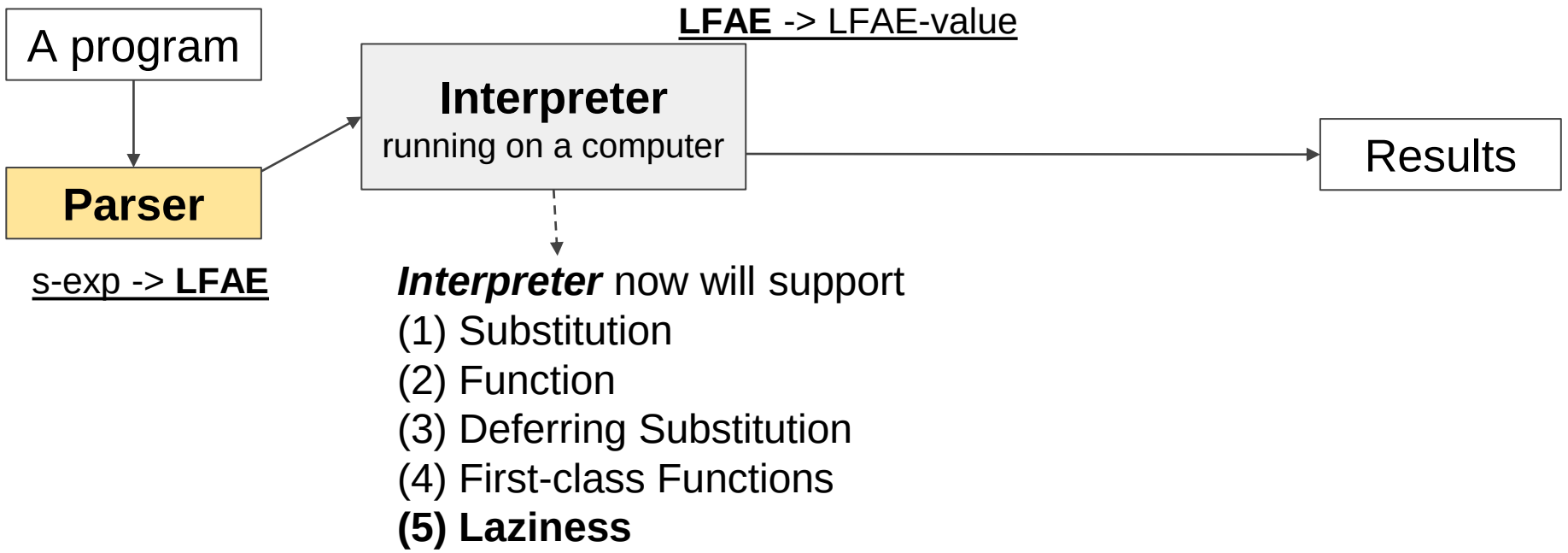
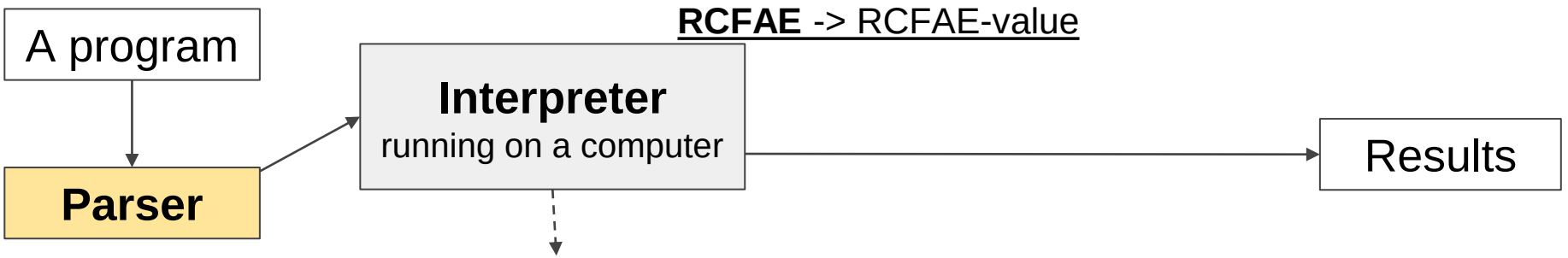
ITP20005
Recursion

Lecture15
JC

Big Picture (modeling languages: substitution)



Big Picture (modeling languages: substitution)



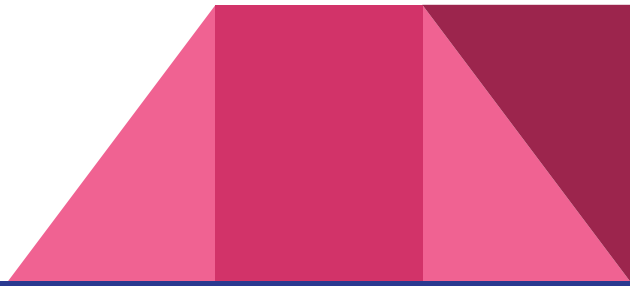
Interpreter now will support

(1) Substitution
(2) Function
(3) Deferring Substitution
(4) First-class Functions
(5) Laziness
(6) Recursion

FAE Values

A way to keep static scope for free ids of a function….

```
(define-type FAE-Value
   [numV      …]
   [closureV  …])
(define-type DefrdSub
   [mtSub]
   [aSub (name symbol?) (value FAE-Value?) (ds DefrdSub?)])
```



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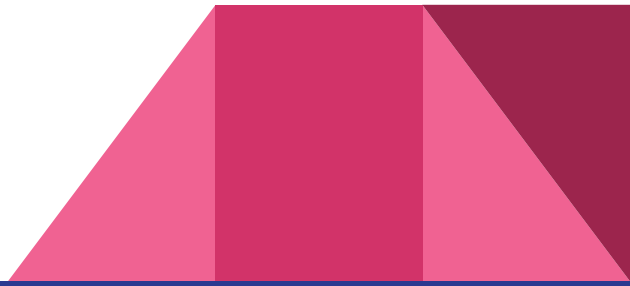

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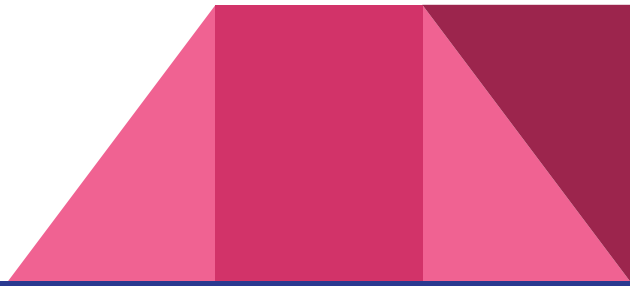
```



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(test (interp (parse '{with {y 10} {fun {x} {+ y x}}}))
      …)
```

FAE Values

A way to keep static scope for free ids of a function….

```
(define-type FAE-Value
   [numV      (n number?)]
   [closureV  (param symbol?) (body FAE?) (ds DefrdSub?)])
(define-type DefrdSub
   [mtSub]
   [aSub (name symbol?) (value FAE-Value?) (ds DefrdSub?)])

(test (interp (parse '{with {y 10} {fun {x} {+ y x}}}))
      (closureV 'x (add (id 'y) (id 'x))
                   (aSub 'y (numV 10) (mtSub))))
```

F1WAE Interpreter with Defrdsub

```
; interp : F1WAE list-of-FucDef DefrdSub -> number
(define (interp f1wae fundefs ds)
 (type-case F1WAE f1wae
  [num  (n)      n]
  [add  (l r)    (+ (interp l fundefs ds) (interp r fundefs ds))]
  [sub  (l r)    (- (interp l fundefs ds) (interp r fundefs ds))]
  [with (i v e)  (interp e fundefs (aSub i (interp v fundefs ds) ds))]
  [id     (s)    (lookup s ds)]
  [app  (f a)    (local
                                [(define a-fundef (lookup-fundef f fundefs))]
                                (interp (fundef-body a-fundef)
                                    fundefs
                                (aSub (fundef-arg-name a-fundef)
                                            (interp a fundefs ds)
                                                (mtSub))
                        ))])))

(test (interp (parse '{f 1}) (list (parse-fd '{deffun (f x) {+ x 3}})) (mtSub)) 4)
```

FAE Interpreter with Deferred Substitution

```
; interp: FAE DefrdSub -> FAE-Value
(define (interp fae ds)
   (type-case FAE fae
     …
      [app    (f a)   (local [(define f-val (interp f ds))
                              (define a-val (interp a ds))]
                        (interp (closureV-body f-val)
                                (aSub (closureV-param f-val)
                                      a-val
                                      ...)))]))
```

FAE Interpreter with Deferred Substitution

```
; interp: FAE DefrdSub -> FAE-Value
(define (interp fae ds)
   (type-case FAE fae
     …
      [app    (f a)   (local [(define f-val (interp f ds))
                              (define a-val (interp a ds))]
                        (interp (closureV-body f-val)
                                (aSub (closureV-param f-val)
                                      a-val
                                      (closureV-ds f-val))))]))
```

Examples

Compare:

```
{with {y 4}
      {{fun {x} {+ y x}} 10}}
```

with:

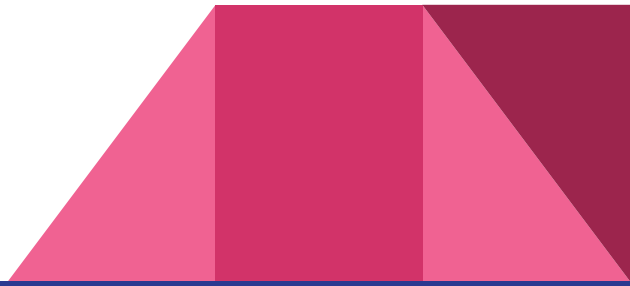
```
{with {f {fun {x} {+ y x}}}
      {with {y 4} {f 10}}}
```

No More 'with'??

```
{with {x 10} x}
```

is the same as

```
{{fun {x} x} 10}
```



No More 'with'

{with {x 10} x}

is the same as

{{fun {x} x} 10}

In general,

{with {<id> <FWAE>$_1$} <FWAE>$_2$}

is the same as

{{fun {<id>} <FWAE>$_2$} <FWAE>$_1$}

No More 'with'

{with {x 10} x}

is the same as

{{fun {x} x} 10}

In general,

{with {<id> <FWAE>$_1$} <FWAE>$_2$}

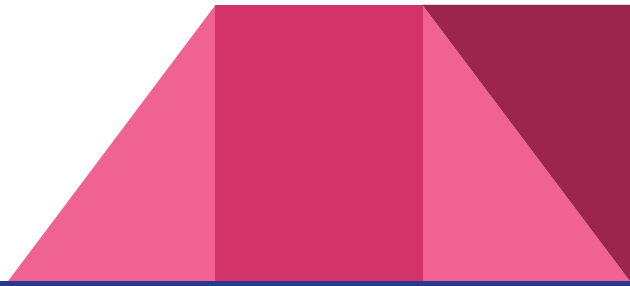
is the same as

{{fun {<id>} <FWAE>$_2$} <FWAE>$_1$}

Let's assume

(with '<id> <FWAE>$_1$ <FWAE>$_2$)

$\Rightarrow$ (app (fun '<id> <FWAE>$_2$) <FWAE>$_1$)



Today, we are going to do logical thinking for recursion.

This will be helpful for implementing recursion for our language in the next lecture.

RCFAE: Concrete Syntax

<RCFAE> ::= <num>
| {+ <RCFAE> <RCFAE>}
| {- <RCFAE> <RCFAE>}
| {* <RCFAE> <RCFAE>}
| <id>
| {fun {<id>} <RCFAE>}
| {<RCFAE> <RCFAE>}
| {if0 <RCFAE> <RCFAE> <RCFAE>}
| {rec {<id> <RCFAE>} <RCFAE>}

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| {<RCFAE> <RCFAE>}
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| {rec {<id> <RCFAE>} <RCFAE>}

Our language now support multiplication for some famous recursive examples ;)

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| <id>
| {fun {<id>} <RCFAE>}
| {<RCFAE> <RCFAE>}
| {if0 <RCFAE> <RCFAE> <RCFAE>}
| {rec {<id> <RCFAE>} <RCFAE>}

For recursive function, we need to support a conditional expression.
if (0==<RCFAE>)
    <RCFAE>
else
    <RCFAE>

RCFAE: Concrete Syntax

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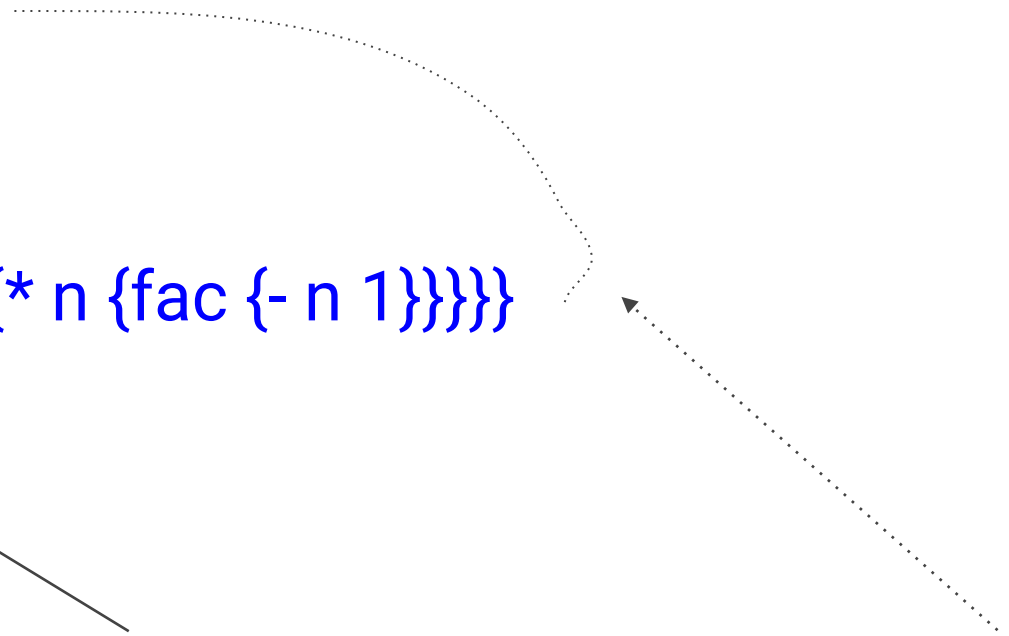
| <id>
| {fun {<id>} <RCFAE>}
| {<RCFAE> <RCFAE>}
| {if0 <RCFAE> <RCFAE> <RCFAE>}
| {rec {<id> <RCFAE>} <RCFAE>}

Syntax for defining a recursive function and its call

Factorial - our language looks like this for recursion

```
{rec {fac {fun {n}
               {if0 n
                    1
                    {* n {fac {- n 1}}}}}}
     {fac 10}}
⇒ 3628800
```

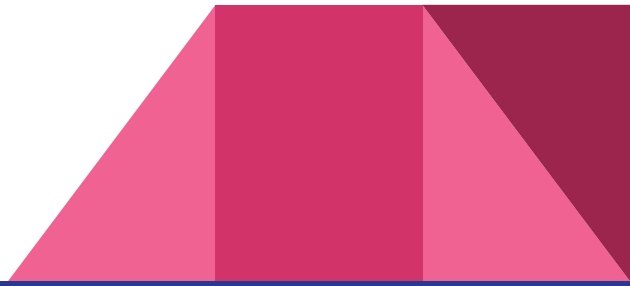


rec binds both in the body expression and in the binding expression.

Factorial

```
{rec {fac {fun {n}
                 {if0 n
                      1
                       {* n {fac {- n 1}}}}}}
     {fac 10}}
```

How about just using 'with'??



Factorial - Let's think this using 'with'?

```
{with {fac {fun {n}
                {if0 n
                     1
                     {* n {fac {- n 1}}}}}}
      {fac 10}}
```

Factorial - Let's think this using 'with'?

```
{with {fac {fun {n}
                {if0 n
                     1
                     {* n {fac {- n 1}}}}}}
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```

⇒ free identifier 'fac'!

Doesn't work: with does not support recursive definitions

Factorial - Let's think this using 'with'?

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      {fac 10}}
```

⇒ free identifier 'fac'!

Doesn't work: with does not support recursive definitions. Still, at the point what we call fac, obviously we have a binding id *fac* for fac…

Factorial - Let's think this using 'with'?

```
{with {fac {fun {n}
                {if0 n
                     1
                     {* n {fac {- n 1}}}}}}
      {fac 10}}
```

⇒ free identifier 'fac'!

Doesn't work: with does not support recursive definitions. Still, at the point what we call fac, obviously we have a binding id *fac* for fac…

… so pass {fac 10} as *an argument*!

Factorial (Assume that a function can have multiple parameters.)

```
{with {facX {fun {facY n}
                 {if0 n
                      1
                       {* n {facY facY {- n 1}}}}}}
      {facX facX 10}}
```

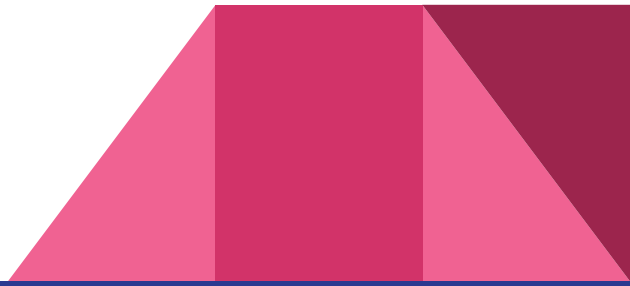
Factorial

```
{with {facX {fun {facY n}
                 {if0 n
                      1
                      {* n {facY facY {- n 1}}}}}}
      {facX facX 10}}
```

Wrap this to fac back...

Factorial

```
{with {fac
         {fun {n}
              {with {facX {fun {facY n}
                               {if0 n
                                    1
                                    {* n {facY facY {- n 1}}}}}}
                    {facX facX n}}}}
       {fac 10}}
```



Factorial

```
{with {fac
           {fun {n}
                {with {facX {fun {facY n}
                                 {if0 n
                                      1
                                      {* n {facY facY {- n 1}}}}}}
                      {facX facX n}}}}
      {fac 10}}
```

But the language we implement has *only single-argument* functions...

From Multi-Arg. to Single-Arg.

```
{with {f {fun {x y z} {+ z {+ y x}}}}
      {f 1 2 3 }}
```

⇒ Rewrite this using a function only with one parameter?

From Multi-Arg. to Single-Arg.

```
{with {f {fun {x y z} {+ z {+ y x}}}}
      {f 1 2 3 }}
```

$\Rightarrow$

```
{with {f {fun {x}
              {fun {y}
                   {fun {z}
                        {+ z {+ y x}}}}}}
      {{{f 1} 2} 3}}
```

Factorial

```
{with {fac
          {fun {n}
               {with {facX
                          {fun {facY n}
                               {if0 n
                                    1
                                    {* n {facY facY {- n 1}}}}}}
                     {facX facX n}}}}
      {fac 10}}
```

Factorial

```
{with {fac
           {fun {n}
                {with {facX
                           {fun {facY}
                                {fun {n}
                                     {if0 n
                                     1
                                     {* n {{facY facY} {- n 1}}}}}}}
                      {{facX facX} n}}}}
      {fac 10}}
```

Factorial

```
{with {fac
          {fun {n}
               {with {facX
                          {fun {facY}
                               {fun {n}
                                    {if0 n
                                    1
                                    {* n {{facY facY} {- n 1}}}}}}}
                     {{facX facX} n}}}}
      {fac 10}}
```

Simplify: {fun {n} {with {f ...} {{f f} n}}}

⇒ {with {f ...} {f f}} by "η reduction"

η reduction (eta reduction)

- If two functions lead to the same result, they are the same functions.
- {fun {n} {{fun {x} x} n}}
  {{fun {n} {{fun {x} x} n}} 2} Result: 2
- {fun {x} x}
  {{fun {x} x} 2} Result: 2
- {fun {n} {e n}} ⇒ e where n is not free in e.
  {fun {n} {{fun {x} {+ n x} n}
  {fun {x} {+ n x}} η reduction is not possible as n is free in {fun {x} n}.

η reduction (eta reduction)

- If two functions lead to the same result, they are the same functions.
- {fun {n} {{fun {x} x} n}}
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- {fun {n} {e n}} ⇒ e where n is not free in e.
  {fun {n} {{fun {x} {+ n x} n}
  {fun {x} {+ n x}}

η reduction is not possible as n is free in {fun {x} {+ n x}.

Free id: we cannot do η reduction for this code.

Factorial

```
{with {fac
           {fun {n}
                {with {facX
                           {fun {facY}
                                {fun {n}
                                     {if0 n
                                     1
                                     {* n {{facY facY} {- n 1}}}}}}}
                      {{facX facX} n}}}}
      {fac 10}}
```

"η reduction"

{fun {n} {e n}} ⇒ e where n is not free in e.

Factorial

```
{with {fac
         {fun {n}
              {with {facX
                        {fun {facY}
                             {fun {n}
                                  {if0 n
                                       1
                                       {* n {{facY facY} {- n 1}}}}}}}
                    {{facX facX} n}}}}
       {fac 10}}
```

This 'n' is a binding id.

"η reduction"

{fun {n} {e n}} ⇒ e where n is not free in e.

Factorial

```
{with {fac
          {fun {n}
               {with {facX
                          {fun {facY}
                               {fun {n}
                                    {if0 n
                                    1
                                    {* n {{facY facY} {- n 1}}}}}}}}
                     {{facX facX} n}}}}
          {fac 10}}
```

"η reduction"

{fun {n} {e n}} ⇒ e where n is not free in e.

Factorial

```
{with {fac
           {with {facX
                      {fun {facY}
                           {fun {n}
                                {if0 n
                                     1
                                     {* n {{facY facY} {- n 1}}}}}}}}
                 {facX facX}}}
       {fac 10}}
```

Factorial

```
{with {fac
          {with {facX
                    {fun {facY}     ; Almost original fac
                         {fun {n}
                              {if0 n
                                   1
                                   {* n {{facY facY} {- n 1}}}}}}}
                {facX facX}}}
      {fac 10}}
```

Factorial - Original

```
{with {fac
        {fun {n}
             {if0 n
                  1
                  {* n {fac {- n 1}}}}}}
  {fac 10}}
```

Factorial

```
{with {fac
        {with {facX
                {fun {facY}     ; Almost original fac
                  {fun {n}
                    {if0 n
                         1
                         {* n {{facY facY} {- n 1}}}}}}}
              {facX facX}}}
       {fac 10}}
```

Make this to be substituted by fac.

More like original: introduce a local binding for {facY facY}...

Factorial

```
{with {fac
          {with {facX
                    {fun {facY}
                         {with {fac {facY facY}}
                               ; Exactly like original fac
                               {fun {n}
                               {if0 n
                                    1
                                    {* n {fac {- n 1}}}}}}}}
                    {facX facX}}}
      {fac 10}}
```

Factorial

```
{with {fac
          {with {facX
                    {fun {facY}
                         {with {fac {facY facY}}
                               ; Exactly like original fac
                               {fun {n}
                               {if0 n
                                    1
                                    {* n {fac {- n 1}}}}}}}}
                    {facX facX}}}
       {fac 10}}
```

Opps! - this is an infinite loop

We used to evaluate {facY facY} only when n is non-zero.

Delay {facY facY}...

Can you improve our language to support 'if0'?

This code can run after delaying {facY facY}

(But we will implement a complete and general interpreter in the next class)

We may apply same logic for other recursive examples.

Factorial

```
{with {fac
          {with {facX
                    {fun {facY}
                         {with {fac {fun {x} {{facY facY} x}}}
                               ; Exactly like original fac
                               {fun {n}
                                    {if0 n
                                         1
                                         {* n {fac {- n 1}}}}}}}
                    {facX facX}}}
      {fac 10}}
```

Factorial

```
{with {fac
          {with {facX
                    {fun {facY}
                         {with {fac {fun {x} {{facY facY} x}}}
                               ; Exactly like original fac
                               {fun {n}
                                    {if0 n
                                         1
                                         {* n {fac {- n 1}}}}}}}
                    {facX facX}}}
       {fac 10}}
```

Now, what about fib, sum, etc.?

Abstract over the fac-specific part...

Make-Recursive and Factorial

```
{with {mk-rec {fac {body-proc}
                   {with {fX {fun {fY}
                                  {with {f {fun {x}
                                                {{fY fY} x}}}
                                        {body-proc f}}}}
                         {fX fX}}}}
      {with {fac {mk-rec
                         {fun {fac}
                              ; Exactly like original fac
                              {fun {n}
                                   {if0 n
                                        1
                                        {* n {fac {- n 1}}}}}}}}
            {facX facX}}}
```

Fibonacci

```
{with {fib {mk-rec
                    {fun {fib}
                         ; Usual fib
                         {fun {n}
                              {if {or {= n 0} {= n 1}}
                                  1
                                   {+ {fib {- n 1}}
                                      {fib {- n 2}}}}}}}}
      {fib 5}}
```

Sum

```
{with {sum {mk-rec
                    {fun {sum}
                         ; Usual sum
                         {fun {l}
                              {if {empty? l}
                                  0
                                   {+ {first l}
                                       {sum {rest l}}}}}}}}}
      {sum '{1 2 3 4}}}
```

Do you want to use recursion in such a complicated way?

RCFAE: Concrete Syntax

<RCFAE> ::= <num>
| {+ <RCFAE> <RCFAE>}
| {- <RCFAE> <RCFAE>}
| {* <RCFAE> <RCFAE>}
| <id>
| {fun {<id>} <RCFAE>}
| {<RCFAE> <RCFAE>}
| {if0 <RCFAE> <RCFAE> RCFAE>}
| {rec {<id> <RCFAE>} <RCFAE>}

Syntax for defining a recursive function and its call

Factorial

```
{rec {fac {fun {n}
                 {if0 n
                      1
                       {* n {fac {- n 1}}}}}}
     {fac 10}}
```

Topics we cover and schedule (tentative)

- Racket tutorials (L2,3, HW)
- Modeling languages (L4,5, HW
- Interpreting arithmetic (L5)
- Language principles
  - **Substitution** (L6, HW)
  - **Function** (L7)
  - **Deferring Substitution** (L8,L9)
  - **First-class Functions** (L10-12)
  - **Laziness** (L13, L14)
  - **Recursion** (**L15**, L16)
  - Representation choices
  - Mutable data structures
  - Variables
  - Continuations
  - Garbage collection
  - Semantics
  - Type
- Guest Video Lecture

No class: October 2 (Fri, Chuseok), October 9 (Fri, Hangul day)
Online only class can be provided.

TODO

Read Chapter 9. Implementing Recursion

JC

jcnam@handong.edu

https://lifove.github.io

* Slides are from Prof. Sukyoung Ryu's PL class in 2018 Spring or modified/created by JC based on the main text book.