Question 1:
Plot the region of absolute stability for Euler's Method and the Midpoint method. Explain how to derive the equations that you plot.

The situation is similar for other one-step methods. In general, a function Q exists with the property that the difference method, when applied to the test equation, gives

$$w_{i+1} = Q(h\lambda)w_i. \tag{5.66}$$

Definition 5.25 The **region *R* of absolute stability** for a one-step method is $R = \{h\lambda \in \mathcal{C} \mid |Q(h\lambda)| < 1\}$, and for a multistep method, it is $R = \{h\lambda \in \mathcal{C} \mid |\beta_k| < 1$, for all zeros β_k of $Q(z, h\lambda)\}$. ■

Forward Euler

$$w_{i+1} = w_i + h f(t_i, w_i)$$

$$f(t,w) = \lambda w$$

$$w_{i+1} = w_i + h\lambda w_i = (1+h\lambda) w_i$$

$R = \{h\lambda \in C \mid |1+h\lambda| < 1\}$ ← Region of absolute stability for Euler.

Midpoint

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)$$
$$= w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h\lambda}{2} w_i\right)$$
$$= w_i + h\lambda\left(w_i + \frac{h\lambda}{2} w_i\right)$$
$$= \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right) w_i$$

$$R = \left\{h\lambda \in C \;\middle|\; \left|1 + h\lambda + \frac{(h\lambda)^2}{2}\right| < 1\right\}$$

$z = h\lambda$

$|1+z| < 1 \qquad z \in \mathbb{C} \qquad z = a+ib \qquad a, b \in \mathbb{R}$

$|1+a+ib| < 1$

$|a+ib| = \sqrt{a^2+b^2}$

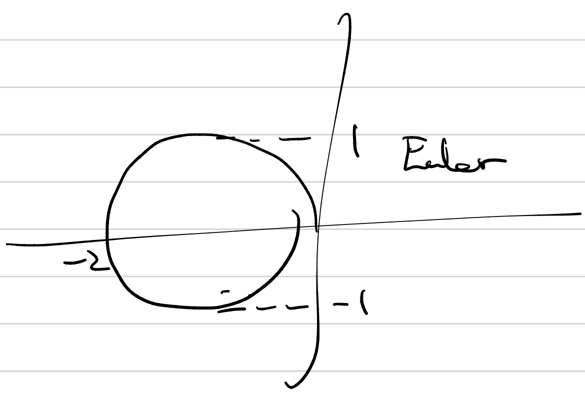
$\sqrt{(1+a)^2 + b^2} < 1$

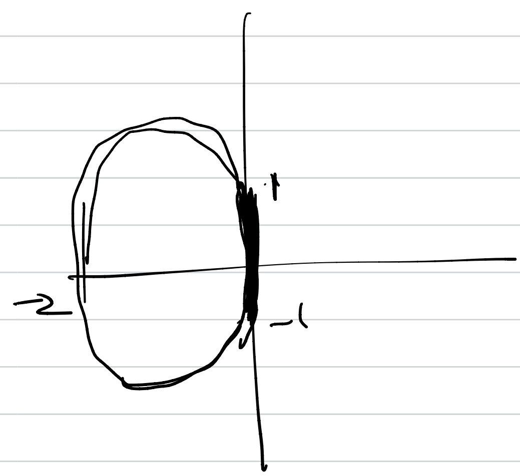
$\left|1+a+ib+\frac{(a+ib)^2}{2}\right| < 1$

$= \left|1+a+ib+\frac{a^2-b^2+2iab}{2}\right|$

$= \left|\left(1+a+\frac{a^2-b^2}{2}\right)+(b+ab)\,i\right| < 1$

$\left(\left(1+a+\frac{a^2-b^2}{2}\right)^2 + (b+ab)^2\right)^{1/2} < 1$



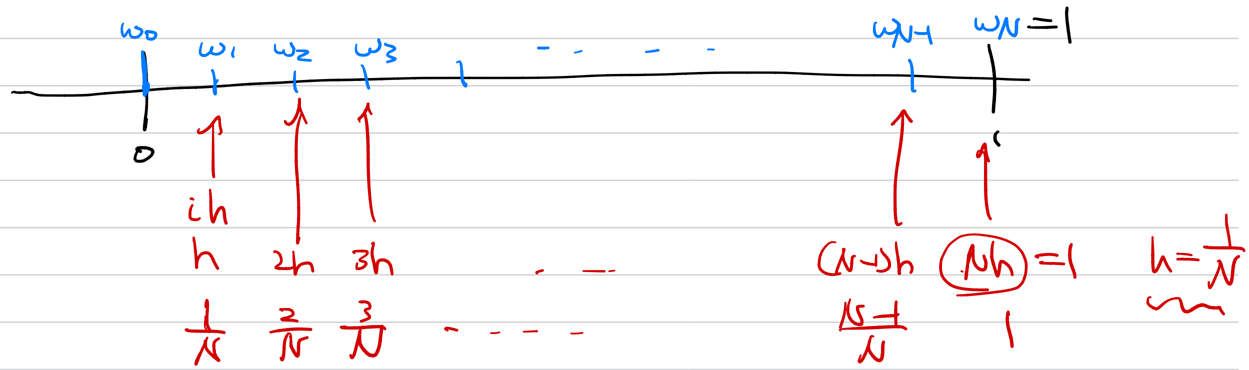


Question 4:
Consider a linear BVP,

$$y'' = 4y - 4x \ , 0 \le x \le 1 \tag{1}$$

with boundary conditions $y'(0) = 0$ and $y(1) = 1$, that is, the slope of the function at $x = 0$ should be zero, rather than the value. Use the finite difference method for linear BVPs to solve this problem. For an approximation with N grid-points, set up a system of $(N+1) \times (N+1)$ equations, where the first equation come from a backward difference approximation to $y(0)$ that is accurate to $O(h^2)$, and the last equation is simply $w_N = 1$. Then $i = 2, 3, ..., ..N-1$ equations should come from the central-finite-difference discretisation of the differential equation. Derive the formula you use for the backward finite difference approximation to $y'(0)$ (i.e. use the values w_0, w_1 and w_2 approximate $y'(0)$). Show and explain how to set up the matrix equation.

endpoints & interior points.



$$(w_i)' = \frac{w_{i+1} - w_i}{h}$$

$$\left((w_i)'\right)' = \frac{(w_i)' - (w_{i-1})'}{h} = \frac{\frac{w_{i+1} - w_i}{h} - \frac{w_i - w_{i-1}}{h}}{h}$$

$$= \frac{w_{i+1} - w_i - w_i + w_{i-1}}{h^2}$$

$$= \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$$

$$y''(x) - 4y(x) = -4x \qquad x = \frac{i}{N}$$

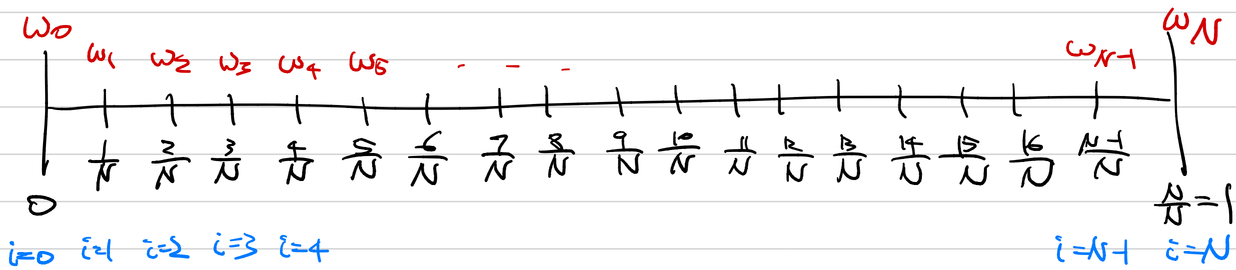
$$y''\left(\frac{i}{N}\right) - 4y\left(\frac{i}{N}\right) = -4\frac{i}{N}$$

$$w_i'' - 4w_i = -\frac{4i}{N}$$

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} - 4w_i = -4\left(\frac{i}{N}\right)$$

interior points.
$i = 1, \cdots, N-1$

1D



$y(0) \approx w_0 \qquad x=0$

$x=\frac{i}{N} \qquad y\left(\frac{i}{N}\right)=y(x)\approx w_i$

$\boxed{\text{Boundary}}$

$w_N = 1$

$$y(x+h) = y(x) + h\,y'(x) + \frac{h^2}{2}y''(x) + O(h^3)$$

Goal
approximate $y'(0) \approx w_0'$

$$y'(x) = \frac{y(x+h)-y(x)}{h} - \frac{h}{2}y''(x)$$

$$w_i' = \frac{w_{i+1}-w_i}{h} - \frac{h}{2}w_i''$$

$$= \frac{w_{i+1}-w_i}{h} - \frac{h}{2}\left(\frac{w_2-2w_1+w_0}{h^2}\right) = \frac{w_1-w_0}{h} - \frac{w_2-2w_1+w_0}{2h} = 0$$

$$(w_i') = \frac{w_{i+1}-w_i}{h}$$

$$((w_i)')' = \frac{(w_{i+1})'-(w_i)'}{h} = \frac{\frac{w_{i+2}-w_{i+1}}{h} - \frac{w_{i+1}-w_i}{h}}{h} = \frac{w_{i+2}-2w_{i+1}+w_i}{h^2}$$

$$(w_0)'' = \frac{w_2-2w_1+w_0}{h^2}$$

$$\begin{cases} i=0 & \dfrac{w_1-w_0}{h} - \dfrac{w_2-2w_1+w_0}{2h} = 0 \\ i=1\cdots N-1 & \dfrac{w_{i+1}-2w_i+w_{i-1}}{h^2} - 4w_i = -4\left(\dfrac{i}{N}\right) \\ i=N & w_N = 1 \end{cases}$$

$$\frac{1}{2h}\left(2w_1-2w_0-w_2+2w_1-w_0\right) = \frac{1}{2h}\left(-3w_0+4w_1-w_2\right)$$

$$= -\frac{3}{2h}w_0 + \frac{2}{h}w_1 - \frac{1}{2h}w_2 = 0$$

$$\frac{1}{h^2}w_{i-1} + \left(-4-\frac{2}{h^2}\right)w_i + \frac{1}{h^2}w_{i+1} = -\frac{4i}{N}$$

$$\frac{1}{h^2}w_0 + \left(-4-\frac{2}{h^2}\right)w_1 + \frac{1}{h^2}w_2 = -\frac{4}{N}$$

$$\begin{pmatrix} -\frac{3}{2h} & \frac{2}{h} & -\frac{1}{2h} & 0 & \cdots & 0 & \cdots & 0 \\ \frac{1}{h^2} & \left(-4-\frac{2}{h^2}\right) & \frac{1}{h^2} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \frac{1}{h^2} & \left(-4-\frac{2}{h^2}\right) & \frac{1}{h^2} & 0 & \cdots & & 0 \\ & \ddots & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & \cdots & \cdots & 0 & \frac{1}{h^2} & \left(-4-\frac{2}{h^2}\right) & \frac{1}{h^2} \\ 0 & 0 & \cdots & & 0 & 0 & 0\;0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ \vdots \\ w_{N-2} \\ w_{N-1} \\ w_N \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{4}{N} \\ -\frac{8}{N} \\ \vdots \\ \vdots \\ -\frac{4(N-1)}{N} \\ 1 \end{pmatrix} \begin{array}{l} i=0 \\ i=1 \\ i=2 \\ \vdots \\ i=N-1 \\ i=N \end{array}$$

Question 2: Use the finite difference method for nonlinear BVPs to solve

$$y'' = 2y^3, \quad -1 \le x \le 0, \quad y(-1) = \frac{1}{2}, \quad y(0) = \frac{1}{3},$$

using Broyden's Method to solve the system of equations. Explain how to derive the system of equations, and provide your code (you can use the Broyden's Method function from CCLE). Using $N = 7$, give your approximation for $y(-1/2)$.

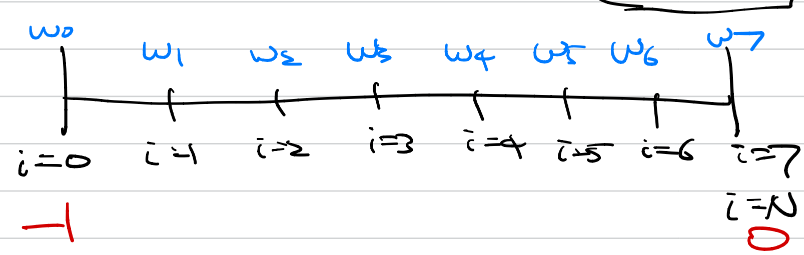
Goal: ①

$$\underbrace{\begin{pmatrix} & & \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} \\ \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} \\ \end{pmatrix}}_{b}$$

$$F(x) = Ax - b = 0$$

② Use Broyden's method $\quad F(x) = 0$

$$Ax - b = 0$$



$i = 1, \dots, 6 \qquad y'' - 2y^3 = 0$

$$\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - 2(w_i)^3 = 0$$

$i = 0 \qquad w_0 - \frac{1}{2} = 0$

$i = 7 \qquad w_7 - \frac{1}{3} = 0$

$$F(w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7) = 0$$

$$J(x)_{ij} = \frac{\partial f_i}{\partial x_j}(x)$$

$$\begin{pmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_0}{\partial x_1} & \frac{\partial f_0}{\partial x_2} & \frac{\partial f_0}{\partial x_3} & \cdots & \frac{\partial f_0}{\partial x_7} \\ \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_7} \\ \frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_7} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_7}{\partial x_0} & \frac{\partial f_7}{\partial x_1} & \frac{\partial f_7}{\partial x_2} & \frac{\partial f_7}{\partial x_3} & \cdots & \frac{\partial f_7}{\partial x_7} \end{pmatrix}$$

$$f_0(w) = w_0 - \frac{1}{2}$$

$$\frac{\partial f_0}{\partial x_0} = 1 \quad \frac{\partial f_0}{\partial x_1} = 0 \quad \frac{\partial f_0}{\partial x_2} = 0 \quad \cdots \quad \frac{\partial f_0}{\partial x_7} = 0$$

$$\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - 2(w_i)^3 = 0$$

$i=1$ $$\frac{w_0 - 2w_1 + w_2}{h^2} - 2w_1^3 = 0$$

$$\frac{\partial f_1}{\partial w_0} = \frac{1}{h^2} \quad \frac{\partial f_1}{\partial w_1} = -\frac{2}{h^2} - 6w_1^2 \quad \frac{\partial f_1}{\partial w_2} = \frac{1}{h^2} \quad \frac{\partial f_1}{\partial w_3} = 0 \quad \cdots \quad \frac{\partial f_1}{\partial w_7} = 0$$

$i=2$ $$\frac{w_1 - 2w_2 + w_3}{h^2} - 2w_2^3 = 0$$

$$\frac{\partial f_2}{\partial w_0} = 0 \quad \frac{\partial f_2}{\partial w_1} = \frac{1}{h^2} \quad \frac{\partial f_2}{\partial w_2} = -\frac{2}{h^2} - 6w_2^2 \quad \frac{\partial f_2}{\partial w_3} = \frac{1}{h^2} \quad \cdots$$

$i=6$

$\cdots$

$i=7$ $$f_7(w) = w_7 - \frac{1}{3} = 0$$

$$\frac{\partial f_7}{\partial w_0} = 0 \quad \cdots \quad \frac{\partial f_7}{\partial w_6} = 0 \quad \frac{\partial f_7}{\partial w_7} = 1$$

$$J(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2}-6\omega_1^2 & \frac{1}{h^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h^2} & -\frac{2}{h^2}-6\omega_2^2 & \frac{1}{h^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{h^2} & -\frac{2}{h^2}-6\omega_3^2 & \frac{1}{h^2} & 0 & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{h^2} & -\frac{2}{h^2}-6\omega_6^2 & \frac{1}{h^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Question 3: Use the method of steepest descent to to approximate the solution of

$$\begin{aligned} f_1(x) \quad & x_1^3 + x_1^2 x_2 - x_1 x_3 + 6 = 0 \\ f_2(x) \quad & e^{x_1} + e^{x_2} - x_3 = 0 \\ f_3(x) \quad & x_2^2 - 2x_1 x_3 - 4 = 0. \end{aligned}$$

with tolerance 0.01, and a starting guess $(1,1,1)^T$. Verify your answer by computing $\mathbf{F}(\mathbf{x})$.

If the tolerance is made much smaller, the algorithm will run for many more iterations. Describe how the approximations converge to the final answer if the tolerance is decreased to 10^{-5}.

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}$$

$$(x_1^3 + x_1^2 x_2 - x_1 x_3 + 6)^2 + (e^{x_1} + e^{x_2} - x_3)^2 + (x_2^2 - 2x_1 x_2 - 4)^2$$

$$g(x) = \sum_i f_i^2 = f_1^2 + f_2^2 + f_3^2$$

$$\nabla g = \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_3} \end{pmatrix} = \Big(2(x_1^3 + x_1^2 x_2 - x_1 x_2 + 6)^2 (2x_1 x_2 - x_3) + 2(\cdots$$

$$\frac{1}{h^2}\Big(y(x+h) - 2y(x) + y(x-h)\Big) =$$

$$\frac{1}{h^2}\left(\begin{array}{l} y(x) + h y'(x) + \frac{h^2}{2} y''(x) + \frac{h^3}{6} y'''(x) + O(h^4) \\ -2y(x) \\ +y(x) - h y'(x) + \frac{h^2}{2} y''(x) - \frac{h^3}{6} y'''(x) + O(h^4) \end{array}\right)$$

$$= \frac{1}{h^2}\left(h^2 y''(x) + O(h^4)\right) = y''(x) + \underline{O(h^2)}$$

Order 2.

For HW5 Q3, getting an equation for boundary condition at $x=0$.

$$y(2h) = y(0) + 2h\, y'(0) + \frac{4h^2}{2} y''(0) + \frac{8h^3}{6} y'''(0) + O(h^4)$$

$$y(h) = y(0) + h\, y'(0) + \frac{h^2}{2} y''(0) + \frac{h^3}{6} y'''(0) + O(h^4)$$

$$y(0) = y(0)$$

Find a, b, c such that

$$a y(2h) + b y(h) + c y(0) = h y'(0) + O(h^3).$$

Thus,

$$\left.\begin{array}{ll} a+b+c & =0 \\ 2a+b & =1 \\ 2a+\frac{1}{2}b & =0 \end{array}\right) \Rightarrow \quad a=-\frac{1}{2} \quad b=2 \quad c=-\frac{3}{2}$$

$$\Rightarrow \boxed{y'(0) = \frac{1}{h}\left(-\frac{1}{2} y(2h) + 2y(h) - \frac{3}{2} y(h)\right) + O(h^2).}$$

2nd order approximation of $y'(0)$.