Bottom Up Parsing

PITT CS 1622

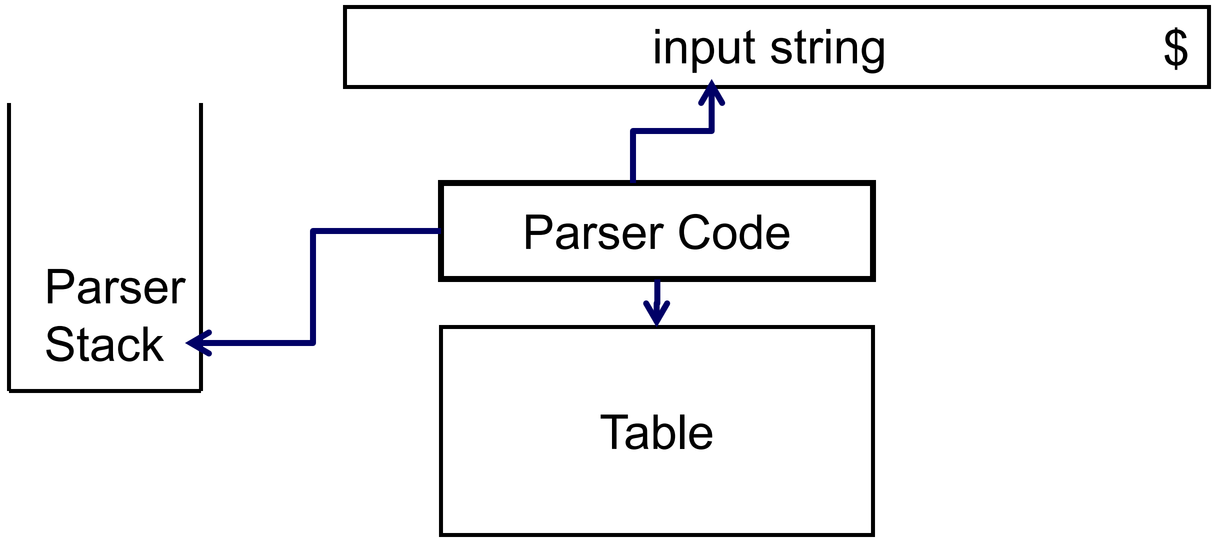
Bottom Up Parsing

- More powerful than top down
  - Don’t need left factored grammars
  - Can handle left recursion
  - Can parse a larger set of grammars (and languages)

- Begins at leaves and works to the top
  - In reverse order of rightmost derivation
    (In effect, builds tree from left to right)

- Also known as Shift-Reduce parsing
  - Involves two types of operations: shift and reduce

Parser Implementation

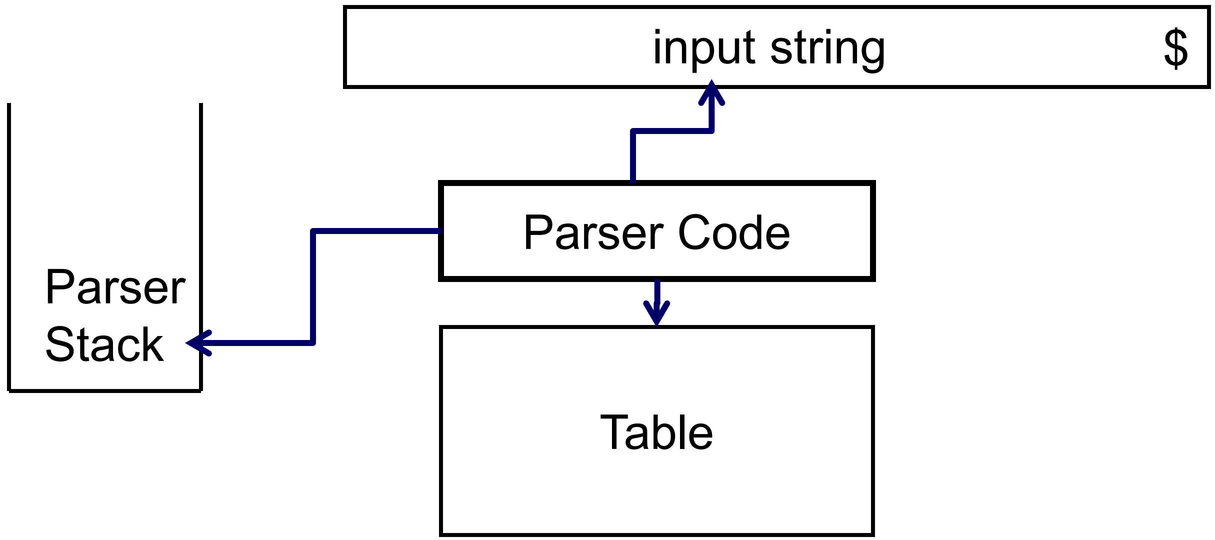


Parser Stack – holds consumed portion of derivation string

Table – "actions" performed based on rules of grammar, and current state of stack and input string

Parser Code – next action based on **[stack top, lookahead token(s)]**

Parser Implementation



Actions
1. **Shift** – consume input symbol and push symbol onto the stack
2. **Reduce** – pop RHS at stack top and push LHS of a production rule, reducing stack contents
3. Accept – success (when reduced to start symbol and input at $)
4. Error

Bottom-Up compared to Top-Down

- Conceptual difference in how the stack works:

	Stack Content At Input Start	Stack Content At Input End	Stack Represents	Key Operations
Top-Down	**Start Symbol**	**Nothing**	Unconsumed input	Match / Expand
Bottom-up	**Nothing**	**Start Symbol**	Consumed input	Shift / Reduce

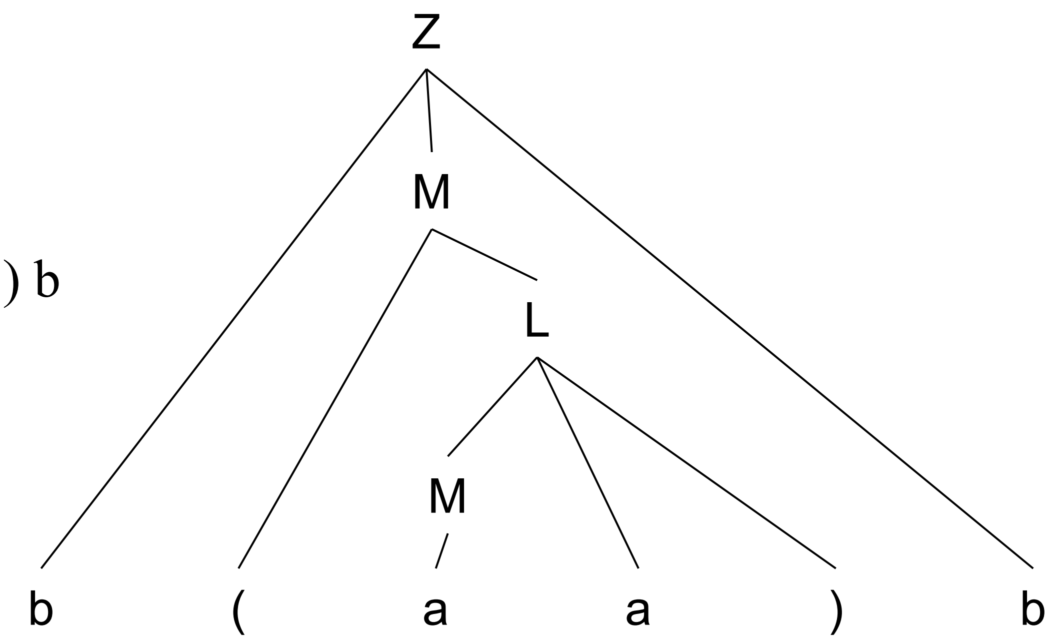
- But both use a stack to parse languages with nested structures
  - Not surprising since CFGs are parsed using Pushdown Automata!

$Z \to b\ M\ b$

$M \to (\ L \mid a$

$L \to M\ a\ ) \mid)$

Considering string: w = b (a a) b



The rightmost derivation of this parse tree:

$Z \Rightarrow b\ M\ b \Rightarrow b\ (\ L\ b \Rightarrow b\ (\ M\ a\ )\ b \Rightarrow b\ (\ a\ a\ )\ b$

Bottom up parsing involves finding "handles" (RHSs) to reduce

$b\ (\ a\ a\ )\ b \Rightarrow b\ (\ M\ a\ )\ b \Rightarrow b\ (\ L\ b \Rightarrow b\ M\ b \Rightarrow Z$

$Z \rightarrow b\ M\ b$
$M \rightarrow (\ L \mid a$
$L \rightarrow M\ a\ ) \mid)$

String
b (a a) $

Stack	Input	Action
$	b (a a) b $	shift
$ b	(a a) b $	shift
$ b (	a a) b $	shift
$ b (a	a) b $	reduce
$ b (M	a) b $	shift
$ b (M a	) b $	shift
$ b (M a)	b $	reduce
$ b (L	b $	reduce
$ b M	b $	shift
$ b M b	$	reduce
$ Z	$	accept

Sentential Form and Handle

- **Sentential form**: Any string derivable from the start symbol
- **Handle**: RHS of a production rule that, when reduced to LHS in a sentential form, will lead to another sentential form

- Definition:
  - Let $\alpha\beta w$ be a sentential form where
    α, β is a string of terminals and non-terminals
    w is a string of terminals
    $X\rightarrow\beta$ is a production rule
    Then β is a handle of $\alpha\beta w$ if
    $S \Rightarrow^* \alpha X w \Rightarrow \alpha\beta w$ by a rightmost derivation
  - Handles formalize the intuition "β should be reduced to X for a successful parse", but does not really say how to find them

Single Pass Left-to-Right Scan

- Note in the formulation of a handle $S \Rightarrow^* \alpha X w \Rightarrow \alpha\beta w$
  - α is a string of terminals and non-terminals
  - w is a string of only terminals
  - Why is this so?
- Proof by example
  - Given $S \rightarrow ABCD$, $A \rightarrow a$, $B \rightarrow b$, $C \rightarrow c$, $D \rightarrow d$
  - $S \Rightarrow ABCD \Rightarrow ABCd \Rightarrow ABcd \Rightarrow Abcd \Rightarrow abcd$
- Mathematical proof
  - Let's assume w contained a non-terminal Y ($w = w_1 Y w_2$)
  - Then, $S \Rightarrow^* \alpha X w_1 Y w_2 \Rightarrow \alpha\beta w_1 Y w_2$
  - Above is not a rightmost derivation (you derived X before Y)
  - Contradiction!

Single Pass Left-to-Right Scan

- Note in the formulation of a handle $S \Rightarrow^* \alpha X w \Rightarrow \alpha\beta w$
  - α is a string of terminals and non-terminals
  - w is a string of only terminals

- Why is this important?
  - $\alpha\beta$ is consumed input in the stack and w is unconsumed input
  - The reduced handle β is always at the top of the stack
  - No need to view middle of the stack to reduce!
  - No need to view unconsumed input to reduce!

  → Amenable to single pass left-to-right scan using a stack

Handle Always Occurs at Top of Stack

- Grammar

  $E \rightarrow E+E$

  $E \rightarrow E*E$

  $E \rightarrow (E)$

  $E \rightarrow id$

Sentential form	Handle	Production
$id_1 + id_2 * id_3$	id_1	$E \rightarrow id$
$E + id_2 * id_3$	id_2	$E \rightarrow id$
$E + E * id_3$	id_3	$E \rightarrow id$
$E + E * E$	$E*E$	$E \rightarrow E*E$
$E + E$	$E+E$	$E \rightarrow E+E$
E		

- # indicates top of stack (at the frontier of reduction where the handle is)
  Left of # : stack contents, Right of # : unconsumed input string

  $id_1 \# + id_2 * id_3 \Rightarrow E \# + id_2 * id_3 \Rightarrow E + \# id_2 * id_3 \Rightarrow E + id_2 \# * id_3$

  $\Rightarrow E + E \# * id_3 \Rightarrow E + E * id_3 \# \Rightarrow E + E * E \# \Rightarrow E + E \# \Rightarrow E$

- Stack works because the reduction $X \rightarrow \beta$ always happens at the top of the stack

Handle Always Occurs at Top of Stack

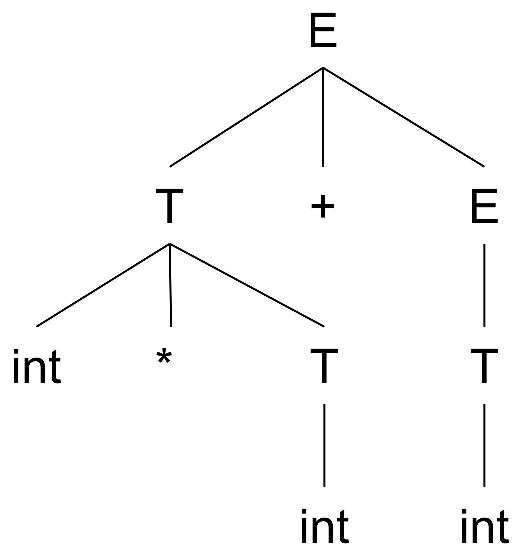
- Consider our usual grammar

$$E \rightarrow T + E \mid T$$

$$T \rightarrow \text{int} * T \mid \text{int} \mid (E)$$

Consider the string: int * int + int

sentential form	*production*
int * int # + int	$T \rightarrow$ int
int * T # + int	$T \rightarrow$ int * T
T + int #	$T \rightarrow$ int
T + T #	$E \rightarrow T$
T + E #	$E \rightarrow T + E$
E #	



- Reduction of a handle always happens at the top of the stack

Ambiguous Grammars

- Conflicts arise with ambiguous grammars
  - Just like LL parsing, bottom-up parsing tries to predict the correct action
  - But if there are multiple correct actions, conflicts arise
- Example:
  - Consider the ambiguous grammar

$$E \rightarrow E * E \mid E + E \mid (E) \mid int$$

Sentential form	Actions
int * int + int	shift
…	…
E * E # + int	**reduce E → E * E**
E # + int	shift
E + # int	shift
E + int #	reduce E → int
E + E #	reduce E → E + E
E #	

Sentential form	Actions
int * int + int	shift
…	…
E * E # + int	**shift**
E * E + # int	shift
E * E + int #	reduce E → int
E * E + E #	reduce E → E + E
E * E #	reduce E → E * E
E #	

Ambiguity

- Previous shift-reduce conflict occurred because of ambiguity
  - Due to lack of precedence between + and * in the grammar
  - Ambiguity shows up as "conflicts" in the parsing table
    (More than one action in parse table, just like for LL parsers)

- Shift-reduce conflict also occurs with input "int + int + int"
  - Due to ambiguous associativity of * and +

- Can always rewrite to encode precedence and associativity
  - But can sometimes result in convoluted grammars
  - Tools have other means to encode precedence and associativity

    %left '+' '–'
    %left '*' '/'

Properties of Bottom Up Parsing

- Handles always appear at the top of the stack
  - Never in middle of stack
- Easily generalized shift – reduce strategy
  - If there is a handle at top of stack, reduce to LHS non-terminal
  - If there is no handle at the top of the stack, shift input token
  - Easy to automate parser using a table [stack top, lookahead token(s)]
- Can have conflicts
  - If it is legal to either shift or reduce then there is a shift-reduce conflict.
  - If there are two legal reductions, then there is a reduce-reduce conflict.
  - Most often occur because of ambiguous grammars
    - In rare cases, because of non-ambiguous grammars not amenable to parser

Types of Bottom Up Parsers

- Types of bottom up parsers
  - Simple precedence parsers
  - Operator precedence parsers
  - LR family parsers
  - …

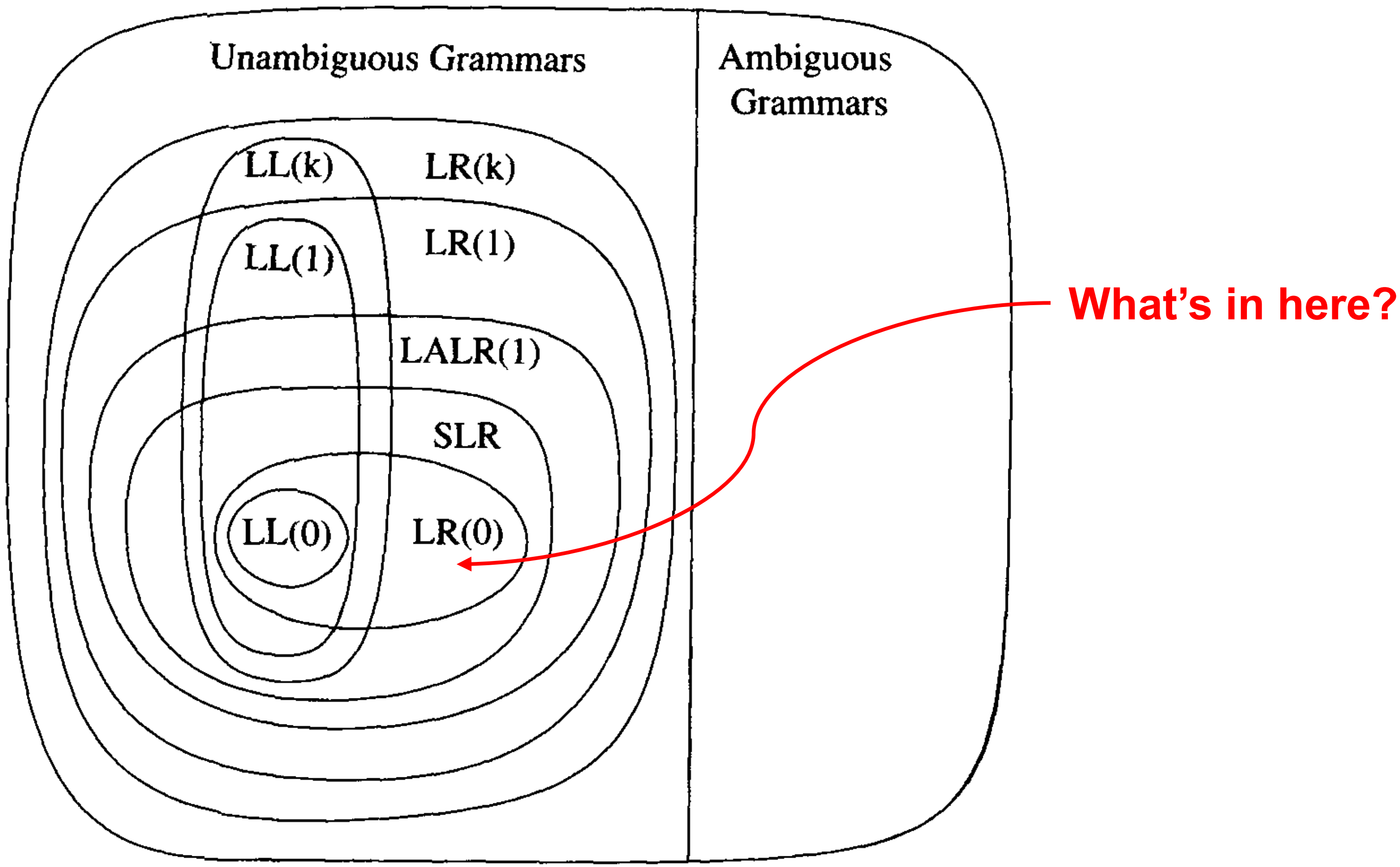
- In this course, we will only discuss LR family parsers
  - Most automated tools generate either LL or LR parsers
  - Precedence parsers are weaker siblings of LR parsers

LR Parsers are more powerful than LL

- LR family of parsers
  - LR(k) L – left to right scan
    R – rightmost derivation in reverse
    k elements of look ahead
- Pros in comparison to LL(k)

1. More powerful than LL(k)
   - Handles more grammars: no left recursion removal, no left factoring needed
   - Handles more languages: LL(k) $\subset$ LR(k)
2. As efficient as LL(k)
   - Linear in time and space to length of input (same as LL(k))
3. As convenient as LL(k)
   - Can generate automatically from grammar – YACC, Bison

A Hierarchy of Grammar Classes



LR Parsers are harder to deal with

- Cons in comparison to LL(k)
  1. More complex in structure compared to LL(k)
     - Structure of parser looks nothing like grammar
     - Parse conflicts are hard to understand and debug
  2. Harder to emit informative error messages and recover from errors
     - LR is a bottom-up while LL is a top-down parser
     - When parse error occurs,
       LR: Knows only of currently reduced non-terminal
       LL: Knows how upper levels of tree look like and context of error

     → LL can emit smart error messages referring to context of error

     → LL can perform better error recovery according to context

Implementation
-- LR Parsing

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Viable Prefix

- Definition: α is a viable prefix if
  - There is a w where αw is a rightmost sentential form, where w is the unconsumed input string
  - In other words, if there is a w where α#w is a configuration of a shift-reduce parser

  b (a # a) b $\Rightarrow$ b (M # a) b $\Rightarrow$ b (L # b $\Rightarrow$ b M # b $\Rightarrow$ Z #

- If contents of parse stack is a viable prefix, that means the parser is on the right track (at least for the consumed input)

- Shift-reduce parsing is the process of massaging the contents of the parse stack from viable prefix to viable prefix
  - Error if neither shifting or reducing results in a viable prefix

Massaging into a Viable Prefix

- How do you know what results in a viable prefix?
  - Example grammar
    $S \rightarrow a\ B\ S \mid b$
    $B \rightarrow b$
  - Example shift and reduce on: a # b b
    Shift: a # b b $\Rightarrow$ a b # b   How do you know shifting is the answer?
    Reduce: a b # b $\Rightarrow$ a B # b   Should I apply $B \rightarrow b$ (and not $S \rightarrow b$)?
- You need to keep track of where you are on the RHS of rules
  - In example
    Shift: a # b b $\Rightarrow$ a b # b
    $S \rightarrow a\ \#\ B\ S,\ B \rightarrow \#\ b \Rightarrow S \rightarrow a\ \#\ B\ S,\ B \rightarrow b\ \#$
    Reduce: a b # b $\Rightarrow$ a B # b
    $S \rightarrow a\ \#\ B\ S,\ B \rightarrow b\ \# \Rightarrow S \rightarrow a\ B\ \#\ S,\ S \rightarrow \#\ a\ B\ S,\ S \rightarrow \#\ b$

LR(0) Item Notation

- LR(0) Item: a production + a dot on the RHS
  - Dot indicates extent of production already seen
  - In example grammar

    Items for production $S \rightarrow a\ B\ S$

    $S \rightarrow .\ a\ B\ S$

    $S \rightarrow a\ .\ B\ S$

    $S \rightarrow a\ B\ .\ S$

    $S \rightarrow a\ B\ S\ .$
- Items denote the idea of the viable prefix. E.g.
  - $S \rightarrow .\ a\ B\ S$: to be a viable prefix, terminal 'a' needs to be shifted
  - $S \rightarrow a\ .\ B\ S$: to be a viable prefix, a set of terminals need to be shifted and reduced to non-terminal 'B'

States in the LR Parser DFA

- LR parser constructs a DFA to detect viable prefixes
  - Each state represents where we are in the RHS production rules
  - State is denoted by a set of LR(0) items
- Why a *set* of LR(0) items?
  - There may be multiple candidate RHSs for the prefix. E.g.

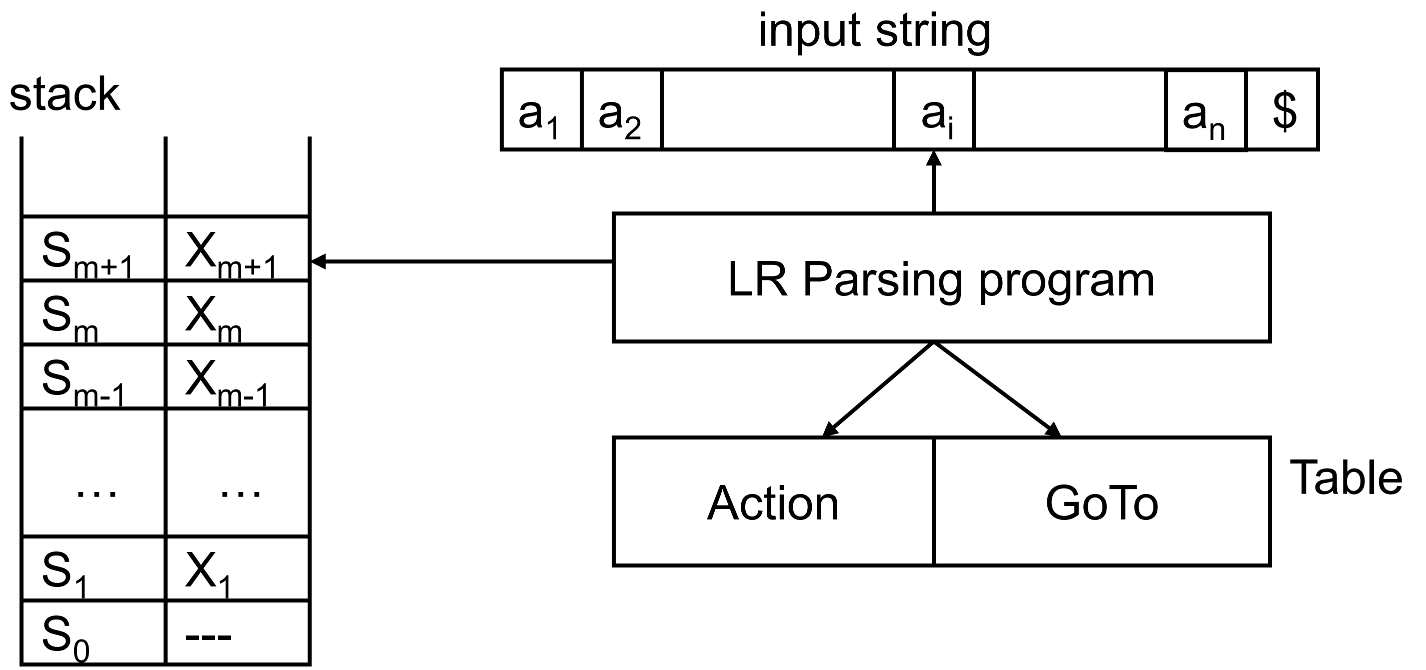
    Given grammar S → a b | a c, and given prefix "a"

    S → a **.** b and S → a **.** c would be items in current state
  - If dot is before a non-terminal, it may start another RHS. E.g.

    Given grammar S → a B, B → b, and give prefix "a"

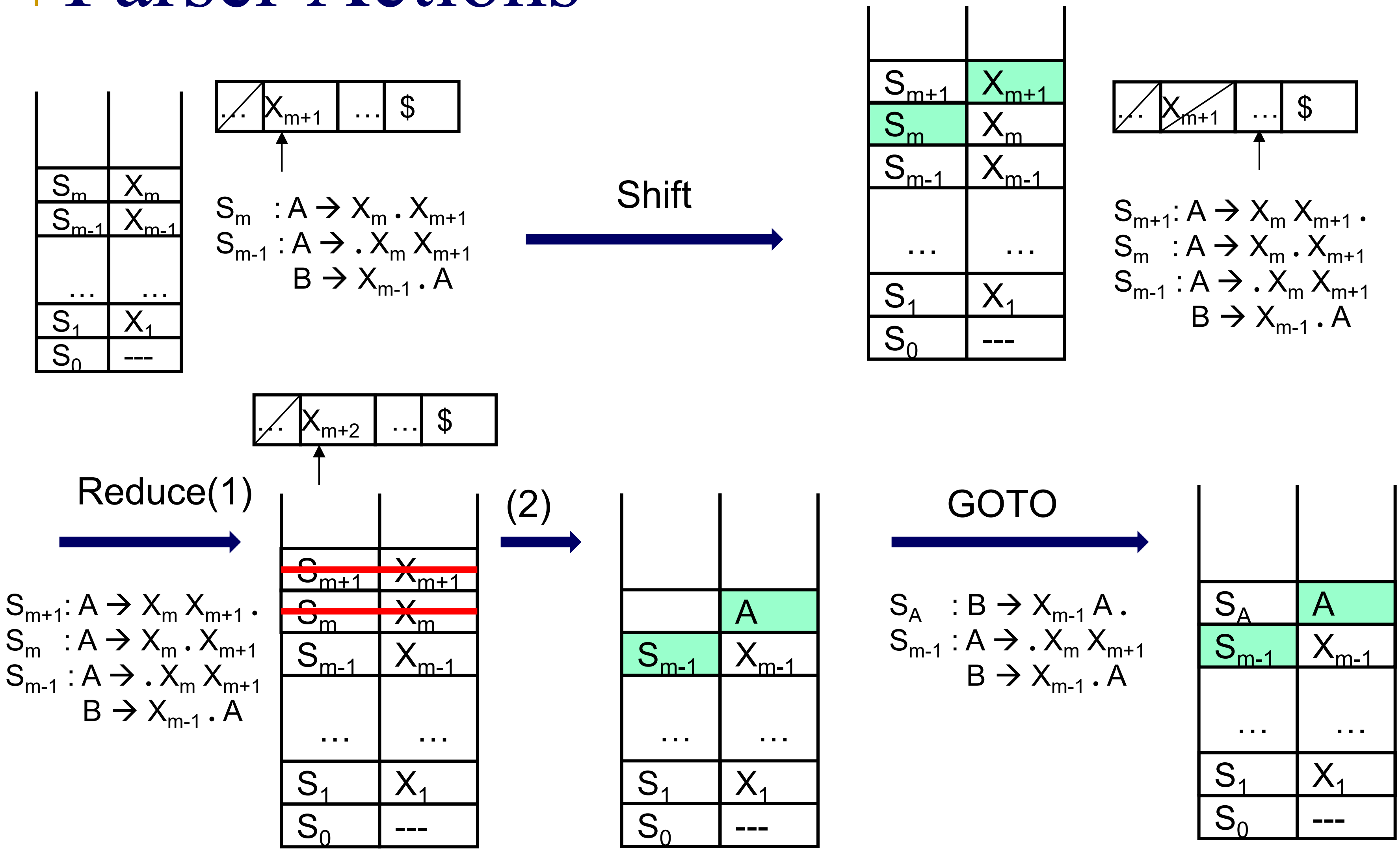
    S → a **.** B and B → **.** b would be items in current state
- LR parser keeps track of states alongside symbols in stack
  - State informs the next set of possible actions parser can take

Parser Implementation in More Detail



- Each grammar symbol X_i is associated with a state S_i
- Contents of stack ($X_1X_2\ldots X_m$) is a viable prefix
- Contents of stack + input ($X_1X_2\ldots X_ma_i\ldots a_n$) is a right sentential form
  - If the input string is a member of the language
- Uses **state** at the top of stack and current input to index into parsing table to determine whether to shift or reduce

Parser Actions



Parser Actions

- Assume configuration = $S_0X_1S_1X_2S_2\ldots X_mS_m\#a_ia_{i+1}\ldots a_n\$$
- Actions can be one of:
  1. Shift input a_i and push new state S
     - New configuration = $S_0X_1S_1X_2S_2\ldots X_mS_m\ a_i\ S\ \#\ a_{i+1}\ldots\$)$
     - Where Action $[S_m, a_i]$ = s[S]
  2. Reduce using Rule R ($A \rightarrow \beta$) and push new state S
     - Let k = $|\ \beta\ |$, pop 2*k symbols and push A
     - New configuration = $S_0X_1S_1\ldots X_{m\text{-}k}S_{m\text{-}k}A\ S\ \#\ a_i\ a_{i+1}\ldots\$$
     - Where Action $[S_m, a_i]$ = r[R] and GoTo $[S_{m\text{-}k}, A]$ = [S]
  3. Accept – parsing is complete (Action $[S_m, a_i]$ = accept)
  4. Error – report and stop (Action $[S_m, a_i]$ = error)

Parse Table: Action and Goto

- Action $[S_m, a_i]$ can be one of:
  - s[S]: shift input symbol a_i and push state S
    (One item in S_m must be of the form $A \rightarrow \alpha \,.\, a_i \,\beta$)
  - r[R]: reduce using rule R on seeing input symbol a_i
    (One item in S_m must be R: $A \rightarrow \alpha \,.\,$, where $a_i \in$ Follow(A))
    - Use GoTo $[S_{m-|\alpha|}, A]$ to figure out state to push with A
  - Accept (One item in S_m must be $S' \rightarrow S\,.$ where S is the original start symbol, and a_i must be $)
  - Error (Cannot shift, reduce, accept on symbol a_i in state S_m)
- GoTo $[S_m, X_i]$ is [S]:
  - Next state to push when pushing nonterminal X_i from a reduction
    (At least one item in S_m must be of the form $A \rightarrow \alpha \,.\, X_i \,\beta$)
  - Similar to shifting input except now we are “shifting” a nonterminal

- Grammar
  1. S→E
  2. E→E+T
  3. E→T
  4. T→id
  5. T→(E)

Non-terminal	Follow
S	$
E	+) $
T	+) $

ACTION

	+	id	(	)	$
S0		s3	s4		
S1	s7				accept
S2	r3			r3	r3
S3	r4			r4	r4
S4		s3	s4		
S5	s7			s6	
S6	r5			r5	r5
S7		s3	s4		
S8	r2			r2	r2

GOTO

	E	T	S
S0	1	2	
S1			
S2			
S3			
S4	5	2	
S5			
S6			
S7		8	
S8			

- Grammar
  1. $S \rightarrow E$
  2. $E \rightarrow E+T$
  3. $E \rightarrow T$
  4. $T \rightarrow id$
  5. $T \rightarrow (E)$

Non-terminal	Follow
S	$
E	+) $
T	+) $

ACTION

	+	id	(	)	$
S0		s3	s4		
S1	s7				accept
S2	r3			r3	r3
S3	r4			r4	r4
S4		s3	s4		
S5	s7			s6	
S6	r5			r5	r5
S7		s3	s4		
S8	r2			r2	r2

GOTO

	E	T	S
S0	1	2	
S1			
S2			
S3			
S4	5	2	
S5			
S6			
S7		8	
S8			

Parse Table in Action

- Example input string

  id + id + id

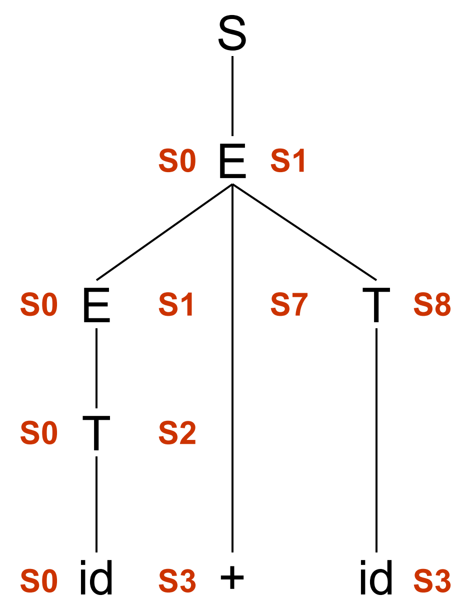
- Parser actions

Stack	Input	Actions
S0	id + id + id $	Action[S0, id] (s3): Shift "id", Push S3
S0 id S3	+ id + id $	Action[S3, +] (r4): Reduce rule 4 (T→id) GoTo[S0, T] (2): Push S2
S0 T S2	+ id + id $	Action[S2, +] (r3): Reduce rule 3 (E→T) GoTo[S0, E] (1): Push S1
S0 E S1	+ id + id $	Action[S1, +] (s6): Shift "+", Push S7
S0 E S1 + S7	id + id $	Action[S7, id] (s3): Shift "id", Push S3
…	…	…
S0 E S1 + S7 T S8	+ id $	Action[S8, +] (r1): Reduce rule 2 (E→E+T) GoTo[S0, E] (1): Push S1
…	…	…

Power Added to DFA by Stack

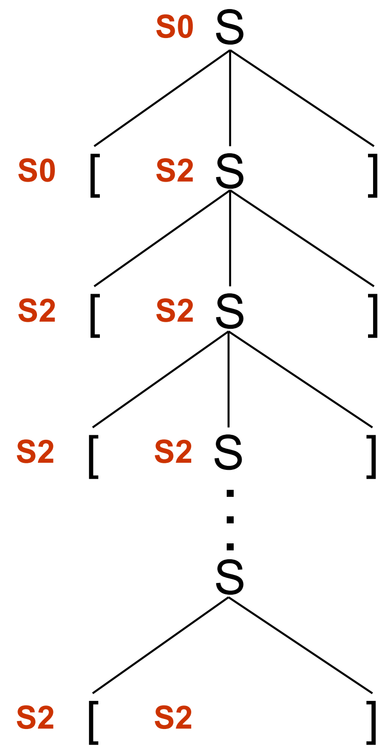
- LR parser is basically DFA+Stack (Pushdown Automaton)
- DFA: can only remember one state (“dot” in current rule)
- DFA + Stack: remembers current state and all past states (“dots” in rules higher up in the tree waiting for next symbol)

Stack	Input	Action
S0	id + id $	s3
S0 id S3	+ id $	r4, goto[S0, T]
S0 T S2	+ id $	r3, goto[S0, E]
S0 E S1	+ id $	s7
S0 E S1 + S7	id $	s3
S0 E S1 + S7 id S3	$	r4, goto[S7, T]
S0 E S1 + S7 T S8	$	r2, goto[S0, E]
S0 E S1	$	Accept



Power Added to DFA by Stack

- Remember the following CFG for the language $\{ [^i]^i \mid i >= 1\}$?
  $S \rightarrow [\ S\ ] \mid [\ ]$
- Regular grammars (or DFAs) could not recognize language because the state machine had to "count"
- LR parser stack counts number of **[symbols**



- Q: Is this language LL(1)?
  - Yes. After left-factoring.
    $S \rightarrow [\ S', S' \rightarrow S\ ] \mid]$
  - LL parser stack counts number of **] symbols**
  - Same pushdown automaton but different usage

LR Parse Table Construction

- Must be able to decide on action from:
  - State at the top of stack
  - Next k input symbols (In practice, k = 1 is often sufficient)

- To construct LR parse table from grammar
  1. Build deterministic finite automaton (DFA) using LR(0) items
  2. Express DFA using Action and GoTo tables

- State: Where we are currently in the structure of the grammar
  - Expressed as a set of LR(0) items
  - Each item expresses position in the RHS of a rule using a dot

Construction of LR States

1. Create augmented grammar G' for G
   - Given G: $S \rightarrow \alpha \mid \beta$, create G': $S' \rightarrow S \quad S \rightarrow \alpha \mid \beta$
   - Creates a single rule $S' \rightarrow S$ that when reduced, signals acceptance
2. Create first state by performing a *closure* on initial item $S' \rightarrow .\, S$
   - Closure(I): computes set of items expressing the same position as I
   - $\text{Closure}(\{S' \rightarrow .\, S\}) = \{S' \rightarrow .\, S,\ S \rightarrow .\, \alpha,\ S \rightarrow .\, \beta\}$
3. Create additional states by performing a *goto* on each symbol
   - Goto(I, X): creates state that can be reached by advancing X
   - If α was single symbol, the following new state would be created:
     $\text{Goto}(\{S' \rightarrow .\, S,\ S \rightarrow .\, \alpha,\ S \rightarrow .\, \beta\}, \alpha) =$
     $\text{Closure}(\{S \rightarrow \alpha\, .\}) = \{S \rightarrow \alpha\, .\}$
4. Repeatedly perform gotos until there are no more states to add

Closure Function

- Closure(I) where I is a set of items
  - Returns the state (set of items) that express the same position as I
  - Items in I are called kernel items
  - Rest of items in closure(I) are called non-kernel items
- Let N be a non-terminal
  - If dot is in front of N, then add each production for that N and put dot at the beginning of the RHS
    - $A \rightarrow \alpha \,.\, B\,\beta$ is in I ; we expect to see a string derived from B
    - $B \rightarrow \,.\, \gamma$ is added to the closure, where $B \rightarrow \gamma$ is a production
    - Apply rule until nothing is added
  - Given

$$
\begin{aligned}
&S \rightarrow E\\
&E \rightarrow E + T\\
&E \rightarrow T\\
&T \rightarrow id \mid (E)
\end{aligned}
$$

Closure({ $S \rightarrow .\,E$ }) = { $S \rightarrow .E$, $E \rightarrow .\,E + T$, $E \rightarrow .\,T$, $T \rightarrow .\,id$, $T \rightarrow .\,(E)$ }

Kernel and Non-kernel Items

- Two kinds of items
  - Kernel items
    - Items that act as "**seed**" items when creating a state
    - What items act as seed items when states are created?
      - Initial state: S' $\rightarrow$. S
      - Additional state: from goto(I, X) so has X at left of dot
    - Besides S' $\rightarrow$. S, all kernel items have **dot in the middle of RHS**
  - Non-kernel items
    - Items added during the **closure** of kernel items
    - All non-kernel items have **dot at the beginning of RHS**

Goto Function

- Goto (I, X) where I is a set of items and X is a symbol
  - Returns state (set of items) that can be reached by advancing X
  - For each $\underline{A \rightarrow \alpha \,.\, X \beta}$ in I,
    $\underline{\text{Closure}(A \rightarrow \alpha\ X \,.\, \beta)}$ is added to $\underline{\text{goto}(I, X)}$
  - X can be a terminal or non-terminal
    - Terminal if obtained from input string by shifting
    - Non-terminal if obtained from reduction
  - Example
    - $\text{Goto}(\{T \rightarrow .\ (\ E\ )\},\ (\ ) = \text{closure}(\ \{T \rightarrow (\ .\ E\ )\}\ )$

- Ensures every symbol consumption results in a viable prefix

Construction of DFA

- Algorithm to compute set C (set of all states in DFA)

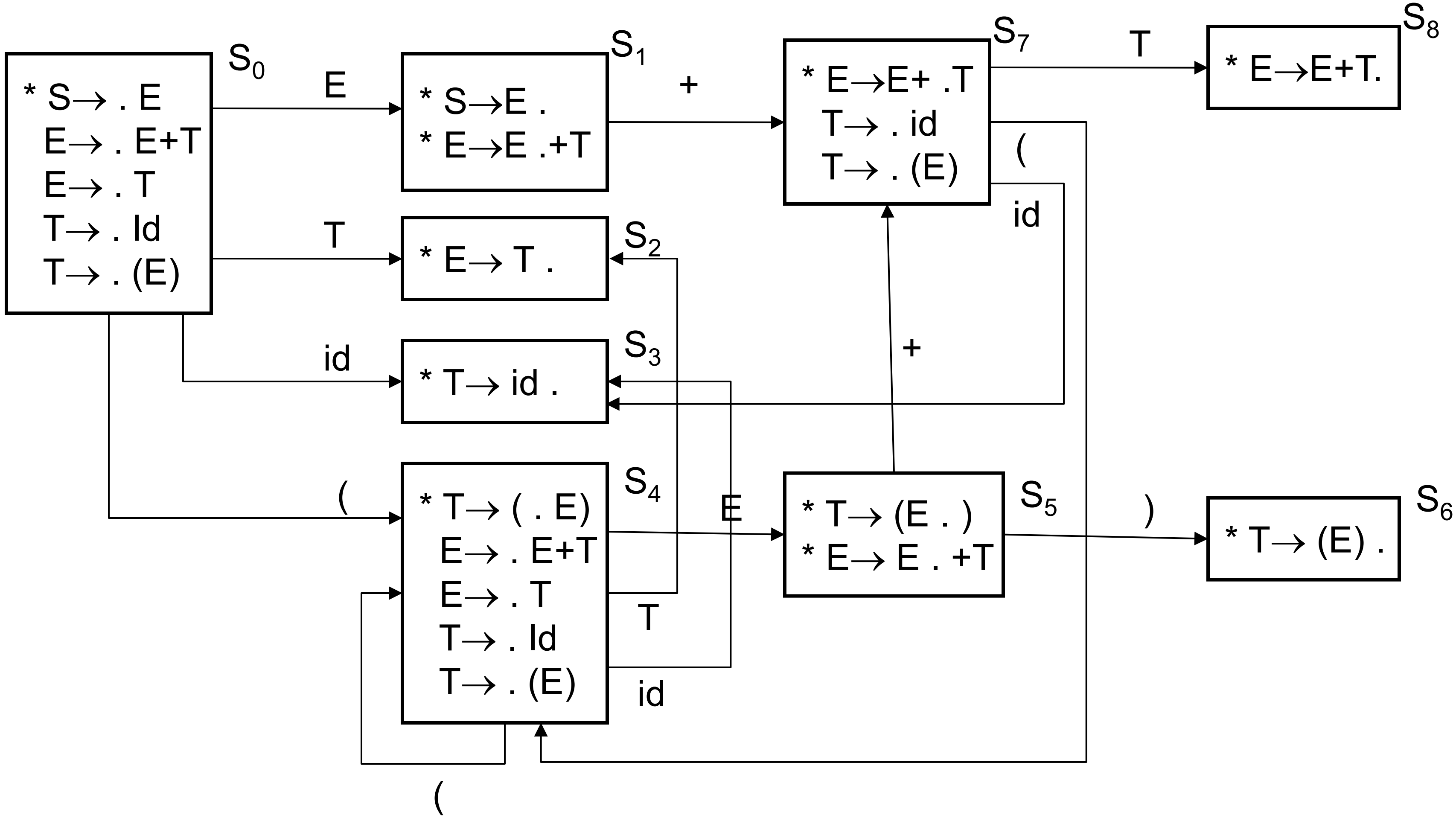
```
void constructDFA (G') {
  C = {closure({S'→ . S})}  // Add initial state to C
  repeat
    for (each state I in C)
      for (each grammar symbol X)
        if (goto(I, X) is not empty and not in C)
          add goto(I, X) to C
  until no new states are added to C
}
```

- Add transitions from I to goto(I, X) on symbol X

- Example: $S \rightarrow E$
  $E \rightarrow E + T \mid T$
  $T \rightarrow id \mid (E)$

- S_0= closure ({$S \rightarrow . E$})
  = {$S \rightarrow . E$, $E \rightarrow . E + T$, $E \rightarrow . T$, $T \rightarrow . id$, $T \rightarrow . (E)$}
- goto(S_0, E) = closure ({$S \rightarrow E .$, $E \rightarrow E . + T$})
  S_1 = {$S \rightarrow E .$, $E \rightarrow E . + T$}
- goto(S_0, T) = closure ({$E \rightarrow T .$})
  S_2 = {$E \rightarrow T .$ }
- goto(S_0, id) = closure ({$T \rightarrow id .$})
  S_3 = {$T \rightarrow id .$ }
- ……
- S_8 = …

- DFA for the previous grammar
  (* are closures applied to kernel items)



Building Parse Table from DFA

- ACTION [state, terminal symbol]
- GOTO [state, non-terminal symbol]

❑ Filling in the ACTION and GOTO cells

1. If $[A \rightarrow \alpha \bullet a \beta]$ is in S_i and $goto(S_i, a) = S_j$, where "a" is a terminal
   then $ACTION[S_i, a]$ = shift j
2. If $[A \rightarrow \alpha \bullet A \beta]$ is in S_i and $goto(S_i, A) = S_j$, where "A" is a non-terminal
   then $GOTO[S_i, A] = S_j$
3. If $[A \rightarrow \alpha \bullet]$ is in S_i
   then $ACTION[S_i, a]$ = reduce $A \rightarrow \alpha$ for all $a \in Follow(A)$
4. If $[S' \rightarrow S_0 \bullet]$ is in S_i
   then $ACTION[S_i, \$]$ = accept

❑ Two potential prediction conflicts

- ➢ Reduce-reduce conflict: when an ACTION cell has two 3s
- ➢ Shift-reduce conflict: when an ACTION cell has both 1 and 3
- ☞ More lookahead in Follow(A) may improve prediction accuracy

- Grammar
  1. S→E
  2. E→E+T
  3. E→T
  4. T→id
  5. T→(E)

Non-terminal	Follow
S	$
E	+) $
T	+) $

ACTION

	+	id	(	)	$
S0		s3	s4		
S1	s7				accept
S2	r3			r3	r3
S3	r4			r4	r4
S4		s3	s4		
S5	s7			s6	
S6	r5			r5	r5
S7		s3	s4		
S8	r2			r2	r2

GOTO

	E	T	S
S0	1	2	
S1			
S2			
S3			
S4	5	2	
S5			
S6			
S7		8	
S8			

Types of LR Parsers

- SLR – simple LR (what we saw so far was SLR(1))
  - Small parse table
  - Not as powerful
- Canonical LR
  - Much larger parse table
  - More powerful (can parse more grammars)
- LALR
  - Look ahead LR
  - In between the 2 previous ones in power and overhead

Overall parsing algorithm is the same – table is different

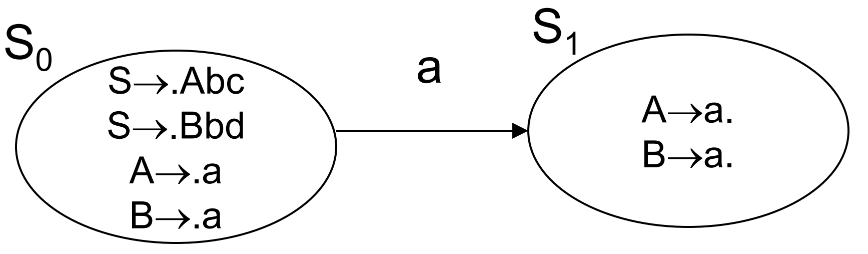
Conflict due to not enough lookahead

- Consider the grammar G

  $S \rightarrow A\ b\ c \mid B\ b\ d$

  $A \rightarrow a$ $\quad$ $b \in Follow(A)$ and also $b \in Follow(B)$

  $B \rightarrow a$



- What is reduced when "a b" is seen? reduce to A or B?
  - Reduce-reduce conflict
- G is not SLR(1) but SLR(2)
  - We need 2 symbols of look ahead to look past b:

    b c – reduce to A

    b d – reduce to B
  - Possible to extend SLR(1) to k symbols of look ahead – allows larger class of CFGs to be parsed

SLR(k)

- Extend SLR(1) definition to SLR(k) as follows
  let $\alpha, \beta \in V^*$

  - $First_k(\alpha) = \{ x \in V_T^* \mid (\alpha \Rightarrow^* x\beta \text{ where } |x|=k) \text{ or }$
    $(\alpha \Rightarrow^* x \text{ where } |x| <= k)\}$
    - all k-symbol terminal prefixes of strings derivable from α

  - $Follow_k(B) = \{w \in V_T^* \mid S \Rightarrow^* \alpha B \gamma \text{ and } w \in First_k(\gamma)\}$
    - all k symbol terminal strings that can follow B in some derivation

SLR(k) Parse Table

Let S be a state and lookahead $b \in V_T{}^*$ such that $|b| \leq k$

1. If $A \rightarrow \alpha. \in S$ and $b \in Follow_k(A)$ then
   - Action(S,b) – reduce using production $A \rightarrow \alpha$,
2. If $D \rightarrow \alpha.a\,\gamma \in S$ and $a \in V_T$ and $b \in First_k(a\,\gamma\ Follow_k(D))$
   - Action(S,b) = shift "a" and push state goto(S,a)

For k =1, this definition reduces to SLR(1)
Reduce: Trivially true
Shift: $First_1(a\,\gamma\ Follow_1(D)) = \{a\}$

SLR(k-1) $\subset$ SLR(k)

- Consider

  $S \rightarrow A\ b^{k-1}\ c \mid B\ b^{k-1}\ d$

  $A \rightarrow a$

  $B \rightarrow a$

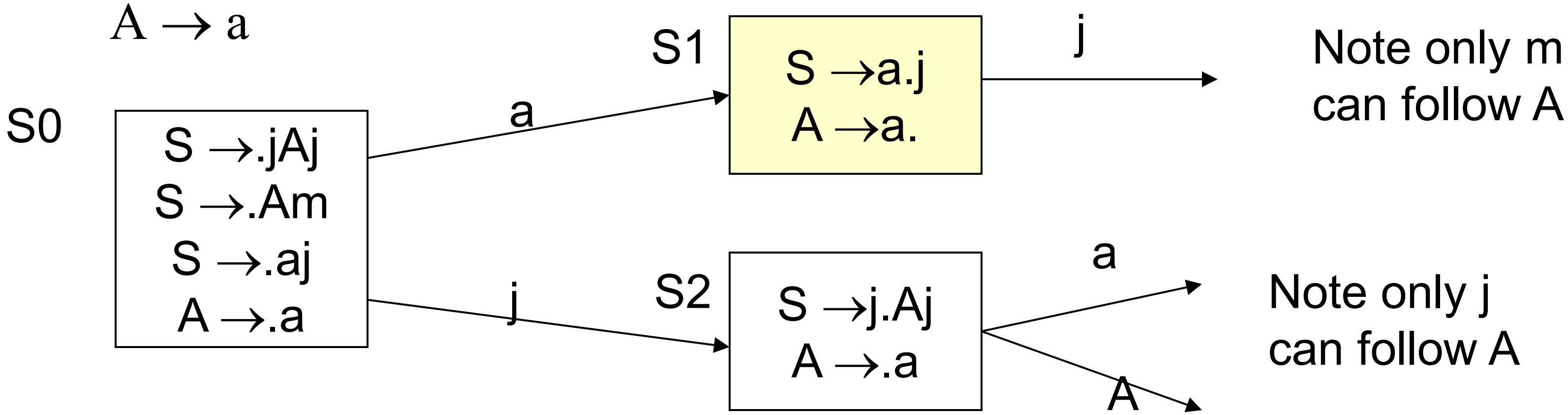
SLR(k) not SLR(k-1)

- cannot decide what to reduce,
- reduce a to A or B depends the next k symbols

  $b^{k-1}\ c$ or $b^{k-1}\ d$

Non-SLR(k) for any k

- Consider another Grammar G
  - $S \rightarrow j\ A\ j \mid A\ m \mid a\ j$
    $A \rightarrow a$



Follow(A) = {j, m}

State S1: [A→a.] – reduce using this production (on j or m)

[S→a.j] – shift j ➔ shift-reduce conflict ➔ not SLR(1)

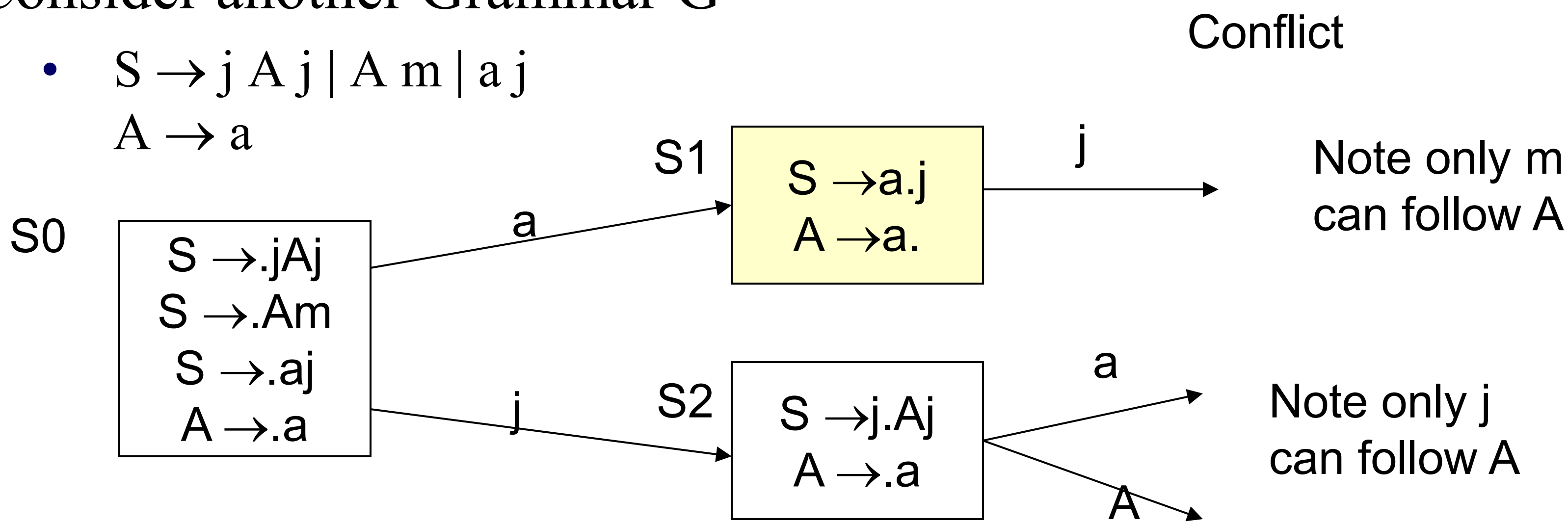
? SLR(k) for some k > 1?

For reducing A→a.: $Follow_k(A) = First_k(jFollow_k(S)) + First_k(mFollow_k(S))$ = {j\$, m\$},

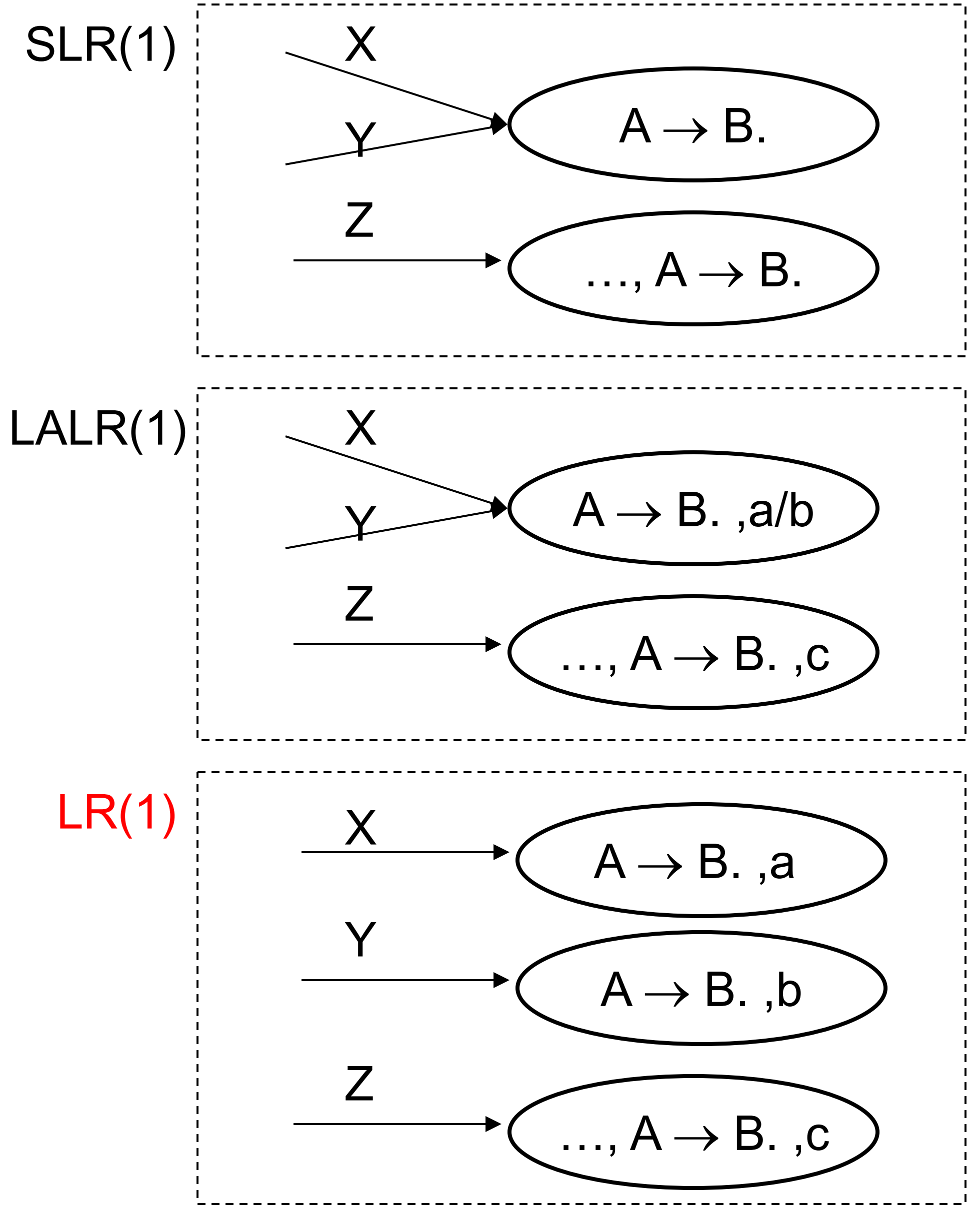
For shifting S →a.j: $First_k(jFollow_k(S))$ = {j\$} so not SLR(k) for any k !!!

Non-SLR(k) for any k

- Consider another Grammar G
  - $S \rightarrow j\ A\ j \mid A\ m \mid a\ j$
    $A \rightarrow a$



- Problem: Follow(A) = {j, m} is too imprecise
  - In S1, we should reduce A only when lookahead is {m}
  - The fact that {j} can follow A in another context is irrelevant
- Canonical LR:
  - Encode appropriate lookahead for the reduction of each LR item
  - Appropriate lookahead is the follow set in the given context



Follow(A)={a,b,c}

state merging

state splitting

Constructing Canonical LR

- LR(1) item: LR item with one lookahead
  - [$A \rightarrow \alpha.\beta$, a] where $A \rightarrow \alpha\beta$ is a production and a is a terminal or \$
    - Meaning: Only terminal a can follow A in this context
    - When we reach [$A \rightarrow \alpha\beta.$, a], reduce A only if lookahead matches terminal a
  - [$A \rightarrow \alpha.\beta$, a/b]: means both a and b can follow A in this context
    - {a, b} $\subseteq$ Follow(A), a more precise version of the follow set

- LR(k) item: LR item with k lookahead
  - [$A \rightarrow \alpha.\beta$, a/b]: a, $b \in V_T^*$ such that $|a| \leq k$, $|b| \leq k$

Constructing Canonical LR

- Essentially the same as LR(0) items only adding lookahead
  - Modify closure and goto function

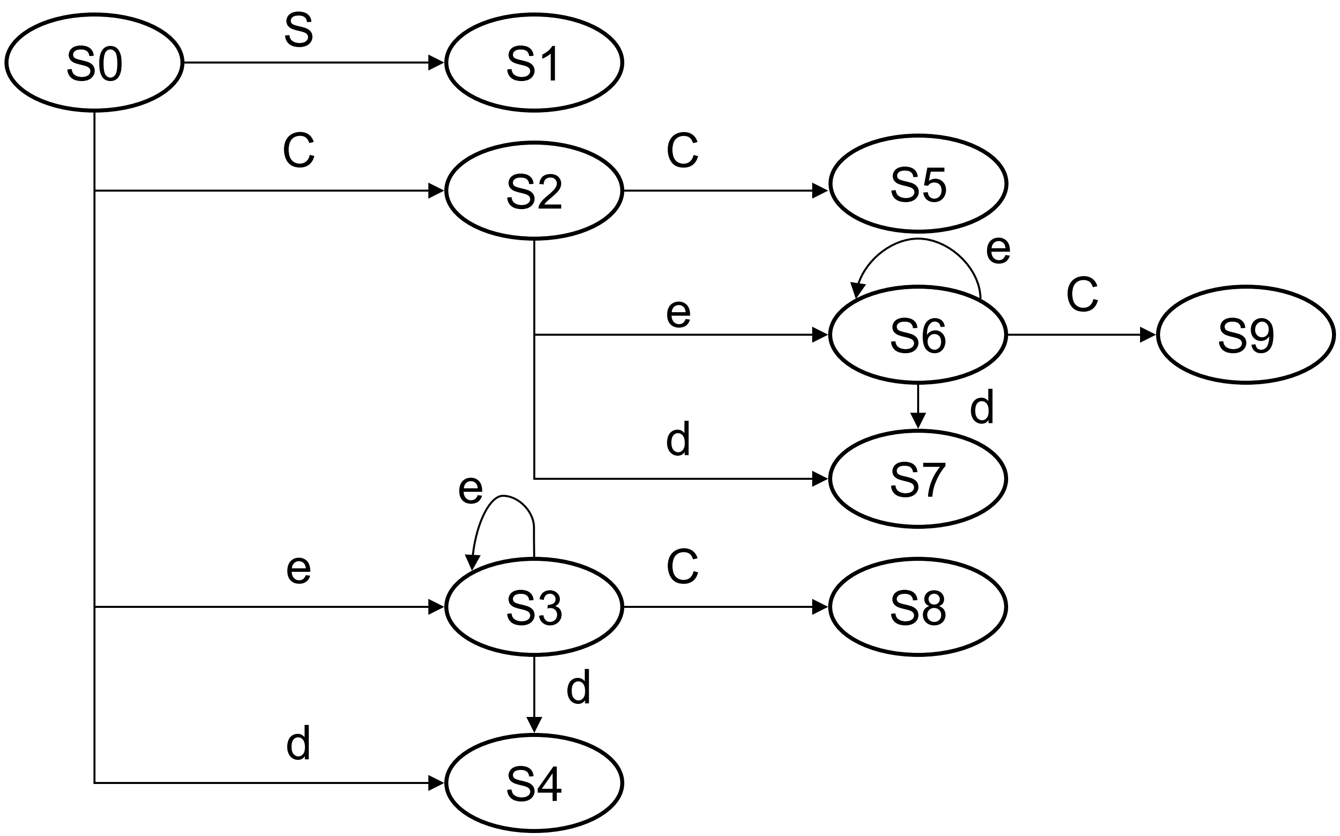
- Changes for **closure** function
  - Initialize lookahead:
    If $[A \rightarrow \alpha.B\beta, a]$ and $B \rightarrow \delta$,
    then $[B \rightarrow .\ \delta, c] \in$ closure($[A \rightarrow \alpha.B\beta, a]$), where $c \in$ First(βa)

- Changes for **goto** function
  - Carry over lookahead:
    if $[A \rightarrow \alpha.X\beta, a] \in I$, then goto $(I, X) = [A \rightarrow \alpha\ X.\ \beta, a]$

Example

- Grammar
  $S' \rightarrow S$
  $S \rightarrow CC$
  $C \rightarrow eC \mid d$
- S0: closure($S' \rightarrow .S$, \$)
  - [$S' \rightarrow .S$,\$]
  - [$S \rightarrow .CC$, \$] first(ε\$)={\$}
  - [$C \rightarrow .eC$, e/d] first(C\$)={e,d}
  - [$C \rightarrow .d$, e/d] first(C\$)={e,d}
- S1: goto(S0, S) = closure($S' \rightarrow S.$, \$)
  - [$S' \rightarrow S.$, \$]
- S2: goto(S0, C) = closure($S \rightarrow C.C$, \$)
  - [$S \rightarrow C.C$, \$]
  - [$C \rightarrow .eC$, \$] first(ε\$)={\$}
  - [$C \rightarrow .d$, \$] first(ε\$)={\$}

- S3: goto(S0,e) = closure(C→e.C, e/d)
  - [C →e.C, e/d]
  - [C →.eC, e/d]   first(εe/d) = {e,d}
  - [C →.d, e/d]   first(εe/d) = {e,d}
- S4: goto(S0, d) = closure(C→d., e/d)
  - [C→d., e/d]
- S5: goto(S2, C) = closure(S→CC., $)
  - [S→CC., $]
- S6: goto(S2,e) = closure(C→e.C, $)
  - [C →e.C, $]
  - [C →.eC, $]   first(ε$) = {$}
  - [C →.d, $]   first(ε$) = {$}
- S7: goto(S2, d) = closure(C→d., $)
  - [C→d., $]
- S8: goto(S3, C) = closure(C→eC., e/d)
  - [C→eC., e/d]
- S9: goto(S6,C) = closure(C→eC., $)
  - [C→eC., $]



Note S3, S6 are same except for lookahead (also true for S4, S7 and S8, S9)
In SLR(1) – one state represents both

Constructing Canonical LR Parse Table

- Shifting: same as before
- Reducing:
  - Don't use follow set (too coarse grain)
  - Reduce only if input matches lookahead for item
- Action and GOTO

1. if $[A \to \alpha \bullet a\beta, b] \in Si$ and goto(Si, a) = Sj,
   Action[I,a] = s[Sj] – shift and goto state j if input matches a

   Note: same as SLR

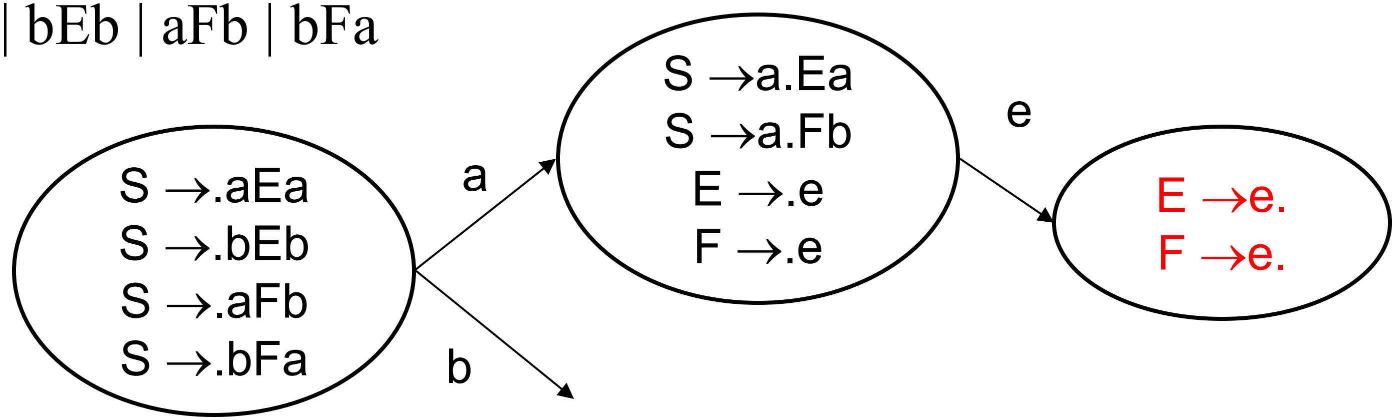
2. if $[A \to \alpha \bullet, a] \in Si$
   Action[I,a] = r[R] – reduce R: $A \to \alpha$ if input matches a

   Note: for SLR, reduced if input matches Follow(A)

- Revisit SLR and LR
  - $S \rightarrow aEa \mid bEb \mid aFb \mid bFa$

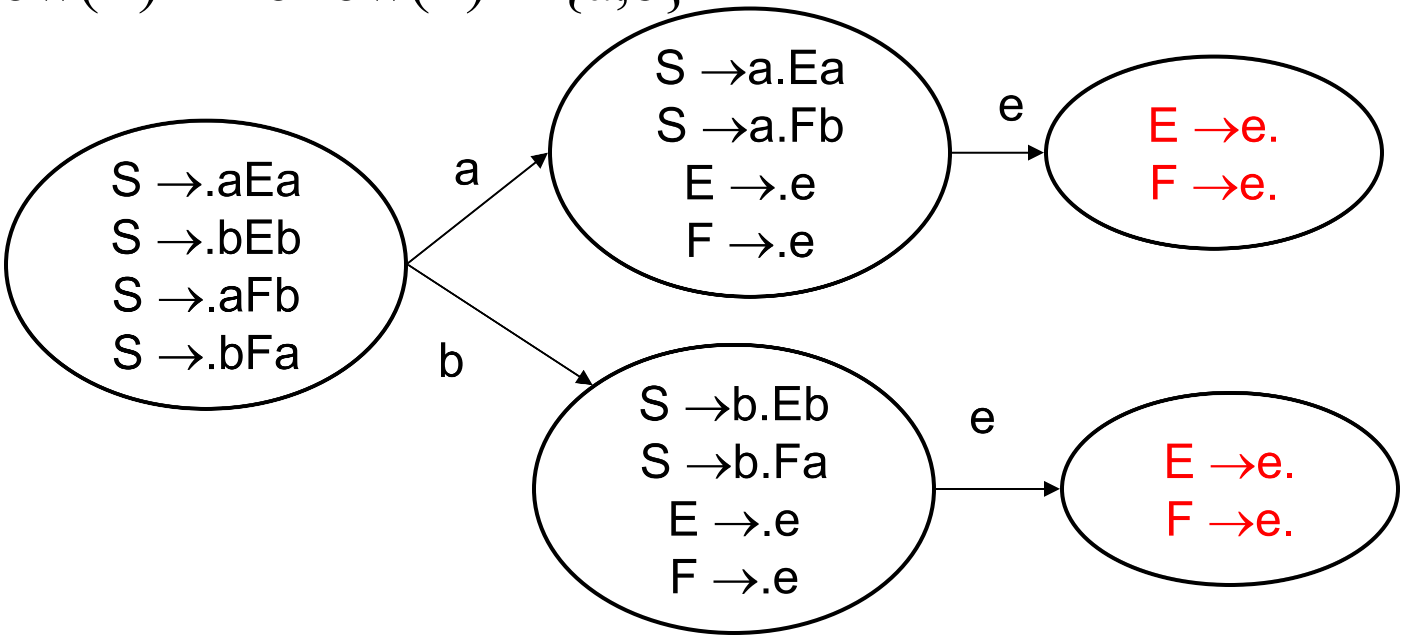
    $E \rightarrow e$

    $F \rightarrow e$

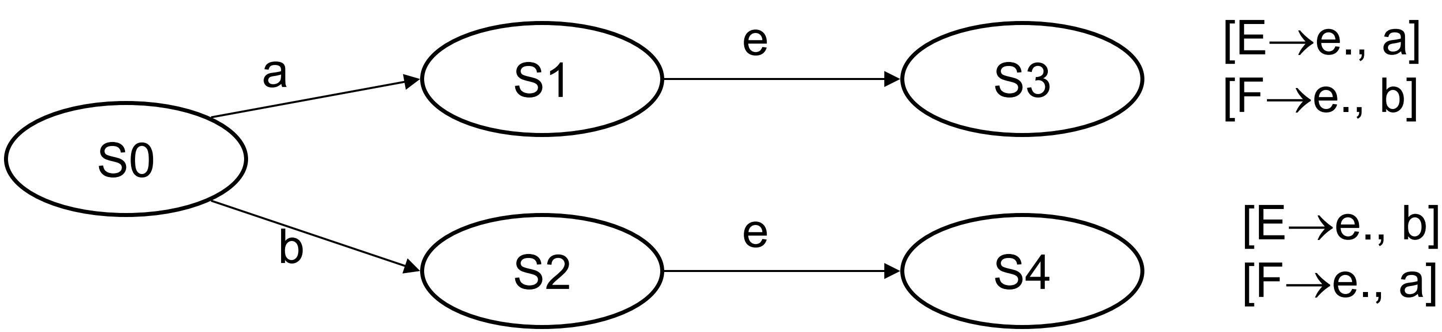


- Not SLR(1): reduce/reduce conflict.
  - Follow(E) = Follow(F) = {a,b}
- LR(1): no conflict because state is split to account for context
  - Follow(E) = {a} only if preceded by a
  - Follow(E) = {b} only if preceded by b
  - Follow(F) = {a} only if preceded by b
  - Follow(E) = {b} only if preceded by a

- SLR: Follow(E) = Follow(F) = {a,b}



- LR: Follow sets more precise

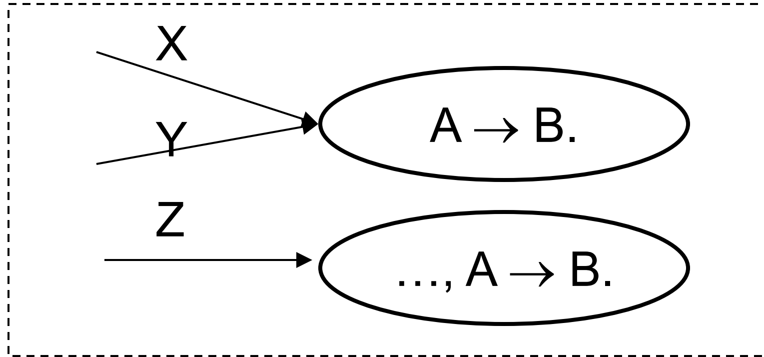


SLR(1) and LR(1)

- LR(1) more powerful than SLR(1) – can parse more grammars
- But LR(1) may end up with many more states than SLR(1)
  - One LR(0) item may split up to many LR(1) items (Potentially as many as the powerset of the entire alphabet)

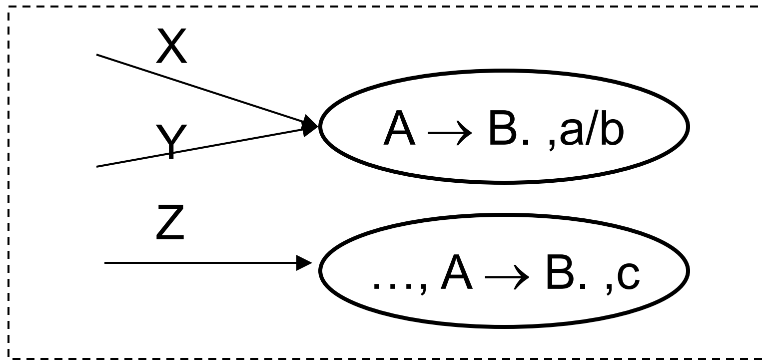
- LALR(1) – compromise between LR(1) and SLR(1)
  - Constructed by merging LR(1) states with the same core
    - Ends up with same number of states as SLR(1)
    - But items still retain some lookahead info – still better than SLR(1)
  - Used in practice because most programming language syntactic structures can be represented by LALR (not true for SLR)

SLR(1)



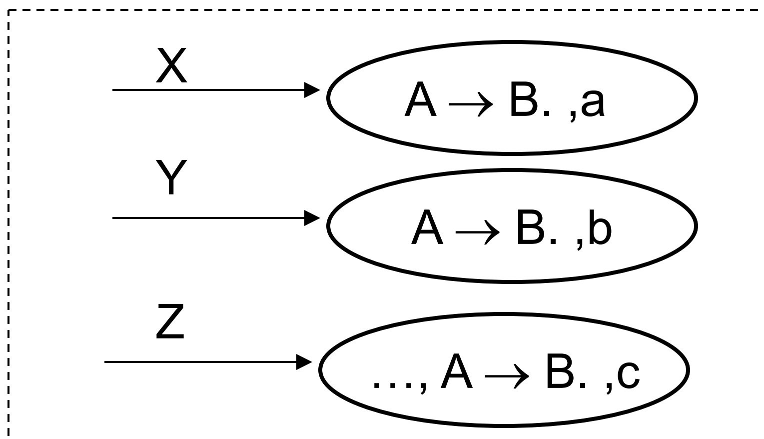
Follow(A)={a,b,c}

LALR(1)

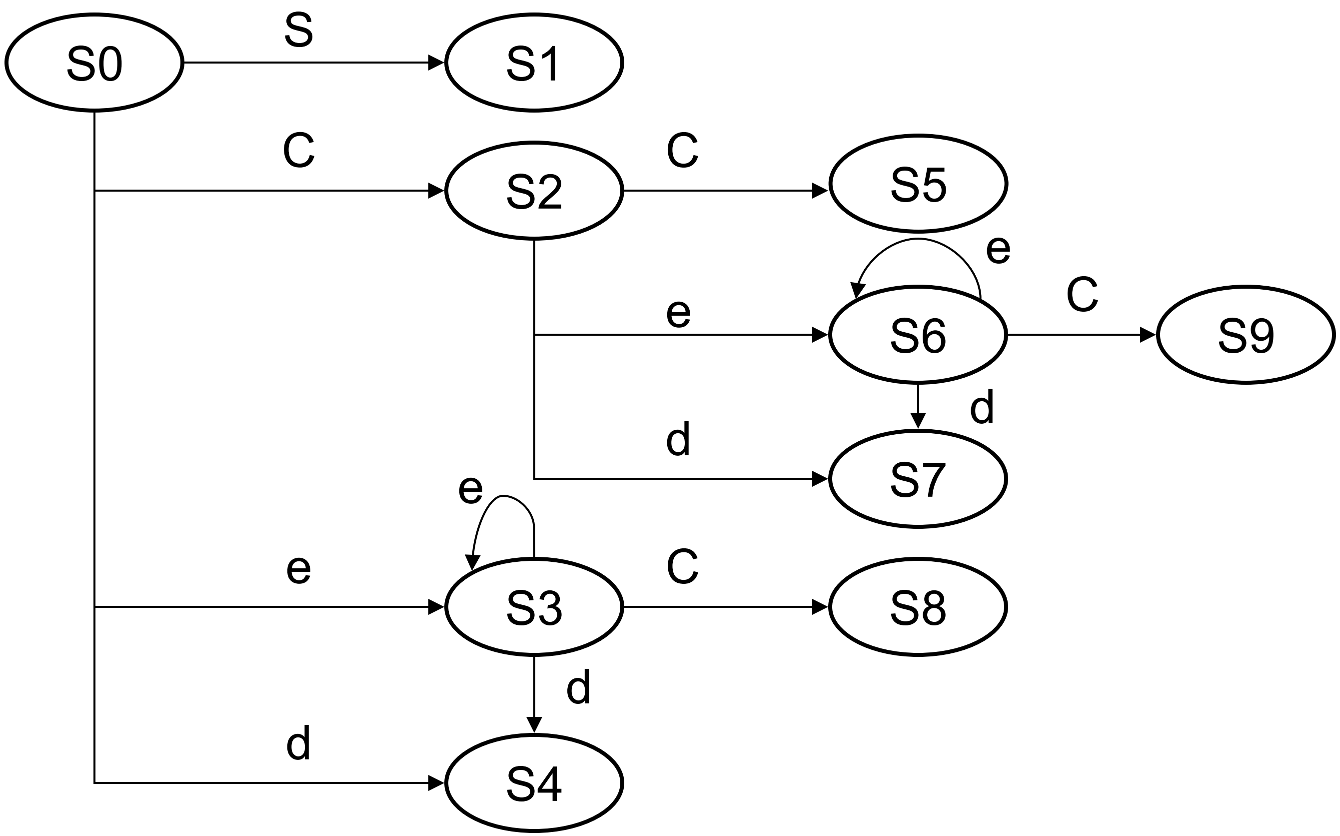


state merging

LR(1)



state splitting



Note S3 / S6, S4 / S7, S8 / S9 have same core (same except for lookahead). In an SLR(1) parser, one state represents both states.

Example

- Grammar

$S' \to S$

$S \to CC$

$C \to eC \mid d$

S3: goto(S0,e)=closure(C→e.C, e/d)
[C →e.C, e/d]
[C →.eC, e/d]
[C →.d, e/d]

S4: goto(S0, d)=closure(C→d., e/d)
[C→d., e/d]

S8: goto(S3, C)=closure(C→eC., e/d)
[C→eC., e/d]

S6: goto(S2,e)=closure(C→e.C, $)
[C →e.C, $]
[C →.eC, $]
[C →.d, $]

S7: goto(S2, d)=closure(C→d., $)
[C→d., $]

S9: goto(S6,C)=closure(C→eC., $)
[C→eC., $]

Note S3 / S6, S4 / S7, S8 / S9 have same core (same except for lookahead). In an SLR(1) parser, one state represents both states.

Merging states

- Can merge S3 and S6

S3: goto(S0,e)=closure(C→e.C, e/d)
[C →e.C, e/d]
[C →.eC, e/d]
[C →.d, e/d]

S6: goto(S2,e)=closure(C→e.C, $)
[C →e.C, $]
[C →.eC, $]
[C →.d, $]

S36: [C →e.C, e/d/$]
[C →.eC, e/d/$]
[C →.d, e/d/$]

- Similarly
  - S47: [C→d., e/d/$]
  - S89: [C→eC., e/d/$]

Effect of Merging: Introduces conflicts

1. Merging of states can introduce conflicts
   - cannot introduce shift-reduce conflicts
   - can introduce reduce-reduce conflicts

- Shift-reduce conflicts

  Suppose S_{ij}: $[A \rightarrow \alpha., a/b/c]$ reduce on input a
  $[B \rightarrow \beta.a\delta, x/y/z]$ shift on input a

  formed by merging S_i and S_j

  Then, S_i: $[A \rightarrow \alpha., \text{lookahead}_i]$ S_j: $[A \rightarrow \alpha., \text{lookahead}_j]$
  $[B \rightarrow \beta.a\delta, \ldots]$ $[B \rightarrow \beta.a\delta, \ldots]$

  And, either $a \in \text{lookahead}_i$ or $a \in \text{lookahead}_j$

  ☞ Conflict existed in the first place!

A reduce-reduce conflict due to merging

$S \rightarrow aEa \mid bEb \mid aFb \mid bFa$

$E \rightarrow e$

$F \rightarrow e$

S3: [E→e., a]
    [F→e., b]

S4: [E→e., b]
    [F→e., a]

After merging S34: [E→e., a/b]
    [F→e., a/b]

- Both reductions are applied on lookahead a and b, i.e. reduce-reduce conflict

Effect of Merging: Delays error detection

2. Detection of errors may be delayed
   - On error, LALR parsers will not perform shifts beyond an LR parser but may perform more reductions before finding error
   - Example: $S' \rightarrow S$ $S \rightarrow CC$
     $C \rightarrow eC \mid d$
     and input string eed$

     - Canonical LR: Parse Stack S0 e S3 e S3 d S4
       State S4 on \$ input = error S4: {C →d., e/d}
     - LALR:

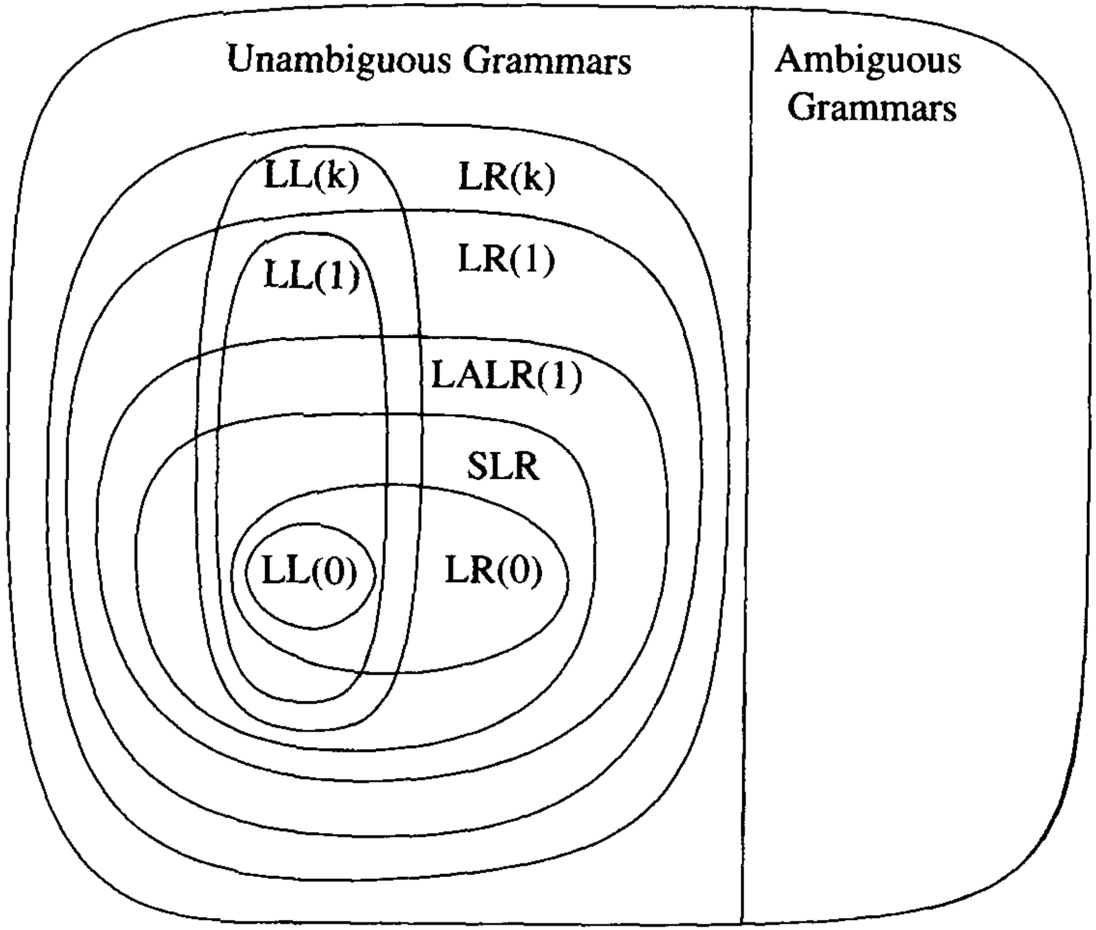
stack: S0 e S36 e S36 d S47 → state S47 input \$, reduce C → d
stack: S0e S36 e S36 C S89 → reduce C →eC
stack: S0 e S36 C S89 → reduce C →eC
stack: S0 C S2 → state S2 on input \$, error

Error Recovery

- To uncover multiple errors, parser must be able to recover from errors.

- Simple error recovery (by discarding offending code sequence)
  1. Decide on non-terminal A: candidate for discarding
     - Typically, an expression, statement, or block of code
  2. Continue to scan down the stack until a state S with a goto on a particular non-terminal A is found
  3. Discard input tokens until a token ‘a’ is found that can follow A
     - E.g. if A is a statement, then ‘a’ would be ‘;’
  4. Push state Goto[a,A] on stack and continue parsing

A Hierarchy of Grammar Classes



LALR(k) ⊂ LR(k)

- LR(k) is strictly more powerful compared to LALR(k)
  - LALR merges states, which can introduce conflicts
- Unlike LL and LR, no formal definition on what is LALR
  - Definition by construction: if LALR parser has no conflicts
  - Conflicts due to state merging are hard to define formally (Hence, they are unpredictable and hard to reason with)

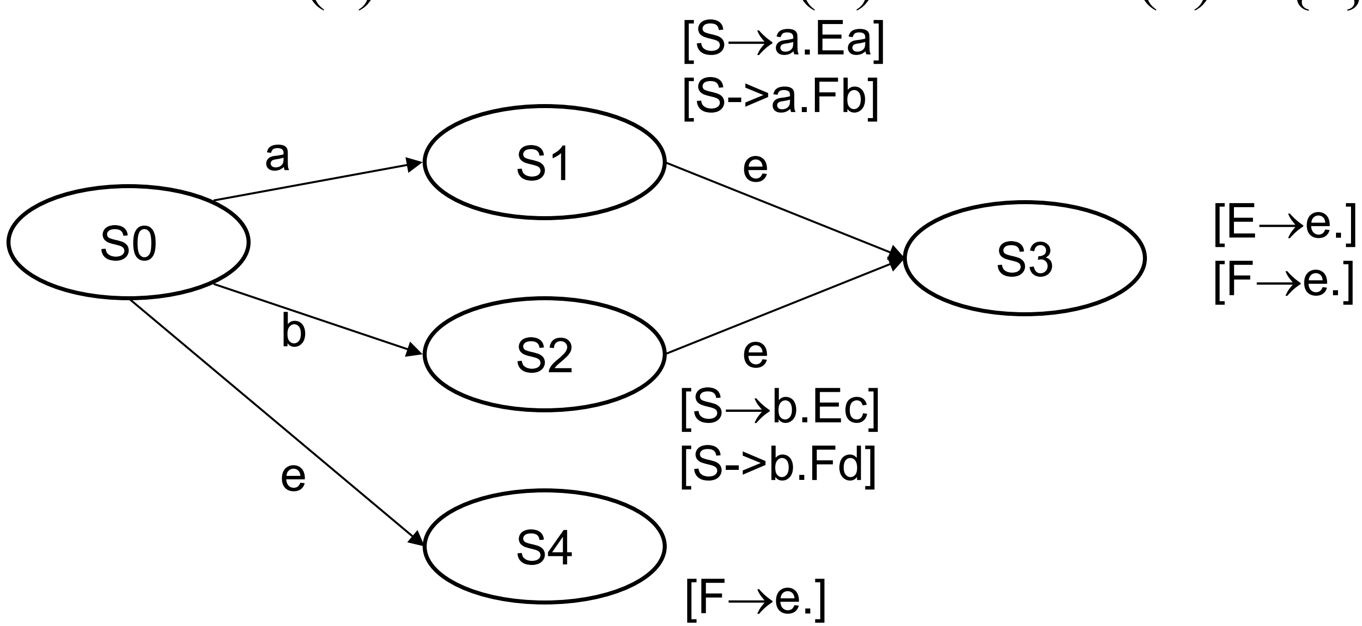
- Nonetheless, LALR(1) has become popular
  - YACC, Bison, etc.
  - Most programming languages have an LALR(1) grammar
  - Reduce-reduce conflicts due to state merging are rare (conflicts are mostly due to ambiguity)

SLR(k) ⊂ LALR(k)

- Let's consider this Grammar G:
  - $S \rightarrow aEa \mid aFb \mid bEc \mid bFd \mid Fa$

    $E \rightarrow e$

    $F \rightarrow e$
- It is non-SLR(1) because Follow(E) ∩ Follow(F) = {a}

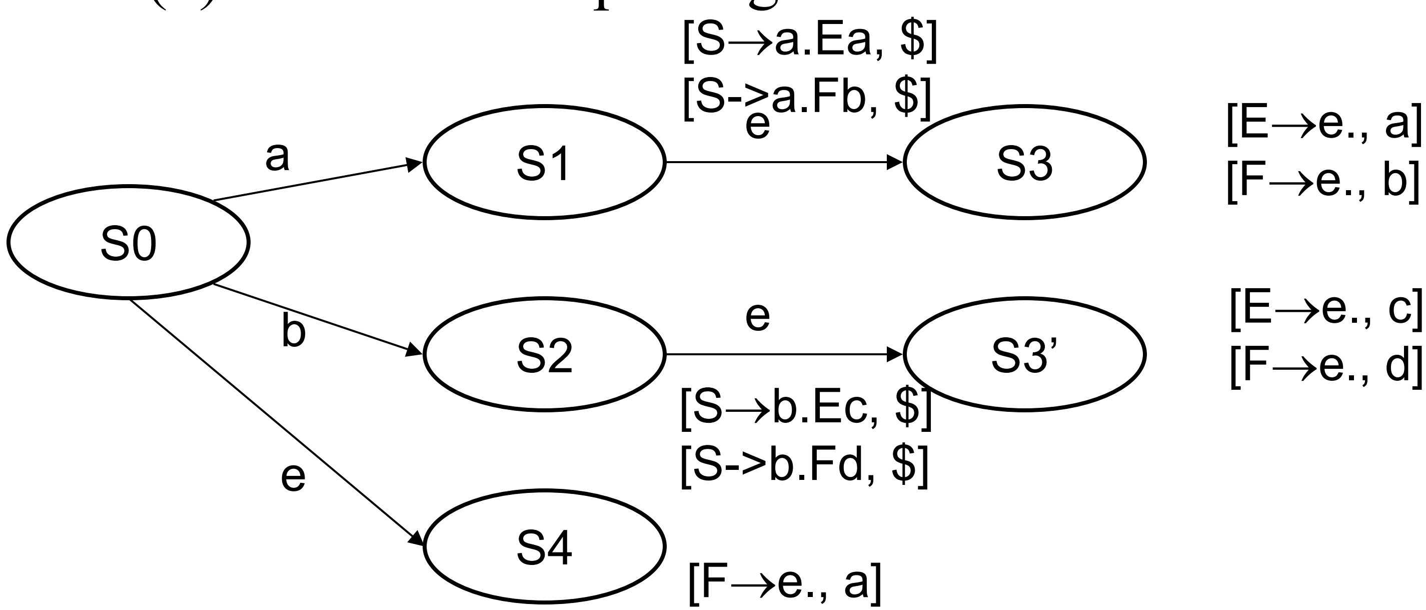


SLR(k) ⊂ LALR(k)

- Let's consider this Grammar G:
  - $S \rightarrow aEa \mid aFb \mid bEc \mid bFd \mid Fa$

    $E \rightarrow e$

    $F \rightarrow e$
- But LR(1) thanks to S3 splitting to S3 and S3'

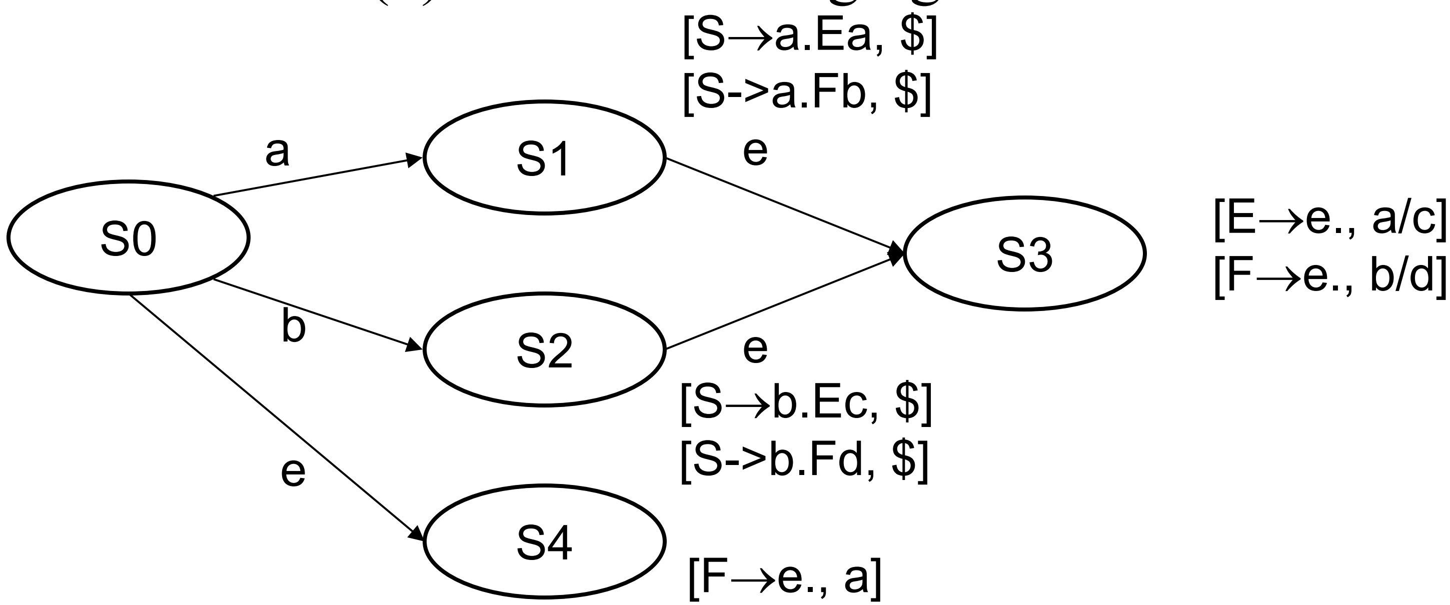


SLR(k) ⊂ LALR(k)

- Let's consider this Grammar G:
  - S→aEa | aFb | bEc | bFd | Fa

    E→e

    F→e
- And also LALR(1) even after merging back S3' and S3



LL(k) $\subset$ LR(k)

- LL(k) parser, each expansion $A \rightarrow \alpha$ is decided on the basis of
  - Current non-terminal at the top of the stack
    - Which LHS to produce
  - k terminals of lookahead at *beginning* of RHS
    - Must guess which RHS by peeking at first few terminals of RHS
- LR(k) parser, each reduction $A \rightarrow \alpha \bullet$ is decided on the basis of
  - RHS at the top of the stack
    - Can postpone choice of RHS until entire RHS is seen
    - Common left factor is okay – waits until entire RHS is seen anyway
    - Left recursion is okay – does not impede forming RHS for reduction
  - k terminals of lookahead *beyond* RHS
    - Can decide on RHS after looking at entire RHS plus lookahead

LL(k) != SLR(k)

- Neither is strictly more powerful than the other
- Advantage of SLR: can delay decision until entire RHS seen
  - LL must decide RHS with a few symbols of lookahead
- Disadvantage of SLR: lookahead applied out of context
  - Consider grammar: $S \rightarrow Bb \mid Cc \mid aBc$, $B \rightarrow \varepsilon$, $C \rightarrow \varepsilon$
  - Initial state $S_0 = \{S \rightarrow .\ Bb \mid .\ Cc \mid .\ aBc,\ B \rightarrow .,\ C \rightarrow .\}$
  - For SLR(1), reduce-reduce conflict on $B \rightarrow .$ and $C \rightarrow .$
    - Follow(B) = {b, **c**} and Follow(C) = {**c**}
  - For LL(1), no conflict
    - First(Bb) = {b}, First(Cc) = {c}, First(aBC) = {a}
- For the same reason, LL != LALR

$LL(0) \subset LR(0) \equiv LALR(0) \equiv SLR(0)$

- $LR(0) \equiv LALR(0) \equiv SLR(0)$
  - $LR(0) \equiv LALR(0) \equiv SLR(0)$ since lookahead is meaningless.
  - If a state has a reduce item, there can be no other items.
    (If there is, it will result in a conflict with the reduce action.)
  - This makes grammars very restrictive and unusable.

- $LL(0) \subset LR(0)$
  - LL(0) can only have one RHS per non-terminal to avoid conflict.
  - LR(0) can still have multiple RHSs per non-terminal.
  - E.g. $S \rightarrow a \mid b$ is not LL(0) but is LR(0).

L(GLR) ≡ L(CFG)

- GLR: Generalized LR parser where L(GLR) ≡ L(CFG)
  - "Parsing Techniques. A Practical Guide." by Grune et al. (2008) https://link.springer.com/book/10.1007/978-0-387-68954-8
  - An LR family parser that does the following on a conflict
    1. Fork the parse stack and follow each action separately
    2. If forked parse stack results in a parse error, discard it
  - Uses any LR table (e.g. SLR, LALR, Canonical LR)
  - GNU Bison: an implementation of GLR https://www.gnu.org/software/bison/

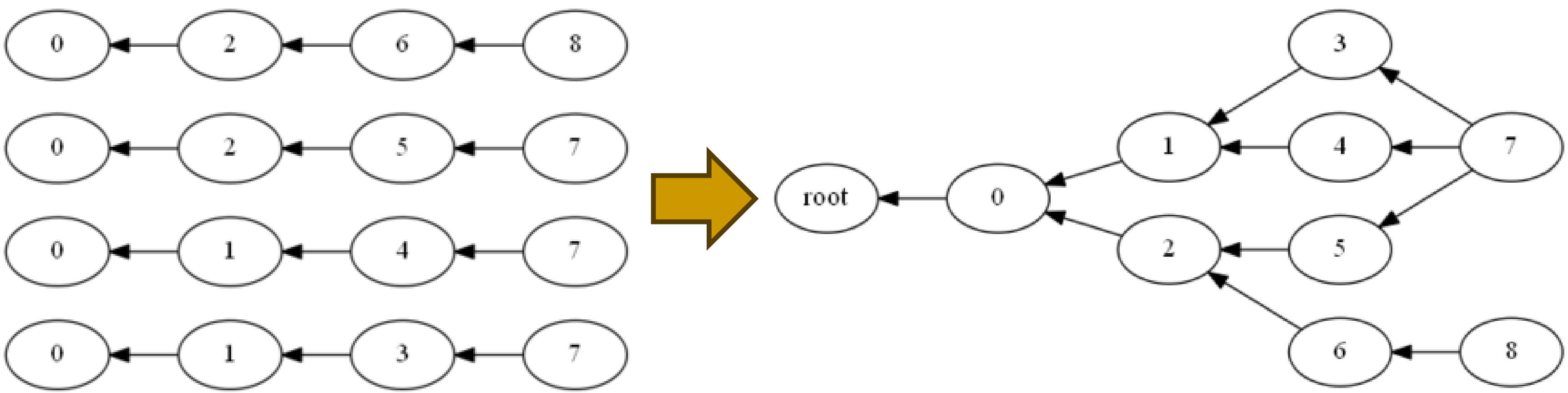
L(GLL) ≡ L(CFG)

- Is there a generalized LL parser that can parse all CFGs?
  - Recall, LL parsers have trouble with left-recursion
- GLL: Generalize LL parser
  - "GLL Parsing" by Scott et al. (2010) https://www.sciencedirect.com/science/article/pii/S1571066110001209
  - How does it deal with left-recursion?
    - Idea similar to GLR: fork stack on every conflict due to left-recursion (And try out all numbers of left-recursion until parse is successful)
    - Difference is, you can potentially end up with many more forked stacks
    - Developed "Graph Structured Stack" to minimize stack memory
  - GoGLL: an implementation of GLL https://github.com/goccmack/gogll

Graph Structured Stack

- “Graph-Structured Stack And Natural Language Parsing” *by* Tomita et al. (1988): https://aclanthology.org/P88-1031/

- Compresses below 4 parallel stacks into one graph:



Using Automatic Tools -- YACC

Pitt, CS 1622

Using a Parser Generator

- YACC is an LALR(1) parser generator
  - YACC: Yet Another Compiler-Compiler
- YACC constructs an LALR(1) table and reports an error when a table entry is multiply defined
  - A shift and a reduce – reports shift/reduce conflict
  - Multiple reduces – reports reduce/reduce conflict
  - Most conflicts are due to ambiguous grammars
  - Must resolve conflicts
    - By specifying associativity or precedence rules
    - By modifying the grammar
    - YACC outputs detail about where the conflict occurred (by default, in the file “y.output”)

Shift/Reduce Conflicts

- Typically due to precedence or associativity ambiguities
- Classic example: the dangling else

  $S \rightarrow$ if E then S | if E then S else S | OTHER

  will have DFA state containing

  [$S \rightarrow$ if E then S. , else]

  [$S \rightarrow$ if E then S. else S , else]

  so on 'else' we can shift or reduce
- Default (YACC, bison, etc.) behavior is to shift
  - Default behavior is the correct one in this case
  - Better not to rely on this and remove ambiguity

More Shift/Reduce Conflicts

- Consider the ambiguous grammar

$$E \to E+E \mid E*E \mid int$$

we will have the states containing

$$[E \to E* \,.\, E,\ +/*] \qquad [E \to E*E \,.\, ,\ +/*]$$

$$[E \to \,.\, E+E,\ +/*] \xRightarrow{E} [E \to E \,.\, +E,\ +/*]$$

$$\ldots \qquad\qquad \ldots$$

Again we have a shift/reduce conflict on input +

- In this case, we need to reduce (* is higher than +)
- Easy (better) solution: declare precedence rules for * and +
- Hard solution: rewrite grammar to be unambiguous

More Shift/Reduce Conflicts

- Declaring precedence and associativity in YACC

  %left '+' '-'

  %left '*' '/'

  - Interpretation:
    - +, -, *, / are left associative
    - +, - have lower precedence compared to *, /
      (associativity declarations are in the order of increasing precedence)
    - Precedence of a candidate rule for reduction is the precedence of the last terminal in that rule (e.g. For 'E→ E+E .' , level is same as '+')
  - Resolve shift/reduce conflict with a shift if:
    - No precedence declared for either rule or terminal
    - Input terminal has higher precedence than the rule
    - The precedence levels are the same and right associative

Use Precedence to Solve S/R Conflict

$$\begin{array}{lcl} [E \rightarrow E* \,.\, E, +/*] & & [E \rightarrow E*E \,.\, , +/*] \\ [E \rightarrow \,.\, E+E, +/*] & \overset{E}{\Rightarrow} & [E \rightarrow E. +E, +/*] \\ \ldots & & \ldots \end{array}$$

- we will choose reduce because precedence of rule $E \rightarrow E*E$ is higher than that of terminal +

$$\begin{array}{lcl} [E \rightarrow E+ \,.\, E, +/*] & & [E \rightarrow E+E \,.\, , +/*] \\ [E \rightarrow \,.\, E+E, +/*] & \overset{E}{\Rightarrow} & [E \rightarrow E. +E, +/*] \\ \ldots & & \ldots \end{array}$$

- we will choose reduce because $E \rightarrow E+E$ and + have the same precedence and + is left-associative

- Back to our dangling else example

  [S → if E then S. , else]

  [S → if E then S. else S, else]

  - Can also eliminate conflict by precedence declarations:
    %nonassoc ‘then’
    %nonassoc ‘else’

  - Perhaps less intuitive compared to arithmetic precedence
  - Use precedence only if it enhances readability of code

Reduce/Reduce Conflicts

- Due to ambiguity stemming from serious flaw in the grammar

- Example: a sequence of identifiers

  $S \rightarrow \varepsilon \mid id \mid id\ S$

  There are two rightmost derivations for the string 'id'

  $S \Rightarrow id$

  $S \Rightarrow id\ S \Rightarrow id$

  How does this ambiguous grammar confuse the parser?

Reduce/Reduce Conflicts

- Consider the states

$[S' \rightarrow .S\ ,\ \$]$		$[S \rightarrow id.\ ,\ \$]$
$[S \rightarrow .\ ,\ \$]$		$[S \rightarrow id.S,\ \$]$
$[S \rightarrow .\ id,\ \$]$	$\overset{id}{\Rightarrow}$	$[S \rightarrow .\ ,\ \$]$
$[S \rightarrow .\ id\ S,\ \$]$		$[S \rightarrow .\ id,\ \$]$
		$[S \rightarrow .\ id\ S,\ \$]$

Reduce/reduce conflict on input “id\$”

$$S' \Rightarrow S \Rightarrow id$$

$$S' \Rightarrow S \Rightarrow id\ S \Rightarrow id$$

Remove ambiguity by rewriting the grammar: $S \rightarrow \varepsilon \mid id\ S$

Semantic Actions

- Semantic actions are implemented for LR parsing
  - keep attributes on the semantic stack – parallel to the parse stack
    - on shift a, push attribute for a on semantic stack
    - on reduce $X \rightarrow \alpha$
      - pop attributes for α
      - compute attribute for X based on attributes for α
      - push it on the semantic stack

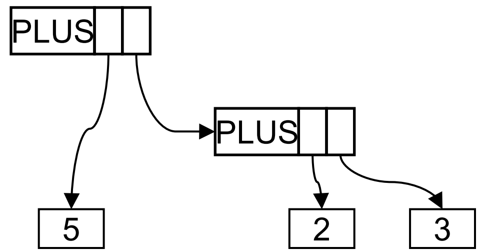
- Creating an AST
  - Bottom up
  - Create leaf node from attribute values of token(s) in RHS
  - Create internal node from subtree(s) passed on from RHS

Performing Semantic Actions

- Example 1: attribute is value of expression

$E \rightarrow T + E$	{$$ = $1 + $2;}
$\quad\mid\ T$	{$$ = $1;}
$T \rightarrow \text{int} * T$	{$$ = $1 * $2;}
$\quad\mid\ \text{int}$	{$$ = $1;}

consider the parsing of the string 3 * 5 + 8



- Example 2: attribute is AST for expression

$E \rightarrow \text{int}$	{$$ = mkleaf($1);}
$\quad\mid$ E+E	{$$ = mktree(plus, $1, $2);}
$\quad\mid$ (E)	{$$ = $1;}

  - a bottom-up evaluation of the ast attribute:

    E.ast = mktree(plus, mkleaf(5),
                mktree(plus, mkleaf(2), mkleaf(3)))