Basic math for big data

Solution of Midterm

Question 1.

(a) 1) proposition 2) not proposition 3) proposition
(b) 1) $\neg\exists x P(x)$ 2) $\exists x\left(P(x) \wedge \forall y\big(P(y) \rightarrow (y = x)\big)\right)$
(c) 1) False 2) True 3) True

Question 2.

(a) This statement is false. Assume $X = \{1\}$ and $Y = \{2\}$. Then, $X \cap Y = \emptyset$, $Y - X = \{2\}$, and $(X \cap Y) \cup (Y - X) = \emptyset \cup \{2\} = \{2\} \neq X$.

(b) This statement is true. Both sides are equal to the set of all pairs (x, y) such that $x \in X, y \in Y$, and $y \notin Z$.

(c) This statement is false. Assume $X = Y = \{1\}$ and $Z = \{2\}$. Then, $Y \cup Z = \{1,2\}$ and $X - (Y \cup Z) = \emptyset$. However, since $X - Y = \emptyset$, $(X - Y) \cup Z = \emptyset \cup \{2\} = \{2\} \neq \emptyset = X - (Y \cup Z)$.

Question 3.

(a) 7
(b) 9+9+90 = 108
(c) $\frac{10!}{3!2!2!} = 151200$
(d) For each sequence of n tosses with k runs, let a_i be the length of i-th run. Then, every sequence of n tosses with k runs defines one sequence $a_1, a_2, \ldots, a_k$ such that $|a_1| + |a_2| + \cdots + |a_k| = n$ for $1 \leq i \leq k$. Each such sequence a_i corresponds to exactly the two following distinct sequences of tosses.

$$\underbrace{T \ldots T}_{a_1}\underbrace{H \ldots H}_{a_2}\underbrace{T \ldots T}_{a_3} \ldots$$
$$\underbrace{H \ldots H}_{a_1}\underbrace{T \ldots T}_{a_2}\underbrace{H \ldots H}_{a_3} \ldots$$

Therefore, the number of sequences of n tosses with k runs is twice the number of sequences a_i. The number of sequences a_i is $\binom{n-1}{k-1}$. Thus, the number of sequences of tosses that contains exactly k runs is $2 \cdot \binom{n-1}{k-1}$.

Question 4.

(a) $\{(1,1), (1,2), (2,1), (2,2)\}$
(b) 1) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(c) 1) Equivalence relation ,2) Not transitive, 3) Not reflexive, not symmetric, not transitive, 4) Equivalence relation, 5) Not reflexive, not transitive

(d) 1) No, 2) Yes, 3) Yes, 4) No