SNU Fourth Industrial Revolution Academy

Basic Math for Big Data

Homework 1

Due: July 6, 10:00 AM

Reminders

- T.A.: Chiwan Park (chiwanpark@snu.ac.kr)
- The points of this homework add up to 100.
- This has to be done individually like all the homeworks.
- Please answer clearly; illegible handwriting may get no points.
- Whenever you are making an assumption, please state it clearly.
- If you have a question about assignments, please upload your question in FIRA portal.

Submissions

- You can submit your homework in the class or via email (only PDFs are accepted).
- Do not submit the homework in a photography form.

Question 1 [12 points]

Let p, q, and r be the propositions:

- p: You get an A on the final exam.
- q: You do every exercise in our textbook.
- r: You get an A in this class.

Write the following propositions using p, q, and r and logical connectives including negations.

a) You get an A in this class, but you do not do every exercise in our textbook.

$$r \land \neg q$$

b) You get an A on the final, you do every exercise in our textbook, and you get an A in this class.

$$p \land q \land r$$

c) To get an A in this class, it is necessary for you to get an A on the final.

$$r \to p$$

d) You get an A on the final, but you do not do every exercise in our textbook; nevertheless, you get an A in this class.

$$p \land \neg q \land r$$

e) Getting an A on the final and doing every exercise in our book is sufficient for getting an A in this class.

$$(p \land q) \to r$$

f) You will get an A in this class if and only if you either do every exercise in our textbook or you get an A on the final.

$$(q \lor p) \leftrightarrow r$$

Question 2 [16 points]

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a duck", "x is one of my poultry", "x is an officer", and "x is willing to waltz", respectively. Express each of the following statements using quantifiers; logical connectives; $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) No ducks are willing to waltz.

$$\forall x\big(P(x) \rightarrow \neg S(x)\big) \text{ or } \neg\exists x\big(P(x) \wedge S(x)\big)$$

b) No officers ever decline to waltz.

$$\forall x\big(R(x) \rightarrow S(x)\big) \text{ or } \neg\exists x\big(R(x) \wedge \neg S(x)\big)$$

c) All my poultry are ducks.

$$\forall x\big(Q(x) \rightarrow P(x)\big) \text{ or } \neg\exists x\big(Q(x) \wedge \neg P(x)\big)$$

d) My poultry are not officers.

$$\forall x\big(Q(x) \rightarrow \neg R(x)\big) \text{ or } \neg\exists x\big(Q(x) \wedge R(x)\big)$$

Question 3 [22 points]

Prove that given a real number x there exist unique numbers n and ϵ such that $x = n - \epsilon$ where n is an integer, and $0 \leq \epsilon < 1$.

We should prove two things, existence and uniqueness.

1) Uniqueness

   Suppose $x = n - \epsilon = n' - \epsilon'$ where n and n' are integers, and $0 \leq \epsilon, \epsilon' < 1$. By subtraction, we find $n - n' = \epsilon - \epsilon'$. Since n and n' are integers and $|\epsilon - \epsilon'| < 1$, $|\epsilon - \epsilon'| = 0$. Thus, $n - \epsilon = n' - \epsilon$ and so $n = n'$. Therefore, the expansion of $x = n - \epsilon$ is unique.

2) Existence

   For a case $-1 < x \leq 0$, the statement is clearly true by taking $n = 0$ and $\epsilon = x$. Now we suppose that we know the theorem for $n - 1 < x' \leq n$ is true. Then, for $n < x \leq n + 1$, we can write $x - 1 = x' = n - \epsilon$. Thus, $x = (n + 1) - \epsilon$ as required. Without loss of generality, the result also holds for a case $n - 2 < x \leq n - 1$. By the two inductions, there is a pair of n and ϵ for any given x.

Question 4 [25 points]

The symmetric difference of sets A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B. Answer the following questions.

a) Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$. [4 points]

$$\{2,5\}$$

b) Show that $A \oplus B = (A \cup B) - (A \cap B)$. [7 points]

$$\begin{aligned} A \oplus B &= \{x | (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\} \\ &= \{x | (x \in A \cup B) \wedge \neg(x \in A \cap B)\} \\ &= \{x | (x \in A \cup B) \wedge (x \notin A \cap B)\} \\ &= \{x | x \in (A \cup B) - (A \cap B)\} \\ &= (A \cup B) - (A \cap B) \end{aligned}$$

c) Show that $A \oplus B = (A - B) \cup (B - A)$. [7 points]

$$\begin{aligned} (A - B) \cup (B - A) &= \left(A \cap \overline{B}\right) \cup \left(B \cap \overline{A}\right) \\ &= \left(\left(A \cap \overline{B}\right) \cup B\right) \cap \left(\left(A \cap \overline{B}\right) \cup \overline{A}\right) \\ &= \left((A \cup B) \cap \left(\overline{B} \cup B\right)\right) \cap \left(\left(A \cup \overline{A}\right) \cap \left(\overline{B} \cup \overline{A}\right)\right) \\ &= (A \cup B) \cap \left(\overline{B} \cup \overline{A}\right) \\ &= (A \cup B) \cap \overline{A \cap B} \\ &= (A \cup B) - (A \cap B) \end{aligned}$$

From above, the set $(A - B) \cup (B - A)$ is equal to the set $(A \cup B) - (A \cap B)$. Therefore, $A \oplus B = (A - B) \cup (B - A)$ by b).

d) Show that $(A \oplus B) \oplus B = A$. [7 points]

We can prove by using a membership table.

A	B	$A \oplus B$	$(A \oplus B) \oplus B$
1	1	0	1
1	0	1	1
0	1	1	0
0	0	0	0

Question 5 [10 points]

A palindrome is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

Let f_n be the number of bit strings of length n are palindromes. Then, $f_1 = 2$. For $n = 2k$, we obtain $f_n = f_k$ because the left half of bit string determines the right half of bit string. For $n = 2k + 1$, we obtain $f_n = 2 \cdot f_k$ because there are two choices (0 and 1) for $(k + 1)$-th bit. Combining the equations, $f_n = 2 \cdot f_{n-2}$ for $n \geq 3$. By the recursive equation, we obtain the following answer.

$$f_n = \begin{cases} 2^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ 2^{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

Question 6 [15 points]

Let $n_1, n_2, \dots n_t$ be positive integers. Show that if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some i where $0 < i \leq t$, the i-th box contains at least n_i objects.

Suppose that the statement is false. Then for each $0 < i \leq t$, the i-th box contains at most $n_i - 1$ objects. Adding all the number of objects in the i-th box, we have at most $(n_1 - 1) + (n_2 - 1) + \cdots + (n_t - 1) = n_1 + n_2 + \cdots + n_t - t$ objects. This contradicts the assumption that there are $n_1 + n_2 + \cdots + n_t - t + 1$ objects in all. Therefore, the statement is true.