Question 1:

Consider the following IVP:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = x - x^2, 0 \le x \le 1 & \\ y(0) = 0 & \text{initial condition} \end{cases} \tag{1}$$

(a) Using Euler's method, find an explicit formula for w_i in terms of the index i and the step-size h.

(b) Show that Euler's method is convergent by showing $\lim_{h \to 0} |w_i - y(x)|$.

Hint: The solution to the IVP is $y(x) = x^2/2 - x^3/3$

$$w_{i+1} = w_i + h f(x_i, w_i) \qquad w_i \approx y(x_i)$$

$$= w_i + h(x_i - x_i^2) \qquad x_i = ih$$

$$= w_i + h(ih - i^2h^2)$$

$$\boxed{w_{i+1} = w_i + ih^2 - i^2h^3}$$

$$w_{i+1} = w_i + ih^2(1 - ih)$$

$$w_{i+1} = w_{i-1} + (i-1)h^2(1 - (i-1)h) + ih^2(1 - ih)$$

$$\vdots$$

$$= w_0 + 1 \cdot h^2(1 - 1 \cdot h) + 2 \cdot h^2(1 - 2 \cdot h) + \cdots + ih^2(1 - ih)$$

$$= h^2\left((1-h) + 2 \cdot (1-2h) + 3 \cdot (1-3h) + \cdots + i \cdot (1 - ih) \right)$$

$$= h^2\left(\begin{array}{l} 1+2+3+4+\cdots+i \\ -h-4h-9h-\cdots-i^2h \end{array} \right)$$

$$i = i+1$$

$$1+4+9+\cdots+i^2 = \frac{i(i+1)(2i+1)}{6}$$

$$1+2+3+\cdots+i = \frac{i(i+1)}{2}$$

$$= h^2\left(\frac{i(i+1)}{2} - h\,\frac{i(i+1)(2i+1)}{6} \right) = \frac{i(i+1)}{2} h^2 \left(1 - h\,\frac{2i+1}{3} \right) = w_{i+1}$$

Question 2:
Write a RKF12 variable step-size method that uses the Modified Euler method to approximate the error using Euler's method, and hence varies the step-size. Write and explain your method, including the details about changing the step size, and present your code. Compare the number of time-steps required to achieve certain levels of accuracy between your RKF12 method, and the RKF45 method we derived in class. Discuss the running time and computational cost of each method for different accuracy levels. You should write at least a few sentences and have some analysis and numerical results to validate your claims.

RKF45:

RK4: $w_{i+1} = w_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5$

RK5: $\tilde{w}_{i+1} = w_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$

RK4, Euler

$t=0$	$t=0$	$w=0$	$h=0.1$
$t=0.2$	$t=0.1$	$w=$ ⋮	
$t=0.4$	$t=0.2$	$w=$ ..	
$t=0.45$	$t=0.3$	$w=$ ..	
$t=0.5$	$t=0.4$	w ...	
$t=0.6$	⋮		
⋮	$t=1$	$w=$ ..	
$t=1$			

RKF 12

RK1 : $w_{i+1} = w_i + k_1$

RK2 : $w_{i+1} = w_i + \boxed{}\,k_1 + \boxed{}\,k_2$

$k_1 = hf(t_i, w_i)$

$k_2 = hf(\ , \boxed{})$