Question 1: Consider the IVP

$$\frac{dy}{dt} = t^2 - 1$$

with $0 \le t \le 1$ and $y(0) = 0$.

(a) Write the Taylor Method of order 4 for this IVP (i.e. truncate the Taylor series at the quartic term).

(b) Use your method to approximate $y(1)$ by setting $h = 1$. Find the exact solution of the IVP and calculate the error in your approximation. Explain your results.

$$y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2} y''(t) + \frac{h^3}{6} y'''(t) + \frac{h^4}{24} y^{(4)}(t) + O(h^5)$$

w: approximated solution using whatever methods.

$y(t_i) \approx w_i \qquad y(t_{i+1}) = y(t_i + h) \approx w_{i+1}$

$$w_{i+1} = w_i + h\,f(t,y) + \frac{h^2}{2}\frac{d}{dt}f(t,y) + \frac{h^3}{6}\frac{d^2}{dt^2}f(t,y) + \frac{h^4}{24}\frac{d^3}{dt^3}f(t,y)$$

$$w_{i+1} = w_i + h(t_i^2 - 1) + \frac{h^2}{2}(2t_i) + \frac{h^3}{6}(2)$$

$$w_{i+1} = w_i + h(t_i^2 - 1) + h^2 t_i + \frac{h^3}{3}$$

$y'(t) = t^2 - 1$

$y''(t) = 2t$

$y'''(t) = 2$

$y^{(4)}(t) = 0$

$\frac{1}{2}f(t,y) + \frac{1}{2}f(t+h, y+hf(t,y))$

Question 2: In this question we investigate an alternative to the Taylor method of order 2 using two separate evaluations of $f(t, y)$.

(a) Pick the coefficients a_1, a_2, α and β so that $a_1 f(t, y)+a_2 f(t+\alpha, y+\beta f(t, y))$ approximates $T^{(2)}(t, y) = f(t, y) + hf_t(t, y)/2 + hf(t, y)f_y(t, y)/2$ (f_ξ refers to a partial derivative of f with respect to the variable ξ).

(b) You should find the solution is non-unique. Set $a_1 = 1/2$, which gives the Modified Euler Method.

Euler 1st
Modife 2nd
Heun 3rd

(c) Find the truncation error of the Modified Euler Method. What is the order of this method? 2 $O(h^2)$

(d) For the IVP from question 1, approximate $y(1)$ with $h = 0.5$ using the Modified Euler Method by hand (include your calculations).

$f_t = \frac{\partial f}{\partial t} \leftarrow f(\cdot, \cdot)$

$f_y = \frac{\partial f}{\partial y} \leftarrow f(\cdot, \cdot)$

Modified Euler Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))], \quad \text{for} \quad i = 0, 1, \ldots, N-1.$$

$a_1 f(t,y) + a_2 f(t+\alpha, y+\beta f(t,y))$

$= a_1 f(t,y) + a_2 \left(f(t, y+\beta f(t,y)) + \alpha f_t(t, y+\beta f(t,y)) + O(\alpha^2) \right)$

$= a_1 f(t,y) + \underbrace{a_2 f(t, y+\beta f(t,y))}_{I} + \underbrace{a_2 \alpha f_t(t, y+\beta f(t,y))}_{J} + O(\alpha^2)$

$I = a_2 f(t, y+\beta f(t,y)) = a_2 \left(f(t,y) + (\beta f(t,y)) f_y(t,y) + O(\beta^2) \right)$

$= a_2 f(t,y) + a_2 \beta f(t,y) f_y(t,y) + O(\beta^2)$

$J = a_2 \alpha f_t(t, y+\beta f(t,y)) = a_2 \alpha \left(f_t(t,y) + (\beta f(t,y)) f_{ty}(t,y) + O(\beta^2) \right)$

$= a_2 \alpha f_t(t,y) + a_2 \alpha \beta f(t,y) f_{ty}(t,y) + O(\beta^2)$

$= a_1 f(t,y) + a_2 f(t,y) + a_2 \beta f(t,y) f_y(t,y) + a_2 \alpha f_t(t,y) + O(\alpha\beta) + O(\alpha^2) + O(\beta^2)$

$= f(t,y) + h f_t(t,y)/2 + h f(t,y) f_y(t,y)/2$

$= (a_1 + a_2) f(t,y) + a_2 \alpha f_t(t,y) + a_2 \beta f(t,y) f_y(t,y)$

$a_1 + a_2 = 1$ $\quad a_1 = \frac{1}{2} \quad a_2 = \frac{1}{2} \quad \alpha = h \quad \beta = h$

$a_2 \alpha = \frac{h}{2}$

$a_2 \beta = \frac{h}{2}$

$$w_{i+1} = w_i + h f(t_i, w_i)$$

$$y(t+h) = y(t) + h f(t,y) + O(h^2)$$

$$\frac{y(t+h) - y(t)}{h} - f(t,y) = O(h)$$

Modified Euler : $\alpha = h, \ \beta = h$

$y(t+h) = y(t) + h f(t,y)$

$y(t+h) = y(t) + \frac{h}{2}\left(f(t,y) + f(t+h, y + hf(t,y))\right)$

$$= y(t) + \frac{h}{2}\left(f(t,y) + f(t+h, y + hf(t,y)) + O(h^2)\right)$$

$$= y(t) + \underbrace{\frac{h}{2}\left(f(t,y) + f(t+h, y + hf(t,y))\right)}_{\boxed{}} + O(h^3)$$

$$\frac{y(t+h) - y(t)}{h} - \boxed{} = O(h^2)$$

Question 3: Heun's Method has error proportional to h^3, and the iterative step is,

$$w_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3\left\{f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right)\right\}\right].$$

(a) Write code to implement Heun's Method. Include the code with your homework submission.

(b) Verify numerically that Heun's Method has error proportional to h^3 by comparing the error in to Euler's Method and the Modified Euler Method and produce a plot showing the errors in each as h is varied.

$y' = -y^2 \qquad y(0) = 5$

$y = \frac{5}{5t+1} \qquad 0 \le t \le 1$

$y(t) = t^5 - \frac{1}{2}t^4 + t \qquad y(0) = 0$

$\frac{dy}{dt} = 5t^4 - 2t^3 + 1 = f(t, y)$

$y(t) = t^2 \sin(t) \qquad y(0) = 0$

$\frac{dy}{dt} = 2t \sin(t) + t^2 \cos(t)$

$= 2\frac{y}{t} + t^2 \cos(t)$