 4e = L$$

Theorem 5.6 Suppose $D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

is well-posed. ■

Question 1: Determine whether the following IVP's are well-posed.

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1+y}{t}, \quad 1 \le t \le 2, \quad y(1) = 2$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\cos(t), \quad 0 \le t \le 1, \quad y(0) = 1$$

$D = \{ (t, y) : -1 \le t \le 2, \quad y \in \mathbb{R} \}$

$f(t,y) = \frac{1+y}{t}$

$\left| \frac{\partial f(t,y)}{\partial y} \right| = \left| \frac{1}{t} \right| \le 1 = L$

f is continuous in D and satisfies Lip condition on D in the variable y. By Thm 5.6, the IVP is well-posed.

f is not continuous at $t=0$. Thus not well-posed.

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in I, \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$

$|f(x) - f(y)| < L|x-y|$

$|x-y| < \frac{\varepsilon}{L}$

Question 2: Consider the IVP

$$\frac{dy}{dt} = \frac{1+t}{1+y} = f(t,y)$$

with $1 \le t \le 2$ and $y(1) = 2$.

(a) By hand, compute an approximation to $y(2)$ using Euler's method with $h = 0.5$.

(b) Using code, approximate $y(2)$ using Euler's method with $h = 0.5, 0.2, 0.1, 0.01$ and record your results. (You do not have to submit your code).

(c) The exact solution to the IVP is $y(t) = \sqrt{t^2 + 2t + 6} - 1$. Compare your approximations with the exact result $y(2)$, and interpret your results.

$$y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2!} y''(t) + \frac{h^3}{3!} y'''(t) + \cdots$$

$$y(t+h) = y(t) + h\,y'(t) + O(h^2)$$

$$y(t+h) = y(t) + h\,f(t,y) + O(h^2)$$

$$w(t+h) = w(t) + h\,f(t,w)$$

$$y(1.5) = y(1) + 0.5\,f(1, y(1))$$

$$= 2 + 0.5 \cdot \frac{2}{3} = 2 + \frac{1}{3} = \frac{7}{3}$$

$$y(2) = y(1.5) + 0.5\,f(1.5, y(1.5))$$

$$= \frac{7}{3} + 0.5\,f\left(1.5, \frac{7}{3}\right) = \frac{7}{3} + 0.5 \cdot \frac{1+1.5}{1+\frac{7}{3}} = \boxed{}$$

Question 3: To derive Euler's method, we truncated the Taylor series expan-sion of y_{i+1} at the linear term. We could truncate at the quadratic term instead, giving the so-called Taylor method of order 2. This method approximates y_{i+1} by

$$w_{i+1} = w_i + hf(t_i, w_i) + \frac{h^2}{2}\frac{\mathrm{d}f}{\mathrm{d}t}\bigg|_{(t_i,w_i)}.$$

(a) For the IVP $y'(t) = y^2e^{-t}$, $0 \le t \le 1$, $y(0) = 1$, calculate $\frac{\mathrm{d}f}{\mathrm{d}t}$. (Remember $\frac{\mathrm{d}f}{\mathrm{d}t}$ is different to $\frac{\partial f}{\partial t}$ since y is a function of t!)

(b) By hand, use both Euler's method and the Taylor method of order 2 to approximate $y(1)$ with $h = 0.5$.

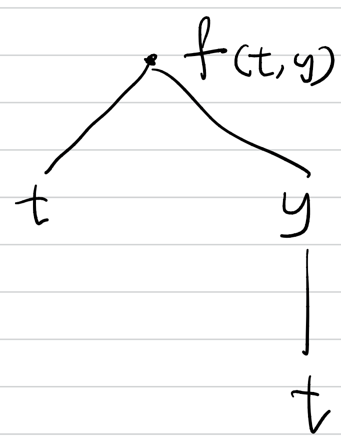
(c) Modify your code from 2(b) to implement the Taylor method of order 2 to approximate $y(1)$ using $h = 0.5, 0.1, 0.01$. Also use Euler's method to calculate the same approximations. Provide the code your wrote for this question.

(d) The exact solution to the IVP is $y(t) = e^t$. Use this to calculate the errors (where error $e = |w_n - y(1)|$) in your approximations from part (c). Summarise your results, comparing Euler's method to the Taylor method of order 2.

$$y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t) + \cdots$$

$$w(t+h) = w(t) + h\,w'(t)$$

$$w(t+h) = w(t) + h\,w'(t) + \frac{h^2}{2}w''(t)$$



$y'(t) = y^2e^{-t}$

$f(t,y) = y^2e^{-t}$

$$\frac{d}{dt}f(t,y) = \frac{\partial}{\partial t}f(t,y) + \frac{\partial f(t,y)}{\partial y}\frac{\partial y}{\partial t}$$

$$= -y^2e^{-t} + \left(2ye^{-t}\right)\left(y^2e^{-t}\right)$$

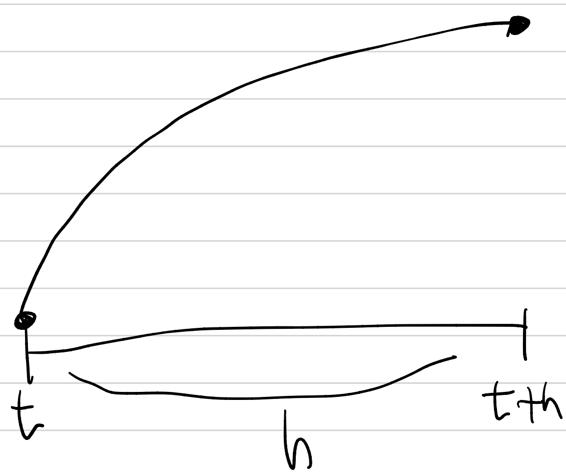
$$= -y^2e^{-t} + 2y^3e^{-2t}$$

Euler method $\quad w(t+h) = w(t) + h f(t, w(t))$

$$= w(t) + h\,(w(t))^2 e^{-t}$$

Taylor's method 2 $\quad w(t+h) = w(t) + h f(t, w(t)) + \frac{h^2}{2} \frac{df}{dt}(t, w(t))$

$$= w(t) + h\,(w(t))^2 e^{-t} + \frac{h^2}{2}\left(- (w(t))^2 e^{-t} + 2 (w(t))^3 e^{-2t} \right)$$



$w_{i+1} = w_i + w_{i-1} + w_{i-2} + h\phi(t_i, t_{i-1}, w_i, w_{i-1})$

Definition 5.11 The difference method

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1,$$

has **local truncation error**

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \ldots, N-1$, where y_i and y_{i+1} denote the solution at t_i and t_{i+1}, respectively. ■

Definition 5.18 A one-step difference-equation method with local truncation error $\tau_i(h)$ at the ith step is said to be **consistent** with the differential equation it approximates if

$$\lim_{h\to 0} \max_{1\le i\le N} |\tau_i(h)| = 0. \quad ■$$

A method is convergent if the solution to the difference equation approaches the solution to the differential equation as the step size goes to zero.

Definition 5.19 A one-step difference-equation method is said to be **convergent** with respect to the differential equation it approximates if

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| = 0,$$

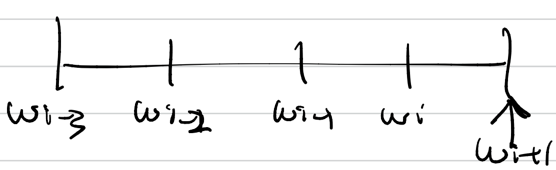
where $y(t_i)$ denotes the exact value of the solution of the differential equation and w_i is the approximation obtained from the difference method at the ith step. ■

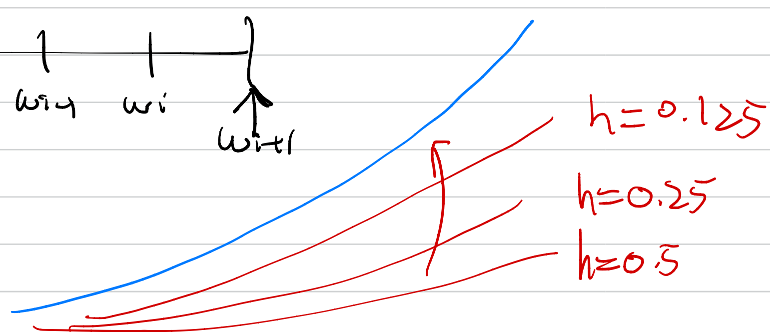
Definition of stable

A method is stable when the results depend continuously on the initial data.

To analyze this situation, at least partially, we will try to determine which methods are **stable**, in the sense that small changes or perturbations in the initial conditions produce correspondingly small changes in the subsequent approximations.

The concept of stability of a one-step difference equation is somewhat analogous to the condition of a differential equation being well-posed, so it is not surprising that the Lipschitz condition appears here, as it did in the corresponding theorem for differential equations, Theorem 5.6 in Section 5.1.

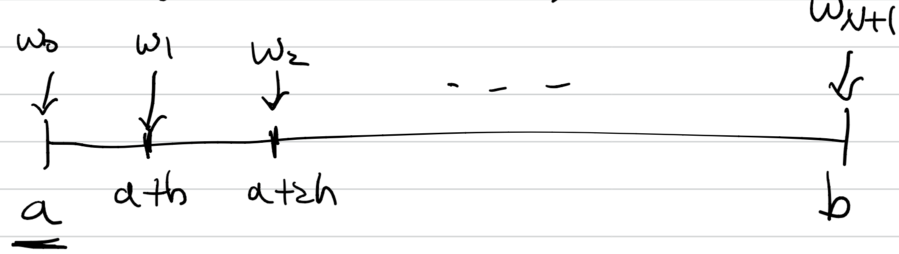




Q: Find the local truncation error of Euler's method.

$t_i = a + ih$

$w_i = w(a+ih)$



$$N = \frac{(b-a)}{h}$$

$$w_{i+1} = w_i + h f(t_i, w_i)$$

$$y(t+h) = y(t) + h y'(t) + \frac{h^2}{2} y''(t) + \underbrace{\frac{h^3}{6} y'''(t) + \frac{h^4}{24} y^{(4)}(t) + \cdots}_{O(h^3)}$$

$$y(t+h) = y(t) + h y'(t) + \frac{h^2}{2} y''(t) + O(h^3)$$

$$y(t+h) - y(t) - h y'(t) = \frac{h^2}{2} y''(t) + O(h^3)$$

$$\frac{y(t+h) - y(t)}{h} - y'(t) = \frac{h}{2} y''(t) + O(h^2) = O(h)$$

$$\left| \frac{h}{2} y''(t) + O(h^2) \right| = \text{local truncation error.}$$

$$\left| \frac{h}{2} \frac{d}{dt} f(t,y) + O(h^2) \right| = O(h)$$

Example 1 Show that Euler's method is convergent.

$$y(t_{N+1}) = y(t_N) + h f(t_N, y(t_N)) + \frac{h^2}{2} \frac{d}{dt} f(t_N, y(t_N)) + O(h^3)$$

$$w_{N+1} = w_N + h f(t_N, w_N)$$

$$|y(t_{N+1}) - w_{N+1}| = \left| (y(t_N) - w_N) + h\left(f(t_N, y(t_N)) - f(t_N, w_N)\right) + \frac{h^2}{2}\frac{d}{dt} f + \cdot \right|$$

$$\leq |y(t_N) - w_N| + h\left|f(t_N, y(t_N)) - f(t_N, w_N)\right| + \left|\frac{h^2}{2}\frac{d}{dt} f\right|$$

$$\boxed{\begin{array}{l}\text{Assumption: ① } f \text{ is Lip in the var } y. \\ \text{② } \left|\frac{d}{dt} f\right| \leq M. \text{ for some positive constant } M\end{array}}$$

$$\leq |y(t_N) - w_N| + hL|y(t_N) - w_N| + \frac{h^2 M}{2}$$

$$|y(t_{N+1}) - w_{N+1}| \leq (1+hL)\, |y(t_N) - w_N| + \frac{h^2 M}{2}$$

$$\leq (1+hL)\left((1+hL)\, |y(t_{N-1}) - w_{N-1}| + \frac{h^2 M}{2} \right) + \frac{h^2 M}{2}$$

$$= (1+hL)^2\, |y(t_{N-1}) - w_{N-1}| + \frac{h^2 M}{2}\left((1+hL) + 1 \right)$$

$$\leq (1+hL)^{N+1}\, |y(t_0) - w_0| + \frac{h^2 M}{2}\left((1+hL)^N + (1+hL)^{N-1} + \cdots + 1 \right)$$

$$= \frac{h^2 M}{2}\left((1+hL)^N + (1+hL)^{N-1} + \cdots + 1 \right)$$

$$r^N + r^{N-1} + \cdots + 1 = \frac{1 - r^{N+1}}{1 - r}$$

$$= \frac{h^2 M}{2} \frac{1 - (1+hL)^{N+1}}{1 - (1+hL)}$$

$$= \frac{h^2 M}{2} \frac{1 - (1+hL)^{N+1}}{-hL}$$

$$= \frac{h^2 M}{2} \frac{(1+hL)^{N+1} - 1}{hL}$$

$$= \frac{hM}{2L} \left((1+hL)^{N+1} - 1 \right) \qquad (1+x)^m \approx e^{xm}$$

$$\approx \frac{hM}{2L} \left(e^{hL(N+1)} - 1 \right)$$

$$\downarrow$$

$$1$$

Theorem 5.20 Suppose the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is approximated by a one-step difference method in the form

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h\phi(t_i, w_i, h).$$

Suppose also that a number $h_0 > 0$ exists and that $\phi(t, w, h)$ is continuous and satisfies a Lipschitz condition in the variable w with Lipschitz constant L on the set

$$D = \{(t, w, h) \mid a \le t \le b \text{ and } -\infty < w < \infty, 0 \le h \le h_0\}.$$

Then

(i) The method is stable;

(ii) The difference method is convergent if and only if it is consistent, which is equivalent to

$$\phi(t, y, 0) = f(t, y), \quad \text{for all } a \le t \le b;$$

(iii) If a function τ exists and, for each $i = 1, 2, \ldots, N$, the local truncation error $\tau_i(h)$ satisfies $|\tau_i(h)| \le \tau(h)$ whenever $0 \le h \le h_0$, then

$$|y(t_i) - w_i| \le \frac{\tau(h)}{L} e^{L(t_i - a)}.$$ ■

Example 2 The Modified Euler method is given by $w_0 = \alpha$,

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right], \quad \text{for } i = 0, 1, \ldots, N-1.$$

Verify that this method is stable by showing that it satisfies the hypothesis of Theorem 5.20.

↳ Suppose f is continuous and satisfies Lipschitz condition in the variable w.